New ICMI Study Series

Celia Hoyles
Jean-Baptiste Lagrange
Editors

Mathematics Education and Technology-Rethinking the Terrain

The 17th ICMI Study

International Commission on Mathematical Instruction

Springer

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Chapter 1
Introduction

Celia Hoyles and Jean-Baptiste Lagrange

Abstract This chapter reports on the aims, organisation and outcomes of the 17th ICMI Study and in its final section provides an executive summary of the book as a whole.

1.1 Introduction

This book is the outcome of a decision by the ICMI Executive Committee (EC) in July 2002 to launch an ICMI Study, the 17th, to be called "Technology revisited". The title reflected the fact that the very first ICMI Study, held in Strasbourg in 1985, had focused on the influence of computers and informatics on mathematics and its teaching (Churchhouse 1986; Cornu and Ralston 1992). ICMI considered that the time was ripe to return to this theme. Part of its remit for ICMI Study 17 was to look at what had been achieved over the previous decades in terms of theory development as well as what had been the actual impact of technology on the teaching and learning of mathematics.

Consideration of the first ICMI Study provided a fruitful starting point for the International Programme Committee (IPC) for ICMI Study 17. Even a cursory glance revealed that this earlier Study had representatives from a restricted set of countries (Europe and North America). Additionally the focus of the papers was almost exclusively on using computers to model and explore rather advanced mathematical ideas, for example using "symbolic manipulators" in courses of calculus or linear algebra in order to allow students to focus on conceptual rather than procedural or technical issues. Many authors identified the potential of the systems they described, but several noted that there was little evidence of any

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significant impact on the mathematics curriculum of secondary schools and universities (primary-level mathematics was not considered at all).

The IPC for the new Study wanted to broaden the focus of concern and in particular were urged by ICMI explicitly to consider the situation of developing countries: how technology could be used for the benefit of these countries rather than serve as yet another source of disadvantage.This introductory chapter summarises the main points of the Discussion Document[1] for ICMI Study 17, the background and the challenges. It also provides a brief report of the Study Conference, held in Hanoi in December 2006. Finally, it presents an executive summary of the book.

1.2 Background and Challenges to ICMI study 17

Since the first ICMI Study in 1992, there have been major developments in digital technologies in terms of hardware: computers of all types, calculator and handheld technologies, digital technologies widely used in society at large such as mobile phones and digital cameras and of course the massive influence of the World Wide Web. Aligned to these hardware changes, new software have been developed with potential impact on all phases of education, and on informal contexts of education. By the time of ICMI study 17, digital technologies were becoming ever more ubiquitous and their influence touching most, if not all, education systems. In many countries, it is hard to conceive of a world without high-speed interactivity and connectivity.

These developments have spawned an increasing number and range of studies around the use of digital technologies in mathematics education, some focused on the impact of specific software, others looking more broadly at the interaction of teachers, students and technologies. The goals, objectives and orientations of these studies have shifted with a broadening of the perspectives, theoretical frameworks and methodologies adopted. Their outcomes are challenging for the mathematics education community. As Hoyles and Noss (2003) claimed: "there are major research issues for mathematics education that are shaping and being shaped by the issues confronting 'technologists'". In mathematics education, epistemological studies of mathematics and psychological approaches to learning mathematics have been extensively supplemented by investigations into teacher and classroom practices, but how far studies have taken on board the challenges of the use of digital technologies and their potential for the improvement of mathematics teaching, learning and the curriculum, remains a matter of debate.

This was the background of ICMI Study 17, which aimed to achieve a balance between two, potentially contradicting, aims:

- To reflect on actual uses of technology in mathematics education
- To address the range of hardware and software with the potential to impact upon or contribute to mathematics teaching and learning

[1] See the announcement in (Hoyles and Lagrange 2005).

This Study sought to identify and analyse some of the issues in mathematics teaching and learning, practically and theoretically, in the light of the use of digital technologies. Most digital technologies do not make explicit how they work or how they can be used in mathematics education. This means that taking account of their design, particularly in terms of implications for epistemology, is a central challenge. But, as we attempt to incorporate new technological tools into teaching and learning, we also intend to make progress in trying to understand how the related epistemological structures are mediated by learning communities, and reciprocally, how learning communities are shaped by the artefacts and technologies in use.

This Study also recognised the diversity in available software and hardware for use in mathematics education, but also considered the influences of diverse curricula organisations, from highly centralised to locally autonomous, and the availability of resources in different countries - whether this was access to handheld devices, computers or to the Web. It sought to take account of cultural diversity and how issues of culture alongside those related to teacher beliefs and practice all shape both the way digital technologies are used and their impact upon mathematics and its teaching and learning.

1.3 The Study Conference

The Study Conference for ICMI Study 17 took place in Hanoi, Vietnam and was hosted by the Hanoi Institute of Technology from 3 to 8 December 2006. Choosing to have this conference in Vietnam was for the IPC one way to ensure that the work of the Study would be sensitive to the question of cultural diversity and that the voices of peripheral countries would be heard.

Following the publication of the Discussion Document and the submission of papers around the themes identified, about 100 delegates were invited to participate in the conference with the following regional distribution: Africa 3; Asia 9; Australia, New Zealand 11; Central and South America 8; Europe and Russia 52; Middle Orient 9; USA and Canada 22.

Plenary keynotes were scheduled, one at the beginning and the other at the end of the conference. For the opening plenary we wanted the conference to be addressed by somebody with vision, experience and stature in the fields of mathematics, mathematics education and technology. We were delighted that Seymour Papert agreed to join the conference, speaking to the title *30 years of digital Technologies in Mathematics Education and the Future*. Using the recently prototyped “100 dollar laptop” renamed the “XO” to present his talk, Professor Papert argued that with full and easy access to computers, a new approach to mathematics education would be possible with particular benefits for developing countries. But he challenged us to consider that while it was important to consider how existing knowledge could be addressed in technology-enhanced ways, we should reserve at least 10% of our time and energy to consider what new types of mathematical knowledge and practices might emerge as a result of access to and effective use of digital technologies.

His accident the next day was a terrible shock to all participants, and the conference struggled to survive this loss, even as Professor Papert struggled in hospital. The best tribute we could think of was to try to keep the spirit of his ambition alive in the meeting by asking for participants to consider “Papert’s 10%” in all their sessions and in their subsequent papers: and we hope that the notion of Papert’s 10% will be the central idea that readers of this book take away with them.

Michèle Artigue, now president of ICMI, gave the concluding key-note using her experience as a researcher in the field. She recalled the evolution of the technological landscape and of research since the first ICMI study, mentioning especially how equity issues hardly mentioned twenty years ago now tend to be at the forefront, and pointed to the wealth of theoretical constructs that have emerged in order to address technological issues in mathematics education. She structured her lecture around five perspectives: theories, the teacher, curricula, design, and regions of the world.

Other plenary sessions were organised in the form of panels. Consistent with the Study’s special focus on cultural diversity, one panel was based on presentations from selected continents with the brief that they would show what was being put in place in their regions around the use of digital technologies in mathematics education. The themes of the two other panels were chosen to reflect two issues that we thought likely to make considerable impact in the future; namely the potential of connectivity to enhance teaching and learning mathematics and the crucial influence of design. Both issues were raised in the original Discussion Document but received less attention from the invited participants.

Working groups met in six sessions throughout the Study conference whose major aim was to prepare for this book. The participants were grouped on the basis of their written contributions. Initially seven themes had been prepared as set out in the Discussion Document (1) Mathematics and mathematical practices, (2) Learning and assessing mathematics with and through digital technologies, (3) Teachers and teaching, (4) Design of learning environments and curricula, (5) Implementation of curricula and classroom practice, (6) Access, equity and socio-cultural issues, (7) Connectivity and virtual networks for learning.

Two themes – Mathematics and mathematical practices, and connectivity and virtual networks for learning – had few contributors. Regarding the former, this is a clear difference with the first study, which was mainly concerned with the impact of technology on the mathematical practice. Regarding the latter, it reflects the fact that rather little research had been undertaken on this topic – the reason for the organisation of a plenary panel about the theme of connectivity. Given the number of contributions, two other themes important for the Study, implementation of curricula and classroom practice and access, equity and socio-cultural issues, were combined together into a single working group.

Discussions within each of the four themes were chaired by members of the IPC who would then take a leading role in four themes preparing the four main sections of this book. The proceedings of the Study Conference gathered all the prior written contributions of the invited participants and the abstracts of the plenaries. Thus its content is different from this book. These proceedings were published as a CD-Rom (Hoyles et al. 2006).

Following a request from the local committee for participation from Vietnamese teachers and consistent with the special focus of the Study on developing countries, the IPC decided that Vietnamese teachers would be able to attend all plenary sessions and parallel contributing talks and that activities would be organised for them while working groups addressed the themes of the Study. The workshop that was set up for this purpose was attended by 44 Vietnamese teachers, three teachers from Cambodia and two teachers from Thailand. Six sessions of three parallel laboratory activities were organised for the participants to the workshop (in addition to the two-hour session of parallel software presentation organised for all the participants to the conference). Laboratory sessions were devoted to the presentation and use of educational software.

The evaluation made by the participants of the workshop at the end of the conference was very positive. Beyond the interest of the academic activities per se, most participants pointed out that they had established invaluable contacts with teachers from different institutions all over the region, as well as with overall participants at the conference from the broad international spectrum.

1.4 Summary of the Book

The book starts with Chap. 1, a foreword by the co-chairs of the ICMI Study and the editors of the book. The book is then divided into five sections. Each section starts with a chapter of introduction and overview by the section editors, followed by several chapters that relate to theme of the section.

Section 1: Design of Learning Environments and Curricula

Chapter 2 introduces the first section of the book. The section editors present its purpose and its focus on the issues and challenges involved in designing mathematics learning environments that integrate digital technologies. They point to the complexity of the design process, given that the tools made available in these learning environments shape mathematical activity in ways that are not altogether predicable. In addition to considering the specific affordances and constraints of different digital technologies for structuring mathematical learning experiences, the chapter also considers the implications of design decisions on tools, curriculum, teaching and learning. The editors conclude that appreciating the interdependencies of tools, activities, pedagogies and learning outcomes and designing accordingly is a challenge that mathematics educators will continue to face as digital technologies evolve and extend their reach. It is hoped that the section will help designers and users to take up this challenge. In addition, the content of the section should serve to guide policy and curriculum-level decisions in the development and implementation of technology.

Chapter 3 considers how using digital technologies in “out of school” modelling projects can foster collaboration and motivation among students. The two case studies presented in the chapter illustrate the wide range of mathematical ideas that can be addressed in this way. The authors reflect upon the underlying theoretical principles that guided the design process in their work, in order that the use of technologies would optimise the chances of significant learning experiences for students. The chapter illuminates how different activity structures can take advantage of the affordances of digital technologies, especially through the design and building of animations and games. It also suggests how new representational infrastructures could bring changes to the mathematics content itself.

Chapter 4 focuses on design, this time through the lens of learning different geometries, including Euclidean 3D, co-ordinate geometry and non-Euclidean geometries. The authors stress the non trivial decisions that have to be made when designing for different geometries on the flat computer screen. They analyse the specific design decisions that motivate and differentiate common examples of these digital technologies. Such decisions are not only about the geometry but also about the learner in terms of supporting their perceptions of key features of geometry.

Chapter 5 deals with large-scale implementation projects, and brings to the fore issues in designing and deploying technology-based mathematical activities in different countries and jurisdictions. It analyses the varying beliefs represented in the implementation of projects in different cultures: theoretical frameworks, methodologies, ideas about mathematical literacy, and assumptions about the appropriateness of abilities and the willingness of teachers and students to engage in activities. The authors also note convergences especially towards the development of on-line environments and activities based upon widely accessible Dynamic Geometry environments.

Section 2: Learning and Assessing Mathematics with and Through Digital Technologies

Chapter 6 introduces the second section of the book, which focuses on developing understandings of how technologies can enhance the learning and teaching of mathematics, and the implications for assessment practices. The chapter emphasises some of the considerations elaborated in the chapters of the section, for example how digital technologies might be employed to open windows on learners’ developing knowledge, how interactions with digital tools mediate learning trajectories and the challenges involved in balancing the use of mental, paper-and-pencil and digital tools in both assessment and teaching activities.

The chapter ends with a short reflection on the section as a whole, noting that the major content emphases are algebra and geometry, suggesting therefore that more attention should be given in the future to for example calculus, statistical reasoning and proof. A closer relationship could also usefully be developed between mathematics education research and educational science in general.

Despite these reservations, it is hoped that the chapter presents a thorough synthesis of the research completed since the first ICMI study. As such it should prove invaluable to all mathematics educators whether or not their central interest is the use of digital technology.

In Chap. 7, the central question at stake is what theoretical frameworks are used in technology-related research in the domain of mathematics education and what do these different theoretical frames offer. The chapter first provides a historical overview of the development of theoretical frameworks that are considered to be relevant to the issue of integrating technological tools into mathematics education. Then some current developments are described, with a particular focus on instrumental approaches and semiotic mediation. While discussing future trends, the authors observe theoretical advancements, but also note that there is rather limited articulation of some of the different theoretical frameworks. They also point to what they see as factors that are given insufficient attention, for example the role of language in instrumental genesis, the role of the teacher in technology-rich learning environments, and the influence of the available tools on tasks and task design. They also note that connectivity, both among technologies and among theoretical frameworks, might be a key focus for future studies.

Chapter 8 focuses on mathematical knowledge and practices that result from access to digital technologies. It first describes how technology has influenced the contexts for learning mathematics, and the emergence of a new learning ecology. Second, the mathematical knowledge that "resides" within the different technologies is addressed, and third, changes in mathematical practices in education are considered. As a result of these analyses, the authors propose a transformation of the traditional didactic triangle into a didactic tetrahedron with the introduction of technology as a fourth vertex and conclude by restructuring this model so as to redefine the space in which new mathematical knowledge and practices can emerge.

Chapter 9 addresses the fact that with the significant development and use of digital technologies, diverse routes have opened for learners to construct and comprehend mathematical knowledge and to solve mathematical problems. The authors consider how the availability of digital technologies has allowed intended learning trajectories to be structured in particular forms and how these, coupled with the affordances of engaging mathematical tasks through digital pedagogical media, might shape the actual learning trajectories.

Chapter 10 addresses the issue of automatic assessment supported by digital technologies. Assessment is seen as a fundamental part of the learning cycle, central to learning and often a primary driver of students' activity. Significant technical developments of the last two decades are described through examples of internet-based systems. The authors stress the potential power of computer-aided assessment because of its immediacy and the mathematical sophistication of automatically generated feedback.

The last chapter of the section, Chap. 11, considers the relationships between research on the role of technology in mathematics education and the framework of social learning theories and suggests that social perspectives on teaching and learning with technology have become increasingly prevalent. Four typologies of digital technologies and their role in collaborative practice are identified: technologies designed for both mathematics and collaboration; technologies designed for mathematics; technologies designed for collaboration; and technologies designed for neither mathematics nor collaboration.

Section 3: Teachers and Technology

Chapter 12 introduces the section by noting that despite the fact that teachers have a central role in the mathematics classroom, they have been somewhat neglected players in research considering the relations between digital technologies and mathematics education and that those studies that do exist confirm that modifying teaching practices to include new tools is quite challenging.

The integration of any new artefact into a teaching situation can be expected to alter the situation's existing equilibrium and requires teachers to undergo a complex process of adaptation. In the case of digital technologies, the modifications required of routine practices are likely to be particularly pronounced. The goal of the section is then to give account of this complex reality and to take up the associated challenges by synthesising various research studies. It also aims to address implications of these issues for teacher professional development. It is suggested that by offering a synthesis of current research work, and a perspective of future developments on complex and crucial issues, this section will be of considerable interest for persons involved at every level of teacher management and education, as well as for all mathematics educators aware of the central role of the teacher in the implementation of technology.

Because pedagogical aspects associated with the use of digital tool in mathematics teachers are rather more complex than originally imagined, the need to involve teachers as partners has become increasingly evident. The authors of Chap. 13 assume that this complexity is linked to the fact that tools are a constituent part of culture, hence the introduction of new artefact necessarily involves the establishment of new cultural practices. The central argument permeating this chapter is the importance of forging partnerships with practising mathematics teachers. The focus of the partnership can be the design of learning activities involving the use of digital tools and/or the design of the digital tools themselves.

Chapter 14 points out that the actual take up of technology within mathematics classrooms has progressed rather slowly. Given this scenario, the authors focus on the search for theoretical frameworks that might illuminate the teacher's role in technology-integrated learning environments and clarify the factors that mediate teachers' use of digital technologies. Of the frameworks considered, particular attention is paid to two, the instrumental approach and Valsiner's zone theory.

In Chap. 15, the final chapter of this section, the authors turn their attention to teacher education course in mathematics and technology. They first note that course developers do not yet have access to a robust corpus of literature documenting strategies already tried and tested by others. They then offer a number of dimensions by which teacher education courses might be characterised. In terms of classifying the beliefs underpinning the courses analysed, three areas are considered: views related to the implementation of technology into teaching; beliefs associated with the impact on technology on teaching practices, the teacher's role and teaching activities; and views on how to prepare teachers. In addition, four strategies used by teacher educators across the five courses are identified and discussed.

Section 4: Implementation of Curricula: Issues of Access and Equity

Chapter 16, which introduces the section, starts from the observation that access to, and use of, digital technologies differs between countries, and within countries, according to socio-economic, gender and cultural factors. Thus, the influence and place of digital technology at all levels of mathematics education provides a unique opportunity to examine reform and change in mathematics curricula and teaching by, for example, examining the political, economic, social and cultural factors that promote or impede access to and integration of digital technologies for quality learning in mathematics. Thus chapters in this section seek to understand how cultural practices in technology-integrated mathematics enhance, or erode, equity and access in mathematics education.

In this chapter, the section editors first stress the necessity to distinguish the curricula as intended by authorities, from that which is actually implemented by teachers and from what is attained by the learners. They conclude that despite the fact that the possibilities that rapidly emerging digital and communication technologies can afford for mathematics, the evidence suggests that there will always be a lag between the development of "new mathematics" and its implementation in education systems. They also suggest that implementation of mathematics afforded by digital technologies is more likely to occur when and where there is a shared vision among political leaders, education authorities, mathematicians and mathematics teachers. The potential therefore exists for late but fast-developing countries to by-pass the curriculum experiments and out-dated technologies of earlier periods.

Given the rarity of comprehensive documentation and analysis about the implementation of curricula involving technology and associated issues of access and equity in countries of different economic capacity and cultural heritage, it is hoped that this section will be invaluable for decision makers and researchers at all levels from across the international spectrum.

The second chapter of this section, Chap. 17, provides useful information about the intended and implemented curricula of different countries and regions, including Russia, South Africa, China, Vietnam and several Latin-American nations. The differences in economic capacity and cultural heritage of these countries and of developed western nations is an opportunity to analyse the influence of social, economic, political and cultural factors on the integration of digital technologies in mathematics curricula.

In order to work to the goal that everyone can take advantage of technology-rich mathematics curricula, Chap. 18 begins by defining equity and access in the context of digital technology and mathematics education; that is, equitable distribution of resources, equitable pedagogies and equitable learning outcomes. The authors then synthesise research studies that focus on the role of gender and socio-economic differences in access to, and learning outcomes derived from technology-rich mathematics, and express principles regarding equitable pedagogies with respect to the use of digital technologies in mathematics. They also note a lack of studies involving students with special needs.

In the political, social and cultural factors influencing intended curricula, as well as the resistances to their implementation, the authors of the last chapter of this section, Chap. 19, distinguish those factors concerned with the development of technology literacy from those more focused on taking advantage of the capacity of digital technologies to support or enhance mathematics teaching and learning. All these factors may differ from one level to another and from one culture to another, depending on the role that society assigns to mathematics teaching. They observe that system-level curricula change that involves the integration of digital technologies in high-stakes assessment was more likely to result in widespread implementation.

Section 5: Future Directions

Completing the comprehensive synthesis of the potential and impact of digital technologies and mathematics teaching and learning provided by the first four sections, this section that looks to the future is well placed to conclude the book.

In the introductory chapter (Chap. 20), the editors of this section simply indicate that the three chapters of the section were derived from the plenary sessions at the Study conference, namely the plenary panels and lecture.[2]

Chapter 21 is derived from the panel about design for transformative practices and returns to the issue of design already addressed in Sect. 1. Creators and designers of well-established and widely used software environments were invited to give panel presentations about their own unique expertise of designing and building their environments. Their contributions in this chapter throw light on the type of design decisions that have to be made, how these decisions connect with visions of teaching and learning, and how they can give rise to changes in practice and in future designs.

Chapter 22 is derived from another set of invited panellists who were asked to present[3] their views and experience of the role of connectivity and virtual networks for learning mathematics. This area was thought to have a strong impact upon mathematics education in the future but at the time of the Study conference

[2] We are unfortunately unable to produce a text of the second plenary lecture by Seymour Papert given his tragic accident.

[3] One team participated through a video link.

there was rather little research or practice on which to build. The contributions derived from experiments that took advantage of connectivity within one classroom or across classrooms. As the introduction states, while there is no doubt that connectivity will transform how students interact with each other, yet if and how connectivity, in whatever form, transforms mathematical practices in school is a matter of future investigation.

The final chapter of the book, Chap. 23, is derived from the closing plenary address of Michèle Artigue, President of ICMI. She takes advantage of her personal experience of analysing the evolution of relationship between digital technologies and mathematics education over the last two decades, and situates her reflections about the future in its historical perspective. Then, she focuses on dimensions that she suggests are crucial for thinking about the future: that is dimensions that concern theories, the teacher, the curriculum, and issues of design and equity. She stresses that the reflection issued of the ICMI Study should assist the community in thinking about what educators can do in order to ensure that digital technologies better serve the cause of mathematics education.

1.5 Conclusion

Twenty years after the synthesis brought by the first ICMI study, the scenery of digital technology in mathematics education has radically changed and we look forward to still more dramatic changes. We hope that this book will help mathematics educators to take-up the challenges that technology will continue to bring about, many of which cannot be predicted at this point. “Rethinking the terrain” was urgent. The work could not have been done without the contributions and work of the many experts who participated in the Study conference or are authors of this book. Prominent among these experts is of course Seymour Papert who taught us how reflection on the past can support visions for the future. This book is gratefully dedicated to him.

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Section I
Design of Learning Environments and Curricula

Chapter 2
Introduction to Section 1

Celia Hoyles and Nathalie Sinclair

Abstract In this introduction, we provide a brief overview of the goals of the Design of Learning Environments and Curricula working group. We then describe the three chapters that follow this introduction, and that emerged from the working group discussions, each of which focus on a different aspect of designing environments for mathematics learning, namely, consideration of the mathematics itself (and how it might change with the digital technology), the learner (and her different psychological and cognitive approaches to that content might change) and the curriculum (and its specific exigencies and opportunities).

The purpose of this theme was to focus closely on the issues and challenges that need to be faced in designing mathematics learning environments that integrate digital technologies. We recognised, and wanted to emphasise, that the tools made available in such learning environments shape the mathematical knowledge involved – and sometimes in unpredictable ways. We also wanted to focus on the ways in which the use of digital technologies shape the practices of teaching and the modalities of learning. In addition to considering the specific affordances and constraints of different digital technologies for structuring mathematical learning experiences (including various software packages, hardware configurations and web-based programs), this theme considered the implications of specific design decisions for curriculum, teaching and learning.

We proposed the following questions, which were used to guide the submission of proposals from researchers as well as the discussion sessions at the conference:

1. What theoretical frameworks and methodologies are helpful in understanding how design issues impact upon the teaching and learning of mathematics?

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2. How does the use of different technology-integrated environments both influence the learning of different mathematical concepts and shape the trajectories through which the learning develops?
3. How can technology-integrated environments be designed so as to foster significant mathematical thinking and learning opportunities for students?
4. What kinds of mathematical activities might different digital technologies afford and how can learning experiences (including the tools, the tasks and the settings) be designed to take advantage of these affordances?
5. How can technology-integrated learning environments be designed so as to influence and change curriculum, and how can this be achieved consistently over time?
6. How can technology-integrated learning environments be designed so as to remain sensitive to persistent challenges, for example, swift and inevitable obsolescence and ongoing maintenance costs?
7. How are new types of technology-mediated mathematical knowledge and practices related to current classroom curricula and values, and how should aspects of mathematics curriculum therefore be removed or changed?

Of the papers submitted to this theme, we selected for discussion at the Study Conference ten that most directly attended to the questions listed above (see the *Proceedings* for a complete list). At the conference itself, in order to structure our discussions we split the participants into three groups. Given the principle design interest of Theme 4, the groups were created in order to focus closely on the three separate, but overlapping components involved in the design of digital technologies: the mathematics, the learner, and the curriculum.

Given the strong representation of geometry-based software, the mathematics content group (including Keith Jones, Kate Mackrell and Ian Stevenson) focussed specifically on digital technologies related to geometry, and chose to tackle questions (2) and (4). In particular, they have written about the design of digital technologies for the learning of "different" geometries, including Euclidean 3D, co-ordinate geometry and non-Euclidean geometries. They discuss the specific design decisions that are involved in representing these geometries on the *flat* computer screen, and consider both mathematical and pedagogical implications of these decisions. They also examine the ways in which previous design principles used in dynamic geometry environment have influenced the development of software and of tasks in 3D environments such as *Cabri 3D* and *Autograph* (version 3). They also respond to question (7) above by pointing to ways in which digital technologies might change the knowledge of school mathematics, both in extending it to include different branches of mathematics, and in linking it, through other learning technologies – rather than through the more common mathematics hierarchy – to new mathematical ideas. Finally, they point to a number of design decisions affecting student understanding of different geometries through digital technologies that are frequently overlooked but that are in need of further research. Given the growing privileging of numbers and algebra in school mathematics, especially in North America, this chapter will be crucial in guiding policy- and curriculum-level decisions about the role of geometry, and especially 3d geometry, in developing students' visual and spatial intuitions.

In terms of the learner component, we formed a second group that focussed on ways in which digital technologies can motivate student mathematical learning through long-term engagement in collaborative modelling projects. The authors of Chap. 4 illustrate this type of learner-focussed interaction through two case studies of interactive microworlds: *Lunar Landing* (which involves the creation of a computer-based games) and *Graphs 'n Glyphs* (which involves the creation of animations), both of which target a relatively wide range of mathematical ideas for 11–14 year old learners. The authors compare their approaches in terms of question (1), that is, in terms of the underlying theoretical principles guiding the design and motivation for their work. The chapter also addresses questions (3) by reflecting on how digital technologies can foster significant learning experiences for students (including both the affective and cognitive dimensions of learning). They discuss some of the specific design choices involved in maintaining student engagement in the modelling environment while also maximising opportunities for mathematical expressiveness. This chapter also addresses question (7), by providing two examples of digital technologies that follow Papert's "10% principle" in offering new forms of knowledge for school mathematics.

Our third group gathered together researchers involved in large-scale implementation projects, and brought to the fore issues in designing and deploying technology-based mathematical activities in different countries and jurisdictions. Questions (5)–(7) were prime motivators in guiding the discussion at the conference as well as the writing of the chapter. Of special interest were the varying beliefs reflected by the range of cultures represented in the implementation projects in terms of theoretical frameworks, methodologies, ideals about mathematical literacy, and assumptions about the abilities and willingness of teachers and students. The authors discuss three main themes that emerged from these large-scale implementation projects, and that reflected their conservative approach: the increased focus on the teacher, the high fidelity to existing curriculum content, and the relatively "closed" nature of tasks made available to learners. The chapter discusses these themes within the broader evolution of digital technologies in mathematics education and offers predictions on some of the trends that might arise in the future.

The main thread of the three chapters, from software design around a specific content area, to design considerations around specific learning objectives, to implementation considerations at different scales, allowed us to address most of the questions posed in the discussion document of the ICMI Study and to offer some reflections on important issues in the design of digital technologies for mathematics education. We acknowledge that in the interests of our agreed foci we, of necessity, had to leave out some of the insights reported in the individual papers submitted to Theme 4 and we hope that interested readers will refer for the conference proceedings for more detailed reading. Appreciating the interdependencies of tools, activities, pedagogies and learning outcomes and designing accordingly is a challenge we will continue to face as digital technologies continue to evolve.

Chapter 3
Designing Software for Mathematical Engagement through Modeling

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Abstract The framing theory guiding the work described here is that mathematics learning is facilitated through long-term student engagement in collaborative projects, integration of sustained emphasis on content knowledge, deep engagement of student interests, and support for student experience and progress, and commitment to learning through interactive microworlds that foster modeling and collaboration. We describe two case studies of software design/implementation, one an animation environment, and the other a game and game-design microworld. We describe each case in some detail, and compare the projects' affordances, constraints, and design lessons, and persisting challenges.

Keywords Design • Pedagogical • Epistemological • Bidirectionality • Animation • Momentum • Game • Meta-game • Engagement • Layered design • Professional transitional software

3.1 Introduction

Students are typically told that they must study mathematics in order to keep open their options to pursue quantitatively-oriented careers in mathematics, science, technology, or engineering. For most of them, this is a very distant and abstract motivation, especially for students whose familial network does not include members who currently engage in such work. Indeed, it is estimated that only 10% of students in the United States complete the prerequisites necessary to take Calculus (Roschelle et al. 2000),

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and similarly government statistics for England show that in 2006 only 7%, of all 16-year-olds who had chosen to continue to study to advanced level post-16, chose mathematics. These data provide evidence that these long-term motivational statements are rather unsuccessful in convincing students to persist with mathematics.

Yet, these same students live in a world permeated by technology – the internet, satellite communications and mobile phones. Their lives are managed by numerous technological systems, many completely invisible (transportation, finance and loans, manufacturing, demographics, medicine, and so on (see, for example, Noss et al. (2007)). In order to secure a middle-class income, students must be competent in the use of these new technologies (Murnane and Levy 1998) who refer to this need for competence as a *key technology-knowledge gap*. It is especially ironic in that those countries with the most access to the products of these revolutions often demonstrate the least progress in developing the underlying necessary student proficiencies.

The importance of tapping into youth culture should not be underestimated in motivating and sustaining student educational progress. This is especially true for subjects like science and mathematics, which carry considerable social capital yet are easy for students to dismiss as irrelevant, boring and hard in a world of digital images, animations, easy information retrieval and communication. We need engaging environments, in which the mathematics is actually *needed* for students to achieve goals that *they* find compelling, and *made visible* to students and *expressed in a language* with which they can connect.

Our starting points derive from the position that effective student learning of mathematics is facilitated through: (a) long-term engagement in collaborative projects for which they take individual and collective responsibility (e.g. Harel and Papert 1991), and "tools [that] approach students from an angle that seems interesting and relevant to them" (Rosas et al. 2003); (b) integration of sustained emphasis on content knowledge, deep engagement of student interests, and support for student experience and progress (Jolly et al. 2004); and (c) commitment to learning through interactive microworlds that foster modeling and collaboration.

Modeling, approached in this way, promotes the learning of powerful mathematical ideas through use, in contrast to the conventional way in which mathematics is presented (Papert's Power Principle (Papert 1996)). We build on our preceding research on modeling and the role of sharing and critiquing models (see for example Noss and Hoyles (2006), Simpson et al. (2005), and Ainley et al. (2006) on how modeling places emphasis on the *utility* of a mathematical concept), and draw on the definition of modeling by Confrey and Maloney (2007b):

> Mathematical modeling is the process of encountering an indeterminate situation, problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear to transform the situation. The modeling produces an outcome – a model – which is a description or a representation of the situation, drawn from the mathematical disciplines, in relation to the person's experience, which itself has changed through the modeling process. (p. 60).

Further, we emphasize the importance for accessibility of what we term *layering* of mathematical and scientific principles and abstraction, and embedding increasing problem-solving complexity into the software.

In this chapter, we present two case studies that followed different paths from this common basis. Both adopted similar methodologies with a commitment to iterative design, testing and feedback in multidisciplinary teams. An inevitable consequence of the current curriculum and high-stakes testing environment was that both case studies took place outside of school. One case study focuses on making animations in an environment called *Graphs 'n Glyphs*; the other on designing and developing a game-construction kit around the theme of *Lunar Lander*. The two projects differed in the degree of explicitness of the mathematics inherent in the software, the extent to which a curriculum was planned to support the software, and the long term goals for the technology in relation to schools and curriculum. For example, there was no intention in the case of the UK-based Lunar Lander to 'fit' the game into the curriculum[1]; while the US team envisioned Graphs 'n Glyphs as a software-integrated curriculum unit ultimately for use in the classroom and within programs of teacher professional development in mathematics and technology.

3.2 Case Study 1: Graphs 'n Glyphs: Animation Software for Mathematics Learning

3.2.1 Aims and Description of the Software

Building on experience with the development of *Function Probe* (Confrey and Maloney 2007a) and *Interactive Diagrams* (Confrey et al. 1999; Confrey and Maloney 1999) the Graphs 'n Glyphs animation software project (Confrey and Maloney 2006) took shape from the team's research interests in issues of student learning of both rational numbers and fundamentals of trigonometric reasoning. Using the software, students can model objects and motion in two-dimensional space and also (in the future) from the three-dimensional world in two-dimensional space.

We recognized that students' cultural and economic surroundings are saturated with digital technology, graphics, animation, and video, all displayed in two-dimensional space. The technology and the imaging seem to be compelling, but few understand the mathematics and science that underlie graphics and animation. We developed an animation microworld, Graphs 'n Glyphs, as a model environment for fusing the goals of generating learning research and developing a successful mathematics educational program that takes particular account of the educational needs of at-risk students.

Graphs 'n Glyphs seeks to engage students with a technology while concomitantly teaching content that prepares them for advanced mathematics (i.e. algebra, geometry, and trigonometry). It has the potential to break new ground in the use of an innovative integration of modeling and animation in the context of mathematics instruction, and in investigating the use of an animation environment for mathematical learning. The environment is designed to introduce students to the ways in which computer animations are produced and to permit them to create, edit, and share their animations.

[1] In fact the Lunar Lander was designed to form part of materials within the BBC Jam initiative.

Thus, through this software and project, we invite them to participate in a compelling animation microworld while making the underlying mathematical and computational elements visible and comprehensible. In doing so, we aim to teach students the fundamental mathematical ideas of integer and rational number operations, similarity and scaling, coordinate graphing and tables, basic geometric concepts, transformations, and ratio reasoning. Other learning goals include angles, elementary trigonometry, percents, and decimals. The context of animation provides opportunities to strengthen and connect students' numerical and geometric knowledge, and to build on the foundations established in early childhood (reference the report synthesizing the studies by Clements (2004)). When fully developed[2], the software environment will also facilitate students' learning about optics and acoustics – the science as well as the geometry – and trigonometry-based modeling that underlies three-dimensional images, animations, and their accompanying sound.

3.2.2 Main Issues in Software Design

Theoretical/research base. The software design draws on four major thematic approaches from mathematics learning: (a) modeling, (b) project-based instruction, (c) learning progressions, and (d) microworlds. The work then extends these four theoretical themes by linking the software directly to the functionality and design of software used professionally for animation and graphics. In this way, the work draws upon the study of communities of practice (Lave and Wenger 2002) and on how their practices can be useful to draw students into well-paid careers (Hall 1999).

The underlying philosophy behind the work is the theme of modeling (as mentioned in Sect. 3.1) through the development and revision of inscriptions (graphics, tables, transformation records; Latour 1990) that permit one to render graphical and acoustic animations on the computer. We also build on the work of Lehrer and Schauble (2000, 2006) that conceives of students learning via a continuum of models from physical microcosms to hypothetical-deductive. Our intervention consists of a combination of elements on and off the computer; we emphasize the importance of building different levels of abstraction into the software (Lehrer et al. 2002). The activities and tasks form a learning progression akin to learning trajectories described by Confrey (1990), Simon (1995), Gravemeijer et al. (2004), and Clements and Sarama (2004), and conceptual corridors as described by Confrey (2006).

These elements are drawn together via the concept of microworlds defined first by Papert (1980) and extended by Weir (1987) and then Hoyles et al. (1991). In Graphs 'n Glyphs, we draw upon the definition of Microworld, changed from "teaching computers to solve problems" to "designing learning environments for the appropriation of knowledge and, as a consequence of this change in

[2]Three levels of the software are planned, if funding permits. The first is described herein; the second adds 3D perspective and lighting and the third adds in acoustics and sound.

focus, the transitional object takes on a central role" (Hoyles et al. 1991, p. 2). This signals a shift from the traditional direct-learning paradigm to one that is steeped in experiential learning. As Miller et al. (1999) explains, "In contrast to more traditional educational strategies that try to teach the target knowledge to the student directly, learning by exploration focuses on stimulating the student's initiative in gaining knowledge about the domain. Because microworlds both support exploration and behave according to the laws and constraints of the subject-matter domain, educators believe that students' activities in the microworld produce or foster education about the domain" (p. 305). In Graphs 'n Glyphs, students explore the rich environment of animation and use it as a vehicle for mathematical inquiry. By working through the activities and tasks, they begin to mathematize environments, distance, similarity, scaling, and slope.

Collaborative design. Graphs 'n Glyphs was collaboratively designed by a team of mathematics educators, game and graphics designers, and a programmer, in order to accomplish several goals at once. We required the software to incorporate authentic professional animation software features, and to encompass specified mathematical content, while being robustly learner-centered. These three design goals co-defined each other during the iterations of design, implementation, and revision of the software.

We endeavoured to design software that would be as similar as practicable to professional animation, graphical, and video editing tools such as Adobe Photoshop, Macromedia Freehand, Macromedia Flash, Apple iMovie and 3D Studio Max. One of our aims was to represent this software to student users as software that would prepare them for using professional tools. Whenever possible, the graphical tools, object-manipulation metaphors, commands, file management conventions, and timeline behavior are similar to those used in professional software. At the same time, we sought to ensure that the mathematics underlying such tools and behaviors, instead of invisible as usual behind user-friendly graphical user interfaces, would in this instance be visible to students and, furthermore, would be *required* to construct the animations.

Features of modeling and animation, mathematical content, and learner-centereddesign. The software is designed to allow users to begin to represent 3D objects in motion in 2D screen space. They build objects as sequences of connected points, and animate the images via on-screen mathematical transformations (translation, reflection, rotation, and both aspect preserving and non-preserving scaling).

Students engage in aspects of modeling the 3D world on a 2D space by:

- Creating, modifying, and replicating objects on the 2D plane, using a separate "local coordinate plane" on which objects are constructed, and the "global coordinate plane" on which to place each object and its own local plane.
- Moving the objects by specific transformations on the plane.
- Sequencing the transformations into smooth motion via a timeline.
- Interpreting visual feedback as a form of interactivity.
- Telling a story with the animation, and using visual representations.

Mathematical topics incorporated into the software and accompanying curricular units include:

- Measurement.
- Addition and subtraction of positive and negative integers.
- Coordinate graphing.
- Transformations: translation, rotation, scaling, stretching, reflection.
- Ratio, similarity, and scaling.

Learner-centered design components include:

- Multiple simultaneous representations (graphs, tables, and timeline).
- Movement between representations that can be configured either bidirectionally or unidirectionally.
- Adaptations of professional software features to serve the development by students of conceptual content (for instance, coordinate point representations, pivot point on the local coordinate plane, and scaling "handles").

3.2.3 Notable Characteristics of the Software

The software interface comprises four primary windows and a graphical display for the animations (see Fig. 3.1).

Students build objects in a graphing window that consists of a local (one for each object) and a global plane (on which each object is positioned and moves). Tables

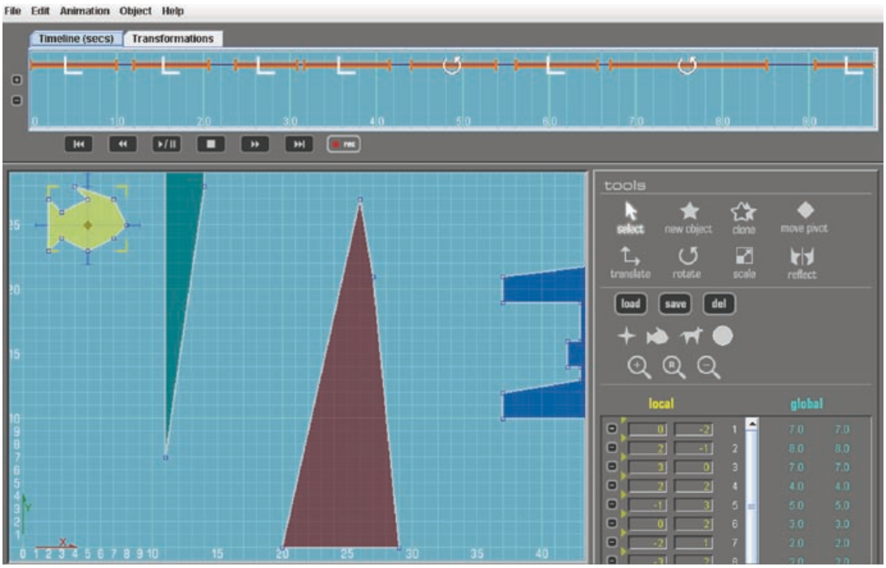

Fig. 3.1 Graphs 'n Glyphs user interface

display the coordinate-pair values of each object's points on both the local and the global planes (and in which they can add or remove points from the object definition). The object palette permits saving and reproducing objects, transformation tools for visually transforming objects and mathematically specifying the transformations. The timeline or sequencer permits students to organize, edit, and run their animation sequences of transformations. The windows are dynamically linked and support predictions, data gathering, and feedback. Visual and numerical feedback, embedded in the multiple representations, facilitates student assessment of their own progress on the various curricular activities. Students who feel they have not mastered certain concepts can choose to keep the detailed feedback available even as they move on to more advanced tasks.

Perhaps the most exciting aspects of the software for students in our 2006 study were the transformation tools and the timeline. Graphs 'n Glyphs incorporates two modes for transformations. One, which we call the "doodle" mode, involves selecting a transformation from the tools palette, and visually carrying out (drag-and-drop) transformations. Doodle mode allows students to conduct visual experiments in transformational motion and values. The values of a transformation are displayed at the right side of the tool area. For example, in Fig. 3.2, the dog has been translated eight units in the positive x direction, and two units in the positive y direction.

Then, to build an animation, the student uses the animation window located above the graph pane. The student specifies the values of each transformation, each of which is recorded within an animation sequence. To build such transformations, the student chooses to insert a new transformation, and, from a dialog box, selects (a) the type of transformation, (b) the object to which it will be applied and (c) the values of the transformation. For example, a student may want to show a dog rearing on its hind legs. Figure 3.3a illustrates how a student could specify moving the dog's pivot point from the center of the dog to its hind foot. An object's pivot point defines the origin of the local plane and is also the point of invariance for scaling and rotation. This is, essentially, a translation of the pivot point by negative two in the x direction and by negative four in the y direction. Figure 3.3b illustrates a clockwise rotation of 50° for the dog, which the student might specify after moving the pivot.

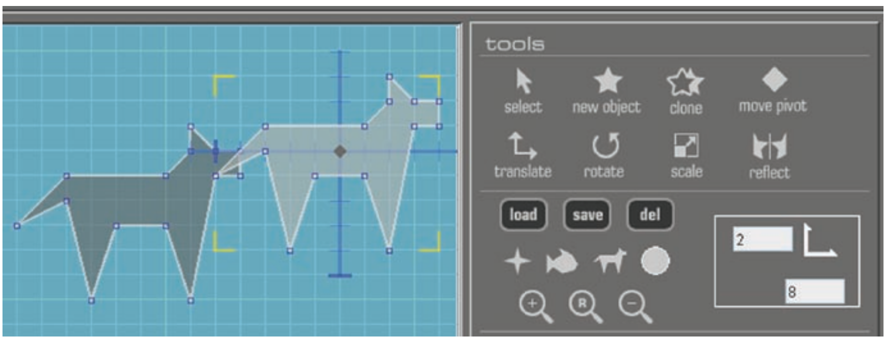

Fig. 3.2 Translation of an object (dog), in "doodle" mode

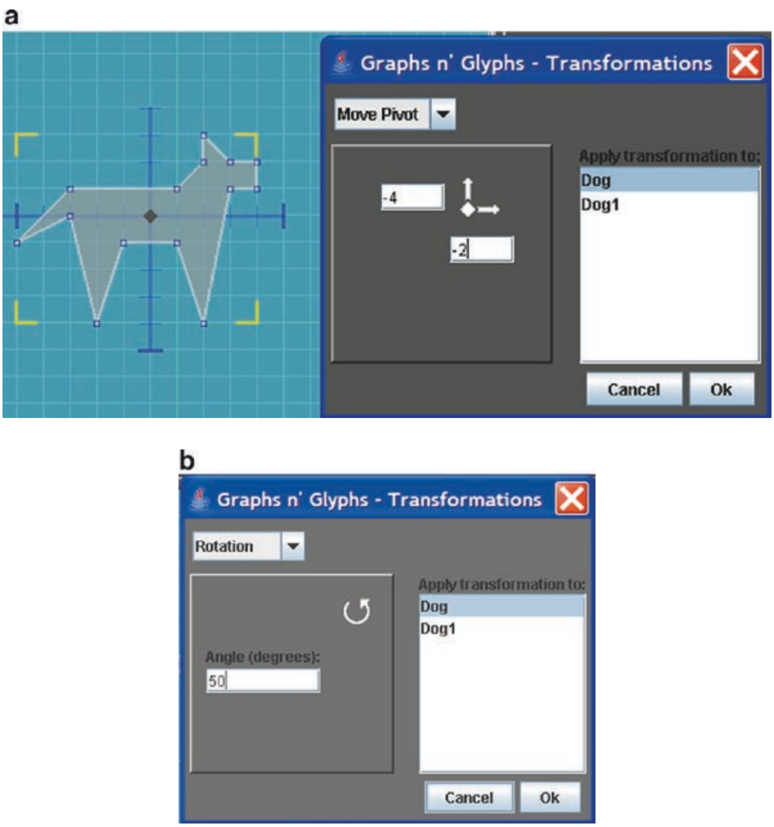

Fig. 3.3 **a** Move-pivot transformation. **b** Rotation

Once a transformation record has been built (Fig. 3.4a), the user views and modifies the transformations in the timeline tool (Fig. 3.4b), a transformation sequencer in which transformations can now be run as smoothly-sequenced animations with standard video-control buttons. Their order, duration, and speed can be changed by simple drag-and-drop; double-clicking re-opens the dialog box for editing of numerical values.

Figure 3.5 shows the timeline of the transformation from Fig. 3.3, and the final state of the dog. Note the final position of the dog's pivot point.

Scaling and distorting an object are illustrated in Fig. 3.6. To make the objects behave as desired, the pivot point's position again becomes a critical animation element. In doodle mode, an object is scaled by manipulating the handles on the scaling frame. To stretch an object with relation to the local *x*- or the *y*-axis, one drags the side or top handles. To scale (preserving aspect ratio), one drags one of the corner handles (Fig. 3.6a). However, when students build an animation sequence, they must use the transformation timeline and must specify each transformation numerically. In the transformation dialog box, the corner entry field specifies the aspect-preserving (scaling) transformation (shown in Fig. 3.6b), and the other two entry fields specify stretches in relation to the two axes.

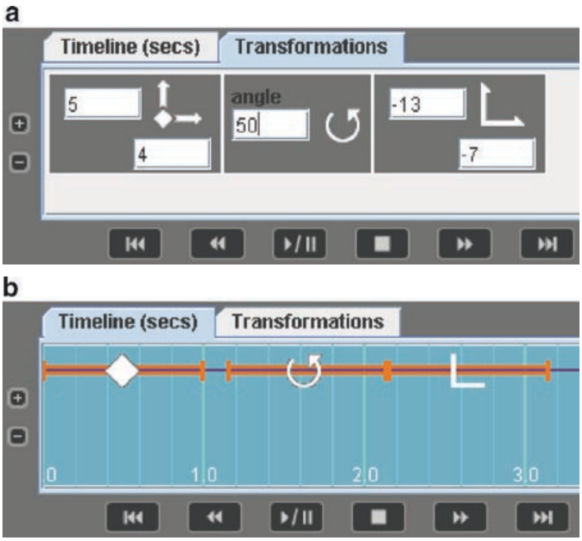

Fig. 3.4 **a** Transformation record, showing three different transformations: a move-pivot, a rotation, and a translation. **b** Same transformations in the Timeline; transformations can be re-ordered, shortened or lengthened (drag and drop)

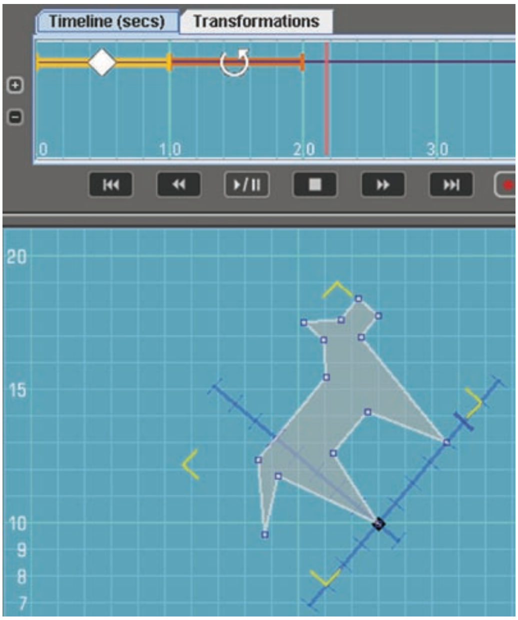

Fig. 3.5 Simple animation of the dog

Software and Curriculum Integration: The design study conducted in summer 2006 involved a group of rising sixth graders (students who had completed fifth grade). Curriculum units were designed to reinforce students' notions of addition, subtraction, and measurement on the number line, and introduce them to Cartesian

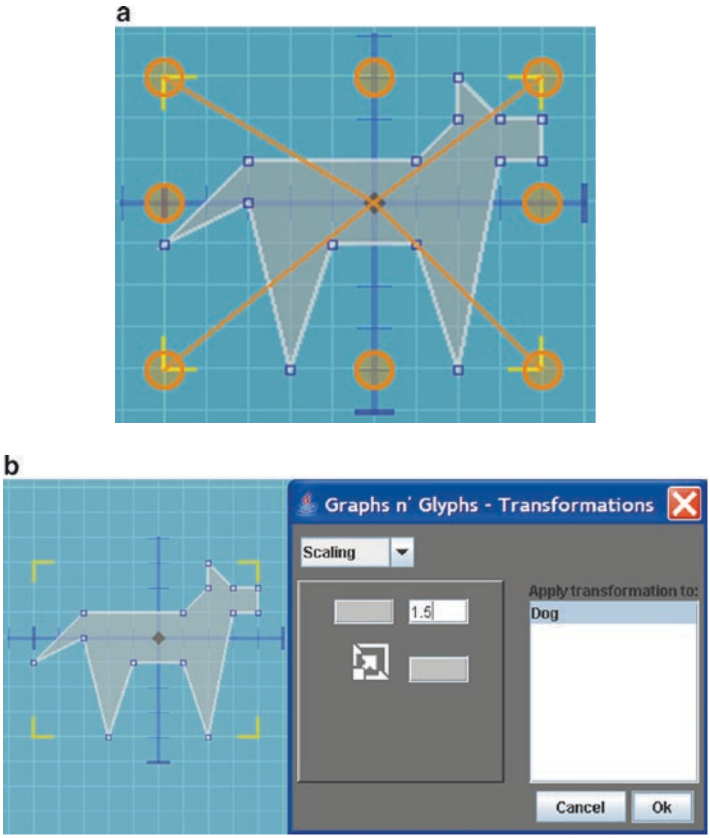

Fig. 3.6 **a** Handles for visual scaling and stretching. **b** Transformation dialog box for scaling or stretching an object

graphing and geometric transformations. Hands-on and paper-and-pencil activities and formative assessments were used during the student workshop. Software activities were carried out in parallel with the mathematical instruction. These included building objects on the Cartesian plane (specifying points, vertical and horizontal distances), taxicab geometry (Krause 1975), matching exercises as instruction in transformations, mazes to reinforce numerical specification of transformation sequences, and, finally, creating their own objects and animations.

3.2.4 *Major Achievements*

Design Study. The 3-week design study in summer 2006 led us to conclude that the animation, modeling, and mathematics instruction that are combined in Graphs 'n Glyphs (GnG) have considerable potential for enhancing student engagement in mathematics learning as well as deepening their understanding of fundamental principles of rational number, graphing, and trigonometry. Several outcomes stand out. (1) Student engagement and persistence was far beyond expectations. By day 3, students were arriving early and staying late, and staying on task for hours at a time. Part of the excitement for the students was their use of the computers. They checked out laptops overnight and used them to connect to the internet themselves at home, show parents their work, and explore online games. (2) In the actual hands-on and computer-based animation activities, the overriding important factor for the children was the story they tried to tell. The creative narrative expression was highly engaging for the children. (3) Most of the students were very weak in addition and subtraction, were unfamiliar with negative numbers, had virtually no concept of subtraction as a way to measure distance. All made progress on these concepts and skills. (4) By the end of the study, the students understood the definitions of the geometric transformations they were using, could work with them on the computer, and had become fluent in verbally specifying and incorporating transformations as they planned their animations, and could describe them to an audience in the final workshop presentations. (5) Subsequent feedback from parents suggests that many of the children were more confident and interested in mathematics at school, and felt that what they learned in the workshop has helped them in the next year's mathematics classes.

Professional Transitional Software (Fig. 3.7). As the project progressed, we explicitly developed the concept of "professional transitional software," of which Graphs 'n Glyphs is an example: learner-centered software in the service of particular educational content (such as mathematics), which incorporates interface design features and functionality that are typical of software used in particular types of professions. The content in the Graphs 'n Glyphs project will eventually comprise algebraic and trigonometric-reasoning fundamentals and more explicit modeling curriculum trajectories. The software also incorporates fundamental metaphors and features of software that is routinely used in computer-based graphic art and animation,

Fig. 3.7 Professional transitional Software schematic

which span a variety of creative pursuits and occupations; familiarity with these will both motivate students and expand their technical experience, interests, and thence their view of occupations to which they can aspire to as they progress into high school and beyond. We believe that Graphs ‘n Glyphs represents progress toward a new model for twenty-first century education, especially in the sense of Jolly et al.’s (2004) principles for successful programs for at-risk students: the content embedded in the software and curriculum is robust, the animation context provides an engaging environment that is relevant to students’ own experiences and interests, and it supports familiarity with tools and movement toward expertise with advanced software that further enhances their ability to express their creativity and create potential opportunities for their adult occupations.

3.2.5 Major Challenges

Several challenges confront the design, implementation, and use of educational software that aims for multiple learner-centered goals. Design that aims simultaneously to facilitate deep reasoning in a discipline such as mathematics, and development of technological skills, while engaging and supporting at-risk students in purposeful creative activity, is essential. The complexity of the design challenges reflect some of the complexity facing education in the current century.

Software design with multidisciplinary teams. Combining mathematics educators who are software designers and professional graphics and animation specialists in the same team led to sometimes-vigorous design and trouble-shooting discussions, and required decisions to be taken that promoted the long-term mathematics educational goals even at the expense of fidelity to professional software. Two major examples illustrate this point. (1) In most professional animation applications, scaling of an object is accomplished by scaling of the entire local plane and preservation of the object’s coordinates. This serves various visual and working needs of animators. However, from a mathematical pedagogical perspective, this was undesirable, because it removes the need for students to grapple with the multiplicative implications of scaling and distortion. We implemented scaling in Graphs ‘n Glyphs so that the local coordinate plane remains fixed, and the coordinates of the scaled object change in the expected multiplicative ways. (2) It is important mathematically to show the point of invariance during scaling. When a student clicks on the scaling tool, rays extending from this point of invariance (i.e. the pivot point) are displayed so that the student can see the reference point for the multiplicative act of scaling.

Energetic design debates between the artist/animators and the mathematics educators/designers during the project were chronicled on the team’s blog. The software’s overarching purpose was to be an environment for learning and utilizing mathematics in the most pedagogically rich way, so such debates were usually resolved in favor of a learner-centered design feature. Nonetheless, these conflicts were few, and the software remains faithful to professional software features in many respects.

Issues of representation and pedagogy. The software’s tools are important in promoting visual reasoning (Confrey and Maloney 2007a), but the graphical user interfaces of professional animation software (on which GnG is based) are deliberately designed to hide the mathematics, making software use more like drawing than like mathematics.

During the iterative design and implementation cycles of GnG's development, we encountered pedagogical issues which are familiar to educational software designers, and which are complicated by the animation and modeling context. One of the most interesting from the standpoints of both learner-centered design and support of content-based reasoning are related to each other: that is directionality of multiple representations and visual feedback. Directionality applies as soon as one uses simultaneous multiple representations and – particularly so for GnG –when tasks can be accomplished using either text/numeric inputs and visual tools (drag-and-drop manipulations), and when one mode of input is dynamically updated on-screen, and vice versa. For example, the team struggled over whether the software should immediately display tabular point coordinates when the student visually (drags) edits an object's points, or whether the student should be required to predict, and then check, the coordinate values in order to reinforce Cartesian graphing skills. Similarly, when a student numerically specifies a particular transformation, should the student immediately see the result, or should the student be permitted to reflect and predict the next position of the object? There are many times when one might want to delay the feedback provided to the student, in order to reinforce mathematical skills or reasoning.

Hence, Graphs 'n Glyphs provides visual transformation tools (the doodle mode mentioned earlier), which allow students to experiment with and learn about individual transformations and which also display the transformation values. The doodle mode does not record the transformation values, however. When students build sequences of transformations into animations, they must record the individual transformations, and they must do so by specifying the type of transformation as well as the values of the transformation parameters.

We continue to evaluate situations in which any pair of representations (for instance, an object on the graph and the display of its points' coordinates in the table) are bidirectionally (simultaneously displayed) or only accessible to the user in one of the representations.

In the curriculum we developed to scaffold children's learning with the software, we also sought to implement instruction consistent with conceptual corridors in the way tasks are sequenced. We employed paper-and-pencil tasks as well as on-screen tasks in our instruction. Practice was encouraged, and assessments were periodically conducted. The finalized implementation of Graphs 'n Glyphs, as with any educational software-curriculum system, will depend on feedback from settings where student interactions are encouraged and instructors guide and monitor the students' work.

3.2.5.1 Reflections on Design

We are actively considering the viability of the software for both engaging and sustaining student interest and the effectiveness of the implemented software and curriculum units in enhancing student learning of and confidence with mathematics.

The animation and modeling context, driven in large measure by mathematical concepts and tools, appears to be viable for student engagement and learning. The students in our initial design study were not classified as gifted, and many lacked confidence. Most of these students lacked competence in the mathematics that the state curricular standards suggested they should have mastered. For instance, the Missouri Grade Level Expectations indicate that by the end of fifth grade students should have already mastered addition and subtraction of positive and negative integers as well as multiplication of double digit numbers (Missouri Department of Elementary and Secondary Education 2004). However, it was clear from our pre-test results that students in our studies were still struggling with addition and subtraction. Feedback from students and parents suggested that many of the design study participants retained a measure of improved confidence and interest in math during the ensuing school year.

Level 1 of Graphs 'n Glyphs has been discussed here. Design and implementation of levels 2 (three-dimensional graphics, incorporating models of light transmission, reflection, and shadow) and 3 (incorporating sound creation and reproduction) will be undertaken upon availability of resources.

3.3 Case Study 2: Lunar Lander, a Prototype Web-Based Space Travel Games Construction Kit

3.3.1 Aims and Description of the Software

The aims of this work were to design a prototype space travel game construction kit based on the familiar "Lunar Lander" theme that would (a) be *engaging* for 11–14 year old students, (b) motivate the interaction with ideas of mass, gravity, velocity and acceleration and the relationships between them in the *design* of the game, and (c) stimulate in the playing of the game the use of different *representations* of these mathematical and scientific ideas in order to optimize the chances of winning. The work drew on our previous research that had investigated the potential of student-construction of computer games for learning. Thus, in the Playground Project (see, for example, Hoyles et al. (2002)) we attempted to tap in to children's games culture by adding a new dimension whereby they built their *own* games, or modified the games of others by editing play objects or playing with the rules of the games (Harel 1988; Kafai 1995). The Playground project set out to design and try out computational worlds in which the objects in a game *and* the means for expressing them were engaging, in which the programming of a game was itself a game. Thus we designed tools that gave to children the opportunity to construct creative and fun games, and at the same time, offered them an appreciation of – and a language for – the rules that underpinned their games.

Our attachment to constructionism as an orienting framework, along with our commitment to a modeling approach within microworld design, meant that we acknowledged that effective learning would 'not come from finding better ways for the teacher to instruct but from giving the learner better opportunities to construct' (Papert 1991, p. 3). Our belief that building things is a locus of significant educational change drove the initial rationale for microworlds and provides a crucial organising distinction between systems which put the child in the role of builder and thinker, and those which place him or her in the role of listener or receiver (see for example Hoyles, 1993 on design of microworlds, and Noss and Hoyles 1996). But construction is not enough. As with Graphs 'n Glyphs, the Lunar Lander work had to grapple with the problem of knowledge – what mathematical ideas would the student encounter – as well as the challenge of designing engaging game-like situations that simultaneously fostered learning and engendered ownership, in which the mathematical ideas we planted could become a source of power for the student.

This challenge has occupied us for many years. We have, on occasions, left the mathematics relatively inexplicit. At other times, we have been more prescriptive about the mathematics underpinning our design. For example, the WebLabs project sought directly to engage 11–14 year old students in deep mathematical challenges (such as the cardinality of infinite sets; see Kahn et al. (2005), Mor et al. (2006), Simpson et al. (2007)). We will return to the question of explicitness in the concluding section.

Overview of the game: The software employs narrative as a key device. This was not a simple matter for design: after all, story-telling does not immediately come to mind when devising an activity structure for engaging with difficult mathematical ideas, based as they are on precision and rigour. (For some general observations on this question, see Mor and Noss (2008)). The game starts with a player being hired as a game developer at a game company. The player is presented with a goal and is able to interact with virtual teammates in order to acquire both the needed components and the knowledge to proceed (actually, encountering the knowledge is not a necessary condition for proceeding). The design of this game about making games – the meta-game – aimed to provide structure, background information, guidance, and a gradual introduction to features and capabilities. The team of simulated experts includes a programmer, a scientist, an historian, an assistant game designer, and an animator.

The meta-game embodies the design of a learning sequence. It was introduced as a way of orienting students to the issues in the game and its design while avoiding prescriptive instructions. The response of each teammate to a visit by the player is scripted, but also depends upon both the current state of the game being constructed by the player and the history of the player's interactions with all the teammates. This gives the player freedom to visit the teammates in any order and with any frequency. Furthermore, each game component has an associated help button. When a component's help button is pressed, the player is informed which teammates have something to say about it. For example, the programmer, the scientist, the historian, and the game designer all have a unique perspective on the component which implements gravity.

Following the meta-game, the player can work through a suggested activity sequence and in so doing builds up the design skills and knowledge to build a Lunar Lander and know how to control it to optimum effect.

3.3.2 The Activity Sequence

In a first phase, the player is presented with a series of challenges. The first challenge is to make a game in which an astronaut is adrift and needs to reach her space ship. The means to do this is to acquire program fragments and artwork, which can be accomplished using only components involving horizontal motion that is achieved by 'throwing' rocks (previously collected by the astronaut). Challenges that have to be faced include agreeing on constraints (e.g. will the astronaut get back safely? Is she going too fast when she hits the spaceship?). The second challenge is to invent a new game. For example, the spaceship has started to move off in a vertical direction. Can the astronaut reach it now? This clearly means that the astronaut has to move 'diagonally' and in such a way so as to intercept the spaceship.

Once these game-making challenges have been encountered, students are ready to build a Lunar Lander, and a game panel with images of the astronaut and lander is presented (see Fig. 3.8). Beside it is a control panel with a start button. Pushing the start button initially does nothing, since none of the game elements have been given programs (the design imperative –'nothing should happen unless the student makes it happen!' – was a contentious design decision that is yet to be resolved).

The control panel also has a button that causes the *behavior gadgets* panel to appear (Fig. 3.9). It contains gadgets that consist of one or more code boxes. Any

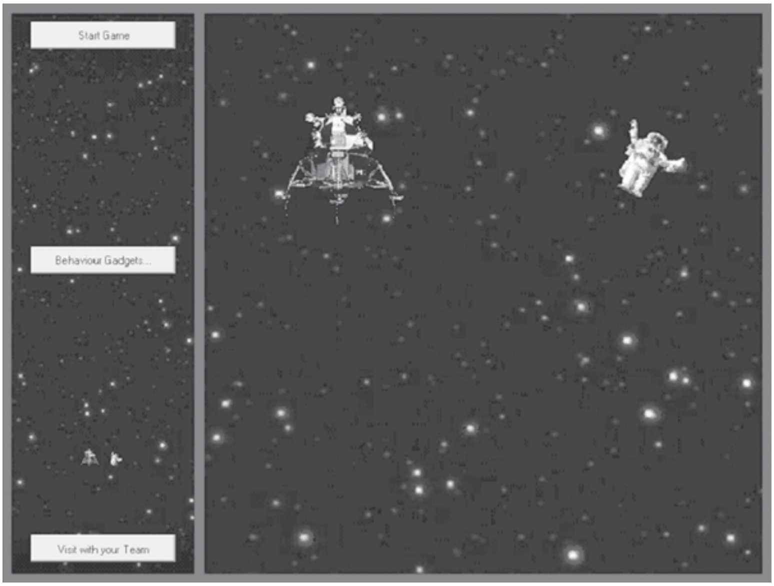

Fig. 3.8 The initial game construction page

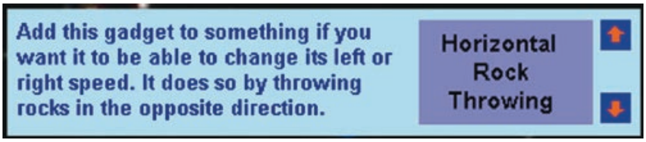

Fig. 3.9 Behavior gadgets are dragged from the behavior gadgets panel, which initially contains only a horizontal velocity gadget and a horizontal rock throwing gadget. As the meta-game progresses, the behavior gadgets panel acquires more elements

picture-object can be given a behavior by placing a behavior gadget on its 'back' (the back of an object is accessed by flipping it over). The behavior can be altered by setting sliders on the gadget's settings page. The code boxes of a behavior gadget can be removed, whereupon they expand to display the code that implements the behavior. Portions of the code that can safely be edited without programming expertise are colour highlighted.

The total mass of rocks (i.e. total fuel), the largest rock (the maximum rate of fuel usage), and the rock velocity (the propellant velocity) can all be adjusted by moving sliders. As one does so, the system calculates and displays derived values such as force, and performs unit conversions where appropriate. These parameters reflect real engineering tradeoffs. For example, adding more rocks/fuel does increase the duration of manoeuverability but at the cost of a greater total mass and hence a smaller acceleration from identical rock throws.

Landing on the moon involves several game-design decisions: what speed constitutes a safe landing? Should a criterion of 'winning' be minimising the use of fuel as well as how quickly a safe landing can be made? Once the design parameters are agreed by the players, the different teams need to think how they can use the tools provided to support their play: for example the dynamically configurable "gauges" (Kalas 2007) that can measure and monitor any of 13 values in graphical or numerical displays (see Fig. 3.10), including velocity, acceleration, remaining fuel, total mass, the application of thrust (by throwing rocks out in the opposite direction), and graph these values against time. Subsequent activities comprise the generation of multiplayer games in which players compete over the network in a race to land on the moon. The aim of this activity is to encourage reflection on the design of optimum strategies to win the game: for example, through reflection on the symbolic trace of a landing produced by the autopilot facility. This is a recording of all the changes to thrusters made manually during a landing and thus captures the settings of any 'manual' landing, settings whose parameters can be tweaked to produce an optimal landing. Finally students are challenged to invent new games to play using their own criteria for winning.

Limitations of space prevent us from illustrating how students used this range of panels, behaviors, and other objects that control instrumentation. For example, how they used the gauges to ensure a safe landing or how the two-player version of the game typically involved a race to be the first to land safely on the moon.

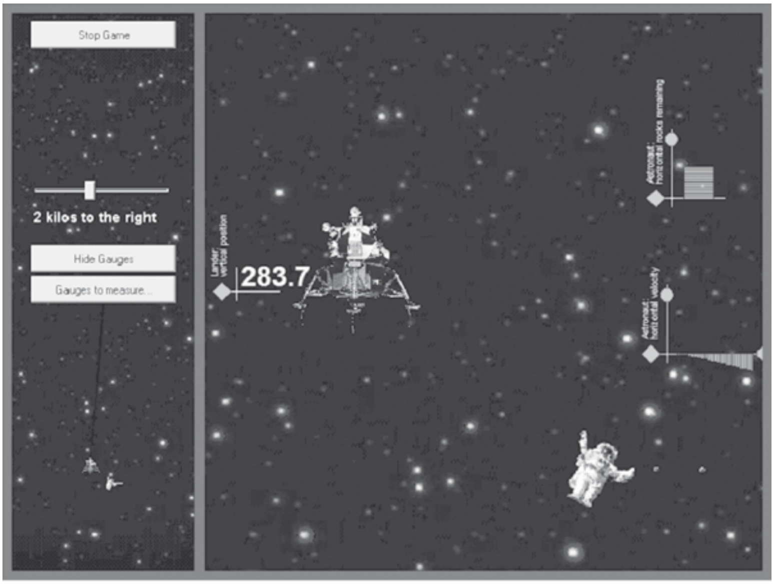

Fig. 3.10 A snapshot of game play with three active gauges

3.3.3 Main Issues in Software Design

There have been numerous attempts to design a programming-based approach to learning over the years. The most successful have achieved tangible learning outcomes across various topics, such as music, mathematics, language, and physics. (Some of these are discussed in Noss and Hoyles (2006).) They have also provided important pointers to the possibilities of learning that transcends the procedural and superficial by encouraging a playful – yet mindful – spirit of enquiry on the part of learners, aiming to break down the curricular silos that so often characterise traditional schooling. For the most part, what has been missing has been generalised success in tapping into students' own interests on a wide scale and engaging them in debate, investigation, and production.

In Playground, we had found it helpful to introduce the distinction between *platform* and *superstructure*. By platform, we meant the base level at which it was possible for users (rather than professional programmers) to interact. A platform would include high level programming languages but not for example machine code. In most cases, users interact with the platform because the designer *expects* them to do so. Most software takes pains to make the platform level *completely invisible*, and, in general, make a virtue out of this perceived necessity on the grounds that only programmers need to know how to program. Superstructure, on the other hand, describes the objects in the microworld and ways to manipulate them. The idea of superstructure raises new dilemmas. How visible is the platform level? What is the appropriate grain size for objects and relationships at the superstructural level? How easy is it for the user to 'descend' to the platform level? How permeable is the barrier that separates superstructure from platform? How rich is the potentiality of modifying tools at the superstructural level along with the interactions that go with it? How familiar can and should the user be with platform tools?

Our approach in Lunar Lander owed much to our early experiences with Playground. In both Playground and WebLabs, we created a class of playground objects called 'behaviors' that were portable components packaging the functionality of the programs into manageable pieces while allowing them to be inspected. These behaviors could be dropped on 'objects' so as to give them the functionality required, thus bypassing the need to program from scratch (although, crucially, the symbolic representation could still be accessed and edited). Building games with behaviors or modifying behaviors in the games of others was the usual way that students interacted in Playground (or undertook their explorations in WebLabs).

This antecedent work pointed to the motivational benefits of this layered approach and the substantial commitment students often showed to their products if we could manage to design the grain size of interaction appropriately. We also noted that student-authored games tended not to suffer the difficulty that other educational games suffer around poor production values: much to our surprise, we found that when students are constructing their own games, they are relatively forgiving if the look and feel of the game is quite crude.

3.3.4 Notable Characteristics of the Software

There were a number of notable design characteristics that we now specify:

A straightforward interface for the *composition of pre-built program fragments*; this was made possible by an underlying computation model that simplified the process of composing fragments by building upon multiple independent processes. We have found over a number of projects that this object-oriented (from the learner's point of view) approach is valuable – thinking about things-with-properties allows the learner to have something to reflect upon, discuss with others, and build understandings incrementally.

An *easy means to parameterise the code*, either by directly manipulating inspectable fragments of the code itself, or by accessing pages where the values of parameters (e.g. how strong is gravity on this planet?) could be directly altered.

An *underlying physics model* based upon conservation of momentum that is simpler than the one based upon $F = Ma$ and first and second derivatives of position and the first derivative of momentum. From a technical point of view, this gained us modularity and composibility of the concurrent processes underlying the system. From a pedagogical point of view, this approach bought us a concrete and discrete way to think about forces, in which the mechanism underlying rocket thrust became transparent: Throwing a one kilogram rock once per second is the same mechanism as actual rockets that "throw" trillions of trillions of mini-rocks (molecules) per second.

The concept of a "*meta-game*" in which games are made within an overarching narrative game structure. We decided to bring the task to life for learners, by putting them in the position of the game designer – what would they have to know about physics, about animation, programming, sound, to design the system? By doing this, we sought to provide structure, background information, guidance, and a gradual introduction to new features and capabilities in a reasonably natural way. The different characters reacted with a very modest degree of intelligence: the response of each to a visit by a player is scripted but does depend upon both the current state of the game being constructed by the player and the player's history of interactions with all the teammates.

An *autopilot* in which the settings of any manual landing – i.e. the values of all the variables and how they are changed over time – could be captured by an autopilot. This automatic recording and re-running of 'successful' landings allows learners to modify parameters, reflect on their values, and how they are related. This contributes to work at the symbolic level aimed at optimizing their lunar lander.

A range of carefully crafted *gauges* (Kalas 2007) that enable the player to monitor 13 parameter values in graphical or numerical displays, including for example, velocity, acceleration, remaining fuel, total mass of the lander by 'connecting' a graphical display to the lander which changes as the lander moves.

The facility to engage in *multiplayer* games played over the web. Players can see other players' landing attempts and compete with them, and can attach gauges to their own and their opponents' games. A typical Lunar Lander game has players competing to land with the safest landing speed using the least amount of time or fuel. A typical astronaut game has players *cooperating* to rescue the astronaut. Students are also challenged to invent new games to play.

Finally, a fundamental design objective that we named *layered design*. The construction kit provides small program fragments together with tools for customising and composing them. The fundamental idea is that players/learners can choose how far to delve into the workings of the tools. At a first layer, engagement is mainly through reading, watching, and making conjectures based on observation and, for example, describing a motion and reflecting on it. The tools for this layer are basic, perhaps only a handful to control a simulation on video, or the timing of movement. At a second layer, the learner can begin to manipulate motion, and predict and test out the effects of different values. At a third layer, the learner might explore further how variables relate to each other, for example position, velocity, and acceleration, by reference to the values set by sliders. And finally, a fourth layer that engages with these relationships either by modifying existing programming code or by writing new programs or fragments of programs.

3.3.5 Snapshots of Learning

We now very briefly outline some snapshots of what we interpreted as learning that emerged from the iterative design/test cycle with three groups of students: two drawn from a large, urban comprehensive school (one "Year 7" class aged 11–12;

one "Year 8" class, aged 12–13) and a small group of three students (aged 12–14) from a second school in an after-school setting.

Developing understandings of Newton's third law: In the first task the Year 7 students began with a relatively low knowledge of the physics concepts involved. For example, R suggested, "you could throw some rocks away and that would make her lighter so she would move." For the most part, they were unaware that the effect of gravity in space is negligible (in the terms in which the software was devised). Through the course of the activity and experimentation with the horizontal rock thrower, students appeared to develop an appreciation that throwing a rock would develop an opposite movement proportional in velocity. Indeed, later in the session two students worked with the theory that "throwing larger rocks makes her move faster."

Minimising the time for the astronaut to intercept the spaceship, or for the lander to safely land on the moon proved a motivating task, particularly for the Year 7 students. Most students used an iterative strategy, e.g. T and A were delighted to refine their strategy again and again by optimizing the use of the horizontal rock thrower against the speed of reaching the spaceship.

Using the gauges: Throughout the sequence much use was made of the gauges and interpretation of their output. Most of the students found this relatively easy to put in place and tended to refer to them constantly as a guide to their use of the rock thrower or thrust. When landing on the moon some students applied far too much thrust causing the lander to move upwards and disappear off the screen. Reading the vertical velocity gauge, which they had set up for the lander, they predicted how the lander would "keep on getting slower until zero. Then it will fall back again because of gravity." In the two-player game the ability to attach gauges to the opponent's lander was a particularly successful feature, enabling one group to make a close comparison with the other and to adjust the strategy second by second.

Composing horizontal and vertical velocities: Coming up with the hypothesis that to achieve diagonal movement a combination of horizontal and vertical thrusts would be needed appeared almost effortless and was tested using, for example, both horizontal and vertical rock throwers to the astronaut and using both simultaneously.

Gravity: The Year 7 students did not immediately make a connection between the rock-throwing astronaut and the rock throwers for the lander, although Year 8 students seemed to require no prompting. The Year 7's also only had a vague concept of gravity. Only two students volunteered that the lander game would be different from the astronaut game because of a gravitational pull near the surface of the moon.

Collaboration, competition and motivation: Beating previous best scores proved highly motivating, especially for the team of Year 7 boys. Collaboration centred around agreeing what the two teams should have in common: the total mass of the projectiles, an agreed safe landing action, a value for gravity, and the vertical starting position of their landers (for which they sought and found a new gauge, previously not used).

Attempts to minimise fuel use became more sophisticated. The boys realised that for their agreed safe landing speed of 30 m/s, they needed only to keep just below this figure to ensure a safe landing and minimal fuel consumption. Previously they had been trying to reduce velocity to minimum regardless of fuel use.

In summary, the competitive element of the two-player version was an enormous motivation to the students: they loved seeing the opposition's ship on their screen and being able to monitor its progress through gauges. The students became wildly excited during landings.

3.3.6 Challenges and Reflections on Design

Overall, the Space Travel Games Construction Kit was successful as a prototype in allowing access by diverse students at many layers of learning. It stimulated considerable interest and discussion among students, who used quite sophisticated ideas in pursuit of their game making and playing. They generally enjoyed the challenge of the game – individually, in pairs, and also over the web; as a measure of engagement, many came up with inventive and meaningful suggestions for improvement.

At the same time, there remain some challenges that we have as yet been unable to address. We single out just two.

1. *Help provided by the meta-game*

This was only partially successful for three reasons. First, there was rather too much reading required in the initial stages to introduce the different experts; second, some terminology was too complex in places; and, third, although the meta-game provided some 'intelligent' help by suggesting 'who else' the students should consult, this was rather crude and sometimes led them off track.

The students made some design suggestions involving reducing the reading load, that a choice should be offered between reading and listening to instructions, and that more complex instructions – such as showing how to set up a gauge or to use a behavior gadget – might be communicated through demo buttons or tutorials. But we know there is much to do, and in our latest project, we have begun seriously to address the need for real intelligence on the part of the system, aimed not only at supporting students directly, but on helping their teachers. See http://www.tlrp.org/proj/tel/tel_noss.html.

2. *Permeability as a mechanism for layering*

It is one thing to celebrate the virtues of layering as a pedagogic device. It is quite another to find design solutions that invite students to explore the different layers, to see the relationships between them, or to have a sense of what the utilities of the different layers are. Very few students looked beyond the highest drag-and-drop level; and we expect that it would take a far longer induction to create a rationale, and a culture, in which layers would be productively exploited.

Just what is the right level of interaction to maximise the possibility of engaging with – and ultimately learning about – deep ideas? If the game could be made by merely dropping random components onto it, something might be achieved – but not much. After all, it is one thing to know that gravity can be added to the system: quite another to know that it produces a change in momentum (an acceleration) and that this can be characterised by an equation or a piece of programming code. And while it is important for students to acknowledge the existence of gravity – i.e. gravity does something (and not necessarily what might be expected) – acknowledgment is quite different from understanding the quantitative relations that make it work. We have much to do to understand and design effectively to address these issues.

3.4 Conclusions

We finish with some cross-cutting themes, notable either by the difference in the way the theme played out across the two case studies, or by similarities in which two independent design teams resolved issues.

Both teams recognised the need for a back-story. In Lunar Lander, there was reference to a meta-game, where the narrative was that the students would help to build a lunar landing game. In GnG, the hook was the potential to create working animations, with the children's desire to create a narrative highly motivating. The back-story placed the activity squarely in the cultural world of the students and created an initial purpose for the students to engage (see Ainley et al. (2006), who discuss attempts to build purpose into the task, either as an implicit element of the software or more explicitly as part of the task whose completion requires the software as a tool).

Nevertheless, in a modeling activity, the mathematics has to do work if the purposeful activity is to be steered through the design towards the learning of mathematics. In Lunar Lander, one of the main controls for the students was the ability to throw more or less rocks. Although this is certainly an unconventional representation, there is a clear link, at least clear to us as observers already enculturated into the mathematics of dynamics, with the notion of momentum, which was a key idea underpinning the design of Lunar Lander. In GnG, the students were required to control the animation by the explicit defining of transformations, including their parameters, a key idea in the designing of GnG. We have previously observed how such fusion of control and representation of a key mathematical idea is often a successful design principle when designing for abstraction (Pratt et al. submitted).

At the same time though, we note that the representation of momentum was rather more obscure than was the representation of transformation, which had many of the characteristics of transformations as seen in standard text books. It is interesting to consider the implications of this difference. GnG was designed for (eventual) classroom use, predicated on the existence of a teacher and curriculum pieces, and clearly oriented toward mathematics. Lunar Lander, on the other hand, was aimed (at least in initial design) at students who may not be in classrooms, involved no teacher (or even a more capable adult). It is perhaps fair to say that, without the presence of a teacher, the students' rationale for 'playing' the game would rarely include mathematical priorities. This difference affects the nature of the representation/control in opposite directions. Thus, in the case of GnG, the cultural expectation of teachers, parents and educational authorities require explicit reference to the formal mathematics curriculum. In contrast, in the case of Lunar Lander, the cultural expectations of the students require that the software is more playful and less overtly related to formal mathematics.

The approach in Lunar Lander was to leave open access to underlying layers, where the mathematics would be more explicit and more formal, though possibly still unconventional (for example, we would regard programming as formal but programming is not conventionally seen as mathematical). It goes without saying that the extent to which the invisible layers of a program should be hidden from students is much more of a pragmatic rather than principled question. GnG hid details from students in doodle mode; Lunar Lander's approach was to make these layers always-available. Ironically, the explicit transformations in GnG were always used by students as they were an unavoidable control over the process of creating animations, whereas students did not often access the always-available underlying layers in Lunar Lander. Why might this have been? The design team for GnG had a clear strategy that the mathematics of animation-making would only be addressed if the interface features in professional animation software, which tended to hide the mathematics, were removed and replaced by the explicit mathematical controls. Provided the students would still engage, and this was supported by the teacher and the curriculum, the design team could be assured that the mathematics would be encountered. The Lunar Lander design team decided to represent the mathematical ideas in rather less conventional ways, since otherwise the students may not have engaged in the first place. In many cases however, the students did not appear to *need* to access lower layers. It was assumed that the need to refine the landing would be satisfied through the auto-pilot, providing a window onto the lower layers of mathematics. That this did not happen very often is testimony to the complexities of design. Designing for insufficiency seems an important idea when designing top level layers. Nevertheless, for all the similarities in how the two design teams went about their task, the software solutions provide contrasting case studies on how, when and why it might be appropriate to hide or layer the mathematics.

Another interesting theme is that of directionality. Decisions have to be made about whether two representations should be 'hot-wired' or dynamically linked, so that changing one automatically changes another. The danger of making the two representations automatically dependent in that way is that students may not become aware of the connection if they do not need to focus on it. The danger of disconnecting the representations is that first of all a design decision may have to be made about which representation to make primitive, and this is not always obvious. Second, the creative process may be hampered by introducing an additional step, with resulting loss of engagement. GnG's solution was to have a doodle window, in which students were less concerned about formal details and could play more freely, and then a separate animation-creating window, where students needed to focus on using the transformation language to create the expected animation, requiring some visualisation of what would happen. This solution is reminiscent of Logo, in which students are able to play at top-level and see the animation of the turtle as immediate feedback but then, to create a more complex program, it was necessary to build procedures, for which the behavior of the turtle needed to be visualised. In both Logo and GnG, the principle of insufficiency seems to apply. The more formal reflective mode must be utilised by the students as the freer mode is insufficient to complete the task satisfactorily. In Lunar Lander, students could play relatively freely and their actions would automatically create code in the auto-pilot. The code could then be modified to refine the landing process. In one direction, the two representations were hot-wired whereas, in the other direction, the code would need to be amended before its exact consequence could be seen in the animated landing.As discussed above, autopilot was not necessary to complete the Lunar Lander task and so this subtle bi-directionality might not be witnessed by the students themselves.

Gauging success with respect to explicitness and directionality is difficult: it raises, not just the question of *whether* mathematical knowledge was learned, but *what kinds* of knowledge were available for engagement. Here we touch on a key theme of the ICMI Study and this volume: the theme of knowledge – on which Papert focused in his opening keynote and on which the conference tried to focus in its work, even when we had to continue without him. His entreaty that we, as mathematics educators, reserve "10%" of our effort to think about the knowledge that we were designing, a theme he has explored earlier in Papert (1996). If we think of 'explicitness' as a theme, we run the risk of considering knowledge as invariant under different lenses, different degrees of explicitness. But in fact, as we become more or less explicit about mathematics, the knowledge itself is open to change.

As an example, our choice of momentum as a fundamental idea (rather than force) in Lunar Lander, emerged from our wish to open layers of mathematics more broadly to our intended audience. It forced us (or at least encouraged us) to consider ways to make mathematical knowledge accessible, and in doing so, to think about *new* knowledge - at least 'new' in the sense of knowledge that was unconventional. In GnG, the situation was somewhat reversed: here professional software would keep hidden the mathematical layers of the transformations, so the 'newness' of the knowledge was only in relation to the utility and purpose of the mathematical tools in effecting animations, rather than in the newness of the mathematics itself.

Inevitably, these kinds of considerations are complex, and involve pedagogic as well as epistemological challenges. Yet we have no alternative. If we are genuine in our attempt to open mathematics to wider cross-sections of the population, to make mathematics operational in the sense of it achieving some personal purpose, then we have to acknowledge the design challenges, confront them, and somehow respond to them.

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Chapter 4
Designing Digital Technologies and Learning Activities for Different Geometries

Keith Jones, Kate Mackrell and Ian Stevenson

Abstract This chapter focuses on digital technologies and geometry education, a combination of topics that provides a suitable avenue for analysing closely the issues and challenges involved in designing and utilizing digital technologies for learning mathematics. In revealing these issues and challenges, the chapter examines the design of digital technologies and related forms of learning activities for a range of geometries, including Euclidean and co-ordinate geometries in two and three dimensions, and non-Euclidean geometries such as spherical, hyperbolic and fractal geometry. This analysis reveals the decisions that designers take when designing for different geometries on the flat computer screen. Such decisions are not only about the geometry but also about the learner in terms of supporting their perceptions of what are the key features of geometry.

Keywords Design • Digital technologies • ICT • Learning • Geometry • Geometries

4.1 Geometry, Technology, and Teaching and Learning

While forms of algebra software (such as *Derive*, *Macsyma*, *Maple*, *Mathematica*, etc.) were amongst the first mathematics software packages (pre-dating, in many cases, the graphical interface), it is software tools for geometry (beginning with *Logo* and followed by 'dynamic geometry' environments such as *Cabri* and *Sketchpad*) that have emerged as some of the most widely used digital technologies in the mathematics classroom – and arguably amongst the best researched (for reviews, see Clements et al. 2008; Hollebrands et al. 2008; Laborde et al. 2006).

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Our aim in this chapter is to consider the interplay between the design of digital technologies and activities that utilize those technologies for learning mathematics. We focus on geometry, covering both Euclidean and co-ordinate geometry in two and three dimensions, and non-Euclidean geometries such as spherical, hyperbolic and fractal geometry. The reason for considering this span of geometries is to capture key aspects of how the use of digital technologies can and does shape the mathematical activity of the user.

The chapter concludes by reflecting on how some of the key decisions that need to be taken regarding issues of geometry are handled by designers of digital technologies, and by designers of related learning activities, and on the implications for future users of educational digital technologies.

4.2 Working with Different Geometries on the Flat Screen

A distinctive, but perhaps somewhat neglected, characteristic of current digital technologies is ‘flatness’, both of the screen used as the visual medium in the classroom, and the ‘computer mouse’ operating on a flat mouse mat. As we demonstrate in this chapter, ‘flatness’ is problematic when representing and interacting with any geometry, and even introduces design issues when working with plane (two-dimensional) geometry.

That the flat screen presents some difficulties in handling representations of different geometries is nothing new. Artists and mapmakers have wrestled for centuries with trying to present the three-dimensional (3D) world on the two-dimensional (2D) canvas or atlas. In western art, beginning in the fifteenth Century with artists such as Brunelleschi, the use of perspective first found systematic presentation in Alberti’s *Della Pittura* published in 1435. The most common method for representing 3D space on a surface, usually known as linear perspective, is illustrated by Albrecht Dürer in a famous engraving of 1525 reproduced in Fig. 4.1. Here a hook on the wall takes the position of the eyes, and a taut string represents the straight line joining the eyes to a visible spot beyond the frame. This provides one solution to the problem of representing solid (3D) objects on a flat surface in a way that is compatible with human stereographic vision. As such, the idea of linear perspective is a result of taking account of human perceptual apparatus.

In cartography, many forms of map projection have been developed as attempts to portray the surface (or a portion of the surface) of the earth (taken as a sphere) on a flat surface. Each of these projections maintains some geometrical properties (such as distance, area, or shape), but, by their very nature, such projections cannot maintain all such properties simultaneously. What is preserved, geometrically, in any particular cartographic projection, and what is not, is dependent on the purpose for which the 2D map is created (Kreyzig 1991).

A major revolution in geometry came in the nineteenth century with developments that led to consistent non-Euclidean geometries, and the emergence of curvature as a key idea. Work by Euler, Wolfgang and Janos Bolyai, and Lobachevskii, to name but a few, showed that Euclidean geometry was one of many possible geometries: its uniqueness lay with its ‘flatness’, not, as Kant would have it, because it is ‘absolute’.

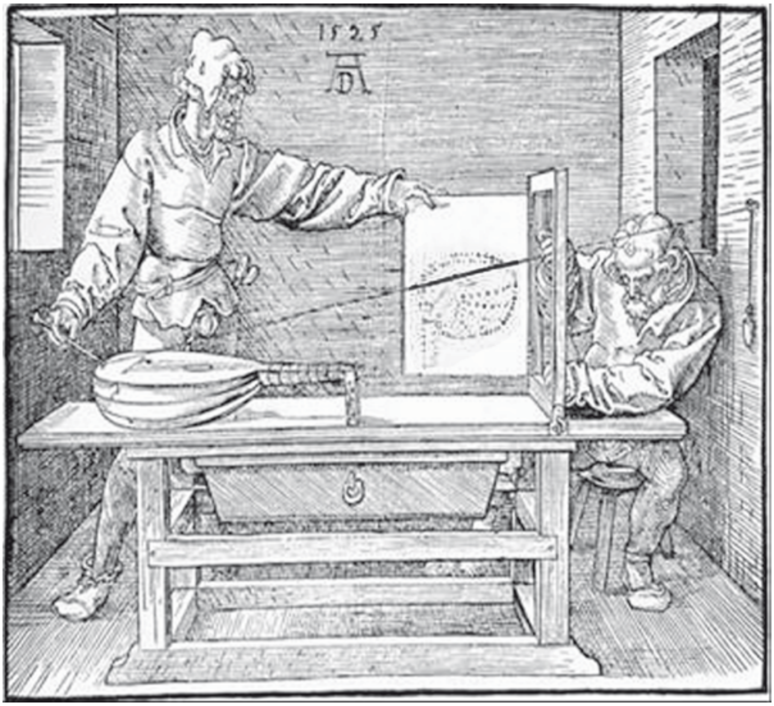

Fig. 4.1 One-point perspective, as illustrated by Albrecht Dürer in 1525. *Source:* Willi Kurth (Ed.) (1963). *The complete woodcuts of Albrecht Dürer*. New York: Dover (illustration 338)

Curvature, as a geometric property, and because it characterizes more geometries than Euclidean, became an active area of research. Gauss and Riemann, for example, showed that curvature is an intrinsic property of surfaces, defined locally rather than globally. Hitherto, curvature had been defined by embedding a non-Euclidean surface in Euclidean (two and three-dimensional) space and using the associated global co-ordinate system. Riemann's introduction of a local description of geometry removed the need for projections, a technique which, as noted above, arose out of human (perceptual) need rather than mathematical necessity. Yet the price of this advance into a range of geometries can be the loss of visual intuition that we need, as humans, to understand our experience of space.

In architecture and in many branches of engineering, prior to the development of computer-assisted design and manufacture (CAD/CAM), the 'distorting' nature of the forms of projective geometry used in cartography was circumvented through the use of orthographic projection (as developed by Monge in the late eighteenth century; see Bessot 1996) in which several 2D views of the object (often referred to as front, side, and plan elevation) are utilized instead of a single view. With the development of CAD/CAM, 3D modeling became possible – first through a 2D-to-3D paradigm (whereby the 3D object is built up from 2D objects) and more recently through the use of new geometric forms (including grid-like polygonal subdivisions of surfaces known as 'meshes' and curves in 3D space defined by control points known as ‘splines’), assisted, at times, by the use of a 3D input device (rather than the usual mouse on the 2D plane).

What this short historical introduction indicates is that projections of various kinds are the result of human needs, sometimes dependent on the available technological medium – such as the ‘flat screen’ of the canvas or atlas – and sometimes because of human stereographic vision. As such, projections need to be understood in relation to the problem that led to their creation.

Introducing digital technologies has enabled us to interact with more forms of geometrical objects, and this underlines the need to understand the conventions of the flat screen and how that medium alters our appreciation of the translated logical geometric structures (Euclidean or otherwise). What digital technologies may offer is a way of building, and developing, our visual intuition across a range of geometries. Yet we need to be much clearer as to the affordances and constraints of such technologies in the teaching/learning process. It is these issues that we turn to next.

4.3 Designing Digital Technologies for Different Geometries

In examining decisions about representations and interactions when designing for different geometries for the flat screen, we focus on three geometry technologies that are common to mathematics classrooms: 2D ‘dynamic geometry’ environments (such as *Cabri* and *Sketchpad*), software for 3D geometry (with 3D Euclidean geometry illustrated by *Cabri 3D*, and 3D coordinate geometry software illustrated by *Autograph*), and software suitable for various non-Euclidean geometries (illustrated by the use of *Logo*).

4.3.1 2D Dynamic Geometry Environments

Over the years since the first ICMI study on technology (Howson and Kahane 1986) when users had to rely solely on text-based input via the keyboard, major innovations have involved the introduction of direct manipulation graphical capabilities that have become synonymous with contemporary computers (Norman and Draper 1986). Such changes have impacted particularly on geometry education with the development of ‘dynamic geometry’ environments (DGEs) such as *Cabri* and *Sketchpad* (and many others).

At first glance, a DGE is nothing more than a graphics editor enabling geometrical figures to be drawn on the computer screen. Yet there is more to it than this because with a DGE the user can utilize the mouse to ‘grasp’ an element of the on-screen figure and drag it about. As this ‘dragging’ takes place, the diagram on the screen changes in such a way that the geometrical relations specified (or implied) in its construction are maintained. Such digital environments are called ‘dynamic’ for this reason.

Yet the way in which a DGE figure moves when it is dragged is not solely to do with geometry. Even though, as Goldenberg and Cuoco (1998) explain, an over-riding principle in DGE interface design has been to try to ensure that the behavior of geometrical objects constructed on-screen conform as closely as possible to how users would naively expect them to behave (in Euclidean 2D geometry), there is an unavoidable tension for DGE designers between the need for objects to move continuously when dragged, and the need for the position of the constructed elements to be uniquely determined (Gawlick 2004). The problem for DGE designers is that no DGE can be fully continuous and fully deterministic at the same time. For deterministic DGEs (and most currently available DGEs are deterministic) while on-screen figures are completely determined by the given points, the result is that some constructions can jump or behave unexpectedly when a particular point is dragged. With continuous DGEs (the minority at the moment), dragging any point does produce a continuous motion of the construction (through the use, usually, of a heuristic 'near-to' approach) but it can happen that when a dragged point is moved back to the original position, the resulting construction might be different from the original. Gawlick (2004) provides illustrations of both cases.

The result of such issues is that users of DGEs need to learn to distinguish between changes in the on-screen image (as objects are dragged) that are a consequence of geometry and those that are the result of decisions of the software designer. A seemingly trivial example is that, in some DGEs, objects that look the same may not act the same (for instance, some points may be dragged while others cannot). Yet even this apparently trivial issue can leave beginning DGE users wondering why not (Jones 1999). Another design decision involves deciding whether an arbitrary point on a line segment might maintain the ratio to the endpoints when either is dragged – or whether, for instance, the point jumps to another arbitrary position (since it is an arbitrary point), or whether it maintains a fixed distance to one or other of the endpoints. The common decision by DGE designers seems to be to maintain the ratio to the endpoints when either is dragged. Yet this is a decision of the DGE designers; it is not something governed completely by geometric theory. For more on the decisions of DGE designers, see Goldenberg et al. (2008); Laborde and Laborde (2008); Scher (2000).

In graphing software such as *Autograph* (which shares some aspects of a DGE), every object is defined relative to a coordinate system. This means that changing the relative scale of the axes changes the appearance of objects. The consequence is that, for example, lines which have been defined as perpendicular will no longer 'look' perpendicular when one axis scale is changed (though, of course, in the mathematical sense, the lines remain perpendicular). In contrast, in some DGEs (such as *Cabri* or *Sketchpad*), objects are not necessarily defined in relationship to coordinate axes. In such DGEs, a circle (defined, in effect, as the locus of points that are a fixed distance from a fixed point) retains the appearance of a circle on-screen even when either coordinate axis is changed. The impact of such decisions regarding the role of coordinate systems in the representation of objects on learners (especially beginners) is currently under-researched.

Whatever the DGE, another design decision relates to the provision of menu items (Goldenberg et al. 2008). Providing too few means that more things need to be constructed, something which becomes very tedious. Yet providing too many menu items produces undue complexity (rather than 'user-friendliness') and could mean that teaching opportunities are lost. Finding the balance between these two aspects is a key design decision in any educational application – and is something that simultaneously involves technical and pedagogical issues (Hoyles et al. 2002). In tackling the issue of too many, or too few, menu item, many DGEs, while necessarily prescribing a selection of provided constructions, also allow some menu items to be 'hidden' (thus allowing the software interface to be simplified) while, at the same time, featuring a macro or 'script' facility for user-defined constructions to be automated (thus allowing new idiosyncratic menu items to be added). How this adjusting of menus is used by teachers, and the impact on learners, is currently under-researched.

What this section illustrates is that there is a range of issues that add complexity for the technology user when it might be assumed that plane geometry on a flat computer screen would be the most straightforward case of doing geometry with digital technology.

4.3.2 Software for 3D Geometry

From a purely mathematical perspective, it is perfectly possible to use common 2D geometry software to create '3D' objects, figures, and graphs. Yet it is complicated and time-consuming to do so. As a consequence, recent software development has provided a range of geometry environments in which learners can manipulate 3D objects directly on-screen. Such environments include *Cabri 3D* and *Autograph* (version 3).

The issue of representing 3D objects on a flat screen means that a number of design decisions, unique to 3D software, need to be made by software developers. One key decision is how the opening software screen both orients the user to 3D space, *and* provides a framework for the creation of 3D figures and structures. This has been tackled in different ways by different software developers. The opening screen for *Cabri 3D*, for example, shows part of a plane, with, at its center, three unit vectors representing the x, y and z directions (see Fig. 4.2). This initial viewing angle was chosen so that the plane and vectors would have an appearance compatible with the usual textbook representation of 3D space, with the base (or reference) plane deliberately chosen so as metaphorically to represent the ground (in order to orient the user).

The opening 3D screen of *Autograph* (version 3) shows a framework of a cube bounding 3D space from –4 to 4 on each axis (see Fig. 4.2). This design was chosen as being likely to encompass most objects of interest at the relevant level of school mathematics. The scale and numbering of the axes is given along the edges of the framework so that labels do not 'float' through objects created within the cube. When objects are created, only the parts of the objects within this bounding box are displayed on the screen, the bounding box being chosen as a means of making this active area of the screen visible.

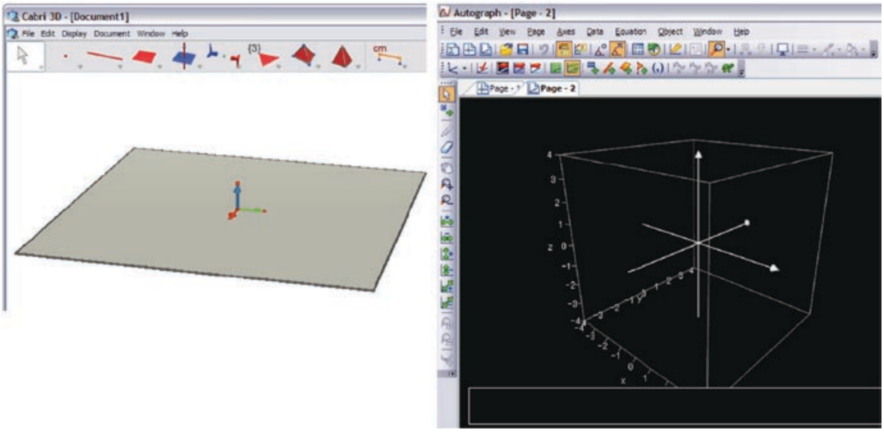

Fig. 4.2 Opening screens of *Cabri 3D* (*left*) and *Autograph3* (*right*)

Even more so than with 2D software, the designers of 3D geometry software have to make a number of decisions about the ways in which objects are seen on-screen. For example, given that in 3D software a point in space is created by clicking in any empty screen location, a decision has to be made about the location in space of such a point, as a screen location does not define a unique point in space. In *Cabri 3D*, such a point is positioned on the base plane at the position of the cursor. *Autograph* locates such a point halfway through the bounding cube and along the observer's inferred line of sight through the cursor position when the point is created.

Another set of decisions is about the way 3D objects 'look' on-screen. For an object and its surfaces to have a 3D appearance, use is made of perspective and 'rendering' (the computer graphics term for the ways in which the visual appearance of a 3D on-screen object depends not only upon its geometry but also upon the viewpoint by making use of lighting, shading, and, where appropriate, texture). In terms of perspective, the default for *Cabri 3D* and for *Autograph3* is one-point perspective. In *Cabri 3D*, the default viewing distance is 50 cm, representing the screen at arm's length from the viewer's eye, chosen as it was thought to be 'natural'. The viewing distance was more subjectively chosen for *Autograph 3* and is shorter. In terms of 'rendering', both *Cabri 3D* and *Autograph 3* use shading (by which the brightness of a surface is dependent on the direction in which it is facing relative to the inferred observer); *Cabri 3D* also uses 'fogging', a computer graphics terms for the effect by which objects 'at a distance' appear to be fainter than objects 'close at hand'.

A further set of decisions relate to dragging objects using the mouse. Given that dragging on a flat screen can only give motion in two dimensions, in *Cabri 3D* a decision was made that 'ordinary' dragging would move a free point (or object) parallel to the base plane, while pressing 'shift' at the same time as dragging would move the point (or object) perpendicular to the base plane. In *Autograph* (version 3), dragging a free point continues to position it halfway through the bounding cube along the line of sight of the observer.

Given the centrality of 'dragging' in 2D DGE and its implications for developing different types of reasoning (Arzarello et al. 2002), and as dragging is something which might make motion in 3D (on the 2D screen) more difficult to interpret by the user, the various aspects of dragging in 3D DGE are issues that could usefully be the focus for research.

4.3.3 Software for Various Non-Euclidean Geometries

The 'turtle geometry' of *Logo* can give rise to several types of non-Euclidean geometry, each of which can be made available on the usual 2D computer screen. (Abelson and diSessa 1980). In *Logo*, a turtle's 'state' is defined intrinsically (by reference to its own movement of forward–backward and its heading of turn left or right by so many degrees) and locally (since measures of steps and amount of turn are referred only to the turtle, not external coordinates). As a result, curvature in turtle geometry is turn per step, and is intrinsic to the turtle's behavior.

While the turtle is 'viewed' through the Euclidean lens of the flat computer screen, if the screen's metric is changed so that the turtle's steps are lengthened or shortened in each step (with its turns unaffected), then there is the basis for non-Euclidean geometries. The turtle still responds to forward and right in the same way, irrespective of the geometry, but adjusting the screen metric alters its behavior as if the turtle were in spherical or hyperbolic space. The effect is that the screen can be thought of as having a variable 'temperature' (Gray 1989): from this perspective, spherical geometry has a screen that increases in 'temperature' as the turtle moves towards the screen's edge, while hyperbolic geometries get 'cooler' towards the edge. By 'dashing' the turtle's path (see Fig. 4.3) so that the dashes grow longer or shorter according to the geometry, the turtle's steps are expanded or contracted by the 'temperature' of the screen. A corresponding speeding up or slowing down of the turtle's movement occurs as it leaves dashes as it is moved. Angles are preserved in these worlds, so that they sum appropriately to more (or less) than 180° in a triangle, depending on whether the screen gets 'hotter' (spherical) or 'colder' (hyperbolic) at the screen's edge, respectively.

The dynamic features provided via *Logo* are thought to play a significant part in helping learners to understand what is happening geometrically when exploring non-Euclidean geometries (Stevenson and Noss 1999; Stevenson 2000). Given that non-Euclidean models are obtainable through stereographic projection of a sphere or a hyperboloid onto the flat screen plane, this aspect of such models is thought to be critical in helping learners to understand the screen images. As illustrated in the next section of this chapter, such features can be used in the design of related learning activities.

Another form of non-Euclidean geometry that can be explored through utilizing the *Logo* turtle is fractal or 'broken' geometry (Mandelbrot 1975), formally defined as geometry in a space of a non-integer dimension and illustrated by objects such as the 'tree' and 'snowflake' in turtle geometry (Abelson and diSessa 1980).

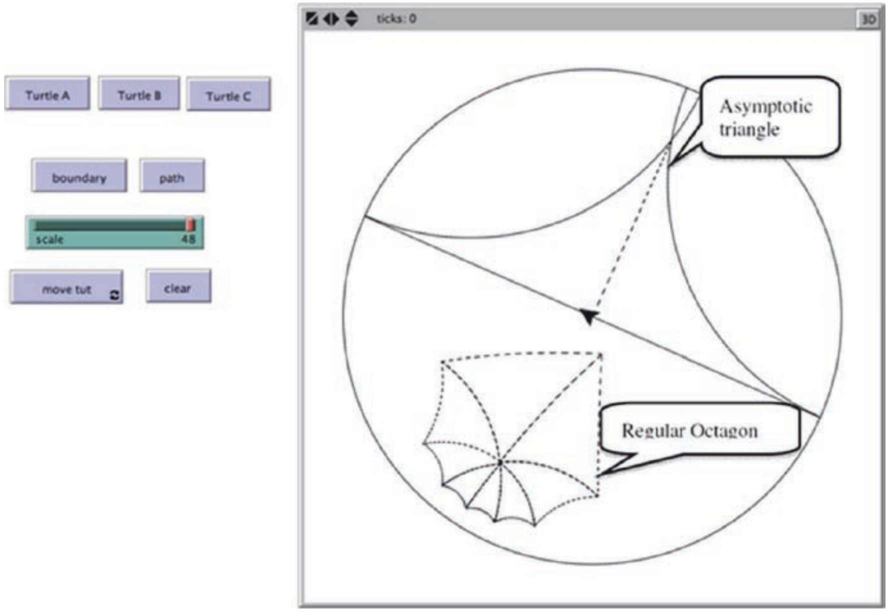

Fig. 4.3 Using *Logo* to create an asymptotic triangle in hyperbolic geometry

A snowflake, for instance, has an infinite perimeter, but a finite area – classic properties of fractal objects – leading to the scale-independent complexity of objects like the Mandelbrot set (Blanchard 1984; Mandelbrot 1980). Exploring fractals with Turtle geometry is thought to be powerful because such objects can be defined entirely in terms of four basic *Logo* commands (forward, back, right, left) and recursion.

Given the issues involved in the design of software for different geometries, we now turn to the issues in designing learning activities that attempt to realize the affordances, but take account of the constraints that are part and parcel of such software environments.

4.4 Designing Learning Activities to Engage Students with Different Geometries

As, when using 2D DGEs, dragging provides learners with an interactive way of validating their own constructions, much effort in task design has focused on encouraging learner conjecturing and on developing sequences of tasks that move pupils from conjectures to proofs (for examples, see Laborde et al. 2006). One interesting form of task is akin to a 'black box' (see Laborde 1998) by which learners are provided with a DGE figure for which they do not know the construction. The task is to construct a figure which has identical behaviour when dragged.

Such a task is not possible with paper-and-pencil technology. This illustrates the powerful affordances of 2D DGEs. For more on task design for 2D DGEs, see, for example, Garry (1997) and Laborde (1995, 2001).

In designing learning activities for 3D geometry software (both Euclidean and co-ordinate), the complexity of the on-screen image, and the need for learners to orient themselves to a flat-screen representation of 3D, need to be taken into account. There may also be issues for users moving from 2D DGE to 3D software. For example, in 2D DGE the 'perpendicular' tool produces a line, while in *Cabri 3D* the 'perpendicular' does not produce a *line* perpendicular to a chosen line because the perpendicular to a line in 3D is a plane (and the perpendicular to a plane is a line).

Given such issues, the ways that the tools available in 3D software mediate the learners' understanding of geometry are only just being researched (see, for example, Accascina and Rogora 2006). In designing learning tasks, Mackrell (2008), for example, has found that Grade 7 and 8 students can be highly motivated to use *Cabri 3D* to create their own structures. Such structures included models of 'real-world' objects and/or objects that moved, with the creation of such structures necessitating the use of a range of mathematics. The 'flat' representation on the screen appeared to have an influence on student use. For instance, in order for an object to have a particular visual property when viewed from all angles (such as a segment being perpendicular to the base plane) the object needs to be constructed using the mathematical tool which creates the desired relationship (in this case the Perpendicular tool). Animation also appeared to be important in that it is only points that can be animated and hence other moving objects need to be constructed in relationship to the points.

Research on the use of software such as *Autograph* appears to be more limited, though teaching ideas involving the intersections of planes, and volumes of revolution (in Calculus) are provided by Butler (2006). More systematic studies of the use of software packages such as *Autograph* are needed.

In terms of the research on constructing a Turtle-based microworld for non-Euclidean geometry, several principles illustrate the importance of the interplay between design and learning, especially the learner-centered development of tools and activities that mediate understanding in specific geometries (Stevenson and Noss 1999; Stevenson 2000). In Stevenson's research, Papert's (1980, 1991) principle of finding links to cognitive development was a central design feature. These links emerged by working with learners to find what engaged them with the structures of the new geometries. Three types of links were needed to help learners connect with non-Euclidean turtle geometry because of the complexity of the screen images: physical surfaces and their projection, metaphors, and on-screen structures. Through tracing paths on the physical surfaces with their fingers, learners were able to make sense of what they saw on screen by metaphorically linking their action with the screen turtle. Utilization of the metaphor 'turtles walk straight paths' helped learners identify 'straight lines' on curved surfaces with straight lines left by the turtle on the screen (Abelson and diSessa 1980). By 'dashing' the turtle's path so that the dashes grew longer or shorter according to the geometry, learners were provided with an on-screen structure that indicated that the turtle's steps were expanded or contracted by the 'temperature' of the screen. A corresponding speeding up or slowing down of the turtle's movement as it left dashes, coupled with a tool that drew the large-scale path which a turtle might take given a particular position and heading, provided a dynamic structure for learners to build up their understanding. The key point here is that these physical, conceptual, and virtual resources emerged through looking for cognitive 'hooks' in these specific geometrical contexts.

Overall, the principle of iterative design (see, for example, van den Akker et al. 2006) is a feature of much work on learner activities as such a perspective pays careful and systematic attention to learners' needs. For example, in Stevenson's research on non-Euclidean geometries, the non-Euclidean microworld emerged through analysis of a series of structured activities and observations based on the relationship between the roles, tools and organization of resources over three cycles of development. It used a combination of didactic intervention, reflective discussions, task-based interviews and non-participatory observation of learners. Each of these roles was applied consciously in designing activities to achieve particular design objectives.

In this section, and in terms of switching attention to how learning activities are designed, what also needs to be acknowledged is how the activities are transformed *in use* by learners and teachers, and that feedback from task design can lead to further modifications of software design. As Harel (1991) points out, learning and designing are intimately connected, both for 'learners' and 'designers'. As a field, mathematics education has benefited from some useful connections between technology designers and users, perhaps no more so than in the area of geometry education.

4.5 Shaping, and Being Shaped by, Digital Technologies

In this chapter we have shown how key decisions taken by designers of digital technologies for mathematics are influenced both by the mathematics involved (in the case of this chapter by geometrical ideas of projection, curvature, local and global co-ordinates, and so on), and by the affordances of the available flat-screen technology. For more examples of the design process see Battista (2008), where a case study of the design of a 2D geometry microworld is presented, and Christou et al. (2006), where the theoretical considerations in the design of a form of 3D geometry software are revealed.

We have also examined the ways in which the design of learning activities is affected by, but also affects, the design of the digital technology. As we have illustrated, the software packages featured in this chapter exemplify how mathematics *and* learner needs influence the design of the digital technology, while, at the same time, the use of these digital technologies undoubtedly shapes the mathematical activity of the user. It is this symbiotic beneficial relationship that is continuing to offer so much - not only in the area of geometry education, but also as fruitful ways are being developed of linking geometry and algebra (Jones 2009).

Given that the book in which this chapter appears follows on from the very first ICMI study (Howson and Kahane 1986), it is appropriate to conclude by looking forward to the follow-up to this present study. It may be that, in another 20 years, we will have moved beyond flat screen technology, perhaps to a spherical screen for spherical geometry, and perhaps to 'virtual reality' (VR) environments which embed the user in space, something that is already being tested (see, for example, Kaufmann et al. 2000; Moustakas et al. 2005). In June 2007, *Flatland the movie*, an animated film inspired by Edwin A. Abbott's classic novel, *Flatland* (originally published as Abbott 1884) was released. Perhaps, in due course, we can look forward to the release of *Flatland the VR game* in which the learner might take part as one of the 'creatures' in *Flatland* and experience (in 'virtual reality') what it is like to 'live' in a flat land.

Perhaps it is fitting to finish with raising the issue of just how 'direct' is what is often called 'direct interaction' when interacting with different geometries using digital technologies. As digital technologies for geometry develop, will users feel that they are interacting directly with geometrical theory; or will rapidly moving dynamic on-screen images seem more like computer-generated imagery (CGI) of the form commonly found in contemporary movies? How, we ask, can interaction with different geometries be facilitated through different digital technologies in a way which successfully builds the visual intuition that we need, as humans, to understand our experience of physical and mathematical space? We look forward to further research on such issues.

4.5.1 Coda

This chapter examines the design of digital technologies and associated forms of learning activities for a range of geometries. The purpose is analyzing how design is influenced by the mathematics involved, by the affordances (and constraints) of the available technology, and by the needs of the learner. If space had permitted an even longer chapter title, then the borrowing of Abbott's (1884) subtitle *a romance of many dimension*s could well be appropriate. While there is no space in this particular chapter for analyses focusing on other areas of mathematics (such as algebra or statistics), such analyses would usefully complement this chapter and are to be encouraged.

4.5.2 Notes

The main geometry software mentioned in this chapter (with publisher or contact in brackets) are as follows:

- *Autograph* (Autograph Maths)
- *Cabri* (Cabrilog)
- *The Geometer's Sketchpad* (Key Curriculum Press)
- *Logo* (Logo Foundation)

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Chapter 5
Implementing Digital Technologies at a National Scale

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Abstract In this chapter we describe a range of digital technology implementation projects that have been undertaken at a national scale in different parts of the world. These projects vary widely in breadth, in the digital technologies involved, in their relation to mandated curriculum and in their involvement of different stakeholders. We compare these different projects with a view to identify some significant trends that are currently developing in such efforts, and also with a view to guide future large-scale implementation work. We also analyse the projects in terms of relevant theories of technology use in mathematics education.

Keywords Implementation • Digital technologies • ICT • Professional development • Curriculum • Policy • Software • Activity design • Teaching practices • Instrumentation

5.1 Introduction

The growing number of large-scale digital technology implementation projects may well be considered a sign of the relatively widespread acceptance of the presence of computer-based tools in the mathematics classroom - acceptance in communities such as teachers, administrators and policy makers. While early work with digital technologies tended to focus on individual learners, and then perhaps classroom – or school-based groups, these large-scale projects demand a

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much more systemic approach that takes into consideration issues such as teacher adoption and curriculum integration. In this chapter, we study a small sample of current large-scale implementation projects with a view to both comparing and contrasting their choices – What technologies have they chosen to implement? How closely do they integrate with the curriculum? How much effort do they dedicate to supporting teachers? – and looking for overall trends across the projects. We also analyse the strengths and weaknesses of these choices.

Readers with a long history of involvement with digital technologies in mathematics education might find it interesting to turn back twenty years and, from that viewpoint, try to predict how large-scale implementation projects would play out. Would they involve the introduction of new mathematical topics especially well-suited to computer use, such as fractals, and the suppression of other topics made redundant by digital technologies, such as long division or plotting graphs by hand? Would they continue to focus on new opportunities for and modalities of student learning? Would implementation projects focus on a particular piece of software, such as Logo, or would they adopt a more pluralistic approach? Would incoming teachers, who have grown up in a computer-based world, respond differently to implementation attempts than more experienced teachers? Would large-scale implementation projects involve equipping all classrooms with computers or would they continue to develop the idea of the computer lab? Readers may well have more extreme predictions to make given the early hope and rhetoric around the use of digital technologies.

In the following section, we provide a brief description of the large-scale implementation projects we considered, and invite readers to refer to the conference proceedings (see Behrooz 2006; Dagiene and Jasutiene 2006; Paola et al. 2006; Jackiw and Sinclair 2006; Trigueros et al. 2006) for more complete descriptions. We then offer a first round of interpretation of these projects that seeks to compare and contrast them on a number of axes that seem pertinent both to traditional concerns of technology implementation and to emerging theories related to the use of technology in teaching and learning. Through this analysis we seek to identify some trends that might be useful in future work on large-scale implementation. Lastly, we take a second round of analysis through a theoretical lens of instrumentation, and offer a different interpretation of the projects, their goals, and their impact.

5.2 Overview of the Projects

We now summarise each of the following projects: Mexico's *Enciclomedia*, Italy's *M@t.abel*; the US's *Sketchpad* for *Young Learners*, Lithuania's *Mathematics 9 and 10 with The Geometer's Sketchpad*, and Iran's *E-content initiative*. As we move into the comparative sections, we will also provide some more detail on the project – the sections below are meant to provide only a flavour of each initiative.

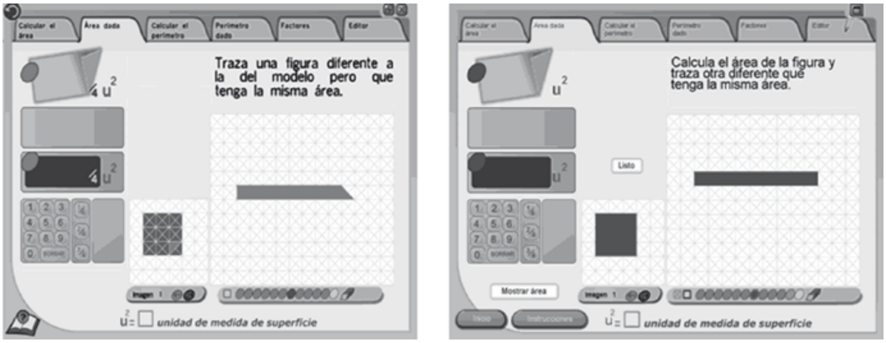

Fig. 5.1 The *perimarea* interactive enciclomedia activity

5.2.1 Enciclomedia

This Mexican national project is aimed at the primary school classrooms (60,000) and intends to complement existing textbook materials with computer programs and teaching resources that are to be used with an interactive whiteboard. The mathematics section is part of a wider initiative known as Enciclomedia, which has produced electronic versions of grades 5 and 6 textbooks that are used in all primary schools. These textbooks have now been complemented to include links to computer tools designed to help teachers with the teaching of all subjects. Enactivism provides the theoretical and methodological underpinnings of the project; methodologically, it supports the iterative design approach of the resources and theoretically it supports the close connection between learning and the use of tools in human action.

The project seeks to identify concepts and problems that students find difficult in the textbooks and complement them with appropriate technology-based tools. The difficulties are identified through existing research, conversations with classroom teachers and in-depth analysis of the textbooks' chapters specifically those that can be problematic. The resulting tools vary in terms of their interactivity, but are closely related to conceptual ideas presented in the textbooks and always pose a problem for students and teacher exploration. Figure 5.1 shows a screenshot of one of the interactive activities that involves the concept of perimeter. These virtual programs are accompanied by teaching guides that advise teachers on the use of the interactive programs and encourage collaborative work.

Research on the classroom use of these programs leads to further refinements of the digital programs. For example, changes in the mode of feedback were made on the *Perimarea* program after researchers observed that students were randomly guessing at answers. The researchers have found that teachers and students like the resources, although many teachers fear their use and that explicit direction for teachers helps to overcome that initial fear.

5.2.2 M@t.abel

The Italian project is the most extensive of the large-scale projects described here. It targets the lower secondary school (grades 6–8) and the first two years of higher secondary school (grades 9–10). M@ implements and supports part of the curriculum that was developed by the Italian Commission for the Learning of Mathematics and focuses particularly on the notion of the mathematical laboratory in the classroom. The project involves both the development of teaching and learning materials, and, perhaps most importantly, the training of teachers to use these materials. Through a train-the-trainer model, the project expects to reach almost all the mathematics teachers in grades 6–10 within a few years. Many types of mathematical software are used, including Excel and Java-based dynamic geometry programs, and are intended to support exploration activities and to mediate, with the orchestration of the teacher, the transition from personal, concrete meanings that the students develop with the proposed activities to the more abstract, scientific meanings.

The project focuses on four thematic areas: numbers and algorithms, geometry, relations and functions, and data handling. There are also three transversal themes: measuring, argumentation and conjecturing, and posing and solving problems. Each activity contains a problem description as well as a diverse set of resources including a worksheet and an applet. The activities revolve around a problem situation (see Brousseau 1997) for which no routine procedures can be used in solving the problem. Students are given different tools that can be used to explore the problem, as in the "Luca" problem in which a boy sees a picture of himself at a younger age and wants to figure out how much he has grown since then. In another problem, students are given a problem involving calculating grains of sands in which they must investigate the powers of two. Figure 5.2 shows how the students can increase the powers of two and see the accompanying graphical representation change accordingly.

One of the most important components of the project involves the training of tutors (100 expert teachers), who are then responsible for the training of almost 4,000 teachers. We will provide some more detail on this component of the project in a later section when we compare the modes of implementation across different projects.

5.2.3 Isfahan Mathematics House: E-Content

Iran's project focuses at the high school level, and particularly on early calculus concepts (definition of functions and limit, continuity and derivatives). The project is housed at the Isfahan Mathematics House, which is not affiliated with the country's ministry of education. The crux of the project's work has been to develop effective teams of teachers, mathematicians and programmers who work together to create Java applets related to the above-mentioned concepts. In particular, the project decided to create a mediator or interface role, which would be played by someone who could bridge the discourses of the teachers, mathematicians and programmers. To date, the project has had little penetration, as teachers are

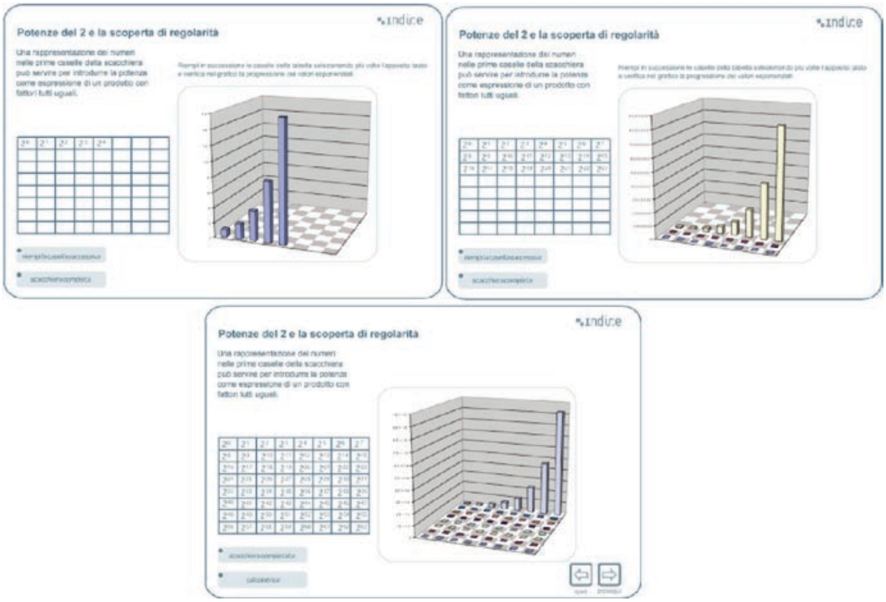

Fig. 5.2 Looking at powers of 2 in M@t-abel

reportedly afraid of using the computer, and of being confronted with unexpected student questions. There is also little support from Iran's educational system for the adoption of these applets.

5.2.4 Mathematics 9 and 10 with The Geometer's Sketchpad

Unlike the previous projects, Lithuania's implementation effort has chosen to focus on one particular software package, namely, *The Geometer's Sketchpad*. The project involves high school level mathematics, specifically focusing on grades 9 and 10, and extends across the entire mathematics curriculum at those grades. Lithuanian teachers have very good access to technology – 27% of schools were already using *Sketchpad* before the project began – but teachers do not always have the time and expertise required to develop appropriate sketches for teaching, and this provided the main impetus for development and implementation work.

Project development began by identifying problematic dimensions of teaching mathematics in schools, and focused on creating resources that could better foster, through dynamic visualization, the understanding of definitions, properties and proofs. This resulted in the creation of over 800 sketches, which can all be used in a whole classroom situation (using a digital projector), covering approximately 50% of the mathematical topics introduced in secondary school (as dictated by the Lithuanian National Curriculum). Students do not typically have direct access to the use of the software, since the teacher manipulates the sketches at the front of the classroom. However, given the high curriculum integration and the large number of sketches, students receive repeated exposure to dynamic visualization.

5.2.5 Sketchpad for Young Learners

As with the previous project, this one involves the exclusive use of *The Geometer's Sketchpad*. However, it is aimed at "young learners" from the primary school grades 3–8. As the U.S. does not have a national curriculum (these are dictated by states or school districts) the SYL project chose to work with a small number of reform curricula and to develop materials that were well-integrated with the mathematical ideas and tasks found in these textbooks. The goal of the project was to develop sketches and activities that could use the dynamic visualization capabilities of *Sketchpad* to help students explore and understand mathematical concepts and problems. Most the activities involve pre-built sketches, where students are invited to drag objects, change parameters, make simple constructions and change style elements. The activities come with extensive teacher notes that suggest how teachers can structure the use of the sketches. These teacher notes also help teachers adapt questions they might have used before to be more appropriate to the new representations offered by dynamic geometry. Most activities are intended for a laboratory situation, though some are designed for whole classroom use. A first set of activities have been made freely available on-line, and the secondary school grades 6–8 subset of these activities have already been downloaded by over 5,000 teachers.

A sample activity from the SYL project is Jump Along, which is targeted at the grades 3–5 level. Students choose different parameters for the number of the jumps and the size of each jump, and then watch as the jumps are made on the screen. This activity enables them to explore concepts around multiplication, factors and covariation (see Fig. 5.3). It focuses attention on visual representations of these concepts, rather than on computational aspects. The dynamic way in which the jumping occurs shows multiplication as a process – one in which 2 × 12 will look very different than 12 × 2, despite the two processes leading to the same place on the number line. Most of the activities developed in SYL involved the use of sketches that modelled, through dynamic geometry, various mathematical concepts.

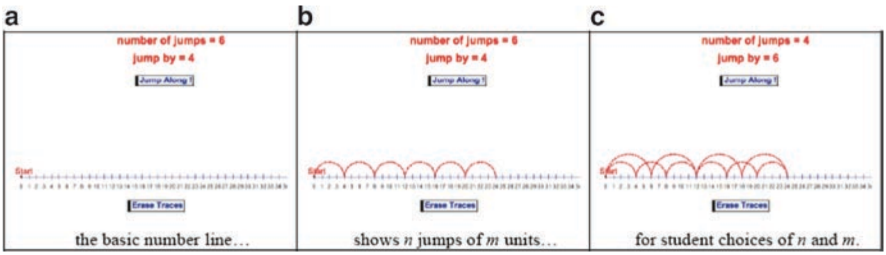

Fig. 5.3 The jump along activity

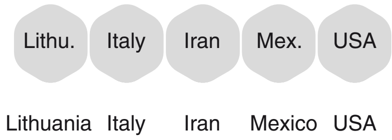

Fig. 5. 4 Legend for each project

5.3 Comparing and Contrasting the Projects

As was evident from their descriptions above, these projects differ in many ways, and our first approach to trying to compare and contrast them was to identify a set of characteristics that could be applied to each project. For example, each project could be placed on a continuum regarding the extent to which the technology-based activities directly supported an existing curriculum. In the following section, we attempt to create an overall picture of the projects by placing them on a set of axes representing basic characteristics of the project. Our goal is to examine the intentions and goals of the projects, rather than their outcomes. Figure 5.4 shows the legend we will use to position each project on the three axes that follow.

5.3.1 Curriculum Content

By their very nature, the use of new tools will have an effect on students' mathematical actions, and will thus change the intended mathematics curriculum, especially when that curriculum is conceived of and designed with only traditional tools in mind. However, increasingly, curricula in different countries are being developed to include the use of digital technologies – with some explicitly designating topics, problems or investigations to be delivered using specific educational software and others simply suggesting or encouraging the use of digital technologies. Changes in content can vary from being quite substantial, involving, say, the addition or new topics (such as fractal geometry) and the deletion of existing topics (long division). They may appear more subtle, involving, for example, changes in reasoning processes, such as using the kind of reasoning by continuity that dynamic geometry software offers. They may entail changes in the kinds of questions teachers ask (for example: with a dynamic number line, teachers can ask primary school students questions involving not only whole numbers, but also integers or real numbers).

In contrast with Papert's (1980) vision expressed several decades ago about learning and teaching *without* curriculum, most of the large-scale projects have chosen to stay *within* the curriculum, and to complement, rather than diverge from, mandated curricula (see Fig. 5.5). However, some projects also strove to encourage new variations of curriculum content. For example, the Enciclomedia project offers activities aimed at scaffolding student learning, that is, activities that either provide more elementary mathematical ideas or that offer different ways of approaching those ideas. This might include a visual animation that helps illuminate a certain difficult idea.

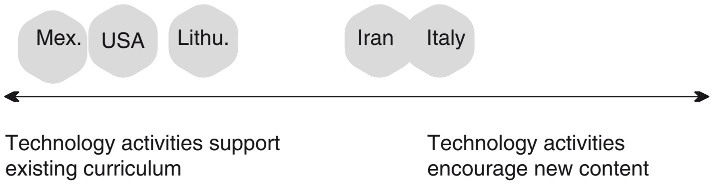

Fig. 5.5 Situating the projects on the curriculum content axis

The visual animation, while supporting the learning of the existing curriculum, can also provide new content, such as geometric explanation of a mathematical idea, or a mathematical connection between an image and an abstract mathematical idea.

The Lithuanian approach is quite different, in that the grades 9–10 project seeks to identify the mathematical ideas that can be directly represented through dynamic visualization, and can thus benefit from technology-based presentation or exploration. Here the expressed goal is to support student learning within the existing curriculum, and to encourage new content through more implicit means, through the use of the tools in the software environment and the modes of interaction.

As with the Lithuanian project, the US project also attempts to identify curriculum-based activities that are well-suited to dynamic visualization. However, it has been more explicit about highlighting and promoting new content opportunities such as focusing on the behaviour of mathematical objects instead of their properties and on promoting the use of the dynamic number line as a fundamental mathematical model. Indeed, in creating some activities, the dynamic nature of the software has led to tensions between the continuous representations that are intrinsic to *Sketchpad* and the more discrete representations often used in the primary school grades, where whole numbers dominate. An example is an activity in which students are asked to break a stick into three parts and determine whether those parts can form a triangle. In the textbook, students consider only whole numbers, but in the Sketchpad-based activity, one can explore the situation in a more continuous manner, where the lengths of each part of the stick might not be a whole number. Some teachers resisted exposing their students to non-whole numbers, even if no calculations were required. Other teachers encouraged their students to use the visual representation of the lengths of the sticks so that they could qualitatively compare them. While mere exposure to advanced concepts at the primary school level does not constitute a change in content, it does present interesting challenges and possibilities.

Like the SYL project, the Italian M@ project finds itself working to support existing curricula, but having to do so in a context in which digital technologies are very rarely used in classrooms, despite being promoted in the formal curriculum. One goal of M@ is to provide teachers with opportunities to use technologies in their classrooms. In this sense the initiative encourages new content according to the philosophy of the Mathematics for the Citizen, namely of a mathematics whose cognitive roots are sought in everyday life and whose theoretical aspects are built from the concrete argumentations of the students.

We find it noteworthy that the projects we have described here are so modest in their ambitions to encourage new content for the mathematics curriculum. Perhaps, following extensive arguments for the benefits and opportunities of doing new mathematics with the new technologies, we are now in a period of restraint in which the goal is to support teachers in making the technologies work within the scope of existing school mathematics. As we shall see, the traditional interest in new content for learners seems to have shifted to emphasise new practices for teachers.

5.3.2 Teaching Practices

On the whole, comparing the axis below with that focused on content, the projects were more intent on endorsing new practices than they were in encouraging new content (see Fig. 5.6). By practices we include a wide variety of normative behaviours that might include ways of structuring interaction in the classroom (lecturing, using individual problem solving, coordinating small-group work), ways of assessing students (homework, quizzes, alternate forms of assessment), and ways of interacting outside the classrooms (developing lessons with colleagues, attending professional development workshops). We provide these just as examples; they are certainly not exhaustive of the range of practices involved in teaching, or targeted by the projects.

The Mexico project provides an example in which the practices being endorsed involved teachers' ways of structuring interaction in the classroom. The technology-based activities designed for Enciclomedia are intended to motivate students to be engaged in mathematical activities by inviting them to take part in games and by providing interesting contexts. Interactive whiteboards also encourage students to take part in the activities. Students can use the programs to validate their solutions to mathematical problems. The software becomes an additional source of knowledge, different from the teacher. Teachers themselves often resort to the programs to experiment with different solutions, and some of them start their lessons by working with an activity within Enciclomedia, without giving a formal explanation to the students.

The Lithuanian project focuses on one new practice to endorse, while leaving the others more or less stable. The new practice that is endorsed is mainly related to providing students with more visual and dynamic means of representing and explaining mathematical ideas. The project materials, which complement the widely-used textbooks, do not require special time for preparation, nor adjustment in classroom interaction structures.

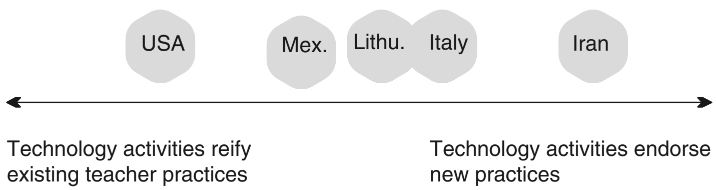

Fig. 5.6 Situating the projects on the teacher practices axis

Italy's M@ project has an explicit aim to endorse new teacher practices; it is one of the main goals of the project. New endorsed practices include not only providing students with more exploratory mathematical tasks mediated by digital technologies, but also participating in a community of colleagues (including tutors and fellow teachers) to discuss the impact of their new initiatives.

Iran's project provides an interesting counterpoint to the ones described above in that the changes in teacher practices that the project endorses are done under a much less developed background of standards. So, for example, while the US may be seen as endorsing new practices relative to the ones most prevalent in elementary school today, it basically reifies the practices that have been advanced by national organizations such as the NCTM – practices that only a small portion of American teachers have adopted (see Jacobs et al. 2006). In the same way, M@ seeks to reify new practices advanced by *Mathematics for the Citizen*, a curriculum designed by the Italian Commission for the Learning of Mathematics. Iran's project seeks to endorse new practices that are not yet mandated at the national level, and that are extremely rare within the teaching population.

The projects differ also in terms of the level and nature of support offered to teachers. The most extensive level of support is found in the M@ project, which includes an explicit train-the-trainer model, in which teachers receive on-site support from a designated tutor who has already used the M@ materials. Teachers also follow an e-course to help them learn about teaching with the new materials, and they are asked to join a group of 12 colleagues with whom they can share questions and difficulties. The goal of the project is to provide teachers with "e-shared practices" through this extensive networking.

On the other end of the spectrum, neither the Lithuanian nor the Iranian project provide targeted teacher support; for the former project, teachers do not seem to require either the mathematical or technological support, whereas for the latter, teachers seem to be lacking basic cultural support from the ministry or their colleagues – and are thus unwilling to try the new technology-based activities proposed to them.

In the middle of these two extremes, we find projects such as Enciclomedia and SYL which have both endeavoured to cater to teachers in terms of providing direct curriculum links, specific suggestions on how and when to use the technologies, and, in the case of SYL, many pre-made sketches that do not require much expertise with the software.

5.3.3 Activity Design

Our previous axis focused primarily on the role of the teacher in each of the projects. However, this next axis considers more closely the design choices made in terms of the students' use of the tasks and tools. Our first interpretation of this axis involved considering the extent to which the particular technology was being used as an expressive medium. Take the following two ways of working with Excel for example. An open use of Excel might involve the student being asked to compare different rates of growth - which would involve the student inputting the necessary formulas and using a graph or table to compare the rates of growth. In a more closed use, the formulas might already be set up, as well as the graphs, and the students would simply change certain parameters in the spreadsheet.

There has long been a tension between the desire to provide students with "open," creative, constructionist ways of working with mathematical ideas and the necessity of restraining and constraining students to specific mathematical learning outcomes. Many technology-based tools for mathematics learning, such as *Logo*, *Boxer*, *Sketchpad*, *Cabri*, *Fathom*, have been stimulated by the attraction of offering students more "open" mathematical experiences, that is, with environments in which students can express themselves mathematically through the particular software's "programming" language. Such "open" activity can (and should!) often lead the students and teacher into unknown territory, and this may lead to tension created by a teacher's desire to maintain curriculum fidelity.

Such "open" experiences stand in contrast with ones in which students follow step-by-step exercises or drills in which students have little effect on the way the problem is defined and little choice on the way it must be solved – technology-based examples might include early example of CAI and many of the Java applets currently available on the internet (see Sinclair 2005). As research has shown, "open" experiences are very difficult to manage and create for teachers, especially under circumstances such as high-stakes testing, poor mathematical knowledge or parental and societal disapproval.

We note that Hoyles et al. (2002), describe some more middle ground interactions using *Logo*-based microworlds in which the students' actions are restricted to certain objects, without compromising their expressive potential. These sorts of microworlds have also been created in *Cabri* and *Sketchpad*, using the macro or custom tool commands. They are much more difficult to create in Excel, Graphing Calculators, or with Java applets due to programming limitations.

With this particular interpretation of "open" vs. "closed" we noticed a strong tendency towards more closed student use of technologies. The pre-made sketches of the Lithuanian project and SYL, the highly authored exploratory resources in the Enciclomedia project, M@ and in Iran – these all contrast significantly with the more "open" activities in which students create a house in *Logo* or constructed a square in *Cabri* in the sense that the student is restricted to operating on pre-defined objects and relationships.

The tendency toward the "closed" side of the continuum (see Fig. 5.7) was interesting in terms of the stated goal of several of the projects to provide maximally "open"

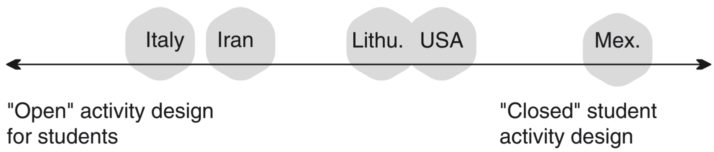

Fig. 5.7 Situating the projects on the activity design axis

environments. It is perhaps not surprising that the endeavour to appeal to a broad range of teachers, students and stake-holders has an effect of tugging toward more specific, restricted tools that are easy to learn and identify. The tendency toward the use of closed activities may also be related to an increase of attention to the role of the teacher in the technology-rich classroom. The designers of these projects had the teacher centrally in mind, and made the decision to offer simpler tools together with teaching guides that explicitly support the process of instrumentation (how to use the tool).

The Enciclomedia project reflects well the tension that results from the interplay between large-scale implementation and "open" design: while it aims for open activities, it must relate to the content of the textbook chapters and the curriculum. The project has favoured the design of tasks where students and teachers can explore mathematical ideas before they attempt to solve the corresponding problems in the textbook. However, in order to address very specific misconceptions that students develop with particular concepts such as fractions or proportionality, these tasks are very directed and focused, thereby limiting students' expressive power.

The Lithuanian project has a slightly more "open" design. This is in part due to the fact that they are using *Sketchpad*, which, in contrast with internet-based applets or animations, offers a wide range of tools that are always available to the student, even with a pre-built sketch. So, while we may not consider it significantly more "open" if a student can change the colour of a particular linear graph, other affordances of the software can indeed allow the student more expressive power. The fact that all the activities consist of pre-built sketches certainly limits the students' scope.

The SYL project combines pre-built activities with more "open" ones in which the students use construction tools to create mathematical models and explore mathematical relationships. While these activities are still quite highly choreographed, they do provide students with additional tools for mathematical expression through which they might approach a problem in multiple ways or pose new problems.

The M@ project encouraged us to consider the "open" vs. "closed" axis somewhat differently. This project consists of an on-line platform in which an entire teaching situation is proposed, including worksheets, Java applets, and pointers to non internet-based communication technologies. Furthermore, the project has adopted the "mathematics laboratory" approach, which suggests a more open environment that involves choosing from a broad range of tools. The project also emphasises the notion of "doing mathematics in order to communicate," which involves not only exploring mathematical ideas using an internet-based applet, but also using communication technologies in order to communicate their mathematical actions and understandings. The teacher is responsible for strongly mediating the use of the technologies in order to foster students' exploration and conjecturing.

While the specific Java applets offered to the students may be thought of as being "closed" in terms of our original distinction, the communicative expectations can be seen as much more open in the sense that students are not confined to acquiring specific content areas but are expected to engage in important process goals of conjecturing, exploring and communicating. The M@ project seems unique in this particular sense of openness.

5.4 Emerging Themes Across the Projects

In this section, we consider some of the critical themes that have emerged in our comparison of the different projects, and use a more theoretical lens to help us elaborate these themes.

5.4.1 Shifts in Audience: Moving Toward More Teacher Participation

As mentioned above, we see a shift in focus in some of the large-scale projects toward increased awareness of and attention to the teacher's role in using and deploying digital technologies in the mathematics classroom.

In order to explore this shift further, we consider two projects that are on opposite extremes in this regard: Lithuania and Italy. On the one hand, the Lithuanian project remains focused on the student, and on improving student understanding directly. The pre-made sketches developed by the project can help teachers in their lesson preparation (in that individual teachers do not have to create the sketches from scratch) and allow them to offer their students new dynamic representations of mathematical ideas. The pre-made sketches can also help teachers make the transition from lecture-style demonstrations to a modality in which students are manipulating the technology themselves. A further aim of the project is to help students learn mathematics on their own – it is envisaged that students can use the sketches to explore axioms, to comprehend features of mathematical objects, and to understand theorems. Teachers are seen as guides who can help students to reach this goal.

In M@, as we have described above, there is extensive attention on the teacher. The project includes direct work with teachers on and with the new technologies. Using the theory of instrumental genesis (Vérillon and Rabardel 1995), which is usually applied in mathematics education to the process of technology use for students (Artigue 2002), we might say that M@ focuses attention on the process of instrumentalization of teachers – that is, the process through which the teachers use and shape the technology. The teacher instrumentalization occurs within a framework in which the processes are strongly focused on the processes of instrumentalization they can promote in their own students when using technologies. There are thus two levels of instrumentalization at work in which the instrumentalization processes in teachers using M@, like a shadow, is nurtured through the corresponding instrumentalization processes of their students.

5.4.2 Shifts in Value: From Pragmatic to Epistemic

In this section we continue our focus on the Lithuanian and Italian projects, but now return our attention to the students. In particular, we look more closely at the changing emphasis on what students gain from their use of digital tools. In discussing the techniques that digital technologies offer for solving tasks (such as graphing or calculating), Artigue (2002) distinguishes two types of values that can be related to these techniques. She notes that "techniques are most often perceived and evaluated in terms of *pragmatic value*, that is to say, by focusing on their productive potential (efficiency, cost, field of validity)" (p. 248). However, she argues that techniques can also have *epistemic value*, as they "contribute to the understanding of the objects they involve" and thus become a "source of questions about mathematical knowledge" (p. 248). We offer a specific example that should help clarify the distinction.

Consider the difference between variables and parameters in a formula such as $y = ax^2$ (where x, y are variables and a is a parameter). The distinction is important, but can be very difficult for students to appreciate. However, a dynamic geometry environment can help students make sense of the difference. In Fig. 5.8, one can use a slider for the parameter a: moving it will change the shape of the parabola. Moving the variable x along the axis will change the location of point P on the curve.

The use of these two instrumented practices, dragging x and dragging a, have pragmatic value since doing them changes both x and a (as well as the location of P and the shape of the parabola). These actions can also help reveal the epistemic aspects of the pragmatic action of dragging. Dragging a changes the whole shape of the curve, while dragging x changes the location of P on the curve. The epistemic value is usually not apparent to the student, and often requires strong teacher mediation (see Schneider 2000).

Although Chevallard (1992) argues that the two values are inseparable, there is often more attention paid to one than to the other. In particular, when the focus is on learning how to use a tool, the pragmatic aspect is privileged. However, since many of the projects are moving towards technology-based activities that decrease the intensity of learning to use the tool, it is possible that the epistemic aspect becomes foregrounded.

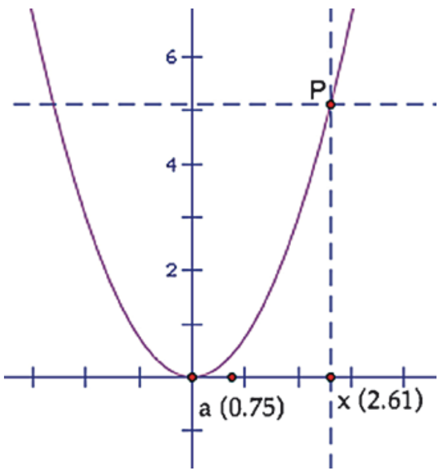

Fig. 5.8 Dragging variables and parameters in dynamic geometry

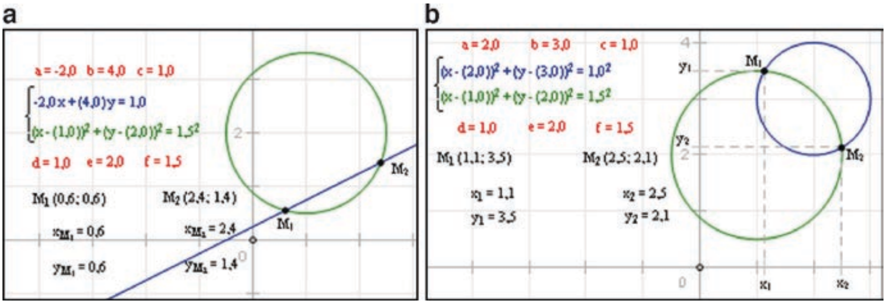

Fig. 5.9 The graphical solution of sets of equations **(a)** system of linear and circle equations, **(b)** system of two circle equations

The Lithuanian project attempts to interweave both the pragmatic and the epistemic. On the one hand, within each sketch, students are required to do something, whether it be changing a parameter or transforming an object (changing the equation, reflecting or rotating, or changing coordinates). On the other hand, students also use the tool to gain a visual appreciation of the mathematics. Figure 5.9 illustrates two sketches intended to help students work with systems of equations (that are not necessarily linear). The goal is to address the question of how many solutions such a system of equations will have.

The technique used is to change the parameters that define the graphs – this is the pragmatic value of using the sketch since the action produces different configurations of the objects that can allow students to assess how many solutions there might be. However, when changing the values of the coefficients, students must try to understand how each coefficient affects the shape, size and location of the corresponding object, and, eventually, what changes can contribute to producing cases with two, one or no solutions – and thus, through the task, the epistemic value of the tool is emphasised.

In the M@ project, the coordination of the two values plays a central role in the teachers' instrumentalization, in part because its underlying philosophy is that the teacher plays a crucial role in mediating the two values for the students. As a result, each suggested practice (proposed to teachers during their discussions with the tutors) always incorporates an epistemic aspect; this aspect is particularly stressed by the tutor in the discussion in the e-learning platform. Influenced also by the anthropological approach of Chevallard, which has drawn attention to the issues of epistemic vs. pragmatic values, the Enciclomedia similarly considers epistemic value in the design of resources and teachings guides. The resources promote the pragmatic value but not by itself, always together with the epistemic value. The distinction has not been as central in the SYL project, given its different theoretical positioning.

5.5 Concluding Remarks

To date, we have minimal evidence regarding the success of these different implementation projects either in terms of their uptake by teachers or their effects on student learning. In looking across projects, it seems that some of the greatest achievements have been a better coordination between researchers and teachers, with the former slowly devising better strategies for increasing the use and effectiveness of digital technologies, strategies that are much more sensitive to the constraints of the curriculum and to teacher instrumentation.

We also note that the types of digital technologies used in these projects (predominantly dynamic geometry environments and Java applets) reflect a very small portion of the digital technologies discussed in this book. While such choices seem sensible given the level of access offered by Java applets and the widespread use and acceptance of dynamic geometry environments, we wonder what effect their large-scale implementation will have on the use and development of other software programs such as *Lunar Landing* or *Graphs 'n Glyphs* (described in Chap. 3). Indeed, it appears that most of the projects (except M@) chose to focus on the use of one multi-purpose digital technology. Such a choice has the advantage of minimizing scaffolding work for teachers and students. In fact, Goldenberg (2000) has coined the "Fluent Tool Use Principle," which states that "learning a few good tools well enough to use them knowledgeably, intelligently, mathematically, confidently, and appropriately" (p. 7) can greatly contribute to a students' mathematical education. On the other hand, it may curb the adoption of newer software programs that can respond to emerging needs in mathematics education.

5.6 Looking Forward

In their three-wave model of the development of digital technologies in mathematics education, Sinclair and Jackiw (2005) describe the evolution of attention from "Wave 1," with its exclusive focus on the relationship between an individual learner and mathematics itself, to "Wave 2" and its broader focus on the context of learning, including the teacher and the curriculum. Examples of Wave 1 technologies include both Logo and multiple-choice testware of the 1970s computer-assisted instruction – both first wave, despite their diametrically opposed theories of learning. Wave 2 technologies include the graphing calculator, spreadsheets, and interactive or dynamic geometry – technologies that have been adopted by several of the projects described in this chapter. Both the exigencies of the national-scale implementation projects (which must look beyond individual students in classrooms) and the very nature of Wave 2 technologies may account for the strong focus we have seen on the teacher and the curriculum in the projects described here.

While Wave 2 technologies are most pervasive today, Sinclair and Jackiw anticipate the emergence of Wave 3 technologies, which involve yet another expansion of the technology's pedagogic focus to include relationships between individual learners, groups of learners, the teacher, the classroom, classroom practices, and the world outside the classroom. They cite proto-third wave technologies such as the "networked calculator", with its emphasis on collaboration and information-sharing. They also point to ways in which third wave technologies might evolve out of "amplifying" technologies such as networking facilities, interactive whiteboards or videoconferencing, which render second wave technologies more sensitive to a social knowledge web.

In our analysis of the large-scale implementation projects presented in this chapter, we have noted time and again how the focus has shifted toward a much greater attention to the teachers and to teacher practices (including teacher instrumentation), as well as on the interplay between digital technologies and the curriculum. When the focus is on the student interacting with the technology, as is the case in the expressed effort to re-balance epistemic and pragmatic values, the motivating concern is much broader than individual student learning; it relates to a greater awareness of the dynamic between conceptual and technical work, and the status of each in a given institution.

If there is a relationship between the types of digital technologies we now use in mathematics education and the trends we have seen emerge in current large-scale implementation projects, then it stands to reason that further changes in technologies – as in the move to the third wave – might affect the predominant concerns and goals of future implementation projects. Such projects might focus more on, for example, extending mathematical learning into students' homes (which may begin to affect *the role of the parent*), or extending the confines of the usual classroom (to include other students that may be only virtually present), or introducing new modes of communication that are consonant with cultural trends (deploying graphing calculators through cell phones). In cases such as these, we might expect to encounter a new set of axes – different from the one we proposed for this chapter – that relate to the extent to which the digital technology is accessible and pervasive (outside the classroom) or the extent to which the implementation involves activity development, teacher instrumentation, parent instrumentation, or infrastructural support.

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Section 2
Learning and Assessing Mathematics with and through Digital Technologies

Chapter 6
Introduction to Section 2

Paul Drijvers, Maria-Alessandra Mariotti, John Olive and Ana Isabel Sacristán

Abstract This introduction sets the scene for the volume section 2 on the theme of learning and assessing mathematics with and through digital technologies. It first describes the section's points of departure. Then each of the chapters of the section is briefly addressed. The introduction ends with a short reflection on the section as a whole, noting that the major content emphases are on algebra and geometry, with only limited attention to calculus, statistical reasoning, and proof. In closing, we call for a closer relationship between mathematics education research and educational science in general.

Keywords Mathematics education • Learning • Digital technology • Assessment • Learning trajectories

6.1 The Points of Departure

One of the themes that served to frame the 17th ICMI Study "Digital technologies and mathematics teaching and learning: Rethinking the terrain" and which was distinguished as a pivotal issue in the study, was that of *learning and assessing mathematics with and through digital technologies*. In the study's Discussion Document (IPC 2005), this theme was described as follows:

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> This theme will concentrate on developing understandings of how students learn mathematics with digital technologies and the implications of the integration of technological tools into mathematics teaching for assessment practices. Its foci will include consideration of how digital technologies might be employed to open windows on learners' developing knowledge, and on how interactions with digital tools mediate learning trajectories. Additionally, the theme will address the challenges involved in balancing use of mental, paper-and-pencil, and digital tools in both assessment and teaching activities. (IPC 2005, p. 356)

In the same document, the following six questions and issues to address were formulated:

1. What theoretical approaches and methodologies help to illuminate students' learning of mathematics in technology-integrated environments? What are the relationships between these approaches and how do they compare or contrast with other theories of mathematics learning?
2. How does the use of different digital technologies influence the learning of different mathematical concepts and the shape of the trajectories through which the learning develops?
3. How can technology-integrated environments be designed so as to capture significant moments of learning?
4. How can the assessment of students' mathematical learning be designed to take into account the integration of digital technologies and the ways that digital technologies might have been used in the learning of mathematics?
5. How can and should learning and assessment practices reflect differences in resource level and in cultural heritage?
6. How can the benefits of existing technologies be maximized for the benefit of mathematics teaching and learning?

In the next part of this introduction, we will provide a short "guided tour" through each of the chapters of this section of the study volume. Next, we will briefly reflect on these chapters from the perspective of the above questions. This introductory chapter concludes with some final remarks on the mathematical topics addressed and the relationship between mathematics education research and educational research in general.

6.2 A Guided Tour Through the Chapters

The first full chapter of this section (Chap. 7) is entitled "Integrating Technology into Mathematics Education: Theoretical Perspectives." The central question at stake in this chapter is: What theoretical frames are used in technology-related research in the domain of mathematics education and what do these theoretical perspectives offer? The chapter first provides an historical overview of the development of theoretical frameworks that are considered to be relevant to the issue of integrating technological tools into mathematics education. Then some current developments are described, with a particular focus on instrumental approaches and semiotic mediation. While discussing future trends, the authors observe theoretical advancements; still, the articulation of different theoretical frameworks is not realized. Also, some aspects remain underexposed, such as the role of language in instrumental genesis, the role of the teacher in technology-rich learning environments, and the influence of the available tools on tasks and task design. Connectivity, both among technologies and among theoretical frameworks, might be a key focus for future studies. A plea is made for the development of integrative theoretical frameworks that allow for the articulation of different theoretical perspectives.

The next chapter (Chap. 8) focuses on mathematical knowledge and practices resulting from access to digital technologies. It first describes how technology has influenced the contexts for learning mathematics, and the emergence of a new learning ecology. Notions of cognitive fidelity and mathematical fidelity are introduced to define criteria for the use of technological tools. Then, mathematical knowledge that "resides" within the different technologies is addressed. As a third issue, the changes in mathematical practices in education are considered. It is argued that interactions among students, teachers, tasks, and technologies can bring about a shift in empowerment from teacher or external authority to the students as generators of mathematical knowledge and practices; and that feedback provided through the use of different technologies can contribute to students' learning. As a result, the authors propose a transformation of the traditional didactic triangle into a didactic tetrahedron through the introduction of technology as a fourth vertex and conclude by restructuring this model so as to redefine the space in which new mathematical knowledge and practices can emerge.

The third full chapter of this volume section (Chap. 9) is entitled "The Influence, and Shaping, of Digital Technologies on the Learning – and Learning Trajectories – of Mathematical Concepts." The significant development and use of digital technologies has opened up diverse routes for learners to construct and comprehend mathematical knowledge and to solve problems. This implies a revision of the pedagogical landscape in terms of the ways in which students engage in learning, and how understandings emerge. The authors consider how the availability of digital technologies has allowed intended learning trajectories to be structured in particular forms and how these, coupled with the affordances of engaging mathematical tasks through digital pedagogical media, might shape the actual learning trajectories. The chapter begins with a brief theoretical overview to inform the various perspectives that frame the subsequent sections, including, in particular, a presentation of the construct "learning trajectory." This prefaces later sections addressing hypothetical learning trajectories and the affordances of digital technologies as pedagogical media. How digital technologies influence the evolution of learning trajectories is then considered through the use of examples that contextualize the construction of hypothetical learning trajectories, learning trajectories within and across various platforms, the emergence of actual learning trajectories, and the possibilities for earlier engagement with powerful ideas as afforded by digital technologies. The concluding section draws on these aspects and examples, to consider the manner in which the learning experience is transformed through the engagement of digital technologies. It also attends to the consequential influence of this alternative engagement on the evolution of the leaning trajectories and hence on learning.

The next chapter (Chap. 10) addresses the issue of automatic assessment supported by digital technologies. The authors describe computer aided assessment of mathematics by focusing on the micro-level of automatically assessing students' answers. Assessment is seen as a fundamental part of the learning cycle, central to learning and often a primary driver of students' activity. Significant technical developments of the last two decades are described through examples of internet based systems. As a conclusion, the authors stress the strength of computer-aided assessment through its immediacy and the mathematical sophistication of automatically generated feedback.

The final chapter of this volume section (Chap. 11) is entitled "Technology, Communication, and Collaboration: Re-thinking Communities of Inquiry, Learning and Practice." This chapter addresses the relationships between research on the role of technology in mathematics education and the framework of social learning theories, suggesting that social perspectives on teaching and learning with technology have become increasingly prevalent. A review of recent literature adds further support to the view that there is growing interest among the mathematics education community in how digital technologies can enhance mathematics teaching and learning through attention to social aspects of coming to know and understand. Four typologies of digital technologies and their role in collaborative practice are identified: technologies designed for both mathematics and collaboration; technologies designed for mathematics; technologies designed for collaboration; and technologies designed for neither mathematics nor collaboration. As new technologies continue to be developed and refined, they offer new ways to construe communication, collaboration and social interaction and thus change the availability and feasibility of different kinds of communities of practice. This has implications for both research and practice.

6.3 Looking Back at the Original Issues

After the above global chapter description, we wonder if we really contributed to the six issues as they were formulated in the Study's Discussion Document. To investigate this, we briefly go through these items, link them to the content of the chapters and summarize the main findings.

1. What theoretical approaches and methodologies help to illuminate students' learning of mathematics in technology-integrated environments? What are the relationships between these approaches and how do they compare or contrast with other theories of mathematics learning?

   Theoretical approaches receive considerable attention in this section. This item is addressed in Chaps. 7–9 and 11. While Chap. 7 provides a general overview of theoretical ideas, the main theoretical frames in that chapter are the instrumental approaches and the notion of semiotic mediation. In Chap. 8, a didactic tetrahedron is introduced so as to redefine the space in which new mathematical knowledge and practices can emerge from technology-rich mathematics education. Chapter 9 examines the theoretical construct of learning trajectories, considers it in relation to the use of digital technologies and attempts to connect it with one of the theoretical definitions of the idea of microworld. Chapter 11 focuses on the relationship between social learning theories and connecting or connective technologies. As one of the important future issues, Chap. 7 ends with recommendations for the development of integrative theoretical frameworks that allow for the articulation of different theoretical perspectives. Some first steps in this direction are sketched.

2. How does the use of different digital technologies influence the learning of different mathematical concepts and the shape of the trajectories through which the learning develops?

   The issue of learning trajectories is addressed in Chap. 9. The chapter shows how the availability of digital technologies informs the shape of learning trajectories, both from a theoretical perspective and in concrete examples. Other chapters (e.g. Chap. 8) contain practical examples and address implementation issues as well. Still, the redesign of learning trajectories for conceptual understanding in the context of digital technology remains an issue.

3. How can technology-integrated environments be designed so as to capture significant moments of learning?

   We take the word "capture" here not as the technological feature of recording students' actions while working in a technological environment, but rather as technology being a catalyst for significant moments of learning to happen. The examples throughout the chapters provide some answers to this question. However, a general answer is not given. This may constitute an open problem and a focus for future research. As the issue of the design of digital technologies is left to Sect. 5 of this volume, we refer to Chap. 21 in that section for more elaboration.

4. How can the assessment of students' mathematical learning be designed to take into account the integration of digital technologies and the ways that digital technologies might have been used in the learning of mathematics?

   Automatic assessment supported by digital technologies is addressed in Chap. 10. The strength of computer aided assessment lies in its immediacy and the mathematical sophistication of automatically generated feedback, which can be seen as the backbone of the system, but probably also as its Achilles' tendon. However, it could prove promising to investigate the potential offered by this new tool for enhancing teachers' work. Assessment that takes into account the ways in which digital technologies might have been used in the learning of mathematics is not explicitly addressed in Chap. 10. Chapters 8 and 9 tangentially touch on this issue in their discussion of how different uses of different technologies can affect learning and learning trajectories; but an explicit discussion of how to take into account those different uses for assessment purposes, or how to develop assessment methods that evaluate the learning that is developed through the use of digital technologies, is not included, and remains an area that requires still much research.

5. How can and should learning and assessment practices reflect differences in resource level and in cultural heritage?

   Item 5 is not addressed in this section but it is in Sect. 4, Chap. 18 and 19.

6. How can the benefits of existing technologies be maximized for the benefit of mathematics teaching and learning?

   This issue is addressed in Chaps. 7 and 8 and others, through the presentation of theoretical notions as orchestration and models such as the didactical tetrahedron, and through the discussion of concrete teaching examples. And the benefits, affordances and potentials of digital technologies for teaching and learning are amply discussed throughout the section, including considerations of how to integrate these into task-design (Chap. 9). Chapter 11 also suggests ways in which the new communication technologies can enhance learning opportunities through new modes of social interaction. Still, the issue of how to maximize the benefits of the integration of technology is hard to capture in overarching guidelines.

6.4 Concluding Remarks

This chapter ends with some concluding remarks on the mathematical topics addressed in this section and the relationship between mathematics education research and educational science in general.

If we consider the examples presented in this volume section, we notice a focus on (early) algebra, followed by geometry. Probably, this reflects the importance of algebra as a mathematical topic, and of algebraic skills being central in national and international discussions on the future of mathematics learning and teaching. Also, technological tools for supporting the learning of algebra are widely available. For geometry, the immense popularity of dynamic geometry systems has led to a huge library of teaching materials and research papers.

Probability and statistics, calculus and advanced mathematical thinking, are addressed to a lesser extent. Chapter 8 mentions some emerging research on using technology for the learning and teaching of statistics, and dynamic statistics software such as *Fathom* and *Tinkerplots* is becoming more widely used. In spite of this, research literature on this area is limited. As far as advanced mathematical thinking is concerned – in contrast with the first ICMI study (Howson and Kahane 1986) which focused mainly on developments at the tertiary level – calculus learning with technology is hardly discussed in this section, with the reports on SimCalc (see Chaps. 8 and 9) as exceptions; there is also some research presented on the development of infinity-related ideas with technology (see Chap. 9), but these are mostly isolated examples. Another area of advanced mathematical thinking that is mentioned, is that related to the development, through technology, of processes leading to the learning and construction of mathematical proofs; however, available research in this area, in particular that with the use of dynamic geometry, is only briefly mentioned in this section of the study volume. We look forward to the ICMI Study 19 for a thorough review of current research on the teaching and learning of proof in mathematics.

Our final remark concerns the relationship between mathematics education research and educational science in general. In the past, mathematics education research has been quite domain-specific. Research on the use of technology for the learning and teaching of mathematics, in particular, focused on the specific mathematical insights that might be provoked through the use of techniques within mathematical software environments. There seemed to be a distance between the domain-specific research on technology in mathematics education and educational science in general. However, some issues in this volume section suggest that this distance is decreasing. At the end of Chap. 7, a plea is made for integration of theoretical approaches. Chapter 11 deals with issues that are at present central in educational research in general, such as learning in communities and collaborative learning. This research interest, in which domain-specific and general educational perspectives reinforce each other are promising for future advancements.

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Chapter 7
Integrating Technology into Mathematics Education: Theoretical Perspectives

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Abstract The central question at stake in this chapter is: What theoretical frames are used in technology-related research in the domain of mathematics education and what do these theoretical perspectives offer? An historical overview of the development of theoretical frameworks that are considered to be relevant to the issue of integrating technological tools into mathematics education is provided. Instrumental approaches and the notion of semiotic mediation are discussed in more detail. A plea is made for the development of integrative theoretical frameworks that allow for the articulation of different theoretical perspectives.

Keywords Mathematics education • Technology • Instrumentation • Semiotic mediation • Theoretical perspectives in technology-related research in mathematics education

7.1 Introduction

For as long as mathematics education has been considered to be a serious scientific domain, researchers, educators, and teachers have been theorizing about the learning and teaching of mathematics. This has led to an overwhelmingly broad spectrum of theoretical approaches, ranging from the philosophical to the practical, from the global to the local, some focusing on learning in general and others very much based in mathematical knowledge. This body of theoretical knowledge is still growing.

Now that the issue of integrating technological tools into the teaching and learning of mathematics has become urgent, one can wonder what the existing theoretical

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perspectives have to offer. Can they be applied to this new context? Or do we need specific theories appropriate for the specific situation of using tools for doing – and learning – mathematics? If yes, what is so specific about the integration of technology that justifies the need for such new paradigms? Are these paradigms to be considered as part of the body of knowledge of mathematics education, or do they rather belong to theories about humans' interactions with technology? What kinds of problems do we want the theoretical frameworks, old or new, to solve for us?

It is our conviction that theoretical frameworks are needed in order to guide the design of teaching, to understand learning, and to improve mathematics education. Therefore, the central question at stake in this chapter is:

> *What theoretical frames are used in technology-related research in the domain of mathematics education and what do these theoretical perspectives offer?*

In an attempt to answer this question, the chapter aims at providing an historical overview of the development of theoretical frameworks that are considered to be relevant to the issue of integrating technological tools into mathematics education. As such, it tries to present a synthesis of the state-of-the-art. Links with existing, general frameworks are established and new, technology-related, theoretical developments are described.

The chapter has the following more-or-less chronological structure. Section 7.2 provides an overview of important theoretical approaches developed in the past, that is, during the period up to the 1990s. Both technology-related theories (Sect. 7.2.2) and more general theories (Sect. 7.2.3) are addressed. In Sect. 7.3, more recent developments are described. After a more general presentation of recent learning theories from mathematical didactics, two frameworks that are relevant for tool use in mathematics education are addressed in more detail: the theory of instrumental approaches (Sect. 7.3.2) and the notion of semiotic mediation (Sect. 7.3.3). In Sect. 7.4, finally, we try to summarize the situation and to sketch some of the possible challenges for future development.

7.2 Looking Back

In this section we consider the theoretical frames that were used in the technology-related research in mathematics education in the period from the 1960s to the 1990s. But before doing that, we look briefly at the development of theory in mathematics education research in general.

In his 1981 plenary at the Psychology of Mathematics Education conference in France, Kilpatrick remarked that:

> A lack of attention to theory is characteristic of US research in this field.… One of our greatest needs in research on mathematical learning and thinking is for conceptual, theory-building analyses of the constructs and assumptions we are using. (Kilpatrick 1981, pp. 23–24)

Although Kilpatrick was noting his concerns about US research, the phenomenon was more widespread. In 1980, Bauersfeld argued that the field at that time lacked a theoretical orientation (Bauersfeld 1980). But, the situation was soon to begin to change. Steiner, who spearheaded the forming of the group *Theory of Mathematics Education* at ICME-5 in 1984, wrote in 1987 that recent developments in mathematics education had shown a "new dynamics in the field":

> New philosophies and epistemological theories have entered the scene: the theory of epistemological obstacles, a synthesis of Kuhnian theory dynamics and Piagetian genetic epistemology,… the epistemology of "microworlds" and of the "society of mind" based on cognitive studies within research on artificial intelligence. (Steiner 1987, p. 7)

Theorizing continued to evolve during the 1990s, as suggested by Lerman's et al. (2002) analysis of the theories used by mathematics education researchers in the papers published in *Educational Studies in Mathematics* during the period 1990–2001. While theorizing in mathematics education research was still an emergent activity during the 1960s–1990s, in contrast, technology use was rapidly evolving. To capture this development, and more particularly the birth of theorizing related to the role of technological tools in mathematics teaching and learning, we first go back to the early days of technology use in mathematics education.

7.2.1 The Evolution of Technology and Its Use in the Mathematics Education Community

From the time of the development of the mainframe computer in 1942, the first four-function calculator in 1967, the microcomputer in 1978, and the graphing calculator in 1985 (Kelly 2003), both mathematicians and mathematics educators have been intrigued by the possibilities offered by technology. However, it was not until the late 1960s when, according to Fey (1984), mathematicians and mathematics educators began to feel that computing could have significant effects on the content and emphases of school-level and university-level mathematics.

Among the earliest applications of the new technology to mathematical learning in schools was Computer Assisted Instruction – the design of individualized student-paced modules that were said to promote a more active form of student learning. Perhaps the most well known is the PLATO project (Dugdale and Kibbey 1980; Dugdale 2007).

The next wave in technology-based approaches to mathematics learning involved programming, in particular, in Logo and BASIC. The development of the Logo programming language by (Feurzeig and Papert 1968; Papert 1980) was instrumental in this regard. Papert, a mathematician who was influenced by the theories of Piaget, was interested in the learning activities of young children and how the computer could enhance those activities (see, e.g., Papert 1970, for descriptions of children and junior high school students learning to program the M.I.T. "turtle" computer). In his 1972 article, entitled *Teaching children to be mathematicians versus teaching about mathematics*, Papert promoted "putting children in a better position to *do* mathematics rather than merely learn *about it* (Papert 1972)." At the time, programming in BASIC was also considered a means for enhancing students' mathematical problem-solving abilities (Hatfield and Kieren 1972), even for students as young as first graders (Shumway 1984).

The arrival of the microcomputer in the late 1970s not only increased the interest in programming activity, but also led to the development of more specialized pieces of software. Some of these specialized software tools were created specifically for mathematics learning (e.g., *CABRI Geometry* developed by Laborde 1990, and *Function Probe* developed by Confrey 1991), while others were adapted for use in the mathematics classroom (e.g., spreadsheets and computer algebra systems). The microcomputer and the graphing calculator also fed the growth of functional approaches in algebra and interest in multiple representations of mathematical objects (Fey and Good 1985; Heid 1988; Schwartz et al. 1991). However, by the 1990s, technological tools were still not widespread in mathematics classrooms, nor was there an abundance of qualitatively good software available (Kaput 1992).

Against the above technological scene in mathematics education during the years from the 1960s to the early 1990s, we now examine the question of the theoretical frames that were used in the technology-related research in mathematics education during the same period. We begin with the Proceedings of the 1985 ICMI Study on technology.

7.2.2 The Emergence of Theory from the Integration of Technology Within Mathematics Education

In 1985, the first ICMI Study was held in Strasbourg, France, with the theme, "The Influence of Computers and Informatics on Mathematics and Its Teaching at University and Senior High School Level." While research on the learning and teaching of mathematics was not a main thrust of the questions addressed by the Study group, we thought it might be illuminating to peruse the Proceedings of the Study (Howson and Kahane 1986) for an indirect glimpse at the kind of theories figuring in the discussions of the Study group participants.

In the opening general report that synthesized the Study papers and discussions, the editors of the proceedings, Howson and Kahane, emphasized the roles that computers could play in the learning of mathematics, such as, "advantages to be derived from the use of computer graphics" (p. 20), "the design of software to encourage the discovery and exploration of concepts" (p. 20), and "the active involvement of students in their own learning through the writing of short programs" (p. 20). The activities of exploration and discovery were particularly pointed to. However, one cannot but be struck by the way in which the papers emphasized the educational potentialities and capabilities of computing technology – such as, visualizing, modeling, and programming – with an optimism that was not yet supported by evidence.

Included in the Proceedings was a set of 11 "Supporting Papers" selected from those that had been presented at the study conference. The papers, mostly of the essay variety, included deliberations on the synergy between mathematics and computers, and considerations of the potentialities and limitations of the computer. The only theoretical discussions that could be said to be present in any of the 11 papers concerned epistemological issues involving the nature of mathematics and that of computer science – but at a rather general level. Theorizing and theory on the role of technology in the teaching and learning of mathematics were clearly neither the aim nor the by-product of the study meeting. As Burkhardt (1986) remarked in his paper: "[This is] a conference for conjectures" (p. 147).

While not reflected in the ICMI Study papers, theory with respect to technology and its use in mathematics education was nevertheless developing during the 1980s. However, the first examples to emerge tended to be rather descriptive models of the roles being played by technology than research tools for designing learning environments or for testing hypotheses about the possible enhancement of mathematical learning and teaching. These theoretical beginnings focused on specific issues related to integrating technology into education. They had a local feel to them that was based in large measure on the characteristics of the technology and which suggested certain uses and forms of mathematical activity. Often, the theoretical notions were related to specific types of software and were not related to more general theories on learning. As examples of these first steps in theorizing, we briefly discuss the Tutor-Tool-Tutee distinction, the White Box–Black Box idea, the notion of Microworlds and Constructionism, and the Amplifier–Reorganizer duality.

7.2.2.1 Tutor, Tool, Tutee

With the arrival of the microcomputer and its increasing proliferation, a new framework was developed, which classified educational computing activity according to three modes or roles of the computer: tutor, tool, and tutee (Taylor 1980). To function as a *tutor*: "The computer presents some subject material, the student responds, the computer evaluates the response, and, from the results of the evaluation, determines what to present next" (p. 3). To function as a *tool*, the computer requires, according to Taylor, much less in the way of expert programming than is required for the computer as tutor and can be used in a variety of ways (e.g., as a calculator in math, a map-making tool in geography,…). The third mode of educational computing activity, that of *tutee*, was described by Taylor as follows: "To use the computer as *tutee* is to tutor the computer; for that the student or teacher doing the tutoring must learn to program, to talk to the computer in a language it understands" (p. 4). The rationale behind this mode of computing activity was that the human tutor would learn what s/he was trying to teach the computer and, thus, that learners would gain new insights into their thinking through learning to program.

7.2.2.2 White Box – Black Box

A theoretical idea that focused on the interaction between the knowledge of the learner and the characteristics of the technological tool was the White Box/Black Box (WBBB) notion put forward by Buchberger (1990). According to Buchberger, the technology is being used as a white box when students are aware of the mathematics they are asking the technology to carry out; otherwise the technology is being used as a black box. He argued that the use of symbolic manipulation software (i.e., CAS) as a black box can be "disastrous" (p. 13) for students when they are initially learning some new area of mathematics – a usage that is akin to the Tool mode within the Tutor-Tool-Tutee framework. However, other researchers (e.g., Heid 1988; Berry et al. 1994) have shown that students can develop conceptual understanding in CAS environments before mastering by-hand manipulation techniques. While the WBBB idea is pitched in terms of two extreme positions, others (e.g., Cedillo and Kieran 2003) have taken this notion and adapted it in their development of "gray-box" teaching approaches.

7.2.2.3 Microworlds and Constructionism

Papert and Harel (1991) encapsulated the theoretical ideas underlying the educational goals of microworlds (e.g., the Turtle environment in Logo) in the notion of *constructionism*, that is, "learning-by-making." In terms of the Tutor-Tool-Tutee framework, we are now in Tutee mode. Papert and Harel (ibid.) have described constructionism as follows:

> Constructionism – the N word as opposed to the V word – shares constructivism's connotation of learning as "building knowledge structures" irrespective of the circumstances of the learning. It then adds the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity, whether it's a sand castle on the beach or a theory of the universe. (p. 1)

While admitting in 1991 that the concept itself was in evolution, Papert and Harel provided examples of studies that Papert himself was involved with during the 20 years previous and that fed the early evolution of the idea. Microworlds, such as turtle geometry, were a central component of the theory:

> The Turtle World was a microworld, a "place," a "province of Mathland," where certain kinds of mathematical thinking could hatch and grow with particular ease (Papert 1980: p. 125)
>
> The Turtle defines a self-contained world in which certain questions are relevant and others are not… this idea can be developed by constructing many such "microworlds," each with its own set of assumptions and constraints. Children get to know what it is like to explore the properties of a chosen microworld undisturbed by extraneous questions. In doing so they learn to transfer habits of exploration from their personal lives to the formal domain of scientific theory construction. (Papert 1980, p. 117)

Some critics (e.g., Becker 1987) suggested that Papert's theory needed further elaboration. Later development of the notion of "microworld" would not restrict the term to Logo-based environments or even to computer environments (Edwards 1998; Hoyles and Noss 2003). In a paper prepared for the ICMI Study 17, Ainley and Pratt (2006) describe how they drew on Constructionist ideas to develop a framework for task design that involved the linked constructs of *purpose* and *utility*. In bringing this example on Constructionism to a close, we would be remiss if we did not note that the potential of Logo learning environments grabbed the attention, and research activity, of hundreds of researchers in mathematics education during the 1980s (see, for example, the 4 years of *Proceedings of the International Conference for Logo and Mathematics Education* from 1985 to 1989).

7.2.2.4 Amplifier – Reorganizer

Pea (1987) re-elaborated the psychological notion of cognitive tools for the case of technology in education. Computers have the potential for both *amplifying* and *reorganizing* mathematical thinking. However, Pea argued that the one-way amplification perspective, whereby tools allow the user to be more efficient and to increase the speed of learning, misses the more profound two-way reorganizational possibilities afforded by the technology. By this he meant that not only do computers affect people, but also that people affect computers (both by the way they decide on what are appropriate ways of using them and on how in refining educational goals they change the technology to provide a better fit with these goals). Meagher (2006), in his ICMI Study 17 contribution, has described how digital technology introduced into the classroom setting can bring to the fore two-way effects that are unanticipated and that can lead to unintentional subversion of the expressed aims of a given curriculum. He proposes an adaptation of the Rotman (1995) triangular model of mathematical reasoning as a tool for better understanding the complex interaction among student, technology, and mathematics.

Pea's theoretical work also included the development of a taxonomy comprising two types of functions by which information technologies can promote the development of mathematical thinking skills: *purpose functions* and *process functions*. The purpose functions engage students to think mathematically; the process functions aid them once they do so. The purpose functions focus on constructs such as ownership, self-worth, and the use of motivational "real-world" contexts and collaborative learning environments. The process functions include, according to Pea, five categories of examples: "tools for developing conceptual fluency, tools for mathematical exploration, tools for integrating different mathematical representations, tools for learning how to learn, and tools for learning problem-solving methods" (p. 106). Some of this work fed into the development of theories on distributed cognition (Pea 1989) and on situated cognition (e.g., Brown et al. 1989) – the latter construct being taken up in a later section of this chapter on situated abstraction.

7.2.3 Theoretical Ideas Emanating from the Literature on Mathematical Learning

Not only did local theorization concerning the use of new technologies in education begin to grow during these years; gradually, links with recently developed theory from the learning and teaching of mathematics were established. In that an exhaustive coverage is not possible, the following three examples of theoretical ideas emanating from the literature on mathematical learning illustrate some of the ways in which such theories and frameworks were used in research involving technological environments during the years leading up to the early 1990s.

7.2.3.1 Process-Object

Using the process-object frame (Sfard 1989, 1991), Moschkovich et al. (1993) observed the ways in which students came to grips with connections across representations and with different perspectives regarding the given functions themselves. For the concept of function, the researchers described this frame as follows:

> From the *process perspective*, a function is perceived of as linking *x* and *y* values: For each value of *x*, the function has a corresponding *y* value. From the *object perspective*, a function or relation and any of its representations are thought of as entities – for example, algebraically as members of parametrized classes, or in the plane as graphs that, in colloquial language, are thought of as being "picked up whole" and rotated or translated. (p. 71)

For the case of linear relations, the researchers argued that developing competency involves learning which perspectives and representations work best in which situations, and being able to move fluently from one to the other according to the demands of the situation and one's desired goals. Using a computer-based microworld called GRAPHER, the researchers set out to develop a tutoring curriculum that would introduce students to the object and process perspectives within the context of linear functions. It is interesting to notice that the theoretical reflections drove the design of the software, rather than the other way around, as we encountered in the previous section. What Moschkovich, Schoenfeld, and Arcavi found is that developing flexibility between these two perspectives is difficult for students. However, they emphasized that technological tools such as the one used in this study offer students the opportunity to deal with aspects of functions and to develop intuitions that were simply inaccessible prior to the availability of such tools.

Other applications of the process-object framework (later to become APOS theory) were developed by Schwingendorf and Dubinsky (1990). They used the programming language ISETL as an environment for having students construct mathematical concepts as *processes* by means of writing computer programs. Programming tasks that used a function as input and that also yielded a function as the output were considered to help students encapsulate the notion of function as *object*. However, critics have argued that an approach to functions that is based primarily on programming activities, though valuable for emphasizing *process* aspects, may be too closely tied to computability to permit a full-blown *object* conception of functions. The APOS theory, and its element of genetic decomposition, was also used by Repo (1994) in one of the early studies on the pedagogical use of computer algebra.

A third approach to the process-object duality was the notion of procept, introduced by Gray and Tall (1991). The authors suggest that the "the use of the computer to carry out the process, thus enabling the learner to concentrate on the product, significantly improves the learning experience" (p. 137).

7.2.3.2 Visual Thinking vs. Analytical Thinking

The interplay between visual and analytical schemas in mathematical activity and students' tendencies to favor one over the other (Eisenberg and Dreyfus 1986) was a theoretical notion that was adopted in some of the past research studies involving technology. For example, Hillel and Kieran (1987) distinguished between the two, within the context of 11- and 12-year-olds working in turtle geometry Logo environments, as follows:

> By a *visual schema* we refer to Logo constructions of geometric figures where the choice of commands and of inputs is made on visual cues, Rationale for choices is often expressed by, "It looks like…". By an *analytical schema* we refer to solutions based on an attempt to look for exact mathematical and programming relations within the geometry of the figure. (p. 64)

These researchers found that the students did not easily make links between their visualizations and their analytical thinking. While research in nontechnology learning situations had disclosed (older) students' preferences for working with the symbolic mode rather than with the graphical, the advent of graphing technology provided the potential for a shift toward valuing graphical representations and visual thinking (e.g., Eisenberg and Dreyfus 1989). These issues would continue to be explored in the years to come.

7.2.3.3 Representational Issues

Representational issues were very much a part of the theoretical frames of the early research involving technological environments in mathematics education. However, in much of this research, as well as in some of the research that did not involve technology, there was a lack of theoretical precision regarding *visualization*, *mental imagery*, and *representations* (for later precision in these theoretical areas, see, Bishop 1988; Dreyfus 1991; Hitt 2002; Presmeg 2006). The demand for clarification coming from the new technologies and their representational potential contributed to an effort to outline a unifying theoretical frame for representation. As Kaput pointed out to participants at a conference on representation in 1984, "Some mathematics education researchers have, in response to the need for understanding forms of representation in a particular area, developed local theories…; however, a coherent and unifying theoretical context is lacking" (Kaput 1987, p. 19). He proposed that a concept of representation ought to describe the five following components: the represented entity; the representing entity; those particular aspects of the represented entity that are being represented; those aspects of the representing entity that are doing the representing; and the correspondence between the two entities. This framework served as a basis for conceptualizing several "representational studies" involving the three representations of the tabular, the graphical, and the symbolic (e.g., Schwarz and Bruckheimer 1988). While the body of research on students' making connections among the three representations of functions in various technological environments would continue to grow (e.g., Romberg et al. 1993), further developments of a theoretical nature with respect to representations were forecast by Kaput when he spoke of, "the potential of notations in *dynamic interactive media*" (Kaput 1998, p. 271, emphasis added). This particular evolution in theoretical frameworks will be among the ones discussed in an upcoming section.

7.2.4 From Past to Present

A very interesting inventory of the mid-1990s research on technology in mathematics education is the one carried out by Lagrange et al. (2003). In their review of the world-wide corpus of research and innovation publications in the field of Information–Communication–Technology integration, they point out that "the period from 1994 to 1998 appeared particularly worthy of study [662 published works], because during these years the classroom use of technology became more practical, and literature matured, often breaking with initial naïve approaches" (p. 238). In the entire corpus of papers that was reviewed, Lagrange et al. found that the only theoretical convergences were at a general level and touched upon issues related to visualization, connection of representations, and situated knowledge. The study shows that less than half of the publications surveyed appeared to go beyond descriptions of the environment or phenomena being observed – and this literature was intended to reflect a certain maturity in the field.

To summarize this section on the theoretical frames that were used in the technology-related research in mathematics education in the period from the 1960s to the 1990s, we notice an initial concern with the potentials of technology use rather than with theoretical foundations. Gradually, both local technology-driven theories emerged and recently developed theories from mathematics education research were adapted to the case of learning with technology. This overall development can be extrapolated and applied to the current situation, which is the issue at stake in the next section.

7.3 Current Developments

The current proceedings concerning theory in research on technology in mathematics education cannot be dissolved from recent technological developments. On the one hand, technological devices have become smaller and handheld devices such as graphing and symbolic calculators are widespread. On the other hand, communication has become a more integrated part of technology use: software can be distributed using the Internet, and students can work, collaborate, and communicate with peers and teachers in digital learning environments. The content of such learning environments, however, turns out to be not easy to set up. The question of what a digital course should look like, so that it may really benefit from the potentials of technology and exceed the "paper-on-screen" approach, has not yet been answered.

These technological developments are included as background to the discussion of current theoretical developments described in this section. Section 7.3.1 provides an overview of some actual theoretical approaches that have been adapted from existing theories in mathematics education and beyond, but with new emphases that signal a movement toward tailor-made frameworks for investigating mathematical learning and teaching within technological environments. Sections 7.3.2 and 7.3.3 address with a certain degree of elaboration two specific theories on learning using technology applied to mathematics, instrumentation theory and semiotic mediation.

7.3.1 Learning Theories from Mathematical Didactics

As mentioned above, the early 1990s witnessed the beginnings of a more mature literature with respect to the use of technology in mathematical learning. However, this maturing did not occur in a vacuum. Within the mathematics education community at large, not only had theorizing become a more widespread activity, but also the nature of the theories being utilized within research on the learning of mathematics was experiencing a shift.

While Constructivism and its Piagetian roots provided the underpinnings of the theoretical elaborations that emerged in research related to technology use during the previous decades – elaborations that focused primarily on cognitive aspects of learning – the theoretical writing of Vygotsky with its sociocultural emphasis began to percolate through the international mathematics education community in the 1980s (see Streefland 1985, for the first mention within PME[1] Proceedings of Vygotsky's work). The steady growth in the development of sociocultural perspectives by researchers during the ensuing years was reflected in, for example, the scientific program of the 1995 PME Conference where plenary and panel presentations were devoted to Vygotskian theory (Meira and Carraher 1995). One of the first aspects of Vygotskian theory to be appropriated was his zone of proximal development (ZPD). However, later work by mathematics education researchers focused on the role played by language and other mediational tools in the teaching and learning of mathematics (Lerman 1998; Bartolini et al. 1999; Kieran et al. 2001). Yet another theoretical direction appeared during these years with the notion that knowledge is *situated* and is a product of the activity, context, and culture in which it is developed and used (Brown et al 1989).

The sociocultural perspectives that became popular during the late 1980s and 1990s were adopted, and adapted, by researchers with an interest in the role of technological tools in mathematical learning. The first example of a present-day theory used in research on the teaching and learning of mathematics within technological environments that we present in this section deals with a frame that resulted from the adapting of aspects from both sociocultural and situated learning theories, as well as from classic ideas on abstraction: the Webbing and Situated Abstraction frame. A second example, which shares aspects of both Vygotskian and Piagetian theories but is quite distinct from either of these, features Brousseau's Theory of Didactical Situations and illustrates the way in which one of its concepts has served to inform research involving the integration of technology into mathematical learning situations: the concept of *milieu*. The third and last example of this section presents an emerging and still developing theoretical frame, one that was conceptualized for use in research on modeling environments involving physical apparatus: the Perceptuo-Motor Activity frame.

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[1]The International Group for the Psychology of Mathematics Education (PME): http://www.igpme.org/

7.3.1.1 From Scaffolding and Abstraction to Webbing and Situated Abstraction

Noss and Hoyles (1996) describe how they have taken the original notion of *scaffolding* (Wood et al. 1979): "graduated assistance provided by an adult which offers just the right level of support so that a child can voyage successfully into his/her zone of proximal development" (p. 107) – support that is gradually faded as the learner's participation in the learning process increases – and from it have developed the metaphor of *webbing*. This metaphor is considered to more adequately take into account both the openness of computational settings as well as the control exercised by the learner. According to Noss and Hoyles, "the idea of *webbing* is meant to convey the presence of a structure that learners can draw upon *and reconstruct* for support – in ways that they choose as appropriate for their struggle to construct meaning for some mathematics" (p. 108). The concept of webbing is said to integrate, as well, a focus on the *situatedness of learning*.

Noss and Hoyles have drawn the latter construct from the theoretical work of Brown et al. (1989), who in developing the position that learning and cognition are situated, did not however address the issue of how learners might apply the knowledge learned in one setting to another. In fact, Brown et al. suggested that, "an epistemology that begins with activity and perception, which are first and foremost embedded in the world, may simply bypass the classical problem of reference – of mediating conceptual representations" (p. 41). While adoption of the construct of *situatedness* with its disregard for issues of transfer might have been problematic, Noss and Hoyles chose to tackle head-on the thorny question of students' making connections between the situated computational medium and official mathematics: "This process of building connections turns out to be central to making mathematical meanings" (p. 119). Their solution to the potential cul-de-sac of *situatedness* was the elaboration of the construct of *situated abstraction*.

Rather than thinking about mathematical abstraction as a pulling away from a real referent and a focusing on relationships involving the referent-less mathematical object and operations, Noss and Hoyles postulated a mechanism that involves abstracting *within*, not *away from* the situation. *Situated abstraction* "describes how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (Noss and Hoyles 1996, p. 122). They add that learning environments can be designed in such a way that expressing generality can be made a central component of the computational setting and, thus, that webbing and situated abstraction are co-emergent constituents of the mathematical learning process. Hershkowitz et al. (2001) have attempted to develop further the notion of situated abstraction. They have defined it as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" (p. 195) and have proposed three dynamically nested epistemic actions as its principal components: constructing, recognizing, and building-with. However, the application of the theory to mathematical learning in technological environments is less at the forefront of Hershkowitz et al.'s elaboration of situated abstraction than it is for Noss and Hoyles.

What is interesting, and extremely relevant, about the approach taken by Noss and Hoyles is the way in which they took two theoretical concepts from the general mathematics education literature, that is, scaffolding and abstraction, and reworked them so as to better capture some of those aspects that are special about learning in technological environments, as well as how this learning can spark connections with other mathematical settings. In their reporting of empirical work that provides the underpinnings for their theorizing, they have described the work of the following two students modeling in a dynamic geometry environment for whom the computer acted "both as a support for developing new meanings and as a means for transcending that support" (Noss and Hoyles 1996; p. 126).

Cleo and Musha were two 14-year-olds faced with the task of finding the mirror line (i.e., line of symmetry) between two flags (see Fig. 7.1), where one was the reflected image of the other. Figure 7.1a illustrates the two flags; in Fig. 7.1b two corresponding pairs of points have been dragged to coincide; in Fig. 7.1c the students are beginning to drag each point until they coincide with their image. Note that the line of symmetry has been drawn for clarification purposes: it 'appears' only when all pairs of points are dragged together.

According to Noss and Hoyles, the two girls did not know what construction would yield the required line of symmetry; so they began to drag points in order to develop some clues. Eventually they landed on the idea of dragging together the topmost and bottom-most points of the flag and its image. The girls articulated their findings as: "The mirror line is what you see on the screen if you drag points and their reflection together" (p. 116). They then sketched the mirror by fitting a line with two points on top of their skinny flags.

With respect to the mathematical thinking of the two girls and the role played by the technology, Noss and Hoyles have argued the following:

> What they [the two girls] didn't 'know' [initially] was a mathematical model of their intuitive knowledge, a piece of formal, systematised knowledge which would help them to construct it. As we saw, the medium provided some kind of bridge between these two states: the

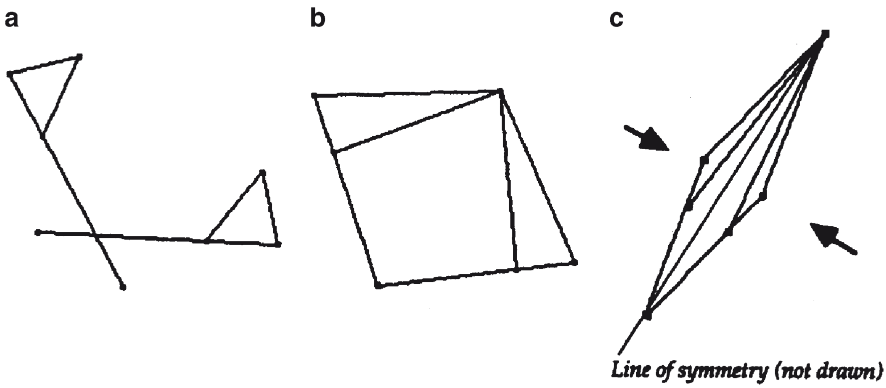

Fig. 7.1 Snapshots of a strategy for locating the line of symmetry of the two flags (from Noss and Hoyles 1996; p. 115)

> representations the students built elaborated and illuminated their knowledge structures and simultaneously provided a window to and a route beyond these structures. … Construction implies an explicit appreciation of the relationships that have to be respected within any situation, a mathematical model of the situation (how else do you know what to focus on, and what to ignore?). The key insight is that parts of this model are built into the fabric of the medium, thus shaping the types of actions that are possible: they do not *only* exist in the mind of learner. (p. 126)

7.3.1.2 Theory of Didactical Situations: the Concept of Milieu

Brousseau's (1998) *Théorie des Situations Didactiques* (TSD) began to take shape in 1970. As it continued to evolve and develop throughout the two decades that followed, it became the theoretical backbone of the French school of research in mathematical didactics. Even as other theoretical frameworks emerged in the 1990s, such as instrumental approaches (e.g., Artigue et al. 1998; see Sect. 7.3.2), some of the basic ideas of TSD were implicitly integrated within them. The central concepts of TSD that have been threaded through recent research related to technological learning environments include that of *milieu* (Floris 1999), *didactical contract* (Gueudet 2006), and *institutionalization* (Trouche 2004). We take up the concept of *milieu* as a paradigmatic example here of the way in which the TSD has been applied, albeit often implicitly, in recent research involving technological tools.

In the late 1970s, Brousseau developed the concept of *milieu*, and described its integration into the learning process as follows:

> The teacher's work therefore consists of proposing a learning situation to the student in such a way that she produces her knowing as a personal answer to a question and uses it or modifies it in order to satisfy the constraints of the *milieu* and not just the teacher's expectations. (Brousseau 1998, p. 228)

The design of the learning situation is thus an integral aspect of the *milieu*. When instruments are part of that learning environment, situations need to be created that provoke the learner to use the tools in the pursuit of some mathematical goal. As pointed out by Artigue (2006), this learning milieu is one that by definition is *antagonistic* – that is, in opposition to the learner and his/her current state of knowledge:

> Within this framework [TSD], the learning outcomes resulting from the use of an instrument at the practical level [that is, when a student is using an instrument to solve a problem] are discussed in terms of the interaction of the learner with the *milieu antagoniste*. … We may consider the learning outcomes as being the result of the adaptation of the learner to the milieu in consequence to the retroactions of the milieu on the learner himself/herself. Thus, if an educator wants to employ an instrument at the educational level, he/she has to set up situations in which the instrument is part of the milieu and is employed by the learner as a means to accomplish the proposed task. (pp. 15–16)

Artigue has contrasted this view with that of the role of instruments within Activity Theory (Engeström 1991) where an instrument is considered as a mediator: "Within this theory [Activity Theory], the learning environment is not considered as antagonist to the subjects (as in the *milieu antagoniste* of the didactic situations theory); on the contrary, it is considered to be a cooperative environment" (Artigue 2006; p. 16).

What then are the implications with respect to designing technological learning environments when the notion of milieu constitutes a part of one's theoretical frame – even if at an implicit level? Artigue (2006) has, for example, discussed the importance of feedback in technological learning environments. Feedback consisting in a simple validation of pupils' answers is considered to be limited, in contrast with more elaborated feedback that is more likely to support the evolution of pupils' strategies and mathematical knowledge development.

Such considerations were of central importance in the design of the tasks and their sequencing in a study reported by Kieran and Drijvers (2006a). While the notion of a technological environment that presented mathematical challenges to the student was not attributed theoretically to the TSD's *milieu*, but rather to the instrumental approach to tool use (whose inheritance includes TSD and which is discussed in an upcoming section), the CAS environments described by Kieran and Drijvers deliberately took students beyond their curricular experience and placed them in challenging situations devised to expose limitations in the thinking frames they were using. For instance, they and their research colleagues adapted the $x^n - 1$ task of Mounier and Aldon (1996) (which is described in Lagrange 2000) to create a learning situation for 10th graders that involved several different kinds of interactions with the CAS instrument, including the forming and testing of conjectures, reflecting on surprise outputs produced by the CAS, and reconciling such outputs with their existing and developing theoretical and technical knowledge.

Examples of some of the task questions used in the Kieran and Drijvers (2006a) study are shown in Figs. 7.2 and 7.3.

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Fig. 7.2 Task in which students confront the completely factored forms produced by the CAS (Kieran and Drijvers 2006a; p. 239)

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:
i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x+1)$ as a factor?
Please explain.

Fig. 7.3 Task in which students examine more closely the nature of the factors produced by the CAS (Kieran and Drijvers 2006a; p. 239)

In designing the sequence of the various task situations that students were to engage in during the project, specific a priori attention was given to variables related to epistemological and cognitive concerns, as well as to those potentially crucial moments in the sequence of tasks when, and the manner in which, teachers might intervene to enrich the milieu (see also Kieran and Saldanha 2007).

7.3.1.3 Perceptuo-Motor Activity in Mathematical Learning

The integration of manipulative tools into mathematical learning environments has a long history (e.g., Gattegno 1963; Dienes 1960). The theoretical underpinnings supporting the use of such learning tools have been continuously evolving over the years. More recently, at a research forum held at the 2003 annual conference of PME, Nemirovsky attempted to lay the groundwork for theorizing about the role of perceptuo-motor activity in mathematical learning. The large variety of technology environments offering dynamic control, including those involving dynamic geometry, would suggest the potentially widespread applicability of such a framework. Drawing on studies of functional brain imaging, perception, and eye movement, Nemirovsky (2003) conjectured the following:

> While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are *perceptuo-motor activities*; furthermore, these activities are bodily distributed across different areas of perception and motor action based in part, on how we have learned and used the subject itself. … We add here that that of which we think emerges from and in these activities themselves. … We think of, say, a quadratic function, by enacting "little thrusts" of what writing its equation, drawing its shape, uttering its name, or whatever else the use of a quadratic function in a particular context might entail. The actions one engages in mathematical work, such as writing down an equation, are as perceptuo-motor acts as the ones of kicking a ball or eating a sandwich; elements of, say, an equation-writing act and other perceptuo-motor activities relevant to the context at hand are not merely accompanying the thought, but are the thought itself as well as the experience of what the thought is about. (pp. 108–109)

Rasmussen and Nemirovsky (2003) have characterized how bodily activity and tool use combine in mathematical learning in their study of calculus students engaging in a number of different tasks involving a physical tool called the water wheel, which was hooked up to a computer that enabled the generation of real-time graphs of angular velocity and angular acceleration. The analysis they reported focuses on the engagement of one student, Monica, with the water wheel during two sessions. During the first session, Monica synchronized the rotation of the water wheel with given graphs of angular velocity versus time. When Rasmussen and Nemirovsky interviewed Monica, she "personified the wheel and imaginatively experienced when the wheel will achieve its maximum and minimum velocity" (p. 129). In the process, she accounted for why these maximum and minimum velocities occurred. The researchers argued that "being" the water wheel engaged both knowing-how to be the wheel and knowing-with the water wheel, thus highlighting the centrality of bodily activity in this process.

Applications of the theoretical frame sketched out by Nemirovsky are also to be found in, for example, the research of Arzarello and Robutti (2001), who have explored the perceptuo-motor components of students' building meaning for functions in an environment involving a calculator connected to a motion sensor (CBR). In one of the activities they researched, students were encouraged to try various running patterns in order to create different graphs. The continuous nature of the CBR graphing allowed students to test conjectures in a direct manner, controllable by their own physical movement. The authors observed that, "students' cognitive activity passes through a complex evolution, which starts in their bodily experience (namely, running in the corridor), goes on with the evocation of the just lived experience through gestures and words, continues connecting it with the data representation, and culminates with the use of algebraic language to write down the relationships between the quantities involved in the experiment" (p. 39).

For our last example in this section on theorizing with respect to perceptuo-motor activity, we choose to refer once again to the case of Cleo and Musha, the two girls whose experience in Cabri Geometry was used to exemplify the theoretical framework of Webbing and Situated Abstraction. We do this for a twofold reason: one, to suggest that the same data might be interpreted according to more than one theoretical framework; and two, to illustrate the role that the bodily experience of dragging in dynamic geometry environments can play in mathematical learning (see also the ICMI Study 17 contribution of Lee et al. 2006, for further examples of dragging in dynamic geometry environments, as well as the example provided in the later section on Instrumentation). Noss and Hoyles (1996) themselves emphasized the role that physical movement played in Cleo and Musha's arriving at a resolution of the line-of-symmetry task: "[their] solution to the problem is *sketched* within the medium by a physical manipulation controlled by perception – seeing that the points and their images coincided" (p. 116). However, as was noted earlier, Noss and Hoyles's theoretical explanation of the learning that took place went beyond the perceptuo-motor to include sociocultural, cognitive, and mathematical considerations.

While the analyses conducted by Arzarello and Robutti (2001), which were pointed to above, found support for the perceptuo-motor frame, as well as evidence for the roles that language, external representations, and instruments play in developing embodied conceptualizations, they have argued that the perceptuo-motor theoretical tools need to be widened and deepened (Robutti and Arzarello 2003; p. 115). They have suggested that connections need to be made to other frames, for example, the embodied approach of Lakoff and Núñez (2000), the notion of ostensives (Bosch and Chevallard 1999), and other perspectives on abstraction and concept building in mathematics (e.g., Vergnaud 1990). As proposed by A. Leung (personal communication, September 2007), connections to the theoretical advances that have been made on the role of visualization in mathematical learning could also be very productive in this regard.

The issue of making connections between existing theoretical frames, and adapting them for use in research on technological environments is a non-negligeable one. Lagrange (2005) has, for example, argued for consideration of "four linked issues – didactical and epistemological analysis, changes in curricula and practices, tool and mathematics relationship, and design – [which would constitute] a 'multidimensional' approach consistent with Lagrange et al. (2003, p. 239) claim that many research studies or reports of innovation about technology in mathematics education fail to be relevant when they consider only one framework" (p. 148).

7.3.1.4 Discussion

The theoretical work done by Noss and Hoyles in adapting existing frames from the broader literature in mathematical didactics so as to take into account the special features offered by technological tools is one side of the coin. This approach could be said to illustrate a mathematics-to-technology direction for developing theoretical frameworks for use in technology environments dedicated to mathematical learning. The other side of the coin, the technology-to-mathematics direction, is highlighted by the efforts of Arzarello and Robutti to use the perceptuo-motor frame – a frame that was conceptualized specifically for use in environments involving bodily movement with physical tools. Their experience led them to suggest that the frame needed to be expanded by considering and integrating additional theories from the broader mathematical didactics literature. As for the third example of theory presented in this section, that of Brousseau's TSD, we have discussed how certain aspects of that frame, such as *milieu*, have been taken just about as is and applied to research in technology environments. In the upcoming section, we will note how some elements of TSD have been used to elaborate more broadly a new theoretical frame – in this case, the theory generally referred to as Instrumentation, in particular the associated frame of instrumental orchestration (see Trouche 2007, for a specific example).

7.3.2 Instrumentation

Many studies on the use of technology in mathematics education refer to an instrumental approach. For example, out of the nine papers submitted to this ICMI Study 17 Conference subgroup on theoretical approaches, not less than seven referred to an instrumental approach as one of the main components of the theoretical framework. Apparently, such an approach seems to be a dominant framework that has much to offer while considering the role of technology in the learning and teaching of mathematics.

However, for several reasons the issue is not that straightforward. First, a discussion in our subgroup on what an instrumental approach is and how it is used, revealed the presence of different accents. Therefore, we speak of “an” instead of “the” instrumental approach (Artigue 2008). Second, language issues play a role. For example, the word *instrument*, as will be explained below, is used in a different sense from its meaning in natural language: in the case of somebody playing the piano – the instrument in daily-life language – the instrument from the theoretical perspective is more than the piano alone and includes the piece that will be played, as well as the schemes the player uses while doing so. Third, confusion arises, particularly in English-speaking countries, because of the earlier theoretical development of the notion of instrumental understanding by Skemp (1976). One is even tempted to claim that instrumental approaches stress the need for reconciliation of Skemp’s instrumental and relational understanding (K. Ruthven, personal communication, February 2007).

As a fourth and final issue, we mention the terms *schemes* and *techniques*, which are used in instrumental approaches, but which refer to different theoretical backgrounds. The theoretical foundations of the instrumental approach to tool use encompass elements from both cognitive ergonomics (Vérillon and Rabardel 1995; Rabardel 2002) and the anthropological theory of didactics (Chevallard 1999). One can distinguish two directions within the instrumental approach, which link up with these two background frameworks (Monaghan 2005). In line with the cognitive ergonomic framework, some researchers see the development of schemes as the heart of instrumental genesis. Although these mental schemes develop in social interaction and with the help of orchestration, the schemes are essentially individual. The main perspective here is psychological and cognitive (Trouche 2000).

More in line with the anthropological approach, other researchers focus on the techniques that students develop while using technological tools and in social interaction. This approach stresses the importance of techniques, which tends to be underestimated in discussions on the integration of technology. It is acknowledged, though, that techniques encompass theoretical notions. The focus on techniques is dominant in the work of Artigue (2002) and Lagrange (2000), who stress in particular the three T’s: task, technique, theory. Artigue (2002; p. 248) notes that technique has to be given a wider meaning than is usual in educational discourse: “A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution[2], each technique is a complex assembly of reasoning and routine work.” This quotation also emphasizes the coordination of the cognitive and institutional dimensions, which is very present in instrumentation theory.

These complicated issues reveal the aim of this section. It is not a very detailed or sophisticated introduction into instrumentation theory; rather, it aims at sketch-

[2]The word ‘institution’ has a broad sense in this theory. Here we consider didactic institutions, devoted to the intentional learning of specific knowledge.

ing the global outline of this theoretical approach, clarifying the main ideas and vocabulary, discussing different points of view, and showing some of its power for research on the use of digital media in mathematics education.

7.3.2.1 Artifact and Instrument

An essential starting point in instrumentation theory is the distinction between artifact and instrument (Rabardel 2002). An *artifact* is the – often but not necessarily physical – object that is used as a tool. Think of a hammer, a piano, a calculator, or a dynamic geometry system on your PC. What exactly is the artifact in a given situation is not always clear. For example, in the case of dynamic geometry software, it is a matter of granularity if one considers the software as a whole as one single artifact, or if one sees it as a collection of artifacts, such as the construction artifact, the measurement artifact, the dragging artifact, and so on (Leung 2008). Also, the notion of artifact is quite wide. A scenario for using computer algebra in algebra teaching that is made available in a digital working space for professional development, for example, is an artifact that the teacher can use to shape his/her teaching (Bueno-Ravel and Gueudet 2007).

The way an artifact is used is nontrivial. As long as I have no idea about what letters stand for, word processors are useless artifacts to me. As soon as I start to learn to write, a pen is no longer an artifact that I use for drawing, but changes into an artifact that I also use for writing. Together with my developing skills, the pen forms an instrument for writing. This brings us to the psychological construct of the instrument being more than an artifact. Following Rabardel (e.g., Rabardel 2002), we speak of an *instrument* if a meaningful relationship exists between the artifact and the user for a specific type of task. Besides the artifact, the instrument also involves the techniques and mental schemes that the user develops and applies while using the artifact. To put this in the form of a somewhat simplified 'formula' we can state: Instrument = Artifact + Schemes and Techniques, for a given type of task.

7.3.2.2 Instrumental Genesis

The process of an artifact becoming part of an instrument in the hands of a user – in our case the student – is called *instrumental genesis*. For a hammer, this process includes skills for not hitting one's fingers, but also the awareness of the type of problems that can and that cannot be solved with a hammer. Instrumental genesis also involves thinking about means to change the artifact and, for example, inventing a hammer with a wedge in its head that can also be used to remove nails.

Instrumental genesis is an ongoing, nontrivial and time-consuming evolution. A bilateral relationship between the artifact and the user is established: while the student's knowledge guides the way the tool is used and in a sense shapes the tool (this is called instrumentalization), the affordances and constraints of the tool influence the student's problem solving strategies and the corresponding emergent conceptions (this is called instrumentation). The dual nature of instrumentation and instrumentalization within instrumental genesis comes down to the student's thinking being shaped by the artifact, but also shaping the artifact (Hoyles and Noss 2003). It should be noted that the word 'instrumentation' has a somewhat double meaning here: in the framework of instrumental approaches, it refers to instrumentation theory as a whole; in the more specific context of instrumental genesis, it refers to the way the artifact affects the student's behavior and thinking, as opposed to instrumentalization, which concerns the way the student's thinking affects the artifact.

As an example, let us consider the graphing calculator. The menu option 'Calculate Intersect' of a TI-84 calculator is an artifact that calculates the coordinates of intersection points of graphs. On the one hand, this artifact enriches the student's view of solving equations with a graphical representation. On the other hand, the artifact may limit the student's conception of solutions, as the results are restricted to rounded-off decimal values instead of exact solutions. Of course, a student might program the calculator so that it will give solutions in the form of radicals, which would be an example of instrumentalization.

7.3.2.3 Schemes and Techniques

If instrumental genesis consists of the development of schemes and techniques, then the question is of course what these schemes and techniques are. The cognitive ergonomy approach and the anthropological approach have different views on schemes and techniques.

In the cognitive ergonomy approach (e.g., see Vérillon and Rabardel 1995) the notion of mental scheme builds on the definition of Vergnaud (1996): a *scheme* is an invariant organization of behavior for a given class of situations. More informally: a scheme is a more or less stable way to deal with specific situations or tasks. As we see a scheme here as part of an instrument, we speak of an instrumentation scheme. Within instrumentation schemes, schemes of instrumented action and utilization schemes are distinguished. Utilization schemes are directly related to the artifact and are building blocks for more integrated schemes of instrumented action, which are more global schemes directed towards an activity with the object (Trouche 2000). In these mental schemes, technical and conceptual aspects are intertwined and codevelop. As we cannot look into the heads of our students – although neuroscientists are advancing! – schemes cannot be observed directly. Therefore, we focus on the observable *instrumented techniques*, which we define as a more or less stable sequence of interactions between the user and the artifact with a particular goal. In this interpretation, the technique can be seen as the observable counterpart of the invisible mental scheme.

In the anthropological approach, techniques are seen as components of praxeologies (Chevallard 1999), and therefore are seen as institutional objects. From this perspective, it is important to consider institutional conditions that enhance instrumental genesis.

Essential in both schemes and techniques is the idea that technical and conceptual aspects coemerge and are closely related. In fact, it is the importance of this relationship that makes instrumentation theory powerful. We consider this to be more important than the difference between technique and scheme.

7.3.2.4 Examples

Let us look at some examples of schemes of instrumented action. The first example concerns the dragging tool in a dynamic geometry environment (DGE), as described by Leung et al. (2006). In line with Laborde and Capponi (1994), the authors write:

> A key feature of DGE is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities. This dynamic tool – dragging – induces potential dialectic between the conceptual realm (abstraction) of mathematical entities and the world of virtual empirical objects. Because of this possibility, dragging has been a major focus of research in DGE resulting in fruitful discussions on promising dragging modalities and strategies that seem to be conducive to knowledge construction. (p. 346)

Based on observations of students' instrumental genesis while working on a task that essentially comes down to the necessary condition of Ceva's theorem (Fig. 7.4), Leung et al. identified the following elements of what they call a Variational Dragging Scheme:

1. Create contrasting experiences by wandering dragging until a dimension of variation is identified.
2. Fix a value (usually a position) for the chosen dimension of variation.
3. Employ different dragging modalities/strategies to separate out critical feature(s) under the fixed value (i.e. a special case for the configuration)
4. Simultaneously focusing, hence "reasoning", on covarying aspects during dragging. A preliminary conjecture is fused together.
5. Attempt to generalize by a change to a different value for the chosen dimension of variation.

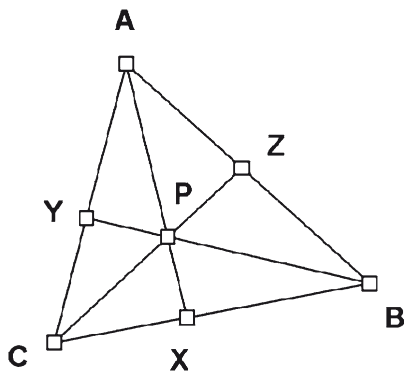

Fig. 7.4 Dragging in a DGE (Leung et al. 2006; p. 350)

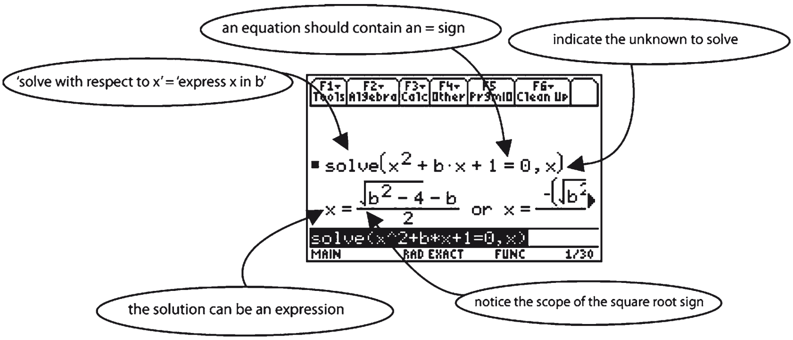

Fig. 7.5 Interrelated technical and conceptual elements (adapted from Drijvers and Gravemeijer 2004; p. 173)

6. Repeat steps 3 and 4 to find compromises or modifications (if necessary) to the conjecture proposed in step 4.
7. Generalization by varying (via different dragging modalities) other dimensions of variation. (Leung et al. 2006; pp. 350–351)

The second example of a scheme of instrumented action concerns the solving of parametric equations in a computer algebra environment. Drijvers and Gravemeijer (2004) claim that solving a parametric equation with computer algebra is a nontrivial issue for grade 10 students. Figure 7.5 sketches some conceptual elements that are involved in "simply" applying the *Solve* command. The authors distinguish the following elements, some of which have a more technical, and others a more conceptual character:

1. Knowing that the *Solve* command can be used to express one of the variables in a parameterized equation in other variables.
2. Remembering the TI-89 syntax of the *Solve* command, that is *Solve* (*equation*, *unknown*).
3. Knowing the difference between an expression and an equation.
4. Realizing that an equation is solved *with respect to* an unknown and being able to identify the unknown in the parameterized problem situation.
5. Being able to type in the *Solve* command correctly on the TI-89.
6. Being able to interpret the result, particularly when it is an expression (Drijvers and Gravemeijer 2004; p. 174)

The third and final example of a scheme of instrumented action is presented by Kieran and Drijvers (2006a, b). It concerns the notion of equivalence of algebraic expressions in combination with the use of a symbolic calculator. What is different here from the previous examples is that a number of techniques are distinguished – in this case, to decide on equivalence. Furthermore, each of the techniques affects the understanding of the notion of equivalence. Altogether, an important component of the instrumentation scheme here is the coordination of the set of available techniques, including the choice of carrying out the technique by hand or with the symbolic calculator.

1. Substituting numerical values
2. Common form – by factoring
3. Common form – by expanding
4. Common form – by automatic simplification
5. Test of equality
6. Solving equations

Even if the three examples are different in the way schemes are described and presented, they share an interest in making tangible the interaction between the techniques involved in using the artifact and mathematical thinking. And that is what instrumentation theory is about.

7.3.2.5 Orchestration

As a final aspect of instrumentation theory, we briefly address the notion of orchestration. So far, we have examined the instrumental genesis of schemes and techniques as an individual process. Different students may develop different schemes for working on the same type of task or for using a similar command in the technological environment. However, instrumental genesis also has a social dimension. Students develop mental schemes within the context of the classroom community, and a process of collective instrumental genesis is taking place in parallel with the individual geneses.

In order to describe this process of collective instrumental genesis and the management of the individual instruments by the teacher in the collective learning process, Trouche (2004) introduced the notion of *instrumental orchestration*. An instrumental orchestration is the intentional and systematic organization of the various artifacts available in a computerized learning environment by the teacher for a given mathematical situation, in order to guide students' instrumental geneses. An instrumental orchestration is defined by didactic configurations (i.e., arrangements of the artifactual environment, according to various stages of the mathematical situation); and exploitation modes of these configurations. Trouche (2007) refers to Brousseau's TSD in defining *situation*, as well as to Chevallard (1999) in discussing the *systematic* nature of orchestrations.

While addressing the ways teachers can orchestrate students' collective instrumentation, it should be noted that the artifacts in use by the students are also to be seen as artifacts that teachers use for their teaching. As such, the teacher's instrumental genesis comes into the picture. Teachers also have their own artifacts such as electronic resources, teaching experiences, and teaching scenarios. From this perspective, instrumentation theory is double-layered and can be a fruitful tool in teacher training on orchestrating technology as well: teachers are themselves involved in a process of instrumental genesis to develop artifacts into instruments for accomplishing their teaching tasks (Bueno-Ravel and Gueudet 2007; Guin and Trouche 2007).

7.3.2.6 Affordances, Constraints, Perspectives

In this final part of the section on instrumentation theory, we discuss what the theory offers, what it doesn't offer, and what different perspectives exist within the theory.

The instrumental approach to tool use was recognized by French mathematics education researchers as a potentially powerful framework in the context of using CAS in mathematics education. Many publications show how valuable this approach is for the understanding of student-CAS interactions and their influence on teaching and learning (Artigue 1997, 2002; Lagrange 2000, 2005; Trouche 2000, 2004; Guin et al. 2004). As the examples show, it has been applied to the integration of computer algebra and dynamic geometry systems into the learning of mathematics, but also to the use of spreadsheets (Haspekian 2005) and applets (Boon and Drijvers 2006). The examples also show what it offers: a lens that enables us to take into account, to investigate, and to assess the subtle emerging relationship between tool use and mathematical thinking. By means of descriptions of a priori hypothetical schemes and their genesis, the researcher captures his/her hypotheses and focuses his/her observations. A posteriori, instrumentation theory guides the data analysis and the conclusions.

However, instrumentation theory cannot of course be "the complete solution to everything" and additional theoretical perspectives may be needed. The seven papers that used instrumentation theory in their theoretical framework also included other elements, such as notions of flexibility (Andresen 2006), semiotic mediation (Mariotti 2006; see next section), elements of mathematical didactics (Dana-Picard and Kidron 2006; Kidron and Dana-Picard 2006), and phenomenography (Leung et al. 2006). It is clear that the instrumental perspective may need to be complemented by, for example, topic-specific notions from mathematical didactics or general notions on collective learning (e.g., Wenger 1998). Combining instrumentation theory with other theoretical perspectives thus can be a fruitful avenue for research on the integration of technology in mathematics learning and teaching.

One of the future challenges for the further development of instrumentation theory is to fine-tune the balance – including both the similarities and the differences – between the cognitive ergonomics frame and the anthropological theory of didactics.

7.3.3 Mediation and Semiotic Mediation

In the previous section the particular theoretical lens of the instrumental approach was used to analyze the relationship between artifacts and mathematical knowledge. Through the notions of utilization schemes and techniques with respect to the solution of a task, both the cognitive and the epistemological dimensions were addressed. The following section aims at further elaborating the relationship between these two dimensions by describing other theoretical models that articulate the relationship between artifacts and mathematical knowledge in an educational context. In recent years, among the various theoretical perspectives that have been used to frame research in mathematics education, some studies have adopted a semiotic perspective – focusing on the role of signs and symbols and their use or interpretation (Sàenz-Ludlow and Presmeg 2006). In the following section, we present an account of the ways in which semiotic perspectives can contribute to the study of the integration of technologies into the classroom.

7.3.3.1 Representation and the Semiotic Approach

Because of its epistemological nature any immediate relationship with mathematics is impossible; any relation passes through a mediation process. Ideal, immaterial, nonperceivable entities such as numbers or figures acquire existence, can be thought of and shared, only through their materialization in a concrete perceivable entity, generally referred to as representation. As discussed in Sect. 7.2.3.3, the potential of new technologies for such mediation appears very promising from early research studies.

In the common practice of experts, representations become so familiar as to achieve complete transparency. Taking an educational perspective, however, we are aware that the complexity of the process underlining the evolution toward transparency cannot be overestimated. Different theoretical perspectives model the functioning of representations and, in particular, the functioning of representations offered by new technological devices with respect to specific educational goals. Drawing on a widely shared assumption about the key role of representation in the development of knowledge, the opportunities offered by new digital technologies have been explored.

> The interaction between a learner and a computer is based on a symbolic interpretation and computation of the learner input, and the feedback of the environment is provided in the proper register allowing its reading as a mathematical phenomenon. (Balacheff and Kaput 1996, p. 470)

Acting within a computer-based environment with representations of mathematical ideas (concepts) occurs in at least two different kinds of modalities that correspond also to different kinds of technological devices. These two modalities can be classified as direct or indirect, according to the nature of the interaction between the user and the machine. In fact, action is accomplished either through communicating with the machine using a particular language (for instance, Logo-based environments) or through directly manipulating objects with the mouse (for instance, Dynamic Geometry environments). Obviously, in most cases both modalities are available and can even interact.

This perspective generated a number of studies that showed the value of such action in enhancing pupils' thinking; in particular, different kinds of animations demonstrated their effectiveness in provoking students to engage in meaningful inquiry involving mathematical ideas (e.g., Nemirovsky et al. 1998; Nemirovsky and Borba 2004).

As was clearly illustrated by Kieran and Yerushalmy (2004), new technologies offer the opportunity of exploiting the coordination of multiple representations of mathematical concepts – both within a given computer environment and between a given computer environment and the traditional paper-and-pencil environment.

Confrey (1992) and Borba (1993, 1994) refer to the coordination of and contrast among representations as an "epistemology of multiple representations" and suggest a model for the meaning of understanding in multiple representational environments.

A comprehensive account of the different theoretical perspectives addressing the issue of representation with respect to digital media is beyond the scope of this section. The European research project ReMath, which aims at coping with the variety and fragmentation of the different theoretical frameworks concerning the role of representations in enhancing mathematical learning with digital media, is addressed later in the chapter.

Below we focus on the idea of representation and on the process of mediation related to the different forms of representation involved in acting and interacting with technological devices.

7.3.3.2 Mediation

Taking a general perspective and assuming that context - in its most comprehensive interpretation - shapes human thinking, a crucial point consists in how one considers the distinction between humans and tools, as particular elements of the context. As Borba and Villarreal (2005, p. 12) claim, statements such as "computers develop students' thinking" and "computers help students to graph" may or may not express a disjunction between humans and tools, depending on the theoretical framework used. This means that the interpretation of sentences of the type "computers develop students' thinking" and the like depends on how one conceives the relationship between human cognition, knowledge, and tools.

Traditionally the dichotomy humans – tools is reflected in the disvaluing[3] of techniques and technology (assumed as the study of techniques). The origin of this disvaluing lies in the assumed lack of creativity of the application of predefined techniques while using a tool, compared to the potentially high creative character of human beings.

On the contrary, assuming a unity between humans and tools or, as Lévy (1990) claims, refusing a dichotomy between humans and technology, may result in a completely different evaluation of the use of tools, and in different approaches to learning issues related to their use.

For instance, the unity between humans and media, humans-with-media (Borba and Villarreal 2005), may be considered a basic goal: the tool becomes transparent (Meira 1998); the violin is one with the violinist (Moreno-Armella and Santos-Trigo 2002). The consequences of this theoretical assumption on the

[3] Stemming from the ancient Greek culture, the distinction between technical and conceptual, and the parallel between practical and ideal, refers to philosophical positions and theoretical oppositions still alive at present; elaborating on this distinction is beyond the goals of this text, but it is useful to keep it in mind in order to understand and overcome the difficulties so often encountered in communicating.

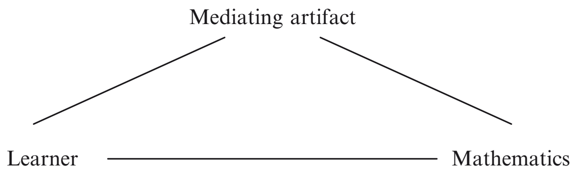

Fig. 7.6 A model of the process of mediation by an artifact (Jones 2000)

educational plane are diverse: the complexity of the school context and the diversity of epistemological approaches shape educational goals and their relationship to curricular requirements with respect to the mediation of a tool, and consequently with respect to mediated action, in a different way.

The general hypothesis stating the basic role of practice in the construction of knowledge does not fully explain the role of the use of artifacts in the particular case of mathematics education. The relationship between artifacts and knowledge is complex and asks for a careful analysis in order to avoid oversimplification. The most insidious risk is that mathematical meanings, rooted in the use of artifacts, might remain extraneous to pupils: they may remain "in the eyes of the observer".

Figure 7.6 illustrates a model of the learning process, inspired by a socio-cultural approach that focuses on the use of an artifact and is expressed in terms of mediation (Jones 2000). This model is often summarized as follows: *artifacts are a means to access* mathematical knowledge. In other words, artifacts are considered not only to be a means to accomplish a concrete action, such as a compass to draw a circle, or a calculator to compute a multiplication, but are also considered to be a means for learning. As just said, the role of artifacts in learning is often expressed metaphorically by the expression "[…] access mathematical knowledge", that is, by evoking the action of entering a place, and in this case, entering mathematical knowledge. This metaphor (Lakoff and Johnson 1980) is widespread in the mathematics education field and brings about a set of assumptions that tend to remain implicit. Although quite effective in expressing the potential of an artifact in mathematics education, such a metaphor does not help one to fully understand how and why the mediating artifact may function to make the learner's access successful, and as such leaves a great part of the story unexpressed (Mariotti 2002).

Through elaboration of the seminal idea of semiotic mediation introduced by Vygotsky (1978), and through the combination of both a semiotic and an educational perspective, more refined theoretical models have been proposed to describe and explain the process that starts with the use of an artifact to accomplish a task and leads to the learning of a particular mathematical content.

7.3.3.3 Mediation According to a Semiotic Approach

The mediating potential of any artifact resides in the double semiotic link that such an artifact has with both the meanings emerging from its use for accomplishing a task, and the mathematical meanings evoked by that use, as recognized by an expert in mathematics. In this respect, any artifact may be considered both from the individual point of view – for instance, the pupil coping with a task and acting with a tool to accomplish it – and from the social point of view – for instance, the corpus of shared meanings recognizable by the community of experts, mathematicians or mathematics teachers. From a socio-cultural perspective, the tension between these two points of view is the motor of the teaching-learning process centered in the use of an artifact.

In this respect any artifact, either belonging to the set of new technologies or belonging to the set of ancient technologies, may offer a valuable support according to its semiotic potential, although the identification of such a potential might require different approaches (Bartolini Bussi and Mariotti 2002).

The double semiotic relationship hinged in the artifact may become the object of an a priori analysis, involving in parallel two interlaced perspectives, the cognitive and the epistemological. The coordination of these two directions of analysis leads one to the identification of the *semiotic potentialof an artifact*, which can be related to particular educational goals.

The determination of its semiotic potential certainly constitutes a basic element for designing any pedagogical plan centered on the use of a given artifact. The construct of instrumental genesis, discussed above, provides a crucial contribution to such analysis. As long as the evolution of personal meanings is related to the accomplishment of a task, it can be analyzed in terms of instrumental genesis, that is, meanings may be related to specific utilization schemes that themselves are related to the specificity of the tasks proposed to students. Thus, an instrumental approach becomes fundamental not only in the identification of semiotic potential but also in the design of appropriate tasks, as well as in the interpretation of pupils' actions and 'speech' acts.

A main hypothesis assumes the contribution of semiotic processes in knowledge construction and the particular role of signs in the internalization process (Vygotsky 1978). As expressed by Radford (2003), artifacts and in particular what he calls semiotic systems contribute not only to accomplishing a task, but also to constructing knowledge:

> In other words, to arrive at the goal the individuals rely on the use and the linking together of several tools, signs, and linguistic devices through which they organize their actions across space and time.
>
> These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call *semiotic means of objectification*. (p. 41)

But how might the student become aware of meanings as related to mathematics? In other words, using a terminology inspired by Leont'ev (1976), how might it happen that personal meanings, which arise in the accomplishment of a task through the use of a certain artifact, become "mathematical meanings"?

Meanings are expressed through representatives of different kinds – words, gestures, drawings, and so on – and even through complex hybrids (Radford 2003; Radford et al. 2004; Arzarello 2006). Meanings emerge through their expression via external representations, so that new signs[4] can be socially shared. The main characteristic of these signs is their strong link with the actions accomplished with the artifact, in so far as the semiotic potential of the artifact has been disclosed.

When this semiotic process takes place in the classroom, social interaction may assume a common goal oriented to teaching/learning mathematics. Both pupils and teacher may be involved in the evolution of signs referring to personal meanings. The teacher's action will be at both the cognitive and the meta-cognitive levels, both fostering the evolution of meanings and guiding pupils to be aware of their mathematical status (Cobb et al. 1993).

In acting at the metacognitive level, the teacher's role becomes crucial: taking the educational goal of introducing pupils to mathematical culture, the teacher plays the role of *cultural mediator*. He/she purposely bridges the individual and the social perspectives and makes the artifact function as a *semiotic mediator* and not simply as a *mediator*. According to a Vygotskian perspective, a basic assumption is that the awareness of the semiotic potential of the artifact, both in terms of mathematical meanings and in terms of personal meanings, allows *the teacher to use the artifactas a tool of semiotic mediation*.

> Thus any artifact will be referred to as tool*of semiotic mediation* as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention. (Bartolini Bussi and Mariotti 2002)

Beyond, but not in contrast with, the objective of making the artifact become transparent or "converting tools into mathematical instruments" (Guin and Trouche 1999), this approach focuses on the learning process related to the use of the artifact using a semiotic lens. A theoretical model is proposed, describing learning as social endeavor, where the evolution of signs is intentionally organized and directed by the teacher. According to a Vygotskian perspective, this evolution corresponds to the move from personal meanings rooted in the context of the artifact to conscious mathematical meanings. Long-term teaching experiments inspired by the perspective of semiotic mediation have largely contributed to clarifying and developing such a semiotic approach; a short account of these studies is given in the following section.

7.3.3.4 Examples of Semiotic Mediation

Consider the relationship between geometrical constructions and drawing procedures in a DGE. Software tools such as Cabri simulate the drawing tools of classic geometry, and nicely reproduce on the screen what for centuries was drawn on various supports, paper, sand, and the like. However, far beyond the correspondence between

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[4]The term *sign* is used consistently with Pierce's characterization: "Something which stands to somebody for something in some respect or capacity" (Pierce 1932; 2.228), taking into account the need for a broad notion of semiotic system. For further discussion, see Bartolini Bussi and Mariotti (2002) or Arzarello (2006).

the graphic results of constructions, for example, the drawing on the screen and the drawing on the paper, the semiotic potential of Cabri resides in the correspondence between the logic of the stability by dragging the Cabri figure and the logic of the geometrical validity of the corresponding construction procedure. The meaning of construction as it emerges from solving construction problems in Cabri has a counterpart in the theoretical meaning of a geometrical construction as it is recognized in geometry.

The core of the dynamics of a DGE figure, as it is realized by the dragging function, consists of preserving its intrinsic logic, that is, the logic of its construction. The elements of a figure are situated in a hierarchy of properties; this hierarchy is defined by the construction procedure and corresponds to a relationship of logical conditionality. At the same time, the intrinsic relationship between a DGE and Euclidean geometry, as described by Laborde and Laborde (1995) amongst others, makes it possible to interpret the control 'by dragging' as corresponding to theoretical control – by proof and definition – within the system of Euclidean geometry.

In summary, as far as Cabri tools are concerned, the semiotic potential is recognizable in a double relationship. On the one hand, Cabri tools are related to the construction tasks and to the utilization schemes that are mobilized by the solver to obtain the Cabri figures on the screen. On the other hand, the Cabri tools are related to the geometrical construction problems that make sense and can be solved within classic Euclidean geometry theory through geometrical theorems.

In accordance with the identification of this semiotic potential, a pedagogical plan was designed for a study by Mariotti (2000, 2001). The main motive of the classroom activities was the evolution of the meaning of *construction*. At the very beginning, 'construction' made sense in the field of experience of Cabri: the solution of drawing problems made sense in relation to the use of particular Cabri tools for producing a Cabri figure, that is, a figure passing the dragging test. Then the meaning slowly evolved towards the theoretical meaning of geometrical construction. The results of extended teaching experiments attested to the emergence of intermediate meanings, rooted in the semantic field of the artifact, as well as their evolution into mathematical meanings, consistent with Euclidean geometry. Such an evolution could be accomplished, under the teacher's guidance, by exploiting the correspondence between, on the one hand, the selection of specific Cabri tools and the elaboration of their utilization schemes and, on the other hand, the assumption of the corresponding Euclidean axioms. The appropriateness of a specific procedure to achieve a "correct" Cabri figure thus had a counterpart in the acceptability of the construction procedure according to geometrical axioms. The conventional constraints of axiomatic assumptions found a parallel in the choice of tools and in the acceptance of their constraints. In this sense, we can talk of semiotic mediation in that Cabri tools and their use could be exploited by the teacher to make sense of axioms and theorems and their functioning within a theoretical system.

The semiotic mediation approach was also used to frame the design and implementation of an elementary symbolic manipulator, *L'Algebrista*, with the aim of developing the meaning of symbolic manipulation as a deduction within algebra theory (Cerulli and Mariotti 2002; Mariotti 2006). The microworld *L'Algebrista* was expected to offer students the opportunity to actively experience the constraints of the deductive rules in play in the algebraic theoretical environment. After this experience, the teacher could exploit the semiotic potential of the artifact to develop the meaning of theory, and of algebra theory in particular.

On the one hand, the constraints defined by the command buttons that are available correspond to the limitations defined by the axioms available in algebra theory. On the other hand, the actions of the commands correspond to the deductive rules of substitution that validate the transformation of one expression into another one that is equivalent. Moreover, the buttons corresponding to the axioms offer the opportunity of objectifying (Radford 2003) these ideas and consequently allow for the emergence of a reflective discourse about their functioning. Furthermore, the design of a specific environment where suitable commands can be used to create a new button, to be added to a personalized list of commands, provides the teacher with suitable tools to approach theory at a metalevel where the meaning of the 'logical status of a statement' makes sense.

The complexity of the didactical model based on the notion of semiotic mediation begs for further investigations that focus on different aspects of the process of semiotic mediation. Recent studies have developed specific theoretical constructs, including the notions of semiotic node (Radford 2003) and of semiotic bundle (Arzarello 2006), which can provide insight into the nature of the signs emerging during instrumented activity. Other promising and ongoing studies are focusing on fine-grained analyses of the action of the teacher in "orchestrating the use of an artifact" (Trouche 2005; p. 123), in particular during collective discussions. Up to now, results have tended to be concerned with the description of types of semiotic games that the teacher can put in place with the aim of exploiting the semiotic potential of an artifact and guiding the evolution of mathematical meanings (e.g., Falcade et al. 2007). Despite these advances, further work in this area is needed.

7.4 Summary and Future Developments

7.4.1 Summary

Earlier in this chapter, Sect. 7.2 addressed the past theoretical thinking on the integration of technology in mathematics education. The review study carried out by Lagrange et al. (2003) indicated that before 1998 the only theoretical convergences were at a general level and touched upon issues related to visualization, connection of representations, and situated knowledge. In 7.3, current theoretical developments were described, with particular attention to learning theories from mathematical didactics, to instrumentation, and to semiotic mediation. Looking back at the chapter so far, we do observe theoretical advancements; still, the overall picture is not quite clear, as the articulation of different theoretical frameworks from different backgrounds is not realized, and probably will not be realized in the future. Also, even if specific theories on tool mediation in students' work on mathematical tasks seem to be fruitful, other aspects remain underexposed, such as the role of language in instrumental genesis, the role of the teacher in technology-rich learning environments, and the influence of the available tools on tasks and task design.

The question we address in this final section, therefore, is how we can extrapolate from the state-of-the-art sketched by the "historical" view of the past and present towards the recognition of relevant issues for the future development of theory in technology-related research in mathematics education.

7.4.2 Technological Developments

As theoretical thinking in this field is not independent from technological proceedings, let us first briefly consider possible future developments of technological tools. Thinking about the evolution of ICT in education, the key expression that comes to the fore is *connectivity*. The interest in personal communication strongly drives the need for connectivity. Even more than is the case nowadays, students and their teachers will communicate in oral or written form through the internet, through electronic learning environments, and through classroom connectivity facilities that allow for gathering students' results from handheld devices and projecting them on an interactive whiteboard. Teachers will monitor students' progress and students will be able to engage in Computer Supported Collaborative Learning (see also within this volume Chap. 11, on communication). Computer tools offer options for file transfer between handheld and desktop devices, and between different types of software applications such as DGE and CAS. Computer tools are integrated into more general mathematical environments that integrate different mathematical topics. Meta-tools are emerging, both as unifying artifacts and as multifunctional tools, where components are connected in an integrated whole. As connectivity gets more and more easy, out-of-school use becomes more important. Students use technology at home and elsewhere, and not only for school purposes. For example, they play computer games (team games!) with peers all over the world. Such games may make use of mathematical elements such as geometrical insights or functional dependencies, but these elements are often implicit and not much is known about possible exploitation of these experiences for educational purposes (Shaffer and Gee 2006).

7.4.3 Theoretical Developments

For the issue of the future development of theory in technology-related research in mathematics education, we are tempted to use the same key word *connectivity*. It is clear that no single theoretical framework can explain all phenomena in the complex setting of learning mathematics in a technology-rich environment. Different theoretical frameworks offer different windows on it, and each view on the landscape can be sound and valuable. In fact, different perspectives may be complementary and as such contribute to 'the whole picture'. Theoretical frameworks that are used to investigate the role of technological tools in society and at the workplace might also be taken into consideration (Kent et al. 2007). Whereas both the instrumental approaches and activity theory, which were addressed in Sect. 7.3, did not emerge within the frame of educational issues, they both contribute to an understanding of the learning of mathematics.

So if the connectivity of different theoretical perspectives is to be an issue, how do we manage to bridge the views of different theoretical perspectives, to understand, articulate, and value the different contributions that each of them offers, and to establish knowledge about their connectivity and their complementarity? To answer this question, an integrated theoretical framework is needed, not in the sense of a unifying "metatheory for everything" but in the sense of a metalanguage that helps us to address, describe, and outline the contributions of the different theoretical perspectives. Even if the need for a metalanguage to discuss and evaluate the different theoretical perspectives is evident, its development is far from trivial and raises many questions:

> Looking for integrative perspectives raises some fundamental questions. What kind of integration can reasonably be aimed at? Does it make sense to look for a unified perspective, an overarching theory or metatheory encompassing the different existing frames? Or is such a perspective unreasonable, due to the incommensurability of most of the existing theoretical frames? What can only make sense would be then to look for structures and languages in order to better understand the characteristics of the corresponding approaches, to organize the communication between these, and to benefit from their respective affordances. If so, can we build such structures and languages, and how can we make these operational? (Artigue et al. 2006, p. 5)

However, if the issues of connecting and articulating theoretical orientations can be dealt with, we hope and expect that the theories on learning mathematics with technological means can contribute to the research on this issue and to the learning and teaching practices of students and teachers in their different educational settings. (In the latter regard, see also Zbiek et al. 2007.)

This chapter closes with a brief description of a first attempt to develop such an integrated theoretical framework.

7.4.4 The Remath Integrative Theoretical Framework

To describe the connectivity of theoretical perspectives by means of such a metalanguage is one of the main, not-too-far-in-the-future challenges in this domain. An interesting and promising example of an attempt to investigate theoretical connectivity in the domain of research on technology in mathematics education is the Integrative Theoretical Frame built by Artigue and colleagues within the ReMath project (Artigue 2006; Artigue et al. 2006).

The research team identified three main dimensions of didactical functionality for ICT tools:

1. Tool characteristics and features
2. Educational goals and associated potential of the tool
3. Modalities of use in a teaching/learning process

For each of these three dimensions, concerns are made explicit by which existing theoretical frameworks can be described and evaluated. For the dimension of tool characteristics, for example (at times referred to as "characteristics of the interactive learning environment" in the report by Artigue et al. 2006), these concerns include the following:

1. The ways *mathematical objects* and their interaction are represented
2. The ways *didactic interactions* are represented
3. The ways *representations* can be acted on
4. Possible interactions and *connections* with other semiotic systems
5. Relationships with institutional or *cultural systems* of representation
6. The rigidity or *evolutive* characteristics of representations (Artigue et al. 2006; p. 48)

While the relationships of some of the various concerns to the overall dimensions have yet to be worked out in this emerging framework (as, e.g., in point (b) above where *didactic interactions* are included), certain representational features of the frame are of potential immediate use. For example, for each of the three dimensions in the ReMath integrative theoretical framework, the concerns are graphically displayed in radar charts. Figure 7.7 shows such a radar chart for the dimension of tool characteristics.

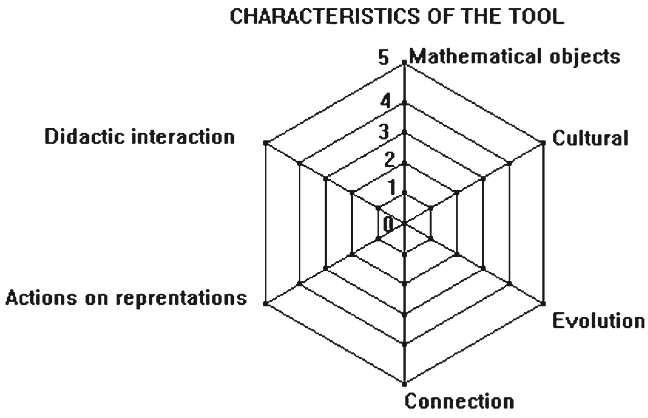

Fig. 7.7 Tool Characteristic radar chart within the ReMath Integrative Theoretical Framework (Artigue et al. 2006; p. 50)

Such radar charts can be used to position different designs and underlying theoretical perspectives, and to compare and articulate them. As an exercise, let us look back at the example of the activity on solving parametric equations presented in Fig. 7.5 (Drijvers and Gravemeijer 2004). We grade each of the above concerns according to the priority it had in this task, with respect to its design and the underlying instrumental theoretical framework (as applied to individual learning). Using grades from 0 (i.e., not considered) to 5 (i.e., high priority), we assign the following scores:

1. The ways *mathematical objects* and their interaction are represented
2. This is certainly an important concern in the task, where the main mathematical object is a parametric equation and the interaction involves solving it with respect to the independent variable. We assign grade 4.
3. The ways *didactic interactions* are represented
4. This is a less important concern in the task because the didactical interactions, for example, between teacher and student and among students, were considered only to a limited extent. We assign grade 2.
5. The ways *representations* can be acted on
6. As we see a parametric equation as an algebraic representation, and the expression of the solution as well, acting on representations is an important aspect in task design. Also, it is important in the process of instrumental genesis, which is a central issue in the theoretical framework. We assign grade 4.
7. Possible interactions and *connections* with other semiotic systems
8. One important focus of the task and the study was to explore the relationship between CAS techniques, paper-and-pencil techniques, and students' conceptual understanding. We see this as a way of focusing on the connections between semiotic systems, and therefore assign grade 5.
9. Relationships with institutional or *cultural systems* of representation
10. In the study and in its focused and limited interpretation of the instrumental approach, institutional and cultural dimensions were hardly considered. Therefore, we assign grade 0.
11. The rigidity or *evolutive* characteristics of representations
12. On the one hand, representations within a computer algebra environment such as the TI-89 are quite rigid. On the other hand, instrumental genesis is an evolutive process. As the study focused on the identification of schemes of instrumented action more than on their genesis, we assign grade 1.

Figure 7.8 shows how these ratings – which do have a degree of subjectivity here and should be established in a more sophisticated way – can be represented in the radar chart, in which the rating 0 for the cultural concern is not displayed. This chart provides an overview of the concerns that played a role in the task design and the theoretical framework as it was used. It allows for comparison with another task and/or another theoretical perspective. It may help researchers to be explicit about their framework and to position their work in relation to other studies in the field.

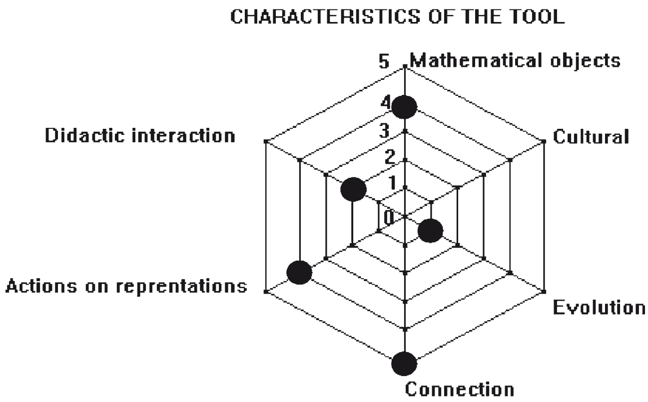

Fig. 7.8 Tool Characteristic radar chart applied to the Drijvers and Gravemeijer (2004) example

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Chapter 8
Mathematical Knowledge and Practices Resulting from Access to Digital Technologies

John Olive and Katie Makar, with Verónica Hoyos, Liew Kee Kor, Olga Kosheleva and Rudolf Sträßer

Abstract Through an extensive review of the literature we indicate how technology has influenced the contexts for learning mathematics, and the emergence of a new learning ecology that results from the integration of technology into these learning contexts. Conversely, we argue that the mathematics on which the technologies are based influences their design, especially the affordances and constraints for learning of the specific design. The literature indicates that interactions among students, teachers, tasks, and technologies can bring about a shift in empowerment from teacher or external authority to the students as generators of mathematical knowledge and practices; and that feedback provided through the use of different technologies can contribute to students' learning. Recent developments in dynamic technologies have the potential to promote new mathematical practices in different contexts: for example, dynamic geometry, statistical education, robotics and digital games. We propose a transformation of the traditional didactic triangle into a didactic tetrahedron through the introduction of technology and conclude by restructuring this model so as to redefine the space in which new mathematical knowledge and practices can emerge.

Keywords Mathematical knowledge • Mathematical practices • Dynamic technologies • Learning ecologies • Didactic triangle • Didactic tetrahedron

8.1 Overview of the Chapter

We have structured this chapter into three major sections: (1) mathematical knowledge and learning that results from the use of technology, (2) mathematical knowledge on which the technologies are based, and (3) mathematical practices that are made

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possible through the use of technology. We preface these three major sections with a look back at the potential of digital technologies to transform the way mathematics could be taught and learned that emanated from the first ICMI study, and comment on the limited realization of that potential. We argue that the assimilation of the technologies to existing classroom practices rather than the technologies provoking an accommodation in those practices has limited that potential. In this preface we suggest a metaphor for how technology could transform the traditional didactic triangle (student, teacher, and mathematics) into a didactic tetrahedron, and use selected faces of that tetrahedron as the focus of subsequent major sections of the chapter.

The first major section begins with a discussion of what we mean by "mathematical knowledge" in a technological world and the different research perspectives on knowledge in a mathematical learning context. Following this discussion, we address the major question of the influence of technology on the nature of mathematical knowledge. We give particular attention to the operational and notational aspects of mathematical knowledge, and then look at how technology has influenced the contexts for learning mathematics, and the emergence of a new learning ecology that results from the integration of technology into these learning contexts. We conclude this section of the chapter with three different case studies that illustrate novel ways of learning mathematics within these different learning ecologies.

The second major section of this chapter discusses the mathematical knowledge that "resides" within the different technologies, or, rather the mathematics on which these technologies are based and that influences their design, especially the affordances and constraints for learning of the specific design. (Design issues are addressed more fully in Sect. 1 of this volume.) This second section concludes with a discussion of how much of this mathematics the user should be aware or even understand.

In the third major section of the chapter we focus on new mathematical practices. We begin with a discussion of the link between knowledge and practice in mathematics learning and teaching. This is followed by a discussion of the interactions among students, teachers, tasks, and technologies, and the resulting shift in empowerment brought about by these interactions. We then look at the role of feedback that can be provided through the use of different technologies. We conclude this section with more detailed descriptions of technologies that have the potential to promote new mathematical practices in different contexts: dynamic geometry, statistical education, robotics and digital games. The chapter concludes with a summary in which we revisit our didactical tetrahedron, restructuring its vertices so as to redefine the space in which new mathematical knowledge and practices can emerge as a result of our review of the literature presented in this chapter.

8.1.1 Preface

As indicated in Chap. 7, the first ICMI study, held in Strasbourg, France in 1985 concerned the influence of computers and informatics on mathematics and its teaching. This first ICMI study reported (with considerable optimism) on the *potential* of these technologies to transform the way mathematics could be taught and learned

(Howson and Kahane 1986). Initial predictions about the influences of technology built up in our minds an image that computing machines would replace arduous tasks, with computational giants freeing the human element (Pacey 1985). The uses of technology in education, however, have often simply replaced paper with computer screens without changing tasks; computers have been used to "simply transfer the traditional curriculum from print to computer screen" (Kaput 1992, p. 516) in ways that resemble traditional worksheets and structured learning environments, rather than working to transform learning (Tyack and Cuban 1995). This limitation, however, is less a limitation of the technology "than a result of limited human imagination and the constraints of old habits and social structures" (Kaput 1992, p. 515).

Piaget (1970) introduced the distinction between the assimilation and accommodation of concepts, contrasting adaptation of the environment to the organism with adaptation of the organism to its environment. In assimilation, learners would attempt to interpret a new idea into their current framework for conceptual understanding. This often meant remaking the concept to fit within their perspective, sometimes at the expense of its intent. Piaget argued that for understanding, it was necessary to sometimes adapt one's framework to take on and make the new concept viable within the environment (accommodation). In some ways, a similar revolution is taking place in mathematics classrooms – some are taking technology innovations and refitting them to retain the viability of the current classroom contexts. For example, many uses of technology take the form of creating electronic worksheets and structured lessons that more or less take the place of current classroom practices. Rather than have the technologies redefine classrooms, they are assimilated into current practice. Alternatively, technology can assist us in considering new forms of practice in profound ways, essentially accommodating new technologies rather than assimilating them. In this chapter we shall attempt to describe those situations in which the use of technology has brought about an accommodation in the ways people teach and learn mathematics and the new kinds of mathematical knowledge that results from such accommodations. Healy (2006), for example, describes the challenges of integrating technology into the Brazilian education system. Built on the assumption that the introduction of computer technology would act as a catalyst for change in classroom practice, researchers came to better understand the complexity of the educational system and in particular the critical role of the teacher in the process of learning, and the reciprocal relationship between technology and meaning-making.

We have developed an adaptation of the "didactic triangle" (Steinbring 2005) that attempts to incorporate the catalytic role of technology in this complex educational system. We add "technology" as a fourth vertex of the didactic triangle, transforming it into a 3D tetrahedron, creating three new triangular faces, each face illustrating possible inter-relationships among student, teacher, mathematical knowledge and technology (see Fig. 8.1). Theme B of this study focuses on teachers and teaching, thus, we shall primarily focus on the Student-Technology-Mathematical Knowledge face of this tetrahedron, realizing, of course, that the teacher is a critical component in any didactical situation. Transforming the didactic triangle into a didactic tetrahedron through the addition of technology is seen as a metaphor for the transforming effects of technology when it is accommodated by the didactical situation rather than assimilated into it. It literally adds a new dimension to the didactical situation.

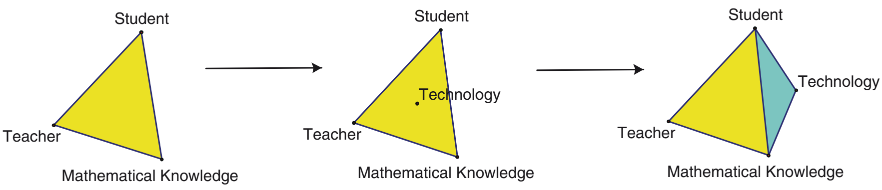

Fig. 8.1 Transforming the didactic triangle into the didactic tetrahedron

8.2 Mathematical Knowledge in a Technological World

8.2.1 What Is Mathematical Knowledge?

Researchers in the field of mathematical education have attempted to distinguish among knowledge of mathematical concepts, knowledge of mathematical procedures and acquisition of skills (e.g. Baroody et al. 2007; Hiebert and Lefevre 1986; Star 2005, 2007; Gray and Tall 1994; Tall et al. 2001). Other researchers (Olive 1999; Olive and Steffe 2002; Steffe and Olive 1996; Thompson 1995; Tzur 1999) have attempted to clarify whose mathematical knowledge they are studying and have articulated these distinctions:

- *Children's mathematics:* The mathematics that children (or learners of any age) construct for themselves and is available to them as they engage in mathematical activity
- *Mathematics for children:* The mathematical activities that curriculum developers/ writers and teachers design to engage students in meaningful mathematical activity
- *Adult mathematics:* The mathematics that adults have constructed through their years of schooling and experience in the world
- *Disciplinary mathematics:* The mathematics created and studied by professional mathematicians

Whether one takes a radical constructivist view of knowledge (Piaget and Szeminska 1965; von Glasersfeld 1995; Steffe 1992) or a social constructivist view (Vygotsky 1978), the question of who's mathematics we are focusing on is relevant when addressing our driving question of what new types of mathematical knowledge emerge as a result of access to digital technologies.

Many researchers have made a distinction between procedural and conceptual knowledge. Baroody et al. (2007) define procedural knowledge as "mental actions or manipulations, including rules, strategies, and algorithms, for completing a task." They define conceptual knowledge as "knowledge about facts, [generalizations], and principles" (p. 123). Following Star (2005), Baroody et al. define these knowledge types independently of the degree of connectedness that may exist within each type of knowledge. They claim that mathematics education researchers:

> [H]ave long contrasted unconnected, disembodied, meaningless, context-bound, or mechanical procedures (what could be called a "weak scheme") with well-connected, contextualized, integrated, meaningful, general, or strategic procedural knowledge (what could be called a "strong scheme" …). Analogously, well-connected conceptual knowledge has been contrasted with sparsely connected conceptual knowledge … For instance,*strong schemas* – which involve generalizations broad in scope, high standards of internal (logical) consistency, principle-driven comprehension, and principled bases for a priori reasoning (i.e. predictions are derived logically) – have been proposed to underlie deep conceptual knowledge.*Weak schemas* – which entail generalizations local in scope, low standards of internal (logical) consistency, precedent-driven comprehension, and no logical basis for a priori reasoning (i.e. predictions are looked up) – have been posited to underlie superficial conceptual knowledge and to explain why younger children's concepts may be less deep and sophisticated (e.g., less general, logical, interconnected, or flexible) than older children's or adults'. (p. 117)

Steffe (1992, 2002, 2004) and Olive (1999; Olive and Steffe 2002; Olive and Vomvoridi 2006) would disagree with the association of younger children's mathematical concepts with weak schemas and superficial conceptual knowledge. They argue, instead, for a distinction between young children's mathematics and older children's mathematics (and adults' mathematics) based on the *nature and content* of their schemes and schemas (Olive and Steffe 2002). Within the constraints of their experiences, young children's schemas can be as strong as adult schemas (or as weak). What distinguishes both their schemes and schemas from those of adults are the mental constructs they have built based on their lived experiences.

Baroody et al. (2007) point out that "The construct of adaptive expertise, for one, unites the notions of deep conceptual knowledge, deep procedural knowledge, and flexibility" (p. 120). Hatano (2003) argues that flexibility and adaptability only seem possible when there are conceptual meanings to provide criteria for selecting among alternative procedures. That is, one needs conceptual knowledge to give meaning to processes. Gray and Tall (1994) put forward the notion of "procept" as a combining of process and concept. They argue that successful students can use symbols as procepts, whereas less successful students are limited to use of procedures.

In the examples that follow later in this chapter, we shall attempt to distinguish among technologies that have the potential to help users combine procedural and conceptual knowledge (proceptual knowledge) and those technologies that enhance the learning of isolated (or disconnected) processes.

8.2.2 The Influence of Technology on the Nature of Mathematical Knowledge

The next step in our investigation of new knowledge comes out of the need to examine the way that learners view mathematics, both its nature and its utility. In keeping with the view that young children can (and do) construct *strong schemas*, Steinbring (2005) notes that "New mathematical knowledge is not merely still unfamiliar, added, finished knowledge, but new mathematical knowledge has ultimately to be understood as an extension of the old knowledge by means of new, extensive relations, which at the same time let the old knowledge shine in a new light and, even generalize the old knowledge" (p. 3). This requires quite a different conceptualization of the nature of mathematical knowledge, both by learners and by teachers. If one considers mathematics to be a fixed body of knowledge to be learned, then the role of technology in this process would be primarily that of an efficiency tool, i.e. helping the learner to do the mathematics more efficiently. However, if we consider the technological tools as providing access to new understandings of relations, processes, and purposes, then the role of technology relates to a conceptual construction kit. In this way, Steinbring argues that for learners, mathematical knowledge is always "on the way," where knowledge bound in their concrete experiences emerges as new generalized understandings developed through ongoing interactions with ideas, relations, processes, structures and patterns viewed in new ways. Therefore, as we shall argue further in Sect. 8.4, knowledge is deeply embedded in practices and experiences.

Within the field of mathematics, many changes have taken place as a result of technological advancements. According to the Australian Academy of Science (2006), however, professionals have rarely taken advantage of new developments in the mathematical sciences (e.g., genetic innovation research, optimization in imaging, stochastic modeling), most of which have emerged out of the intersection of mathematics and new technologies (MASCOS 2004; Australian Academy of Science 2006). Likewise, despite the strong influence of technology on new developments in the field, little has changed in the school mathematics curriculum (e.g., Sorto 2006). These resistances are likely due to conflicts in teachers' and curriculum publishers' beliefs about the nature of mathematics and the goals of school curriculum. The foundations of what and how mathematics should be taught are now being challenged with the infusing of technologies into mathematics education. Reconciling these conflicts requires a re-evaluation of our beliefs about the very nature of mathematics. Most research utilizing new technologies in mathematics education portray mathematics as experimental, challenging, and empowering (e.g. Buteau and Muller 2006; Kaput 1996; Noss and Hoyles 1996; Papert 1972, 1980). These images resonate with philosophical challenges to the nature of mathematics in the last century (Lakatos 1976, 1978).

There is a perception in the general population of mathematics as a field that is "hard, right or wrong, routinised and boring" (Noss and Hoyles 1996, p. 223); this perception is rampant in school mathematics and in the public domain. The divorce of mathematics from its epistemological roots has often created a by-product of perceptions by learners that mathematics is too difficult for ordinary people to grasp. Technological environments potentially reconnect the learner with contexts in which they regain the agency to create meaning. These situations can be authentic contexts supported by technological tools to control complexity or they can be imaginary worlds in which learners can try out ideas.

Borba and Villarreal (2006; drawing on work by Tikhomirov 1981) consider the epistemological role of computers in learning mathematics. They argue that conceptualizing knowledge as atomistic leads to a view that the role of computers in generating knowledge is one where technologies either substitute humans or supplement humans. In schools, this is often the conceptualization that is operationalized – technologies are used to substitute paper-and-pencil calculations or supplement graphing skills. However, they argue that this view is shortsighted. In conceptualizing technologies through a broader complexity framework, one begins to realize the challenges in separating technology's effect on the transformation of knowledge with the transformation of practice. Borba and Villarreal further contend that humans and technologies are often seen as disjoint, assuming that the "cognitive unit" is only the human being, not the humans-with-media perspective that they adopt. "The very idea of considering the human being as the unit that produces knowledge can underestimate the importance of technologies in this knowledge production" (p. 12). Noss and Hoyles (1996) argue that because technologies mediate knowledge construction, they not only alter this construction of knowledge, but the meaning of knowledge for individuals as well.

8.2.3 Mathematical Knowledge: Operational and Notational Aspects

Mathematics in school has frequently focused on *notational* aspects of the discipline. That is, the emphasis is on the symbolic and representational aspects of mathematics (Fey and Good 1985; Kaput 1987, 1998; Hitt 2002), particularly in the areas of algebra, geometry, and statistics. School mathematics has been primarily restricted to routine procedures that could be carried out by hand; time and effort focused on teaching students to calculate and perform procedures by hand. Therefore, the emphasis was on the written, notational, symbolic aspects of mathematics rather than its more operational aspects:

> In the past, teachers and students were confined to a sequential approach to learning these procedures, with the mastery of each step in the procedure necessary before proceeding to the next step. Technology allows a different approach, with more complex procedures (or *macroprocedures*) chunked into a series of simpler procedures (*microprocedures*) (Heid 2003). (Heid 2005, p. 347)

Technology has therefore allowed school mathematics to incorporate a more *operational* focus that adds another dimension to understanding. By an operational focus, we mean an emphasis on the practice and applications of mathematics through visualization, manipulation, modeling, and the use of mathematics in complex situations. With technology, students can use technology to solve an equation before they need to master factoring or the quadratic formula by hand, approximating solutions graphically (e.g., Fey and Heid 1995). In this way, there exist choices regarding which to do first (by hand or with technology) or at all. By operationalizing mathematics, mathematics is distributed between the student and the technology, with the student empowered to decide when and how to use the tool (Heid 2005; Geiger 2006). However, this requires the student to understand and make decisions about what mathematics might be useful and how it might be used. This is quite a new experience after conventional schooling has cued students into knowing which mathematics is needed for a problem up front (Boaler 1997).

This shift to operationalize mathematics is not automatic. Initial uses of technology are often more rote or rely on the technology for simple computational tasks. As facility and understandings develop, the technology becomes more of a thinking tool, what Geiger (2006) and his colleagues (Galbraith et al. 2001) term a technological "partner" or extension of self. As stated by Jere Confrey (in Heid 2005):

> Technology is likely to change not only the content of school mathematics but also the processes of school mathematics and the nature of mathematical understandings. Students in technologically rich classrooms are likely to develop multirepresentational views of mathematics. Some technologies will enable them to develop almost a kinematic understanding of functional relationships. (p. 357)

8.2.4 Contexts for Learning Mathematics

The focus of mathematics in school has been on teaching students the power that mathematics has to generalize and abstract from particular contexts. These abstractions have developed over centuries of work by thousands of mathematicians. Rather than require children to begin this road again, school mathematics allows them to benefit from the toil and uncertainty of previous generations of mathematicians, and instead work with mathematical tools that have already passed the test of viability. The drawback has been that in the process of abstracting from context, the purpose of mathematics as a tool for making meaning has sometimes been forgotten or put aside. Some of this putting aside has been because of efficiency – the contexts in which mathematical meanings can be derived are neither simple to design nor simple in themselves. This is no surprise given the elapse of time over which these ideas have arisen. Some content that students encounter in schools has developed only in the last century (e.g. fractal geometry and iterative or recursive functions) – in comparison to the millennia on which their foundations are built.

Rather than adopt new mathematics, many reform curricula have chosen to embed traditional mathematics into what they term "real-life contexts" (e.g. CMP, Core Plus, Mathematics in Context, in the US). Unlike problems found in the world, however, many of the problems found in these curricula remain "well-defined" in their attempts to simplify the complexity of the situation. These "pseudo-contexts" can actually make mathematics more difficult to learn (especially for lower socio-economic students), as they give children conflicting messages about whether unintended contextual and experiential factors should be ignored or drawn into play (Lubienski 2000). By oversimplifying the intellectual demands required to mathematize and interpret problems, and by trivializing the contribution of mathematics to solving real problems, the perception of mathematics as a subject with limited use outside of school is reinforced. Noss and Hoyles (1996) contend that technologies open the possibility for meaningful mathematics to be created within the context of school rather than simply brought in from the outside.

Contexts allow the learner to reflect on and control for the meaning and reasonableness of their developing ideas. This allows them to ensure that concepts are viable within the situation. Of course, a goal in mathematics is to abstract and generalize across contexts, but enough is not done to encourage meaning-making to begin with. These contexts come out of a diverse allowance of settings. Technological tools allow for one type of setting from which learners can play with ideas. Dynamic software packages can facilitate visualization (Presmeg 2006), connecting informal and formal mathematics (Mariotti 2006), and develop perceptions of mathematics as an instrument rather than an object (Rabardel 2002). In situating the student in a transformative position of agency, these technologies potentially redefine and expand the student's role as knower and creator of mathematical knowledge. Laborde and her colleagues (Laborde et al. 2006) remind us, however, that it is not only the interaction of the student and the machine that matters but also the design of tasks and learning environment (see Sect. 1 of this volume). They argue for an intrinsic link between mathematical knowledge and understanding of its use as a tool. Mackrell (2006) argues, for example, for the powerful influence that dynamic visualization software programs like Cabri 3D (2005) and Geometer's Sketchpad (Jackiw 2001) can have on students' understanding by enabling them to manipulate mathematical objects as tangible entities, observing and debating invariant relationships. Mathematics is, after all, primarily concerned with properties of *invariance* – the characteristics that describe attributes and relationships that remain constant under varying conditions. For example, in Euclidean geometry, the three angle bisectors of a triangle pass through the same point. Algebraic identities describe relationships that remain fixed as the values of the variables change. Coming to know these invariant properties through dynamically changing that which varies (rather than memorizing facts) can contribute to much more stable and powerful mathematical knowledge. Thus, dynamic technologies can become powerful contexts for learning mathematics.

8.2.5 A New Learning Ecology

It is not the technology nor the play themselves that evoke meaning, but rather careful interactions between the task, teacher support, technological environment, classroom and social culture, and mathematics (Noss and Hoyles 1996). We have learned much from the days of thinking the computer would solve the challenges of learning (Cuban 2001), but this does not mean that we should be tempted to err on thinking dichotomously that computers have failed to add value to learning. In the old way of thinking, computers were seen as human tutors and evoked a vision of the teacherless learning environment. New avenues for using technology take advantage of, rather than marginalize, teacher, task, and classroom cultures. For example, there is the question as to whether strong emphasis in school mathematics on developing expertise in symbolic manipulation should continue to be at the forefront of time in secondary mathematics given the accessibility of Computer Algebra Systems (CAS). Within geometry, traditional instruction utilizes a definition-theorem-proof (dtp) approach to teaching geometry, where students are first taught definitions and given theorems and proofs about geometric objects and relationships before having an opportunity to work with and investigate these relationships for themselves. Dynamic geometry environments (DGEs) are challenging this perspective, including the very nature of what counts as a proof, when one considers that students can test a conjectured relationship with thousands of cases to assess its viability. DGE objects and the assessment of their relationships, because they are based on student design in search of a question, depend in new ways on using argumentation and justification. Proof takes on new meaning in this context, and becomes a tool that learners can use to explain what they discover through their dynamic explorations and, thus enable them to convince their peers of their new conjectures – rather than mimicking a mathematician's proof to satisfy an unknown cultural construct (de Villiers 1999). In statistics, new visualization tools enable learners to interact with data through envisioning relationships informally before more formal tools are brought into play. For example, instead of being taught how to calculate a mean, learners might first examine distributions of heights of children of different age groups and look for viable ways to compare and talk about them. Research has found that in technological environments in which children can design their own tools to describe aspects of the data that they find useful, students can envision concepts of center and spread of data by talking about the "clump" in the data (Konold et al. 2002; Makar and Confrey 2005). In an environment supported by worthwhile tasks and a culture of inquiry, learners have an opportunity to operationalize mathematics and use it as a tool for a productive purpose, rather than apply pre-made mathematical concepts to a contrived situation. "The challenge is to focus on the learning ecology as a whole, considering the interactions between different dimensions – epistemological, technological (or perhaps instrumental), cognitive, and pedagogical – concomitantly" (Healy 2006, p. 3). In the following section we present some example cases of technologies that have been successfully used (in different ways) as an integral part of the learning ecology to bring about the construction of (new) mathematical knowledge (for the learners).

8.2.6 Example Cases of Effective Technologies

In this section we present case reports of the uses of three different technologies: Computer microworlds designed with what Zbiek et al. (2007) call high levels of "cognitive fidelity," simulation software and curricula designed to introduce the Mathematics of Change and Variation (MCV) to middle school students, and dynamic geometry environments (DGE). The first case is an example of how young children, within the context of a constructivist teaching experiment, were able to construct powerful fractional schemes through the use of computer-based tools that enabled them to enact their mental operations. The second case reports results of a state-wide implementation of a curriculum unit that made use of specially designed simulation software to improve middle school students understanding of rates, ratios and proportions. The third case reports on the global use of dynamic geometry environments. Zbiek et al. (2007) would categorize the latter two cases as exhibiting high levels of "mathematical fidelity," that is, they provide the users with mathematically accurate visualizations and feedback. We point out, however in Sect. 8.3 of this chapter, that mathematical fidelity cannot be taken as a given with several common technologies used in mathematics classrooms.

8.2.6.1 The Fractions Project: Using Technology with High Levels of "Cognitive Fidelity"

Steffe and Olive (1990) at the University of Georgia (USA) designed and conducted a 3-year constructivist teaching experiment with 12 children (beginning in their third grade in school) in order to develop cognitive models of children's construction of fractions. Computer microworlds called Tools for Interactive Mathematical Activity (TIMA) (Biddlecomb 1994; Olive 2000b; Olive and Steffe 1994; Steffe and Olive 2002) were specifically designed for the teaching experiment and were revised during the teaching experiment based on the children's interactions within these environments. The TIMA provide children with possibilities for enacting their mathematical operations with whole numbers and fractions. They also provide the teacher/researcher with opportunities to provoke perturbations in children's mathematical schemes and observe children's mathematical thinking in action.

The software consists of on-screen manipulatives analogous to counters or beads (regular geometrical shapes that are called "toys"), sticks (line segments), and fraction bars (rectangular regions), together with possible actions that the children can perform on these objects. These possible actions potentially engage the user in the fundamental operations involved in the development of numerical schemes. These operations are unitizing, uniting, fragmenting, segmenting, partitioning, replicating, iterating, disembedding, and measuring. For example, using TIMA Bars, a child can partition a bar into five equal parts, disembed one of the parts by actually pulling it out of the bar (i.e., a copy of the part is lifted from the bar leaving all five parts still in the bar), and then use the REPEAT action to iterate this one part to create a bar that is six times as large as 1/5 of the original bar (see Fig. 8.2).

Fig. 8.2 Making 6/5 of a unit bar by disembedding 1/5 and repeating it six times

The major purpose of this project was theory-building based on in-depth analyses of several case studies of the children's interactions and cognitive constructions over the 3-year period. The TIMA technology provided the children with ways of enacting their mental operations and visualizing the quantitative relations that they constructed. As Olive (2002) points out in his discussion of one student's construction of fractional schemes, the TIMA were critical affordances in the construction of Joe's schemes:

> Being able to make a stick (in TIMA: Sticks) that is "9 times as long as the 1/7-stick" through repetitions of a 1/7-stick, provided Joe with an instantiation of his iterable unit fraction. He had made a modification in his whole-number multiplication scheme that enabled him to use a unit fraction in the same way that he could use units of one with his composite units. The TIMA software had provided Joe with the tools to build a bridge from whole numbers to fractions. (p. 360)

The TIMA software (and later adaptations) has been used by many researchers in different countries since the conclusion of the Fractions Project: Nabors (2003) used the TIMA: Bars software in her study of proportional reasoning; Norton (2005) used TIMA: Bars in his study of eliciting student conjectures; Hackenburg (2007) used a Java version of TIMA: Bars called JavaBars (Olive and Biddlecomb 2001) in her research on middle school students' rational number concepts; Chinnappan (2006) reported using JavaBars in his 2001 study with elementary children in Australia, in which "JavaBars mediated children's cognitive actions" (Chinnappan 2001, p. 102); and Kosheleva et al. (2006) also used JavaBars in their study on the effects of Tablet PC technology on mathematical content knowledge of pre-service teachers, where JavaBars was found to provide "a creative workspace to explore fractions". (p. 298)

Zbiek et al. (2007) categorized the use of the TIMA software in the Fractions Project as having high levels of "cognitive fidelity":

> Cognitive fidelity is a particularly important consideration for researchers. By providing action choices to the learner that faithfully reflect potential cognitive choices, tools such as the TIMA technology can provide to researchers more powerful evidence of patterns in children's thinking. In turn, an improved understanding of children's thinking can better inform continuing development of the tools. (p. 1177)

In addition to this important synergistic relationship between cognitive model building and tool development that tools with high cognitive fidelity provide, they also have the potential to engender the construction of new mathematical knowledge on the part of the user, as Olive and Lobato (2008) reported in their synthesis of the learning of rational number concepts using technology:

> For instance, one way that technology can enhance the learning of rational number concepts is through the use of computer tools that allow students to enact psychological operations that are difficult to perform with physical materials. In order to establish a relation between a part and a whole in a fractional situation, the child needs to mentally disembed the part from the whole. With physical materials it is not possible to remove a part from the whole without destroying the original whole. With static pictures the part is either embedded in the whole or is drawn separate from the whole. … Using a computer tool that provides the child with the ability to dynamically pull a part out of a partitioned whole while leaving the whole intact, the child can enact the disembedding operation that is necessary to make the part-to-whole comparison. (p. 6)

In addition to enabling students to operationalize the part-to-whole relation, the disembedding action, combined with repeating the disembedded part, led to iterating operations that enabled students to construct meanings for fractions greater than one (improper fractions) (Tzur 1999). The ability to enact recursive partitioning led to reversible reasoning and splitting operations, essential for the construction of the "rational numbers of arithmetic" (Olive 1999; Olive and Steffe 2002).

8.2.6.2 The SimCalc Project: Introducing the Mathematics of Change in Middle School – Technology with High Levels of "Mathematical Fidelity"

An implementation of the SimCalc simulation tools, together with the MathWorlds curriculum has been recently tested with more than a thousand middle school students and their teachers in the state of Texas (Roschelle et al. 2007). Developed by Jim Kaput and colleagues at the University of Massachusetts-Dartmouth over the past 15 years, the SimCalc software and MathWorlds curriculum have undergone rigorous cycles of development-field testing-revisions. According to Roschelle et al.,

> SimCalc software engages students in linking visual forms (graphs and simulated motions) to linguistic forms (algebraic symbols and narrative stories of motion) in a highly interactive, expressive context. SimCalc curriculum leverages the cognitive potential of the technology to develop multiple, interrelated mathematical fluencies, including both procedural skill and conceptual understanding. (p. 2)

In terms of the theoretical frameworks outlined in Chap. 7, the SimCalc software acts as a semiotic mediator, linking several different semiotic systems to develop both procedural skills and conceptual understandings. The results of this extensive implementation of the SimCalc MathWorlds curriculum do, indeed, indicate the cognitive potential of the technology, achieving what has been termed the "gold standard" for experimental research, both in design and effects.

The research project involved 120 grade 7 teachers recruited from 8 regions of Texas. A Treatment-Control experimental design was used, with teachers being randomly assigned (by school) to either group. Of the 120 teachers who originally attended the summer workshop, 95 returned complete data for the 2005–2006 school year. At the outset of the experiment the Treatment Group of 48 teachers and the Control Group of 47 teachers did not differ in any significant way. In the summer of 2005 both groups participated in a 2-day professional development workshop focused on rate and proportionality. The Treatment Group received "an integrated replacement unit incorporating SimCalc curriculum, software, and three additional days of teacher training." The Control Group "used their existing curriculum but had the benefit of training and materials on the topic of rate and proportionality" (Roschelle et al.2007, p. 3). Rate and proportionality are typically taught in a 2- to 3-week unit in grade 7 in Texas. The Treatment Group teachers were asked to replace this unit with the SimCalc unit, the Control Group teachers were asked to continue using their existing textbooks, enhanced with professional development support. "The main outcome variable was student learning of concepts of rate and proportionality, measured on identical tests administered before and after the 2- to 3-week rate and proportionality unit" (p. 4). These tests consisted of 30 items: 11 simple and 19 complex items. The simple items were based on items used on the Texas state test and typically asked students to find the missing term in a proportional relationship (e.g. "If $2/25 = n/500$, what is the value of n?"). The complex items addressed understanding of a direct proportional relationship as a function $f(x) = kx$, and the concept of slope of a line graph as an indication of speed in a distance–time relationship. All 30 items went through rigorous validation processes, "including cognitive interviews with students, item-response theory analyses on field test data collected from a large sample of students, and expert panel reviews" (p. 5).

The experiment achieved a highly statistically significant main effect ($p < 0.0001$), indicating that the students in the Treatment Group classrooms learned more than their counterparts in the Control Group classrooms. The difference was most pronounced across the 19 complex items, and held across SES, race and gender groups. Based on these results, the researchers claim the following:

(a) That the SimCalc approach was effective in a wide variety of Texas classrooms
(b) That teachers successfully used these materials with a modest investment in training
(c) That student learning gains were robust despite variation in gender, ethnicity, poverty, and prior achievement. (Roschelle et al. 2007, p. 6)

The researchers make the important point that the gains were accomplished by the Treatment students on the more complex items dealing with proportionality and rate, whereas all students made similar gains on the simpler items. For example, with respect to the comparison of two distance–time graphs on the same coordinate axes, the Treatment students were more likely to use the correct idea of "parallel slope as same speed," whereas Control students were more likely to have the misconception "intersection as same speed." (Roschelle et al. 2007, p. 7)

Other studies on students' conceptions of slope and rate (e.g. Lobato and Siebert 2002; Olive and Çağlayan 2008) have highlighted the difficulties students experience with these concepts; thus, the results obtained through the use of the SimCalc software and curriculum are seen as a breakthrough in this traditionally difficult and important mathematical topic. For the students in the Treatment Group, the use of the SimCalc technology promoted the construction of new mathematical knowledge.

Kaput (1998) pointed out that dynamic, interactive software like SimCalc, that provide bi-directional links between authentic or simulated phenomena and the representations of those phenomena in several notation systems, opens up the *Mathematics of Change and Variation* (MCV) to students who have traditionally been shut out by "the long set of algebraic prerequisites for some kind of formal Calculus, this despite the fact that the bulk of the core curriculum can be regarded as preparation for Calculus" (p. 7). Kaput goes on to state:

> …we can see that while large amounts of curricular capital are invested in teaching numerical, geometric and algebraic ideas and computational techniques in order that the formal symbolic *techniques* of Calculus might be learned, the ways of thinking at the heart of Calculus, including and especially those associated with the Fundamental Theorem, do *not*

> require those formal algebraic techniques to be usefully learned. Indeed, by approaching the rates-totals connections first with constant and piecewise constant rates (and hence linear and piecewise linear totals), and then gradually building the kinds of variation, we have seen the underlying relations of the Fundamental Theorem become obvious to middle school students. (p. 7)

Thus, when we look at the strong results from the Texas implementation of the SimCalc curriculum in light of Kaput's major points concerning access to the important ideas of Calculus, we can look forward to a majority of students creating new kinds of mathematical knowledge concerning change and variation as a result of using such technologies within a well-conceived curriculum, implemented by enthusiastic and well trained teachers.

8.2.6.3 Dynamic Geometry Environments

Dynamic Geometry Environments (DGEs) include any technological medium (both hand-held and desktop computing devices) that provides the user with tools for creating the basic elements of Euclidean geometry (points, lines, line segments, rays, and circles) through direct motion via a pointing device (mouse, touch pad, stylus or arrow keys), and the means to construct geometric relations among these objects. Once constructed, the objects are transformable simply by dragging any one of their constituent parts. Goldenberg and Cuoco (1998) provide an in-depth discussion on the nature of Dynamic Geometry. A common feature of dynamic geometry is that geometric figures can be constructed by connecting their components; thus a triangle can be constructed by connecting three line segments. This triangle, however, is not a single, static instance of a triangle that would be the result of drawing three line segments on paper; it is in essence a prototype for *all possible triangles*. By grasping a vertex of this triangle and moving it with the mouse, the length and orientation of the two sides of the triangle meeting at that vertex will change continuously.

A study by Olive (2000a) describes how a 7-year old child (Nathan) constructed for himself during just 5 min of exploration with the *Geometer's Sketchpad*® a fuller concept of "triangle" than most high-school students ever achieve. He had been shown how to construct a triangle with the segment tool and then experimented by dragging the vertices of this dynamic figure, all the time asking his father if the figure were still a triangle. His father threw the questions back to him and when Nathan responded that the figures were still triangles (fat triangles, skinny triangles, etc.) his father asked him why they were still triangles. Nathan responded "because they still had three sides." But the real surprise came when he moved one vertex onto the opposite side of the triangle, creating the appearance of a single line segment. Nathan again asked his father if this was still a triangle. His father again threw the question back to him. Nathan thought for a while, then held out his hand with his palm facing outwards, vertically, and rotated it to a horizontal position with his palm facing down, while saying: "Yes. It's a triangle lying on its side!" This last comment and accompanying hand-motion indicates intuitions about plane figures that few adults ever acquire: That they have no thickness and that they may be oriented perpendicular to the viewing plane. [Had Nathan entered *Flatland* (Abbott 1884)?] Such intuitions are the result of what Goldenberg et al. (1998) refer to as "visual thinking."

Nathan's use of the dynamic drag feature of this type of computer tool illustrates how such dynamic manipulations of geometric shapes can help young children abstract the essence of a shape from seeing what remains the same as they change the shape. In the case of the triangle, Nathan had abstracted the basic definition: a closed figure with three straight sides. Length and orientation of those sides was irrelevant as the shape remained a triangle no matter how he changed these aspects of the figure. Such dynamic manipulations help in the transition from the first to the second van Hiele level: from "looks like" to an awareness of the properties of a shape (Fuys et al. 1988). For Nathan, this was new mathematical knowledge.

Lehrer et al. (1998) found that children in early elementary school often used "mental morphing" as a justification of similarity between geometric figures. For instance a concave quadrilateral ("chevron") was seen as similar to a triangle because "if you pull the bottom [of the chevron] down, you make it into this [the triangle]" (p. 142). That these researchers found such "natural" occurrences of mental transformations of figures by young children suggests that providing children with a medium in which they can actually carry out these dynamic transformations would be powerfully enabling (as it was for Nathan). It also suggests that young children naturally reason dynamically with spatial configurations as well as making static comparisons of similarity or congruence. The van Hiele (1986) research focused primarily on the static ("looks like") comparisons of young children and did not take into account such dynamic transformations. The use of DGEs with school-age children brings about a need for research on dynamical theories of geometric knowledge.

At the secondary level dynamic geometry environments can (and should) completely transform the teaching and learning of mathematics. Dynamic geometry turns mathematics into a laboratory science rather than one dominated by computation and symbolic manipulation, as it has become in many of our secondary schools. As a laboratory science, mathematics becomes an investigation of interesting phenomena, and the role of the mathematics student becomes that of the scientist: observing, recording, manipulating, predicting, conjecturing and testing, and developing theory as explanations for the phenomena.

Laborde et al. (2006, citing Hoyles 1995), suggest that the process of decision-making and reflection in the interaction between manipulation and outcomes provide students "with hooks they need on which to hang their developing ideas" (p. 292). The software constrains students' actions in ways that require the teacher to conceptualize problems from a student's point of view and encourage students to conceptualize mathematics in new ways. As Balacheff and Sutherland (1994) point out, the teacher needs to understand the "domain of epistemological validity" of a dynamic geometry environment. This can be characterized by "the set of problems which can be posed in a reasonable way, the nature of the possible solutions it permits and the ones it excludes, the nature of its phenomenological interface and the related feedback, and the possible implication on the resulting students' conceptions" (p. 13).

The publication *Geometry Turned On* (King and Schattschneider1997) provides several examples of successful attempts by classroom teachers to integrate dynamic geometry software in their mathematics teaching in ways that generated new mathematics (for the students). Keyton (1997) provides an example that comes closest to that of learning mathematics as a laboratory science. In his Honors Geometry class (grade 9) he provided students with definitions of the eight basic quadrilaterals and some basic parts (e.g. diagonals and medians). He then gave them 3 weeks to explore these quadrilaterals using *Sketchpad*. Students were encouraged to define new parts using their own terms and to develop theorems concerning these quadrilaterals and their parts. Keyton had used this activity with previous classes without the aid of dynamic geometry software. He states:

> In previous years I had obtained an average of about four different theorems per student per day with about eight different theorems per class per day. At the end of the three-week period, students had produced about 125 theorems… In the first year with the use of *Sketchpad*, the number of theorems increased to almost 20 per day for the class, with more than 300 theorems produced for the whole investigation. (p. 65)

Goldenberg and Cuoco (1998) offer a possible explanation for the phenomenal increase in theorems generated by Keyton's students when using *Sketchpad*. Dynamic geometry "allows the students to transgress their own tacit category boundaries without intending to do so, creating a kind of disequilibrium, which they must somehow resolve" (p. 357). They go on to reiterate a point made by de Villiers (1994 cited in Goldenberg and Cuoco 1998), that "To learn the importance and purpose of careful definition, students must be afforded explicit opportunities to participate in definition-making themselves" (p. 357). Marrades and Gutiérrez (2000) found similar results in their studies of secondary school students using Cabri Géomètre in proof-oriented geometry classes. Hadas et al. (2000) also found that designing activities in dynamic geometry to cause surprise and uncertainty was effective in provoking proof on the part of their students.

Keyton's activity with quadrilaterals stays within the bounds of the traditional geometry curriculum, but affords students the opportunity to create their own mathematics within those bounds. Other educators have used dynamic geometry as a catalyst for reshaping the traditional curriculum and injecting "new" mathematics. Cuoco and Goldenberg (1997) see dynamic geometry as a bridge from Euclidean Geometry to Analysis. They advocate an approach to Euclidean geometry that relates back to the "Euclidean tradition of using proportional reasoning to think about real numbers in a way that developed intuitions about continuously changing phenomena" (p. 35). Such an approach involves locus problems, experiments with conic sections and mechanical devices (linkages, pin and string constructions) that give students experience with "moving points" and their paths.

Laborde et al. (2006) state that DGE "has provided access to mathematical ideas by allowing the bypassing of formal representation and access to dynamic graphing which is particularly important for the learning of geometry. … Just as digital technology provides means to by-pass formalism, it may also provide the means to transform the way formalism is put to use by students" (p. 284). DGEs allow for not only manipulations, but also macro-constructions, trace, and locus (Sträßer 2002). In DGE, geometric objects are constrained by their geometric properties (unlike paper-and-pencil sketches which can be distorted to fit expectations), similar to how physical objects are constrained by properties of physics when manipulated within the world. By observing properties of invariance simultaneously with manipulation of the object, there is potential to bridge the gap between experimental and theoretical mathematics as well as the transition from conjecturing to formalizing.

8.2.7 Summary of Students' Mathematical Knowledge in a Technological World

In this first section of the chapter we have attempted to define, deconstruct, and illustrate what we mean by "mathematical knowledge." In this endeavor, we have examined ways in which both the nature and construction of mathematical knowledge have been influenced by the integration of digital technologies in mathematics teaching and learning in ways that create a new learning ecology. We emphasized the different aspects of procedural and conceptual knowledge that have been discussed in the literature, and attempted to illustrate how certain dynamic technologies can enhance the development of "proceptual" knowledge and bring out the operational aspects of mathematics rather than focus on the notational aspects. In particular, we described the use of tools that exhibit "cognitive fidelity" (certain microworlds) and those that embody "mathematical fidelity" (Zbiek et al.2007), such as SimCalc and DGEs. These kinds of technologies have been used successfully by researchers and classroom teachers in varying contexts to promote the learning of new mathematical knowledge (for the learners). As success stories, they illustrate the complex interactions among students, teachers and mathematics, mediated through technology as depicted by our didactic tetrahedron (Fig. 8.1).

We now turn to a discussion of the mathematical knowledge (if any) that is necessary to comprehend how certain technologies function so that we may use them both sensibly and sceptically.

8.3 Mathematical Knowledge"Within" Technologies

Within the general public, it is a common myth that the computer is always right – a perception of its "mathematical fidelity." The main message of this section is that this notion is simply wrong (Sträßer 1992, 2001a). After an initial discussion of this issue, we suggest some pedagogical consequences of the "wrong-doing" of certain technologies.

8.3.1 Numbers and Arithmetic

As long as arithmetic is only done on everyday numbers (reasonably sized integers), the above statement concerning the infallibility of the computer is basically correct – the computer will normally not make mistakes in elementary arithmetic problems. Nevertheless, numerical analysis shows some important restrictions of computer-based arithmetic:

- As every computing machine has finite storage, it is obvious that it cannot correctly represent irrational numbers or rational numbers with more digits than the storage can hold. This is one reason why it is advisable to stay in "algebraic mode" in Computer Algebra Systems (e.g., *Derive*) as long as possible.
- Most computing machines internally work with "floating point arithmetic," often not in a base-ten system but in a base-two or base-16-system. Consequently, the machine is simply unable to correctly represent fractions as simple as 1/3. To give an example widely used, the program *Excel* does not have more than 15 digits plus three digits to represent the exponent, restricting the interval *Excel* can cover from −9,99999 99999 9999E307 to 9,99999 99999 9999E307. Near zero, the "Microsoft knowledge base" offers 1E−307 as the smallest positive number, and −1E−307 as the biggest negative number. An explanation for the maximal exponent "307" (307 being a prime number) could not be found.

From these two pieces of information, it is clear that even elementary arithmetic has its limitations when it is done on a computer. When it comes to very small and very big numbers or to limits and irrational numbers, computer arithmetic has its limitations and can become "wrong." As a consequence, one can never prove the divergence of the harmonic series $a(n) = \Sigma\ (1 + 1/2 + \ldots + 1/n)$ by adding partial sums on a computer; it is necessary to go back to symbolic mathematics.

8.3.2 CAS and Problem Spotting

To avoid these pitfalls and problems, it has been suggested that students use algebra as long as possible when doing complicated calculations, hoping that the algebra postpones rounding errors as long as possible (Sträßer 2001a). For many of the problems related to computer-based arithmetic, this works fine. In Computer Algebra Systems (CAS) the problems normally start as soon as calculations go beyond the simple development of a formula. Spotting problem situations like reducing $(x - 1)^2/(x^2 - 1)$ to $(x - 1)/(x + 1)$ will be correctly handled by most modern CAS programs (recalling that the reduction is invalid for $x = 1$). Without greater understanding about the inner mechanisms of automatic algebraic calculations, the problem of this type of manipulation of equations and formulae serves as a warning to check the algebraic reductions provided by a particular CAS program.

8.3.3 Geometry with Linear Algebra

The algebraic and arithmetical problems seem far away when considering Dynamic Geometry Environments (DGEs) like Cabri, Cinderella, Geometer's Sketchpad or other comparable software. In fact, this is also a wrong assumption.

To our best knowledge, all of the DGEs internally rely on a representation of Geometry by means of a multi-dimensional linear algebra (in most cases, on real numbers; Cinderella relies explicitly on complex numbers in order to avoid singularities). The "tangent monster" (Sträßer 2001b) illustrates the limitations of such a system: The geometric problem of identifying a tangent to a circle is condensed to solving a quadratic equation where the determinant is zero (converging the two solutions of the equation into one). With the problems of computer-based arithmetic in exactly handling zero, it does not come as a surprise that DGEs have difficulties with the number of intersections of a straight line and a circle when the Geometry indicates they must be tangent (implying just one solution of the quadratic equation).

Relying on an internal algebraic representation of Geometry has additional unwanted consequences. They emerge in situations as simple as constructing an angle bisector twice: Construct the angle bisector SB of an angle ASC and then construct the angle bisector of ASB. If one drags the point C, in most DGEs the second angle bisector will jump as soon as the original angle ASC gets bigger than 180°. This action produces the same drawing when you move the point C around to coincide with the original angle, as most DGEs are not "continuous," but "deterministic" (i.e., when you reproduce an initial location of a point after some dragging around, you will get the initial drawing). Cinderella is proud of avoiding this "jump," offering continuous dragging. The negative aspect of this is that when dragging C around S in a full circle, you will end in a different position of the second angle bisector, making Cinderella a non-deterministic DGE (for the example of this concept and more consequences of these design features of DGEs, see Gawlick (2001), who asserts that DGEs have to make a choice between continuity and determinism). There are other trivial examples where continuous and deterministic DGEs differ, but the point here is that there seems to be a need for a design decision in DGEs, which implies a choice that a geometer would like to avoid.

For statistical software like *Fathom*, developed for educational use, or professionally used software like SPSS, the consequences of the inner representation of statistical models are not as well researched as they are for DGEs. There is some interesting work on how "random" the random numbers are – some software packages even tell the user how the "random" numbers are generated. Without going into details, it is clear that these programs internally do not roll an ideal dice. It may be worthwhile to research the issue of consequences for the simulation of stochastic situations.

8.3.4 Who Has to Know What About the Underlying Mathematical Assumptions and Processes of Spreadsheets, DGEs, Statistical Packages and CAS?

This question has not previously been discussed in depth. The position put forward here is a preliminary and tentative one. It seems obvious that typical end-users of these software programs cannot be aware of all the underlying mathematical assumptions and processes when using the software (some details are even impossible to explain to a prototypic end-user), even in considering only the representation of numbers inside a computer. To really "understand" what is going on, one has to be aware of rational and irrational numbers, representation systems by different bases and a developed concept of limits. Only a thorough scientific analysis of phenomena may lead to a fuller understanding, research that is being undertaken by mathematicians in the field of numerical analysis. As a consequence, complete understanding by every end-user is simply not viable, nor desirable. The computer will not be a transparent machine, but will remain a black box (or at best a grey box, cf. Buchberger 1989).

What can be reasonably hoped for is a basic understanding of the inner representation of mathematics (e.g., numbers, equations, stochastics, graphical representations, and geometric figures) within a computer and a global awareness of problems related to the difference between conceptual and computational mathematics. At the very least, teachers of mathematics and computer science should know about these assumptions and processes and be able to react in an appropriate way when the phenomena discussed above occur. The difficulties of such a position should not be underestimated and are open to misuse by being taken as an excuse for not knowing what the machine actually has produced. We suggest such a position because one can prove that the computer is NOT always correct, but in fact makes "mistakes" if compared to theoretical mathematics. In other words, there is a limit to the level of "mathematical fidelity" (Zbiek et al. 2007) for any digital technology.

8.4 New Mathematical Practices

8.4.1 Link Between Knowledge and Practice

In the first two sections of this chapter, we discussed ways in which technologies have influenced the emergence of new mathematical knowledge, and what (if anything) we need to understand about the inner mathematical workings of the different technologies. In this section, we turn to the role of technologies in developing new mathematical practices. Ball (2002) articulates the importance of extending mathematical content to include the development of *mathematical practices* (p. 24):

> Noting that expertise in mathematics, as in any field, involves more than knowledge, we propose an explicit focus on mathematical know-how – what mathematicians and mathematics users *do*. We refer to these things they do as *mathematical practices*. Being able to justify claims, using symbolic notation efficiently, and making generalizations are examples of mathematical practices. Such practices are important in both learning and doing mathematics. Their absence can hamper mathematics learning.

In many cases, the use of technologies in schools has encouraged a closer relationship between mathematical knowledge and mathematical practice, providing learners with opportunities to experiment, visualize, and test emerging mathematical understandings. From the use of digital technologies, a new model of interaction between the student, the mathematical knowledge and the instrument emerges.

Briefly, technological tools can be experimental instruments whereby ideas can be explored and relationships discovered (Duke and Pollard 2004; Hoyos 2006; Hoyos and Capponi 2000), developing greater flexibility in analysis of complex situations. Realistic data, too complex to be used previously, can be brought into the classroom to make mathematics learning more interesting, challenging and practical (Kor and Lim 2004, 2006). Kosheleva et al. (2006) demonstrate how use of state-of-the-art technology (Tablet PC) allowed future teachers in their study to significantly improve learning of mathematical content knowledge through exploration and utilization of technology within their practice teaching.

Technology also can introduce a dynamic aspect to investigating mathematics by giving students new ways to visualize concepts. Bienkowski et al. (2005) contend that technologies can provide "virtual manipulatives" that help in visualization of abstract mathematical concepts through more tangible objects. According to Nemirovsky (in Heid 2005, p. 358):

> There is a huge overlap between what is activated in a brain by thinking about an activity and what is activated when you actually perform that activity. And so I think that for example imagining that a cube rotates in space is deeply rooted in the physical act of rotating cubes with your hand.

Boon (2006) provides descriptions of a variety of virtual activities that lead to new ideas on visualization in learning mathematics. Chiappini and Bottino (1999) allege that visualization allows students to access mathematical knowledge by integrating the symbolic re-constructive approach (the traditional teaching strategy) with the motor-perceptive approach that involves actions and perceptions and produces learning based on doing, touching, moving and seeing. However, in Kosheleva and Giron (2006) it was shown that students using virtual manipulatives often formed math ideas and approaches that were unexpected or unwanted by the teachers and the designers of these virtual manipulatives. For example, in the "Algebra Scales" activity from the National Library of Virtual Manipulatives (Utah State University 2007) children preferred the approaches that represented "shortcuts," requiring less time to get the correct answer, thus circumventing the equation solving process for which the applet was designed.

It must be made clear, however, that it is not the technology itself that facilitates new knowledge and practice, but technology's affordances for development of tasks and processes that forge new pathways. Just as cases of innovative uses of technology have emerged, there are valid concerns and shortcomings in the ways that technology has been used. Tall (1989) expressed his concern that in the process of using technology, the "authority of the machine" might be an impediment to learning, especially in the early stages. He pointed out that students would lose some autonomy in the problem solving process if they ignored their common sense and followed the lead of the machine.

On the other hand, technology has been used "to motivate students to take on, more and more, the responsibility of mediator in their own mathematics learning" (Buteau and Muller 2006, p. 77). This responsibility can lead to engagement with different kinds of mathematical learning practices. For example, Drijvers and Doorman (1996) observed that the use of graphing calculators appeared to stimulate students' interest in participating in explorative activities. Farrell (1990) reported that students who used graphing calculators to learn mathematics were more active in the classroom. More group work, investigations, explorations, and problem solving were also observed among students using graphing calculators. Others, for instance, Dick and Shaughnessy (1988), noticed that there was a shift by teachers to less lecturing and more investigations being conducted by students. Dana-Picard and Kidron (2006), in their study of a computer algebra system (CAS) as an instigator to learn more mathematics, stated "… the implemented Mathematics has to be understood. In order to afford a real understanding of the process, the user has to learn new Mathematics. We called this occurrence a *motivating constraint* of the software" (p. 130). Although these examples are encouraging, it is important to look beyond using technology as a motivational tool and move towards using technology to deepen and extend mathematical learning. This is challenging when developing proficiency with the technology takes time before the technology can become an instrument. This working relationship (instrumental genesis, discussed in Chap. 2) develops through operationalizing the mathematics with the technology. Confrey (in Heid 2005) emphasized that an obstacle in implementing these kinds of mathematical practices lay in the difficulty that (1) the type of reasoning needed to grapple with complexity is not taught in mathematics courses; and (2) students typically do not "own" the problems they work on. Overcoming these difficulties requires attention to the complex interactions among students, teachers, tasks, and technologies.

8.4.2 Interactions Among Students, Teachers, Tasks, and Technologies: Shifts in Empowerment

A key change in students' mathematical practices spurred by technologies is the locus of control in a task. Technologies can often promote student engagement and command in key decision-making junctures during exploration. Capabilities are distributed between the student and the tool, with the user in charge of making decisions about when and how to use the tool (Heid 2005, p. 348). Control shifts more to the student in making decisions about how to utilize the technology in problems that do not "tell" which mathematics is needed up front (Heid 2005). Technologies allow students to check the validity of their answers and assess their own hypotheses. While engaging in explorative activities with the technological tools, students might encounter unexpected strategies that lead them further to ask new questions when working toward a solution (Makar and Confrey 2006).

Drijvers and Doorman (1996) assert that when students alternate between experimentation and reflection, mathematical concepts are strengthened. Artigue (2002) agrees with this assertion in her genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. Olivero (2006) demonstrates that students with considerable dynamic geometry (DGE) experience and average mathematical background seem to use the software with more interactions and explorations, whereas students with stronger mathematical background and little experience with DGEs typically do not fully exploit the possibilities offered by DGEs.

Galbraith et al. (2001) and Geiger (2006) point out, however, that this shift in empowerment depends on how students envision their relationship with technologies. They articulate four metaphors for how technology can mediate learning. The metaphors illustrate the relationship that students might have with their calculator for a particular task and describe the degree of sophistication in which the technology is used:

- *Technology as master*, where the student is subservient to the technology and the relationship is one of dependence. An example is where the technology is used as a timesaving device, but the student does not evaluate whether the output is accurate or useful.
- *Technology as servant*, where the technology is subservient to the student, typically used to replace pen and paper computation and used as a faster means to the output.
- *Technology as partner*, where the technology is used creatively to boost student empowerment, treating the technology almost as a surrogate human partner.
- *Technology as extension of self*, where users draw on their technological expertise as an integral part of their mathematical thinking.

An example of *technology as partner* can be found in the use of scientific probes and sensors (CBLs) to investigate problems. Probes and sensors connected to a computer or a graphics calculator offer opportunities for teachers to focus mathematics teaching on inquiring, understanding and reasoning instead of the drill and practice of routine problems typical in conventional instruction. Probeware open avenues for students to investigate and explore science in a mathematical setting. The data collected in a science experiment using the probeware can be stored and analyzed via the technological tools while conducting the experiment in real-time. The use of electronic probeware as a technological partner is capable of transforming mathematics into an interdisciplinary, authentic and participatory subject (Lyublinskaya 2004, 2006).

Sinclair (2003) noted the importance of the nature of the task in either promoting or discouraging students' engagement with exploration activities and geometric thinking in DGE. Laborde (2001) further distinguished four types of tasks used by teachers in DGE: (1) tasks for which the technology facilitates but does not change the task (e.g., measuring and producing figures); (2) tasks for which the technology facilitates exploration and analysis (e.g., identifying relationships through dragging); (3) tasks that can be done with paper-and-pencil, but in which new approaches can be taken using technology (e.g., a vector or transformational approach); and (4) tasks that cannot be posed without technology (e.g., reconstruct a given dynamic diagram by experimenting with it to identify its properties – the meaning of the task comes through dragging). For the first two types, the task is facilitated by the technology; for the second two, the task is changed by technology. The nature of the task turns into a modeling activity, where deductions are drawn from observations, and solution paths may differ from the mathematics the teacher intends. Kieran and Drijvers (2006), in their study of learning about equivalence, equality and equation in a CAS environment, report on the "intertwining of technique and theory in algebra learning in a CAS environment." (p. 278) Their analysis revealed that the relation between students' theoretical thinking and the techniques they use for solving the tasks, and the confrontation of the CAS output with students' expectations were the two main issues. While this confrontation became one of the most powerful occasions for learning in the classroom, they assert that appropriate management of these complications by the teacher is a necessary precondition to foster learning.

The findings concerning appropriate management by the teacher in Kieran and Drijvers study suggest that tasks with technology should not be studied without careful attention to the classroom environment created by the teacher. Ruthven et al. (2005) found that teachers often restricted students' explorations in order to avoid meeting situations that did not align with the planned learning outcomes. For example, the teacher may structure students' experiences in Cabri to exploit the mathematical fidelity of the Cabri construction with respect to classic Euclidean Geometry (Mariotti 2006).

Laborde and her colleagues (Laborde et al.2006) note how Logo as a "programming tool" can be used to support links between students' actions and symbolic representations because students must express actions in symbolic language to produce objects on the computer. DGEs, on the other hand, are considered as "expressive tools" as students work with them to produce or manipulate geometric objects. "Students move from action and visualization to a theoretical analysis of diagrams and possibly to the expression of conjectures and reasoning" (p. 296). In their survey of research on technology in the teaching and learning of geometry, they indicate how the focus of research has also shifted among students, teachers, and tasks:

> The focus initially was on the learner and his/her interactions with technology, giving rise to theoretical reflections about learning processes in mathematics by means of technology. The focus moved to the design of adequate tasks in order to meet some learning aims and then to the role of the teacher. The integration of technology into the everyday teacher practice became the object of investigation. Finally, the role of the features of software and technology design were also questioned and investigated in order to better understand how the appropriation of the technological environment by students could interfere with the learning of mathematics and how the teacher organizes students' work for managing this interaction between appropriation of the tool and learning. (p. 296)

Thus, the interactions among students, teachers, tasks, and technologies have now become the focus of research in the field. Laborde and her colleagues, however, argue that more research is needed to better understand links between students' instrumentation processes and their growth of mathematical knowledge.

In this section we have seen the emergence of another component that we need to encompass in our didactic tetrahedron: the nature of mathematical tasks. We shall address this additional aspect of the didactic situation in our concluding remarks for this chapter. In the next section we discuss the critical role of feedback that technologies can play in the development of new mathematical practices.

8.4.3 Role of Feedback in Practice

Laborde et al. (2006) indicate that feedback through technology offers a great deal of opportunity for new ways of understanding mathematics; for example, feedback through DGEs and microworlds, generated by manipulating the environment and generalizing/abstracting through reflection on outcomes to actions. Such feedback creates a need to search for another solution if the feedback gives evidence that a solution is inadequate; it can also help students refine their thinking iteratively as they design (rather than at the end of the design process). "The software incorporating knowledge and reacting in a way consistent with theory impacts on the student's learning trajectory in the solving process" (p. 294). The effect of technology on students' learning trajectories is the focus of Chap. 9.

As described in Sect. 8.2, the use of DGEs encourages students to make conjectures and the feedback they get from both measuring and dragging elements of their constructions allows students to rapidly test these conjectures, and, thus, refine them in a recursive cycle of conjecture-test-new conjecture. Hollebrands (2007) distinguishes two different types of strategies employed in students' activities with the Geometer's Sketchpad: reactive and proactive. The critical difference between the two strategies is whether the choice of action is in response to what the computer produced (reactive) or in advance of what the student anticipates the computer is supposed to do (proactive). In either strategy, the feedback provided by the DGE is critical for the students' subsequent actions. Makar and Confrey (2006) found similar distinctions in ways that learners use and respond to feedback in dynamic statistical software (see Sect. 8.4.4.2).

Zbiek and Glass (2001) conjecture that students will most likely reason extensively and deeply when confronted with a technology-generated result that conflicts with their personal mathematical expectations (a result of a proactive strategy). Zbiek and Hollebrands (2007) provide examples from a research study with prospective teachers using the Geometer's Sketchpad to support this conjecture. Students were investigating the effects of varying parameters on graphs of functions of the form:

$$y = \frac{a}{1 + be^{cx}} + d$$

One student conjectured that graphs associated with negative values of a would be reflections of graphs associated with positive values of a.

> When they were not, she proceeded to unravel the mathematical situation, using supplemental lines and symbolic reasoning to explain why a sign change in the numerator of the fraction would not yield the mirror image for the graph. Her reaction to this technology-based surprise led to her deeper understanding of this function and to her enhanced ability to reason in other parameter explorations. (p. 41)

Heid (2005) noted the way that feedback systems in microworlds allow learners to predict behavior and deepen understanding of how things work. This experience, according to Thompson, "demands a very different conception of mathematical inquiry … because microworlds are typically designed to be experimented with, much like you experiment with some physical system" (in Heid 2005, p. 352). Choate (in Heid 2005) discusses how feedback in intelligent tutors and CAS allow for the "playability" (p. 350) of calculus and functions in the way that geometry has become "playable" through dynamic geometry environments. The calculator can perform the microprocedures and let the student focus on the macroprocedures, which require higher-level processes. Gage (2002) observed that the immediate feedback provided by graphing calculators enabled students to challenge misconceptions that may develop and hence minimize their formation. Finally, students can use graphing calculators to explore complex functions in new ways, relying on the feedback generated to deepen their understanding. For example, in a study by Rosihan and Kor (2004), students investigated the following limit graphically, numerically and symbolically:

$$\lim_{x \to 0} f(x) = x \sin(\tfrac{1}{x})$$

The difficulty faced by learners in this problem was to imagine the changes when the x-coordinate approached zero. Graphing calculators allow students to witness the oscillations around $x = 0$ as well as other properties of the graph such as symmetry. Changing the window parameters on the graphing calculator allows the student to capture different sizes and dimensions of the graphed functions. The "TRACE" command can be used to explore the functional values around $x = 0$ and enable students to make the deduction graphically that the limit is zero. These examples illustrate ways that feedback from students' interactions with technology can have a strong impact on their mathematical understandings and practices.

8.4.4 Example Technologies that Promote New Mathematical Practices

In this section we examine, in more detail, several examples of technologies that have promoted new mathematical practices on the part of students and/or teachers. We begin by revisiting dynamic geometry environments, with particular emphases on "dragging" and the new role of proof in DGEs. The second example examines the use of new technologies in the teaching and learning of statistics, with particular emphasis on the new dynamical statistics software programs that have recently made their way into pre-college classrooms. The third section looks at children's activities in robotics and digital game environments, and the potential of these activities to engage children in mathematical practices. We have already described (in previous sections) some of the new practices made possible by graphing and CAS-enabled calculators, and the use of probeware in conjunction with calculators and computers.

8.4.4.1 New Mathematical Practices in Dynamic Geometry Environments

We now revisit the research on DGEs from the perspective of new mathematical practices that have emerged from the numerous applications of different DGEs in many parts of the world. It is well established that this type of software helps learners identify and thoroughly explore properties and relationships between geometrical shapes (Santos-Trigo 2001), and we have already discussed how DGEs can provide critical feedback to the user and the importance of feedback in the development and testing of conjectures (see Sect. 8.4.3). Perhaps the most obvious new practice made possible by DGEs is the ability to drag elements within a construction and thus rapidly visualize many possible examples of the construction as well as to discern what remains invariant under this dragging action (Heid 2005; also see Sect. 8.2.6.3). A group of researchers in Italy led by Ferdinando Arzarello (Arzarello et al. 1998a) classified different modalities of dragging as follows:

> (i) *wandering dragging*, that is dragging (more or less) randomly to find some regularity or interesting configurations; (ii) *lieu muet dragging*, that means a certain locus *C* is built up empirically by dragging a (dragable) point P, in a way which preserves some regularity of certain figures. (p. 3)

They also describe a third modality: *dragging test* that is used to test a conjecture over all possible configurations. Their distinction between wandering dragging and the more focused lieu muet dragging and dragging tests are not unlike Hollebrands' (2007) description of reactive and proactive strategies in DGEs or Makar and Confrey's (2006) *wanderers* and *wonderers* in dynamic statistical software (below). These different types of dragging modalities can be thought of as new mathematical practices that have emerged in the context of dynamic geometry environments.

Leung et al. (2006) extend the notion of dragging modalities in their study of instrumentation/instrumentalization of dragging via functions of variation (contrast, separation, generalization, fusion). From their observations of two pre-service teachers working with a DGE called C.a.R. they hypothesized a utilization scheme they call a *Variational Dragging Scheme* that involves the following components:

1. Create contrasting experiences by wandering dragging until a dimension of variation is identified.
2. Fix a value (usually a position) for the chosen dimension of variation.
3. Employ different dragging modalities/strategies to separate out critical feature(s) under the fixed value (i.e. a special case for the configuration).
4. Simultaneously focusing, hence "reasoning," on co-varying aspects during dragging. A preliminary conjecture is fused together.
5. Attempt to generalize by a change to a different value for the chosen dimension of variation.
6. Repeat steps 3 and 4 to find compromises or modifications (if necessary) to the conjecture proposed in step 4.
7. Generalization by varying (via different dragging modalities) other dimensions of variation. (pp. 350–351)

The group at the University of Grenoble in France have been conducting research studies on the use of Cabri for many years (Laborde 1992, 1993, 1995, 1998). They have focused both on what students are learning when working with Cabri and the constraints both students and teachers face when teaching and learning with Cabri. Laborde (1992, 1993) and Balacheff (1994) conclude that the observation of what varies and what remains invariant when dragging elements of a figure in Cabri, helped break down the separation of deduction and construction that Schoenfeld (1988) found in his study of geometry teaching and learning. Laborde (1998) points out that it takes a long time for teachers to adapt their teaching to take advantage of the technology. She reports three typical reactions that teachers have to the perturbations caused by the introduction of dynamic geometry software into the teaching-learning situation:

- Reaction alpha: ignoring the perturbation
- Reaction beta: integrating the perturbation into the system by means of partial changes
- Reaction gamma: the perturbation is overcome and loses its perturbing character. (p. 2)

It is only in the last stage (reaction gamma) that teachers make an adaptation in their teaching that truly integrates the technology, thus generating both new teaching and new learning practices. Hollebrands et al. (2007) reviewed approximately 200 research studies on the use of technology in secondary school geometry (about half of these studies involved DGEs). The following themes emerged from their review: the role of representation in the construction of geometrical knowledge and DGE diagrams, the design of tasks and the organization of the milieu, students' constructions within a computer environment, and the instrumental genesis and its relationship to construction of knowledge. (This latter theme was discussed in Chap. 2 of Theme C). The importance of studies addressing the question of proof in a dynamic geometry environment was also a major topic in their review.

The primacy of "proof" as the ultimate mathematical practice has been accepted in the teaching and learning of geometry since the time of Euclid. The very nature of DGEs challenges this primacy of proof but also creates new roles for proof as a mathematical practice (de Villiers 1999, 2006). Hoyles and Healy (1999) indicate that explorations of geometrical concepts using DGEs help students to define and identify geometric properties, and the dependencies necessary for the development of a proof; however, they do not necessarily lead to the construction of a proof. Olive (2000a) provides an example of how the interplay between a dynamic geometric construction and the functional dependencies of the dynamic measurements obtained from that construction, did lead to the development of an algebraic proof. In a course for pre-service high school teachers, students found the dimensions of a rectangle with fixed perimeter that contained the largest possible area by constructing a dynamic rectangle in *Sketchpad*, the sum of whose sides was constrained by a fixed line segment. Their construction allowed them to change the base of the rectangle, which in turn, caused the sides to change length (in order to keep the perimeter fixed). By measuring the base of the rectangle and the resulting area, and then plotting a point in *Sketchpad* based on these dynamic measurements (base vs. area), they were able to construct the locus of the plotted point, which was a parabolic curve with a maximum. They discovered that their plotted point reached this maximum area when the rectangle appeared to be a square. However, the dynamic measures of base and height were not exactly the same when the plotted point appeared to be at its maximum. This discrepancy led to an interesting discussion, and a need to prove by algebraic means that the maximum area will be attained when the rectangle becomes a square. Thus, they made the transition from geometric conjecture to algebraic proof.

The above example of finding a solution in dynamic geometry by experiment is analogous to finding roots of a polynomial using a graphing calculator. The solution can be found but the students still have a need to prove that the solution is valid. In the case of the rectangle with maximum area there is a need to prove the conjecture that, for any rectangle with fixed perimeter, the maximum area will be achieved when the rectangle becomes a square. Manipulating the dynamic rectangle can give convincing evidence that the generalization is indeed true. There is a danger here that students may regard this "convincing evidence" as a proof. Michael de Villiers (1997, 1998, 1999) has addressed this concern through a thorough analysis of the role and function of proof in a dynamic geometry environment. de Villiers expands the role and function of proof beyond that of mere verification. If students see proof only as a means of verifying something that is "obviously" true, then they will have little incentive to generate any kind of logical proof once they have verified (through their own experimentation) that something is always so. de Villiers suggests that there are at least five other roles that proof can play in the practice of mathematics: explanation, discovery, systematization, communication, and intellectual challenge. He points out that the conviction that something is true most often comes *before* a formal proof has been obtained. It is this conviction that propels mathematicians to seek a logical *explanation* in the form of a formal proof. Having convinced themselves that something must be true through many examples and counter examples, they want to know *why* it must be true. de Villiers (1999) suggests that it is this role of *explanation* that can motivate students to generate a proof:

> When students have already thoroughly investigated a geometric conjecture through continuous variation with dynamic software like Sketchpad, they have little need for further conviction. So verification serves as little or no motivation for doing a proof. However, I have found it relatively easy to solicit further curiosity by asking students *why* they think a particular result is true; that is, to challenge them to try and *explain* it. (p. 8)

The group in Italy headed by Ferdinando Arzarello (Arzarello et al.1998a) has conducted investigations of students' transitions from exploring to conjecturing and proving when working with Cabri. They applied a theoretical model that they had developed to analyse the transition to formal proofs in geometry (Arzarello et al. 1998b). They found that different modalities of dragging in Cabri (identified above) were crucial for determining a shift from exploration to a more formal approach. Their findings are consistent with the examples given in previous sections of this paper.

8.4.4.2 Technologies that Encourage New Practices in Statistics

The influence of technology on the statistical knowledge and practices of learners has changed enormously in the past 15 years, although not without growing pains (Rubin 2007). One important benefit of access to technology has been the opportunity that students have to work with authentic data sets that are both larger and more complex. Because the difficulty of calculations becomes overly tedious as the size of the data set increases, learners have been constrained previously to working with small, carefully chosen data sets that limit complexity of calculations, but at the same time become overly contrived. According to Finzer et al. (2007):

> This judicious selection deprives students of the experience of data discovery. … What seems to us to be missing are data sets – especially large and highly multivariate data sets – that are ripe for exploration and conjecture driven by the students' intrigue, puzzlement, and desire for discovery. (p. 1).

The graphing calculator and statistical analysis packages designed specifically for *learning* statistics (e.g., Finzer 2007; Konold and Miller 2005; Hancock and Osterweil 2007) support the use of authentic and realistic data and therefore stimulate students towards exploratory activity. This allows for a shift in emphasis from a focus on graphs, calculations, and procedures taught in isolation for their own sake towards the active use of statistics as a tool to solve interesting problems. This move allows learners to focus on the context under study rather than on the statistical tools as the objects of study (Makar and Confrey 2007). Modeling software can support a better understanding of representation and form "a bond between the data and whatever mathematical model you are starting to make" (Finzer, in Heid 2005, p. 357).

Students can collect large sets of real life data for data analysis. They can download, sort, tabulate and store these data rapidly using CBLs or web-based technologies and thus avoid tedious compilations of data tables by hand. Students can manipulate the data with symbolic expressions, solve equations, analyze data, and graph functions to fit the data. They can switch between screens to observe the different representations of the data. They can make conjectures, test their hypotheses and check their estimations. These dynamic technologies allow learners to work flexibly as the rapid display of different graphical representations allows more time for students to explore larger data sets and make comparison between groups, thus making statistics more meaningful and interesting (Kor 2004, 2005). Students can experience and appreciate more the practices of statisticians when they run statistical tests on authentic data that they obtained. Bienkowski et al. (2005) allege that students who engage in investigative activities with data using technologies perform better than those who simply report data.

The opportunities that have arisen have not been without challenges, however. Rubin (2007) reports that graphs are now so easy to create with software, that students have been deprived of the need to think about appropriate axes, scales, and other design issues. The drag-and-drop facility of many software packages like *Fathom* (Finzer 2007) can encourage users to simply wander through a data set aimlessly looking for an interesting pattern to jump out at them and then try to explain the outcome with anecdotal evidence – what Makar and Confrey (2006) call *wanderers*; at the same time, the ease of creating graphs supports others to assess the validity of hunches generated from the "I wonder" type questions that the technology was meant to encourage (what Makar and Confrey 2006, term *wonderers*). Rubin (2007) further worries that access to technology has not necessarily discouraged students from "over-believing" computers and accepting results calculated by software on blind faith.

> This attitude can lead to students accepting implausible results. … [On the other hand], they may also have done everything right and be seeing a relatively unlikely result. In the end, there's a delicate balance to be struck here, given the uncertainty that is at the heart of statistical reasoning. We want students to question what the computer generates, but not to reject results simply because they are not within the expected probability. (pp. 29–30)

Overall, however, the newer technologies that focus on learning statistics discourage the "black box" mentality of previous statistical packages (Meletiou-Mavrotheris et al. 2007; Makar and Confrey 2004). If technologies can continue to encourage greater focus on the utility of statistics for solving problems over an emphasis on the statistics as an end in itself, there is potential to resolve the widely reported use of statistics in situations where they don't make sense (Pfannkuch et al. 2004).

8.4.4.3 Children's Mathematical Practices Using Robotics and Digital Games

The use of robotics in schools is a fairly recent phenomenon. Although the Turtle Geometry of Logo (Papert 1970) was initially developed as a control language for a physical, dome-shaped robot (dubbed the "turtle"), the expense of the physical device and control mechanisms in the late 1970s and early 1980s made the physical robot turtle prohibitive as a classroom-based learning tool. Mass production of similar control systems with small robotic devices for the toy market, have now made the use of robotics a possibility again in K-12 classrooms. Programming robotic vehicles to travel around obstacle courses, or navigate a specific route, while providing a fun, game-like context, has the potential for rich mathematical learning.

Although the potential of digital games as rich learning tools is widely recognized (Sanford 2006), this potential in schools has not yet materialized (Wijekumar et al. 2005). From a practical standpoint the majority of games released are not the kind of games that educators will find value in using as part of their teaching, and while a recent report (MacFarlane and Kirriemuir 2005) describes some of the issues reported by teachers, it also points to a pressing need to establish a better understanding of the value of games in school environments and the difficulties faced by teachers when using them (Sanford 2006). According to Wijekumar et al. (2005), it is still necessary to work on moving students from a game affordance in a computer environment to a mathematical learning situation in which they may use that affordance.

There are some initiatives, however, in which robotics and digital games are used to try to encourage students to learn specific mathematical topics. For example: (1) Using robots to learn angle concepts (Hunscheidt and Koop 2006); (2) using robots to learn linear functions (Fernandes et al. 2006); (3) designing a game construction kit to foster visual reasoning and self-engaging tasks (Kahn et al. 2006); and (4) exploring the affordances of electronic, mathematical board games (Raggi 2006; Rodriguez 2007) that promote general action patterns for solving mathematics or science problems. Following are brief descriptions of each of these examples:

1. With respect to an understanding of the angle concept in primary school, Hunscheidt and Koop (2006) introduced in the classroom a small robot on wheels programmed to move in centimeters, turn in degrees and wait in tenths of a second (p. 229). This artifact (named Pip) enabled the students to estimate and check distances and angles.
2. In their work with eighth graders, Fernandes et al. (2006) used robots in order for their students to learn functions. In the context of being given two pictures of distance–time graphs that represented a robot's possible travel from a given starting point, students were to design programs for the robot to travel the represented routes. To begin, students made hypotheses about the routes represented in the two graphs, then discussed the possibilities of these situations, realizing that one of the graphs was not feasible (as the robot would have to be in different places at the same time) and finally understood that the graphs were not pictures of the robot's route but a representation of the relationship between time and distance of the robot's travel.
3. Using another kind of virtual scenario, Kahn et al. (2006) designed a Space Travel Games Construction Kit (STGCK) to build a variety of games similar to *Lunar Lander* (p. 261). They tested these STGCK with two student classes (one aged 11–12, and the other one aged 12–13) and a small group of three students aged 12–14. The results were that students developed understandings of Newton's Laws, showing engaged activity and active experimentation with the materials. In particular, students solved the different tasks posed using iterative strategies and repeatedly refined their strategy. Kahn et al. (2006) evidenced collaboration, competition and motivation as the most prevalent student activities. In addition, the authors came to realize that students could analyze and use the relationships hidden in the programming code as an integral part of the game, when they gave students easy access to the programming code and provided situations where they would want to analyze and adjust that code.
4. The emergence of mathematical strategies and consecutive refining strategies were also some characteristics of the results obtained by the instrumentation of a computational board game named Domino (Raggi 2006) with two classes of seventh and eighth graders. In this context, symmetry was the underlying mathematical structure for the game. When playing against the computer, the winning strategy is to place your dominos symmetrically opposite the computer-opponent's placements. The computer game was introduced into the classroom as an exploratory material. Each student had to initially play against the computer (named Robi). The task asked of students was to find a way to beat Robi or, if Robi won, to try to explain why Robi was able to beat them. Two groups of seventh and eighth graders were involved in this experience in order to discover possible affordances of the computer game for helping students learn symmetry (Rodriguez 2007).

The purpose of the Domino game (Raggi 2006) is the search for winning strategies that allow the winner to activate the last two consecutive squares on the game board. The game is immersed with symmetry, yet this (mathematical) structure that the game is intended to foment defines a potential organization that the children concretize in different ways once they are engaged in the task (Saxe and Bermudez 1996). For example, a result of seventh graders playing the game against the computer was a rapid turn toward a different winning strategy, one which consisted of trying to leave blank spaces, counting out how many were necessary according to which turn they had. Concerning eighth grade students, it was observed that when they used a strategy that they believed to be a winner, they continued to use it and perfected it as long as it was functional. Moreover, an opponent who began to win was a cause for reflection and the reformulation or construction of a new winning strategy.

As stated at the beginning of this section, the major question to be answered with respect to new mathematical practices that might develop from either the use of robotics or digital games in the school context, is the movement from the game context to mathematical problem solving situations. To connect to another new or learned context, David Shaffer proposes the notion of epistemic frames as "ways of looking at the world associated with the ways of knowing of a particular community" (cited in Sanford 2006, p. 13). Shaffer's epistemic frames can be regarded as a tool for building accounts of students' use of experience that was gained in one context and applied within another different context. According to Sanford (2006), "building on this concept will contribute to an attempt to build an understanding of the ways in which knowledge may be transferred from the game to other domains" (p. 13). The heuristics of students performing in the competitive situations that the mentioned games created corresponded to general action patterns for solving mathematics or science problems (cf. Polya 1945). Nonetheless, the potential of this type of psychological instrument (epistemic frames, Shaffer 2006) to learning specific mathematical topics is still to be determined, for example how it is related with solving specific mathematical problems.

8.4.5 Summary of New Mathematical Practices Made Possible with Technology

We opened this section with findings that suggest that the link between mathematical practices and mathematical knowledge is strengthened in didactical situations that involve effective uses of technology. A major affordance of technology is how it can be used to help students visualize abstract mathematical concepts. Students can model, experiment, and test their emerging mathematical understandings using dynamic visualization software in many mathematical domains. There is a risk, however, that students (and teachers) may relinquish their mathematical authority to the computing machine (see Sect. 8.3 for the inherent danger in relinquishing this authority to machines that are mathematically limited). We emphasized how the technology could be used to motivate students to mediate their own learning, and how it has brought about a shift in teaching practices from lecturing to student-centered investigations.

In Sect. 8.4.2 we focused on ways in which the interactions among students, teachers, tasks and technology have the potential to bring about a shift in empowerment in the didactic situation. We introduced the need to pay particular attention to the design of the mathematical tasks in order to avoid students perceiving the role of the technology as their master rather than their servant or partner. Ultimately, we would like to see students use technology as an extension of themselves (Galbraith et al. 2001; Geiger 2006). The focus here is on where the locus of control lies in a mathematical task. Technology can be used to shift that locus of control towards the students and, thus, empower the students to take more responsibility for their own learning.

Several researchers have focused on the importance of task design (e.g. Sinclair 2003; Laborde 2001) in technological environments. They argue for designing tasks that are transformed by the technology, leading to new mathematical practices (e.g. modeling real-life phenomena, making deductions based on observations), rather than tasks that could be just as easily completed without the technology. One possible outcome with such tasks, however, is that students may engage with mathematics that the teacher did not intend (and with which s/he may not feel competent). The role of the teacher becomes critical in managing these rich didactical situations involving technology. The teacher can attempt to constrain the situation so that students engage with the intended mathematics, or they can be more open and willing to go where the students' investigations lead them.

The nature of different software tools also has a constraining effect on the possible mathematical practices. When computers were first introduced into the mathematics classroom, their use was primarily for teaching programming. With the development of the mouse interface and dynamic visualization software, the advocacy for programming has diminished in favor of what Laborde et al. (2006) call "expressive tools." While programming tools (such as Logo) support the link between students' actions and symbolic representations (programming code), expressive tools (such as DGEs) assist students in the move from action and visualization to conjectures and reasoning. This shift towards expressive tools has brought about a shift in the focus of research on the interactions among students, teachers, tasks and technology.

Research on the role of feedback provided by technological tools suggests that learning is most likely to occur when the feedback is unexpected. Feedback provided by computational tools (such as CAS) can shift the focus of the student from micro-procedures (that the tool performs) towards macro-procedures that involve higher-level cognitive processes. New solution methods are made possible by the graphical feedback provided by graphing calculators and graphing software. For example, the ZOOM feature on most calculators can provide students with visual solutions to the limits of functions at critical points.

In the last part of this section on mathematical practices we examined examples of several different technologies that have been used successfully to generate new mathematical practices. We revisited the research on DGEs from the perspective of new mathematical practices, emphasizing the important aspect of the different dragging modalities and the utilization schemes that students could develop through use of these different dragging modalities. The introduction of DGEs into the didactical situation often created perturbations for the teacher. When teachers overcame these perturbations (rather than ignoring them) they made adaptations in their teaching that more authentically integrated the technology. The use of DGE also brings about new approaches to proof in geometry and an increased emphasis on the role of proof as explanation rather than only verification. Likewise, the use of dynamical statistics software has made it possible for students to work with large, authentic data sets, which they can download or generate through their own experiments. The ease with which students can represent, explore and manipulate data with these tools has brought about a shift in focus from studying statistical processes for their own sake towards the active use of statistics as a tool to solve interesting problems.

We concluded this section with a look at the introduction of robotics and digital games as contexts for learning mathematics. While the potential of these contexts as rich learning situations has been recognized, this potential has not yet been realized in mathematics classrooms. Teachers do not yet see the value of digital games as learning tools. The problem for teachers is finding ways to move students from a game affordance to a mathematical learning situation in which they may use that affordance. Several researchers have suggested that the use of Shaffer's epistemic frames (Sanford 2006) as a theoretical tool could help teachers organize such movement.

8.5 Final Words: An Adaptation of Our Didactical Tetrahedron

We began this chapter with an adaptation of Steinbring's (2005) didactic triangle that portrayed the didactical situation as interactions among student teacher and mathematical knowledge. We suggested that the introduction of technology into the didactic situation could have a transforming effect on the didactical situation that is better represented by a didactic tetrahedron, the four vertices indicating interactions among Teacher, Student and Mathematical Knowledge, mediated by Technology. In the third part of this chapter it became obvious that the nature and design of the learning task was a further interacting variable that must be taken into account in the didactical situation. From a social constructivist viewpoint (see Chap. 11), mathematical knowledge and practices are constructed as a product of the interactions among student, teacher, task and technology, rather than existing apart from them (as a separate vertex of our tetrahedron). We, therefore end this chapter with a new didactical tetrahedron as illustrated in Fig. 8.3. This new model illustrates how interactions among the didactical variables: student, teacher, task and technology (that form the vertices of the tetrahedron) create a space within which new mathematical knowledge and practices may emerge. It is not arbitrary that we place the student at the top of this tetrahedron as, from a constructivist point of view, the student is the one who has to construct the new knowledge and develop the new practices, supported by teacher, task and technology.

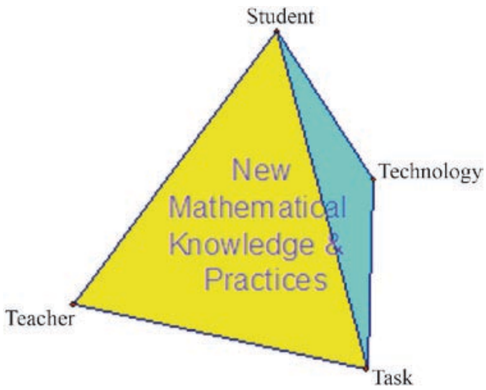

Fig. 8.3 An adaptation of the didactical tetrahedron illustrating how the interactions among student, teacher, task and technology form the space within which new mathematical knowledge and practices may emerge

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Chapter 9
The Influence and Shaping of Digital Technologies on the Learning – and Learning Trajectories – of Mathematical Concepts

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Abstract The significant development and use of digital technologies has opened up diverse routes for learners to construct and comprehend mathematical knowledge and to solve problems. This implies a revision of the pedagogical landscape in terms of the ways in which students engage in learning, and how understandings emerge. In this chapter we consider how the availability of digital technologies has allowed intended learning trajectories to be structured in particular forms and how these, coupled with the affordances of engaging mathematical tasks through digital pedagogical media, might shape the actual learning trajectories. The evolution of hypothetical learning trajectories is examined, while the transitions learners make when traversing these pathways are also considered. Particular instances are illustrated with examples in several settings.

Keywords Digital technologies • Mathematics education • Learning trajectories • Contributions of digital technologies for learning

9.1 Introduction

In this chapter we consider how the use of digital technologies (DT) might influence the learning of mathematical concepts and shape the trajectories through which that learning develops. We discuss and illustrate various key aspects associated with

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the development of learning trajectories when digital technologies are used as pedagogical media. These include intended or hypothetical learning trajectories, the actual trajectories that do emerge, the occurrence of learning trajectories within and across various DT platforms, and the potential shifts in trajectories that are afforded by DT through early engagement in conceptually "advanced" mathematical topics.

We begin by providing an overview of the main theoretical ideas used in this chapter; in particular the meaning of the term "learning trajectories," a construct that is now commonly referred to in mathematics education in general, but that has yet to be redefined for the specific context of learning with and through DT (although some authors have begun to use it in that area, e.g. Clements et al. 2004). We propose that the systematic use of digital technologies plays an important role, not only in the construction of hypothetical learning trajectories (defined in the next section), but also during the students' actual comprehension and development of mathematical concepts. Thus, throughout the chapter, we are interested in discussing themes related to the mediating role of the use of the tools, the multiple representations of mathematical phenomena, and examples in which the use of digital tools can enhance the construction and evolution of students' learning trajectories. One contention to which this position subscribes is that the process of constructing learning trajectories involves transitions between different cognitive and epistemological levels: e.g. from intuitive/informal to formal; from unconscious to conscious, from concrete to abstract; from visual to symbolic/syntactic; from synthetic to analytic; or from situated knowledge towards a more generalizable or abstract knowledge. A learning trajectory may also involve transitions between the technological environment and a non-DT (e.g. paper-and-pencil) environment.

While these transitions are present in any learning trajectory, the use of DT provides opportunities for teachers and students to engage in mathematical activities in ways that make those transitions meaningful, and broadens the range of possibilities for both hypothetical and actual learning trajectories. Later in this chapter we will discuss some of the elements that determine the process of selecting a certain activity to correspond to an intended learning trajectory. One of these is the aspect of the multiple representations that are provided and interconnected (hotlinked) by and within digital environments. This can transform the way in which those transitions take place, providing a structure that learners can draw upon and reconstruct – as per Noss and Hoyles' (1996) idea of *webbing*; described in Chap. 7. Other contributory elements that fashion the transitional process include the context of inquiry; the choice of the specific technological tools, including consideration of the propensity for open investigative practice associated with those tools or environments; the particular affordances DT bring to the learning experience when these tasks are engaged through digital media; the sequencing of tasks within an activity; and the relevance of using more than one tool to enhance or complement the mathematical competences or new knowledge that is generated with the use of DT. Central also are facets derived from taking into account the mathematical content, including the possibility of earlier engagement with concepts and processes that might traditionally be approached in later years.

In summary, the chapter begins with a brief theoretical overview to inform the various perspectives that frame the subsequent sections. This prefaces later sections addressing hypothetical learning trajectories and the affordances of digital technologies as pedagogical media. How DT influences the evolution of learning trajectories is then considered through the use of examples that contextualize the construction of hypothetical learning trajectories, learning trajectories within and across various platforms, the emergence of actual learning trajectories, and the possibilities for earlier engagement with powerful ideas afforded by DT. The concluding section draws on these aspects and examples, to consider the manner in which the learning experience is transformed through the engagement of digital technologies. It also attends to the consequential influence of this alternative engagement on the evolution of the leaning trajectories and hence on learning.

9.2 Theoretical Overview

9.2.1 On Learning Trajectories

In this chapter, we use the construct of *learning trajectory* to structure, organize, and discuss mathematical practices and knowledge derived from the use of digital tools. The notion of learning trajectory has been employed recently in diverse research in mathematical teaching and learning, and as a foundation of innovative mathematics curricula. However, researchers and curriculum developers interpret and use this construct in different ways (Clements and Sarama 2004). In particular, a distinction needs to be made between an *intended* – or *hypothetical –learning trajectory* (HLT) and an *actual learning trajectory*. The first (HLT) serves as a foundation for task design, by characterizing and identifying possible instructional routes to approach mathematical task and develop students' mathematical thinking; whereas the latter indicates the actual pathways followed by students as a result of working on activities or tasks – activities that were possibly set in terms of a hypothetical learning trajectory.

Simon (1995) proposed the term hypothetical learning trajectory to identify and describe relevant aspects associated with a mathematics lesson plan, including: A description of the students' mathematical goals (what is intended for students to learn); the mathematical tasks or problems that students will work on to achieve the goals; and a hypothetical path that describes the students' learning processes. Later, Simon and Tzur (2004) also recognized the importance of selecting and examining the tasks that promote the students development of new mathematical concepts in order to construct a hypothetical learning trajectory to frame students mathematical learning. According to Simon and Tzur, some of the assumptions that justify the use of the HLT construct are:

1. Generation of an HLT is based on understanding of the current knowledge of the students involved.
2. An HLT is a vehicle for planning learning of particular mathematical concepts.
3. Mathematical tasks provide tools for promoting learning of particular mathematical concepts and are, therefore, a key part of the instructional process.
4. Because of the hypothetical and inherently uncertain nature of this process, the teacher is regularly involved in modifying every aspect of the HLT. (p. 93)

The construction of hypothetical learning trajectories can be seen as the tools to guide and foster students' learning. Clements and Sarama (2004) state:

> Extant research is used to identify tasks as effective in promoting the learning of students at each level by encouraging children to construct the concepts and skills that characterize the succeeding level. That is, we hypothesize the specific mental constructions (i.e., mental actions-on-objects) and patterns of thinking that constitute children's thinking at each level. …These tasks are, of course, sequenced corresponding to the order of the developmental progressions to complete the hypothesized learning trajectory. The main theoretical claim is that such tasks will constitute a particularly efficacious educational program. However, there is no implication that the task sequence is the only, or best, path for learning and teaching, only that it is hypothesized to be one fecund route. (p. 84)

Clements and Sarama (2004) suggest that the power of the learning trajectory construct lies in connecting both the students' psychological developmental progression and the instructional sequences to promote mathematical thinking:

> …we conceptualize learning trajectories as description of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p. 83)

9.2.2 The Possible Influence and Mediating Role of Digital Technologies on Learning and Learning Trajectories

Some questions that now arise and that are discussed throughout the chapter, are: What role does the use of digital technologies play in the construction and development of hypothetical and actual learning trajectories? How can learning and its trajectories develop when mathematical tasks and phenomena are explored or analyzed through the use of digital tools as compared to other pedagogical media such as paper-and-pencil? To what extent does the use of DT shape the learners' mathematical ways of reasoning or thinking?

Important in this discussion is the symbiotic relationship between the digital media and the user. In accordance with Vygotsky's (1981) theory of socio-cultural cognition (see Chap. 7), we contend that any cognitive activity is a mediated activity that depends on the continual interaction between the user and the tool. Therefore we consider that computing tools have a mediating role in the learning of mathematics. While the digital medium exerts influences on the student's approach, and hence the understanding that evolves, it is his/her existing knowledge that guides the way the technology is used, and in a sense shapes the technology. The student's engagement is influenced by the medium, but also influences the medium (Hoyles and Noss 2003).

Thus, students' learning may involve different paths or trajectories depending on students' experiences, previous mathematical knowledge, and the tools being used. Furthermore, when using digital tools there are various routes to construct hypothetical learning trajectories and there are different points where learners might deviate from those trajectories. Thus, it is crucial for researchers and teachers to pay attention to the type of knowledge, resources and ways of thinking that students have developed in order to select or design tasks that promote students' mathematical thinking. In developing or formulating HLTs a special emphasis should be put on the promotion of activities in which students have the opportunity of expressing, presenting, using, testing, refining, and revising or adjusting their own ways of thinking (Lesh and Yoon 2004).

9.2.3 Digital Technological Environments as Domains of Abstraction

When using DT, certain activities that were relatively meaningless using paper and pencil, now can have significant mathematical value. For instance, the difference between drawing with a computer-tool such as Dynamic Geometry or Logo, as compared to doing it in a paper-and-pencil setting, may require the recognition of relevant mathematical properties that guide the problem or mathematical object representation. Many studies have shown that the use of DT tools can help learners in the conceptualization of mathematical problems or objects (e.g. Gentle et al. 1994); for instance, there are reports of improved high-level reasoning and problem solving linked to learners investigating in digital environments (Drier 2000; Ploger et al. 1997; Sandholtz et al. 1997).

The study of mathematics involves not only abstract but formal processes. What is the potential role of computer-based activities in the development of abstraction and formalism? Often students follow actions mindlessly without awareness that their results must make sense: it is from the awareness of this problem, and as a means to deal with the abstract nature of mathematics, that researchers proposed computer-based learning environments, or microworlds, where students could express and develop mathematical ideas (Hoyles 1993; see also Chap. 7). Furthermore, Noss and Hoyles (1996) consider a microworld as a *domain of abstraction* that can be understood as a setting in which students can make it possible for their informal ideas to start to coordinate with their more formalized ideas on a subject. The importance of having a domain of abstraction lies in the fact that it provides an environment where a general idea can become visible in the eyes of the students, and where learners can construct *situated abstractions*, that is, "construct mathematical ideas by drawing on the webbing of a particular setting" (Noss and Hoyles 1996, p. 122). Thus, a domain of abstraction supplies the tools so that exploration may be linked to formalization. Constructing bridges between students' mathematical activity and formalization links the mathematical thinking in the classroom with the official mathematical discourse. Computer environments may improve this possibility by enhancing the expressive capacity of students when they can use them (for instance, through the language they may provide) to communicate ideas that are difficult to communicate otherwise due to the lack of a sufficiently developed mathematical language. Related to this is the notion of *situated proofs*, which we will explain in Sect. 9.7.3.1, further below.

9.2.4 Hypothetical Learning Trajectories in DT Environments: Building on the Microworld Idea and Design

Hypothetical learning trajectories in DT environments and computational microworlds are theoretical constructs that share essential characteristics. Nevertheless, they arose within different realms and responded to different needs. Microworlds developed in computational environments; and while they may have been designed with underlying potential didactic routes, these are not always made explicit in their building-up descriptions. On the other hand, the notion of HLT was developed originally within non-DT environments, and making explicit intended paths, needless to say, is one of its identifying elements. In this chapter, we are trying to bring together those two concepts, extending the idea of HLT to a DT environment, in order to better understand and analyze the possible changes and shifts in mathematics teaching and learning that result from the use of digital technologies.

In the next section, by building on theoretical descriptions of the microworld idea, particularly on the definition by Hoyles and Noss (1987), we examine the different components and aspects – together with the affordances of digital technologies – that need to be considered for the design and analysis of DT-based learning trajectories.

9.3 Affordances of Digital Technologies that Might Influence Learning Trajectories, and Considerations for the Design of HLT

Digital technologies, if used appropriately, enable mathematical phenomena to be presented and explored in ways that provide opportunities to initiate and enhance mathematical thinking, and make sense of what is happening. They may give the learner potential to look through the particular to the general (Mason 2005). When the learning experience differs with digital technology (as compared to the experience in traditional settings), we can assume that learning trajectories and understanding will also differ. We will examine these differences through a range of perspectives in the following sections.

We now consider several key aspects to be considered in the development and/or analysis of a learning trajectory when digital technologies are employed. Hoyles and Noss (1987) considered the didactical situation in which the interaction in a microworld takes place, by taking into account the learner, the teacher, the setting and the activity which, in itself, is shaped by the past experiences and intuitions of the learner together with the aims and experiences of the teacher. They described a microworld as composed of four components: the pupil component (concerned with the existing understandings and partial conceptions that the child brings to the learning situation); the technical component (constituted by the software or programming language and a set of tools which provides the representational system for understanding a mathematical structure or a conceptual field); the pedagogical component (the didactical materials and interventions that take place during the computer-based activity); and the contextual component (the social setting of the activities). We would like to expand on their premise, by taking into account aspects to be considered in the design (or analysis) of DT-based HLT, that can be classified approximately within Hoyles and Noss's components, although it is clear that many of these aspects belong to several components (see Fig. 9.1) and there is a strong interaction between all of them.

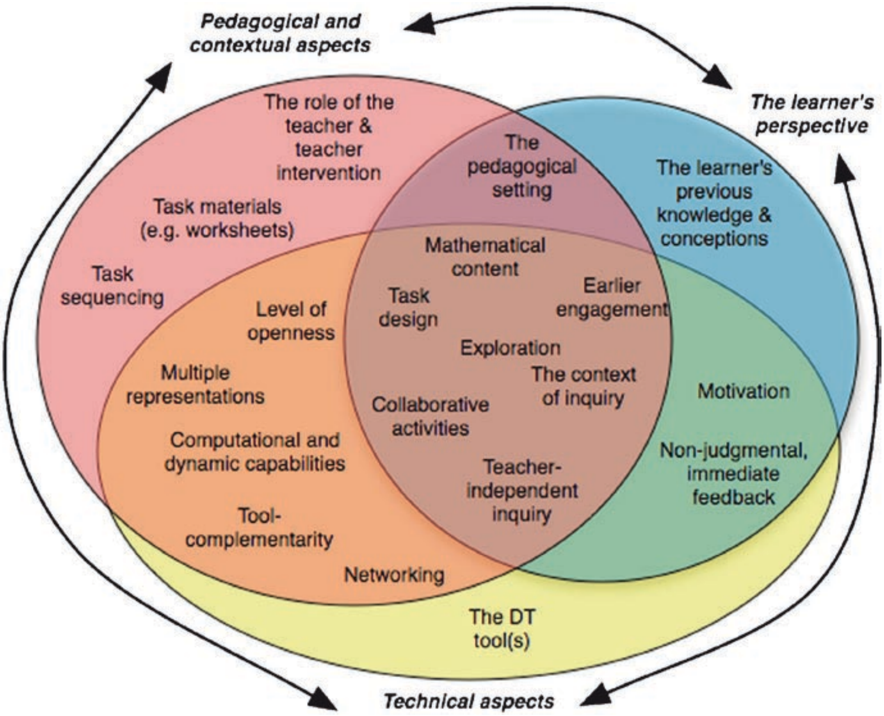

Fig. 9.1 The affordances of digital technologies and the aspects to be considered in the design and analysis of a DT-based learning trajectory

9.3.1 Technical Aspects

9.3.1.1 The Choice of the Technological Tool(s) and Their Design

The choice of the digital tool or environment to be used will have a crucial influence for fostering and promoting mathematical thinking and in the type of learning trajectory that will be developed. As discussed earlier, although a new mathematical knowledge can be generated when using digital technologies, that knowledge is often situated and shaped by the tools and DT environment (Noss and Hoyles 1996; discussed in Chap. 7 of this volume). Furthermore, as Balacheff and Sutherland (1994) indicate, different tools or environments constitute different domains of phenomenology and thus have different epistemological validity (i.e. the knowledge that can be generated is different in different environments). For example, using spreadsheets many mathematical situations can be represented in discrete ways using a table as a means to identify and explore invariants or patterns; whereas the use of dynamic software or CAS allows the problem solver to explore continuous behaviors that often can be modeled through an algebraic representation (Santos-Trigo et al. 2007).

According to Hershkowitz et al. (2002), the selection of appropriate computerized tools for teaching mathematics should take into account the following characteristics: (1) *generality*, i.e., the tool's applicability in different content areas, its availability and its cultural status; (2) *potential to support mathematization*, i.e., the tool's potential for amplification and reorganization (Pea 1985; Dörfler 1993) and of expressing a new "mathematical realism" (Balacheff and Kaput 1996); (3) *communicative power* (or semiotic mediation power), i.e., the tool's power to support the development of mathematical language, and relate to its symbol system and to the symbol system more commonly used in mathematics.

Related to the choice of the tool and/or interface, is its design. One problem that has been observed (Rojano et al. 2009) is that often users are not able to discern the representational elements of the DT environment that are mathematical from those that are not. For example, the learner sometimes focuses his/her attention on the effects and messages produced by the interface that have no mathematical meaning, distracting him/her from the intended mathematical content. This points to the importance of taking into account the design of the components involved in the DT interface, in order to minimize, as much as possible, their potential influence.

In the next sections we consider some of the characteristic technical affordances of DT that may influence the learning experience in these environments and that should also be taken into account when choosing/designing the DT tool or environment, such as the availability of potentially interconnected multiple representations or the computational capabilities of DT.

9.3.1.2 The Role of Representations

One aspect that is central to the contribution of digital environments is the multiple representational registers (Duval 1993) they provide. The ability to link and explore visual, symbolic, and numerical representations simultaneously in a dynamic way has been recognized extensively in research (Borba and Confrey 1996; Mueller-Philipp 1994; Tall 2000; Sacristán and Noss 2008) but still warrants consideration when discussing attributes of various digital media. Multiple representations through interactive digital environments such as applets, and the designing of games have also enhanced the learning process (Boon 2006; Confrey et al. 2006). Associated with this is the idea of visualization. While the debate is inconclusive as to the positioning of visualization in mathematics (e.g. Jorgenson 1996; Thurston 1995), there is greater consensus regarding the positive role of visualization or graphic approaches in the facilitation of understanding in mathematics education (Dreyfus 1991; Olive and Leatham 2000; Borba and Villareal 2005; Calder 2004).

Ainsworth et al. (1998) claim that multiple representations promote learning for the following reasons: (a) they express different aspects more clearly; hence, the information gained from combining representations will be greater than what can be gained from a single representation; (b) they constrain each other, so that the space of permissible operators becomes smaller; and (c) when required to relate multiple representations to each other (as can happen in DT environments), the learner has to engage in activities that promote understanding. In a similar vein, Goldin (2002) emphasizes the role of representations in resolving ambiguity when learning mathematics, and in providing a context for doing mathematics. And Sacristán and Noss (2008) illustrate how computational activities in a carefully designed microworld can lead to a constructive articulation of different representational forms (such as visual, symbolic and numeric); a process that they call *representational moderation*.

Most DT tools incorporate different representational elements that can mediate and support learning. For example, dynamic software can help the learner represent and examine mathematical phenomena or tasks in terms of a functional approach without defining explicitly the function that describes the phenomenon. Instead, the tool can be used to generate a numeric representation that can be analyzed graphically. Based on the graphic representation of the phenomenon the learner can analyze its behavior directly. As a consequence, some types of phenomena can be addressed by students, with the support of these representational infrastructures that may not demand the mastery of algebraic or formal mathematical representations, at an earlier age than in traditional education; this is addressed more fully in Sect. 9.3.2.6.

9.3.1.3 The Computational and Dynamic Capabilities of DT

Digital technologies also have computational and dynamic capabilities that open up and expand almost infinitely the range of possibilities for classroom explorations. For example, DT can manage large amounts of realistic data more easily than paper-and-pencil technology (Ridgeway et al. 2006) allowing students to more readily explore real-life issues through a mathematical lens. They can remove elements of simple, repetitive computation so that more in-depth thinking and consideration of overarching issues could be engaged in (Deaney et al. 2003; Ploger et al. 1997), and often allow the learner flexibility to quickly rearrange information and re-engage with activities from fresh perspectives (Clements 2000; Calder 2005).

In addition to their computational and data-processing capabilities, DT also provide important dynamic perspectives. Dynamic representations of mathematical objects allow learners to visualize problems or mathematical processes in ways that were not possible before. For example, learners can view a process as it develops, rather than trying to analyze it from its fixed initial, partial or end results. The creation and exploration of dynamic models can also enhance students' ability to model mathematically in a reflective way (Borba and Villareal 2005; Zbiek 1998). Or in the case of dynamic geometry, students can engage in explorative activities that result from dragging or moving particular objects within the representation: in such environment, the controlled movement of some elements within a geometric configuration can lead the learner, not only to detect and explore invariants or mathematical relations, but also to explore whether the relation is valid for a family of cases.

9.3.1.4 The Networking Capabilities of DT

Networking capabilities (either local or global, e.g. through the Internet) also open up new possibilities of ways of learning mathematics: Learner communities can be more interactive and collaborative when a network structure is used to share and discuss issues of mathematics and instructional practices (Sinclair 2005), whether within a classroom or beyond. In fact, activities can go beyond the relatively homogeneous environment of a classroom, school or local community, and give opportunity for richer, more diverse global perspectives in mathematics education, and the potential for making sense of, or generalizing, in a different way.

All of the above affordances of DT, which include its computational, dynamical, representational capabilities, as well as the non-judgmental feedback they can provide (discussed in Sect. 9.3.3.2), foster and allow students to explore, experiment, take risks, as well as carry out collaborative work, more independently (from the teacher). These "new" means of working and learning that are facilitated by DT, in turn, can help develop abilities and intuitive thinking that can enhance powerful mental conceptualizations (Meissner 2006). Thus, all of the above are contributions of DT that can be influential for learning and that need to be considered in the pedagogical design of a HLT as discussed in the next section.

9.3.2 Pedagogical and Contextual Aspects (Task Design, the Role of the Teacher and the Didactical Context)

9.3.2.1 The Pedagogical Setting

The simple presence of digital tools in the classroom can change completely the dynamics of that environment, for instance, by offering diverse opportunities for learners to engage in collaborative approaches to data collection and problem solving (as considered in the previous section).

Although the focus of this chapter is on learning, we cannot consider it without taking into account the pedagogical aspects, the role of the teacher and the design of the task. For instance, crucial pedagogical considerations include the way a task is presented to the students, the support of pedagogical materials such as worksheets, and the *milieu* (Brousseau 1997) or learning context of the DT-based tasks. The context, used here in its broad sense, can include the social context that may promote collaborative work and learning; the physical setup of the classroom and equipment; as well as what we refer to as the "context of inquiry," discussed in the next section.

9.3.2.2 Context of Inquiry of the Activity

The use of digital technologies can facilitate the exploration of problems embedded in diverse contexts, making them more interesting for the learner to solve. One context of inquiry are problems set in purely mathematical terms where explorations can be enhanced by DT through the diverse representational systems in which they are situated (as discussed in Sect. 9.3.1). Simulation and modeling can create another valuable context of inquiry. Students can also gather and analyze data and relations (which can, or not, be realistic).

A context-based learning of mathematics considers the use of situations involving mathematical problems as both the starting point, and the main process for understanding concepts and the performance of operations. Bickmore-Brand (cited by Wiest 2001, p. 75) states that "context is paramount to the construction of meaning the whole way through. It is the backdrop against which the parts have to make sense." As stated above, an abstract mathematical problem can also be a context problem (Gravemeijer and Doorman 1999). In any case, a contextual task should be experientially "real" or concrete enough for the student, and should serve as a basis upon which a mathematical concept can be built. In this sense, Wilensky (1991) suggests that abstract objects can become concrete if we have multiple modes of engagement with them.

Mason and his colleagues (1985,1995 p. 36) state that "in order to have clear, confident and automatic mastery of any skill, it is necessary to practice, but the wish to practice will arise naturally from stimulating contexts." A context-based approach has both immediate and general advantages. It facilitates learning processes by providing concrete meaning to an otherwise abstract concept or idea (Heid 1995). It provides points of reference that students can review at a more advanced stage of learning, when work is performed at a more abstract level. It raises student motivation and willingness to become engaged in the learning activity and emphasizes the potential of using models and skills in other fields.

9.3.2.3 Level of Openness of a DT-Based Activity

Digital technologies have the potential to open up the range of possible approaches to investigation, and there is a need to decide on the level of openness of a selected activity. The solution to an open or unstructured task does not require a specific method, a certain representation or an implicitly given sequence of steps, whereas structured problems pose specific requests with regard to the variables mentioned above. Digital technologies facilitate open tasks in ways that were not possible before, since they provide a *scaffolding* (Wood et al. 1976) for students to work more easily on their own and develop their thinking (e.g. Clements and McMillen1996). The open approach (where students have ownership of the development of the task) intends to develop problem-solving skills, to develop creative mathematical thinking, to provide opportunities for students to actually experience investigation, and to achieve a meaningful construction of knowledge (in accordance with constructivist theories, e.g. Bruner 1966). However, for different students, the problem's context will be perceived in different ways, and as a result, students might understand a mathematical concept in ways the teacher didn't anticipate, or follow "unproductive" paths of solution – described by Sutherland et al. (2004) as construction of idiosyncratic knowledge that is at odds with intended learning.

In contrast, the structured approach enables students to pursue a more predictable and planned learning trajectory in the domains of mathematical content and processes of problem solving. Yet, a structured approach imposes a unique and clearly defined learning trajectory that does not necessarily meet the needs or preferences of all students. It is also possible, however, to consider and design semi-open tasks.

9.3.2.4 Sequencing of Tasks Within an Activity

DT-based activities and intended learning trajectories can be structured as particular sequences of tasks and/or explorations. In fact, in a hypothetical learning trajectory, an intended sequence is explicitly designed (which may not always be the case in some computer microworlds), although it needs to be flexible, due to the unpredictable events that DT bring about.

For example, a HLT using a dynamic geometry software could begin with the learner observing and describing invariants or conjectures visually; later those conjectures may be analyzed numerically by measuring and comparing attributes (areas, perimeters, segments, angles, etc.); then a graphic approach can be used to represent and examine the phenomenon and its corresponding relations; finally, those relations or conjectures can be supported in terms of geometric properties or formal arguments.

Many mathematical explorative activities follow an inductive path based on a transition from the investigation of particular cases to pattern generalizations, then to the justification of the evolving pattern and later on, to its implementation in additional cases. Friedlander et al. (1989) recommend a sequence of tasks that follows a path that leads from initial experimentations, to both implicit and explicit generalizations, and then to the use of an explicitly generalized pattern. A similar sequence was also recommended in the domain of data investigation and scientific research (the *Pose-Collect-Analyze-Interpret-Communicate* model proposed by Kader and Perry 1994).

9.3.2.5 Mathematical Content

The mathematical content (i.e., concepts, algorithms, properties, definitions) is one of the main considerations that determine a particular activity. Nowadays, it is recognized that the use of digital tools can offer students the possibility of participating in activities for:

> …(a) gaining insight and intuition, (b) discovering new patterns and relationships, (c) graphing to expose mathematical principles, (d) testing and especially falsifying conjectures, (e) exploring a possible result to see whether it merits formal proof, (f) suggesting approaches for formal proof, (g) replacing lengthy hand derivations with tool computations, and (h) confirming analytically derived results. (Borwein and Bailey 2003, cited in Zbiek et al. 2007, p. 1170)

However, activities that lack a specific mathematical aim, even if they are based on other considerations (such as intended cognitive processes, technological instrumentation or choice of representation) might not have a clear contribution to the learning of mathematics (Wu 1994). It is also important to mention that the use of digital media can help the learner, not only to identify relevant content associated with the construction of hypothetical learning trajectories, but also new mathematical content or connections among diverse approaches to the task, that can emerge as a result of using those tools.

9.3.2.6 The Possibility of Earlier Engagement (Shifts in Trajectories)

The introduction of technology into the educational arena has placed educational systems in the quandary of using technologies to teach the same old thing (more efficiently) or using it as an agent of change (McFarlane 1997). This is a predicament for the field of education in general, but also for mathematics education in particular. Thus, DT-based learning environments can be used either to teach the classical mathematics topics of traditional curricula; or to transform the contents of school mathematics, the means of building mathematical knowledge and the very forms of interaction within the classroom setting. A great deal of research work involving the use of digital technologies has made it clear that promoting this type of transformation is quite feasible.

The access to mathematical ideas and knowledge is strongly related to the representational infrastructures in which that knowledge is expressed. DT tools may provide new representational infrastructures that can improve the learnability of certain mathematical ideas; hence, DT may allow children to become engaged in mathematical topics previously considered as too advanced for them. This is discussed at length by Kaput et al. (2002), and is the premise behind important research projects such as UK's Playground project (http://www.ioe.ac.uk/playground) described in that paper, and the European Weblabs project (http://www.weblabs.eu.com), that aimed to empower young children through DT environments, so that they could access and explore advanced mathematical ideas.

9.3.2.7 The Teacher's Role and the Importance of Appropriate Intervention

The use of DT tools requires a different approach to teaching. The teacher's role may become more that of a mediator, where the teacher not only guides students through their DT tasks, but also intervenes to promote learning. Many (e.g. Clements 2002) have found that the mathematical knowledge constructed in a DT environment can remain hidden or "situated" within the technological context, unless teachers help make that knowledge explicit.

On the other hand, some researchers have also indicated that the affordances provided by DT-environments, *when facilitated appropriately by the teacher*, may lead students to explore powerful ideas in mathematics, to learn to pose problems, and to create explanations of their own. The teachers' appropriate intervention during the development of DT sessions involves guiding the learners to validate mathematical results or relations that emerge when they formulate and explore a problem through the use of the tools. Furthermore, DT attributes, coupled with appropriate teacher intervention, can enable the learner to not only explore problems but to make links between different content areas that may otherwise have developed discretely. This role of the teacher as mediator, bridging the individual and the social perspectives, is discussed in more depth in Chap. 7 of this volume (in the section on mediation and semiotic mediation).

9.3.3 The Learner Perspective

All of the considerations discussed in previous sections affect the learner directly. But learners' fore-conceptions – both mathematical, and the ones which they may have of the digital media – as well as their affective involvement when working with DT, need also be taken into account. We consider these next.

9.3.3.1 Affect and DT: The Role of Engagement and Motivation for Learning

Almost 30 years ago, Papert (1980) emphasized the affective value of computational technologies by observing how learning could be enhanced by developing the motivation of students and providing impersonal non-judgmental feedback. The effect of DT on student engagement and motivation has since been noted. Higgins and Muijs (1999) found much work pertaining to the positive effects on motivation and attitude, and while this enthusiasm might relate to the novelty factor initially, it can't be ignored, given the correlation between students' attitude to learning in mathematics, and their facility to understand. Other researchers have likewise found positive motivational effects through using digital technologies in mathematics programs (e.g. Hoyles 2001; Kulik 1994, in his meta-analysis of computer-based learning; Sandholtz et al. 1997; Schacter and Fagnano 1999; Lancaster 2001; and Ursini and Sacristán 2006).

9.3.3.2 The Role of the Feedback from DT

As mentioned above, Papert (1980) valued the immediate, non-judgmental feedback that can be provided in computer settings. A DT-based activity can involve "trying out something, watching for an effect and responding to the feedback" (Weir 1987, p. 32). The almost instantaneous nature of the response in a digital environment, coupled with the interactive nature of the engagement, allows for the ease of exploration of ideas. Discussion is stimulated, as the results of prediction or conjecture are viewed rapidly and are more easily compared. This promotes reasoning as students investigate deviations from expected output, or the application of procedures. The feedback also helps develop the accuracy required for procedural structures and to be more explicit entering mathematical manipulations (Battista and Van Auken Borrow 1998). Thus, the instantaneous feedback of the computer or digital media can be considered an important factor for enhancing student learning and in the development of learning trajectories. Research results (e.g. Beare 1993; Deaney et al. 2003) support this: For example, Chance et al. (2000) found that the facility of digital media to immediately test and reflect on existing knowledge was an influence on the learning process. Likewise, research into students learning in a CAS environment, identified that tension evoked from differences between actual output (the technological feedback) and students' expectations, probably instigated the most valuable learning (Kieran and Drijvers 2006); more so, perhaps, because in DT settings cognitive conflicts are generally non-judgmental.

9.3.4 Introduction to the Following Sections

So far, we have identified relevant aspects of a framework for developing (and analyzing) DT-based learning trajectories. We recognize that the use of digital tools to foster learners' construction of mathematical concepts and problem solving approaches involves rethinking ways to select, design, and use tasks; and have an influence on the milieu or learning conditions, the teacher's role and students' interactions.

We now consider examples that draw on the affordances offered by engaging mathematical tasks through digital technologies. These examples contextualize several pedagogical elements associated with learning trajectories and are situated in a range of settings involving primary- through tertiary-level students. In the first section we examine an example related to the design of intended (hypothetical) learning trajectories. Then the manner in which learning trajectories evolve across various platforms and the opportunity they offer students to more easily engage in powerful mathematical ideas are illustrated and analyzed to further demonstrate the diversity of learning trajectories through the various media and how they facilitate transitions in the learning process. This is followed by a situation where the affordances of the DT, through interplay with the learner's preconceptions and the mathematical task, influenced the shape of an emerging, actual learning trajectory. We end with examples of how digital technologies can help create a shift in the usual learning trajectory, by allowing learners to engage earlier with mathematical ideas or concepts traditionally considered more advanced, and to develop knowledge and intuitions that can serve as a basis for later formalization.

9.4 An Example of the Design of a Hypothetical Learning Trajectory Through Exploratory Tasks

Students' explorative work with technological tools allows a variety of learning trajectories. The considerations related to the selection of an investigative or explorative activity according to possible hypothetical learning trajectories are connected to theories and research findings on student cognition, learning trajectories and use of technological tools in teaching mathematics. The implementation of exploratory, or investigative, activities provides opportunities for a meaningful learning of mathematical concepts.

Here we provide an example from an activity, *Folding Perimeters* (Friedlander and Arcavi 2005), taken from the domain of beginning algebra. This activity was included in a learning unit on ratio and proportion. This section describes the main characteristics of this activity, and some considerations that led to its selection for a particular hypothetical learning trajectory. These are grouped according to some of the aspects described in Sect. 9.3.

9.4.1 Context of Inquiry

In this activity, students investigate the perimeters of an alternating sequence of squares and rectangles, during a process of repeated folding-in-two (Fig. 9.2). The context of paper folding is simple and familiar on one hand, and rich in mathematical opportunities, on the other. Tourniaire and Pulos (1985) concluded in their review of research on proportional reasoning, that context plays a crucial role in student performance and that use of a wide variety of contexts is needed in the teaching of this notion. This task's context promotes a constructivist path of learning (Hershkowitz et al. 2002) in which students start with a problem situation, investigate the problem and develop the need for appropriate tools and concepts – first at an intuitive level, and later on constructing and analyzing newly formed tools and concepts in a more extended and mathematically formal manner.

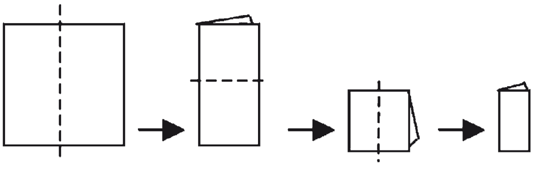

Fig. 9.2 Context of the *folding perimeters* activity

9.4.2 Mathematical Content

Whereas folding paper is a concrete action, the related tasks evoke experience of this action, which provide meaning. For this, the activity integrates various mathematical domains: geometric (squares, rectangles, perimeters, opposite sides, measurement), arithmetic (via numerical tables, operations, difference, ratio), and algebraic (with the use of spreadsheets –*Excel* – via formulas and pattern generalizations). The mathematical content is stated clearly throughout the activity, and is one of the factors that determine the sequence of tasks – the first three tasks require a more geometrical and visual investigation, the next task relates to the differences between the perimeters of two adjacent shapes, and the last two tasks focus on the ratio of two adjacent shapes, and each shape with other shapes in the sequence respectively.

9.4.3 The Choice of the Technological Tool

For the activity, spreadsheets are used to support and promote the processes of generalization and algebraic thinking (Hershkowitz et al. 2002; Friedlander and Tabach 2001). The following considerations led to the selection of spreadsheets for this particular activity:

1. They serve as a powerful tool for data collection, organization and representation
2. They provide continuous and non-judgmental feedback throughout the solution process
3. They present the concept of proportion dynamically, as a sequence of constant ratios obtained by applying the same rule to numerous pairs of numbers or quantities
4. They enable analyzing an extended collection of data
5. They emphasize the meta-cognitive skills of monitoring and interpreting results
6. They allow learners to work simultaneously on various representations
7. They present the algebraic representation as an efficient and meaningful means of constructing data

Spreadsheets fulfill the following criteria: (1) *The generality of the tool:* Spreadsheets have the potential to support the natural and spontaneous creation of numerical series by means of certain kinds of algebraic rules, and to represent numerical data graphically; (2) *The potential to support mathematization*: Spreadsheets have the potential to support students in their development of mathematical processes, by proposing patterns and expressing them via formulas and using the “dragging” (fill down) option (e.g. Kaput 1992); and (3) *The communicative power:* The symbolic language of spreadsheets mediates between the symbolic algebraic language and informal languages (Ainley1996; Filloy and Sutherland 1996; Friedlander and Tabach 2001; Haspekian 2005; Hershkowitz et al. 2002; Sutherland and Rojano 1993; Wilson et al. 2005).

9.4.4 Level of Openness

The activity discussed here addresses the issue of openness by presenting a sequence of tasks, both open and structured: The open tasks require students to identify any properties of the presented sequence of shapes, make predictions, and then look for patterns that describe the change in perimeter. The more directed tasks require the student to collect data for the first ten shapes in the sequence, organize it in a spreadsheet table, present it as a diagram, investigate patterns of perimeter change by considering first the difference and then the ratio between pairs of adjacent shapes, and of shapes placed in the sequence at a distance of two steps. Some other tasks in the activity are less directed with regard to content or solution strategy – for example, students are required to find *any* patterns of perimeter change and justify them.

9.4.5 Representations

In this activity, students are specifically required to present perimeters and perimeter changes in sheets of paper, in drawing, in numerical tables, as algebraic formulas, in bar diagrams, and in verbal descriptions. Some of the tasks focus on the construction and use of a specific representation, whereas others leave this issue open to the students. Figure 9.3 presents a numerical and graphical representation of the data and some of the results obtained by the observed students regarding the alternating sequence of shapes in the activity.

	A	B	C	D	E	F
1	fold #	side	side	perim.	differ.	ratio
2	0	32	32	128	32	3/4
3	1	32	16	96	32	2/3
4	2	16	16	64	16	3/4
5	3	16	8	48	16	2/3
6	4	8	8	32	8	3/4
7	5	8	4	24	8	2/3
8	6	4	4	16	4	3/4
9	7	4	2	12	4	2/3
10	8	2	2	8	2	3/4
11	9	2	1	6	2	2/3
12	10	1	1	4		

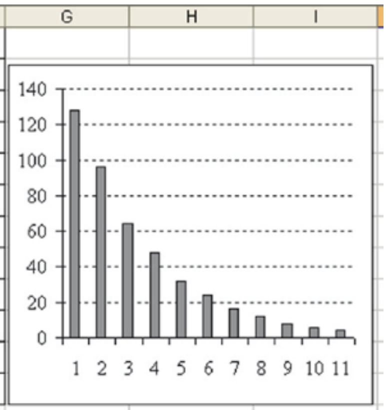

Fig. 9.3 Spreadsheet representation of data and results in the *folding perimeters* activity

9.4.6 Sequencing of Tasks within the Activity

The activity follows a cyclic path. First, the students are required to identify and investigate their own patterns and in the next two cycles, they consider first changes in the difference, and second in the ratio of perimeters of two consecutive shapes. In each case, specific cases are collected, organized, and analyzed, general patterns are formed and conclusions are drawn, interpreted and applied.

9.4.7 Comments

The above example shows the considerations for the design of a HLT. Simon's (1995) hypothetical learning trajectories attempt to constrain the range of learning trajectories that might emerge. However, the nature of the task will determine the actual learning trajectory (the learning paths followed by students while they work on a mathematical activity) to some extent. This actual trajectory can be different from the planned hypothetical learning trajectory and in fact might vary in unexpected ways.

In the following sections we explore, through other examples of both HLT and actual trajectories, how the pedagogical medium is influential on the engagement and subsequent understanding; and how, as such, it informs on how future hypothetical learning trajectories might best evolve.

9.5 Learning Trajectories Within and Across Various Platforms: An Example with Dynamic Geometry and CAS

Earlier, in our brief discussion on the choice of the DT tools, we contend that different tools or environments constitute different domains of knowledge and phenomenology. Thus, it is relevant to reflect on the ways in which the use of different computational tools (that can be complementary) influences the design and/or development of learning trajectories.

Here, we illustrate and trace (through successive episodes) a learning trajectory that emerged from working on a task with the use of two tools: the Cabri-Géomètre dynamic geometry software and a hand-held graphing calculator. A group that includes a mathematics educator, doctoral students, and high school teachers met weekly to construct potential learning trajectories that resulted from examining and exploring mathematical tasks. The analysis of these tasks offers teachers relevant information to think of instructional activities in which their students recognize the advantages of constructing dynamic representations of problems that can lead them to identify and explore not only interesting relations but also diverse problem solving strategies.

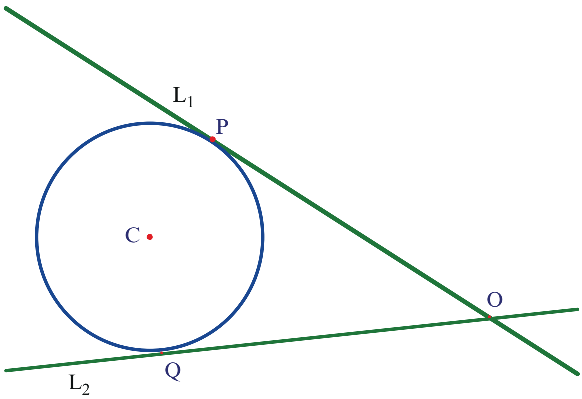

Fig. 9.4 Two intersecting lines and a point P on one line: what properties does the construction of the circle to both lines hold?

The tasks – from the domain of geometry – are similar to those that appear in usual textbooks, such as the following, illustrated in Fig. 9.4:

> Given two intersecting straight lines and a point P marked on one of them, show how to construct, using a straightedge and a compass, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines (Schoenfeld 1985, p. 15).

An important heuristic to approach this task is to assume the existence of a solution. Figure 9.4 shows a circle that is tangent to L_1 at P and also tangent to L_2. The goal is to identify relevant mathematical properties embedded in the figure. Some possible discussion questions are: *Where should the centre of the tangent circle be located? How is the line that passes through the centre of the tangent circle and the tangency point P related to line* L_1*? If P and Q are the points of tangency, what are the relevant properties of triangles PCO and QCO?* To think of problems or mathematical concepts in terms of questions is an important principle in problem solving approaches (Santos-Trigo 2007). In this context, the discussion of the previous questions could provide information to construct a dynamic representation of the problem. This representation can be built with the use of dynamic geometry software.

9.5.1 The Construction of a Dynamic Representation of the Problem

A relevant property of the tangency points on each line is that the centre of a tangent circle must lie on the perpendicular to each line L_1 and L_2 that passes through each tangency point. Using dynamic geometry software, one can create the following construction (Fig. 9.5) to find the centers of the tangent circles:

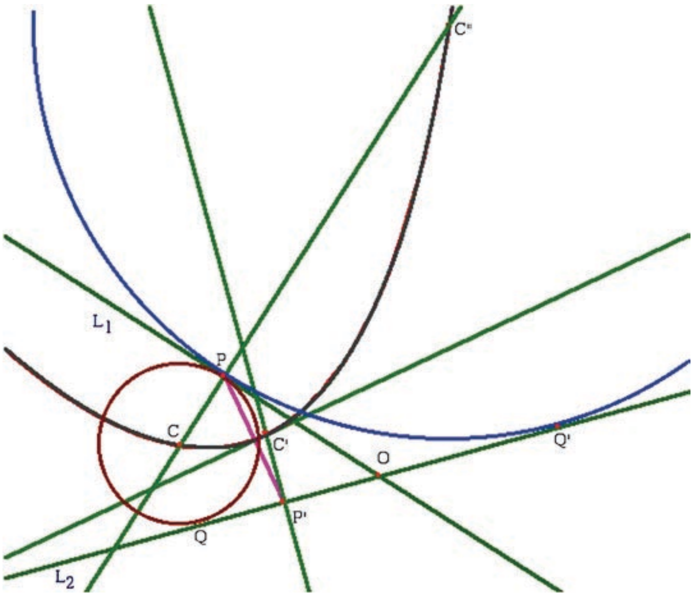

Fig. 9.5 Constructing a dynamic representation of the problem: the locus of point C′ when point P′ is moved along line L_2, depicts a parabola. The center C of the tangent circle will be at the intersection of the parabola and the perpendicular line to L_1 that passes through point P

> Draw a perpendicular to line L_1 that passes through point P and choose point P′ on L_2 and draw a perpendicular line to L_2 that passes through point P′. Then draw the perpendicular bisector L_3 of segment PP′. The perpendicular bisector L_3 cuts the perpendicular line to L_2 that passes through P′ at C′.

What is the locus of point C′ when point P′ is moved along line L_2? One can observe that when point P′ is moved along line L_2, the position of point C′ changes: The path left by point C′ (Fig. 9.5) seems to be a parabola. Indeed, point C′, which generates the locus, is on the perpendicular bisector of segment P′P, thus d(PC′) = d(C′P′). Therefore, the locus is a parabola with focus point P and directrix line L_2. The parabola and the perpendicular line to L_1 that passes through point P intersect at two points C and C″. Points C and C″ are the centers of the two tangent circles to lines L_1 and L_2.

An important property of the representation of the problem is that it is possible to move objects within the configuration and observe the behaviors of other elements of the figure: For example, Fig. 9.6 shows that the locus of point C when point P is moved along line L_1 is the perpendicular bisector of angles POQ and ROQ′ respectively. This is because triangle PCO and triangle QCO are congruent. Similarly, triangle PC″O is congruent to triangle Q′C″O.

Thus, to draw the tangent circles to those lines with P as tangency point the following construction is sufficient:

> Draw a circle with centre O and radius OP, locate, draw the perpendicular line to L_1 that passes through P and the perpendicular lines to L_2 that pass through the intersection points (Q and Q″) of the circle with line L_2.

The intersection points of the perpendicular line to L_1 that passes through P with the perpendicular lines to L_2 that passes by Q and Q′ will be the centers of the tangent circles (Fig. 9.7).

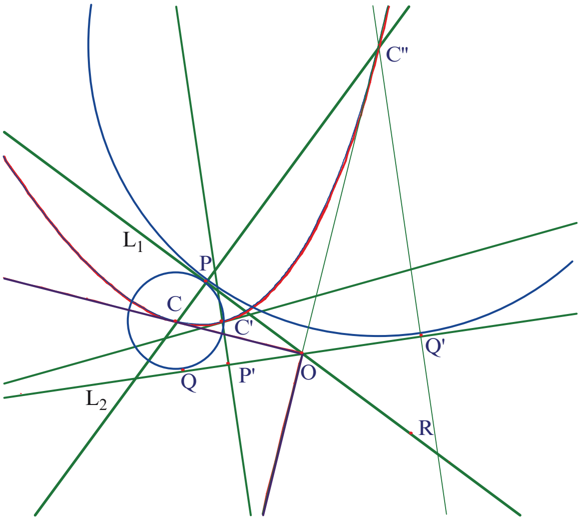

Fig. 9.6 The locus of point C when point P is moved along line L_1 is the angle bisector of angles POQ and QOR

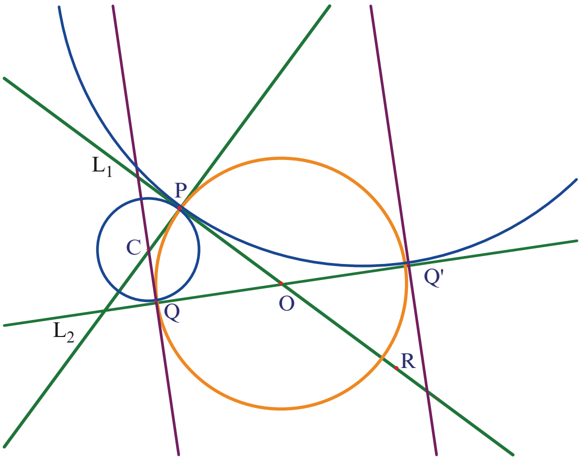

Fig. 9.7 Drawing the tangent circles to line L_1 and L_2 using straightedge and compass

9.5.2 From Geometry to Algebra

An algebraic approach can also be used to find the centre of the tangent circle. Here, again the heuristic that involves the use of the Cartesian system and locating the lines and the intersection point on a proper position will facilitate the process of dealing with the representation of the problem. Line L_2 coincides with the *x*-axis and point P is on line L_1. With the use of the software, one can easily find (Fig. 9.8): the equation of the circle with centre at the origin and that passes through P, the equation of the lines L_1 and L_2, and the perpendicular to them through points P and P′ respectively.

In order to identify the centre of a tangent circle to both lines, one needs to solve the corresponding equations. The use of CAS (a computer algebra system), such as those included on hand-held graphing calculators can be useful. Figure 9.9 shows the algebraic solution of the equations of the circle and line L_1, while Fig. 9.10 shows how the graphic solution can be displayed also by the calculator.

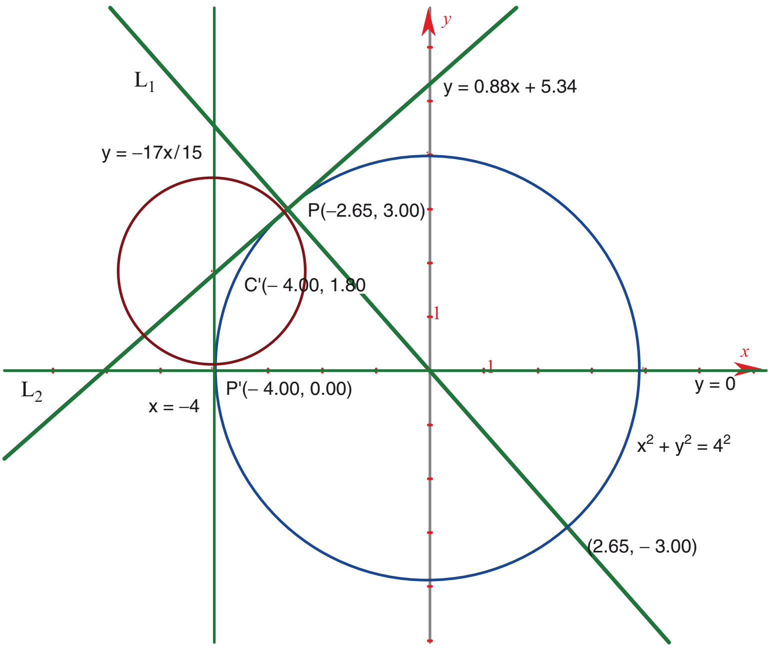

Fig. 9.8 Algebraic representations of relevant objects in the problem

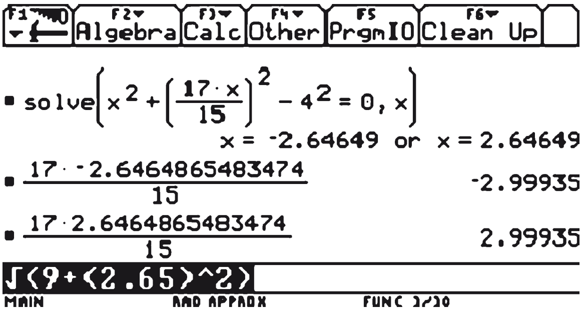

Fig. 9.9 Solving the equation of circle and line L_1

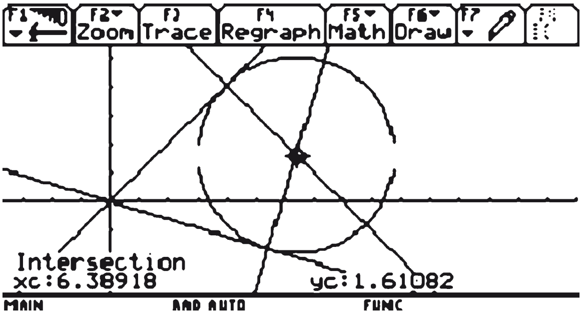

Fig. 9.10 Visual solution on a graphic calculator

9.5.3 Discussion

In the above example, the dynamic geometry software (DG) and CAS calculators helped explore various ways to represent the tasks and search for relations and connections associated with these tasks. In this process, the first tool (DG) offers the opportunity of constructing a dynamic representation of the problem in which a set of heuristics (finding loci, relaxing conditions, assuming the problem solution, using the Cartesian system, looking for invariants) can be used to examine distinct mathematical relations. Whereas the use of the hand-held CAS calculator may facilitate dealing with an algebraic approach to the problem (Santos-Trigo et al. 2006). In this context, the use of more than one tool to represent and examine the task is an opportunity to enhance and complement visual, numeric, geometric, graphic, and algebraic approaches.

In the previous example, if the construction is already made, learners could initiate a trajectory in which, a number of properties can be conjectured. Such conjectures can be inferred from dragging special points along special lines within the dynamic geometry environment. When working with CAS on the graphic calculator, it is possible to switch from a synthetic geometry perspective to an analytical geometry perspective and then to translate the geometric properties found in the DG environment into an algebraic task that can be done using the symbolic manipulator. Since in the calculator, the analytical representation of the elements involved in the construction, and their corresponding graphical representations, are hotlinked, it is then possible to visually check if the construction fulfils the expected properties. This case clearly illustrates how approaching one task across different environments may promote different types of transitions (for example, from intuitive to formal ideas; from a synthetic geometry perspective to an analytical geometry one; from algebraic to graphical representation of functions). These transitions in turn may influence the way learners interpret the task proposed, build up conjectures, and validate them.

9.6 Emergence of Learning Trajectories from the Engagement with DT: An Example with Spreadsheets

In this section we consider the emergence of learning trajectories as mathematical tasks are encountered via digital media. We identify some of the particular features as students engage in mathematical investigations within the DT setting, and reflect on how these features might condition the learner's understanding of mathematical phenomena in particular ways. A key focus is on how informal conjectures emerge, and then evolve for the learner, and how visual perturbations or tensions (caused by the DT feedback – see Sect. 9.3.3.2), and associated discursive networks, give opportunity to enhance the learning experience.

When learners engage in an investigation, interpretation pervades their engagement; social and cultural experiences always condition our perspective (Gallagher 1992) and hence understandings (Cole 1996). Learners' interpretation of the task, their response to it, and the output of their deliberations are filtered by their fore-conceptions of the mathematics, and of the pedagogical medium through which it is encountered (which can be seen as a cultural forms that shape various ways of knowing – Povey 1997). The engagement with the task likewise alters the learner's fore-conceptions, repositioning the learner's viewpoint, and allowing them to re-engage with the task from a fresh perspective. This cyclical process of interpretation, engagement, reflection and re-interpretation continues until some sense of consensus is reached.

This resonates with Borba and Villareal's (2005) notion of humans-with-media, who contend that understanding emerges through the interaction of collectives of learners, media and environmental aspects, with mathematical phenomena: or with Kieran and Drijvers (2006) discussion of the emergence of mental schemes from social interaction. In essence, the mathematical task, the pedagogical medium, the fore-conceptions of the learners, and the dialogue evoked are inextricably linked. It is from their relationship with the learner that understanding evolves, as an interpretation of the situation through those various filters.

Investigation of mathematical phenomena through digital pedagogical media is a distinctive process. The potential for using DT to enhance students' mathematical modeling ability has been established previously (e.g. Zbiek1998; Drier 2000). Providing a multi-representational environment to test ideas, linking the general to the specific, being interactive, and giving students a measure of autonomy in their investigation are also commonalities of DT that facilitate investigation (Calder 2005).

There are, however, opportunities and constraints associated with the process. Typically, when learners investigate in a digital environment, their engagement and the dialogue or reflection evoked, suggests input, which is subsequently entered. The ensuing output (feedback) is produced visually, almost instantaneously (Calder 2004). If this output is at variance to what was anticipated (that is, if the students fore-conceptions suggest a different output), a tension may arise. This ensuing perturbation can either elicit, or alternatively scaffold further reflection and the modification of the learner's perspective. Subsequent re-engagement with the task from this fresh viewpoint can lead to the initiation of informal mathematical conjecturing (Calder et al. 2006). The development of mathematical conjecturing and reasoning can emanate from intuitive beginnings (Dreyfus 1999; Jones 2000; Bergqvist 2005); while the generating and refuting conjectures can be an effective learning strategy (e.g. Lin 2005; Meissner 2003).

The examples below are situated in an on-going study with nine and 10-year-olds exploring how spreadsheets might function as pedagogical media. They illustrate aspects of how particular learning trajectories evolved, and the influence of the tension that emerged from the tool's feedback, in shaping those trajectories. The first set of data refers to an activity investigating the pattern formed by the 101 times table. It was noticeable that the children were willing to immediately enter something into a spreadsheet.

Ben How do you do times?
Awhi There is no times button. Oh no, wait, wait, wait.
Ben There is no times thing. Isn't it the star?
Awhi =A1*101. Enter.

It appears the actual spreadsheet environment provided the impetus to take this initial approach. This approach was confirmed with responses in the interview:

Awhi I preferred thinking something about what I needed to do, then take it and highlight it down and then the whole table is there, which would help me.

Not only did the use of spreadsheets lead them to explore in a particular manner, it also led to an immediate form of generalization. To generate a formula that models a situation is to generalize in its own right, but to consciously look to fill down ("highlight it down"), or create a table of values is also indicative of an implicit cognizance of a pattern; of an iterative structure that is a way into exploring the problem. Awhi and Ben continue:

Ben 202.
Awhi Now let's try this again with three. What number do you think it will equal? 302?
Ben No, 3003. Oh no 303. [The output was 303]

There was a tension between the predictions based on their fore-conceptions (302 and 3003), and the actual output. This relatively minor cognitive conflict initiated a shift in their fore-conceptions, allowing them to interpret the task from a fresh perspective. They accurately predicted the output from other single digit numbers, and were able to predict and confirm in a confident, relatively uninhibited manner. They began to pose conjectures, and test them in an informal approach:

Awhi OK. Now you try a number.
Ben My lucky number 19.

Awhi That'll be one thousand, nine hundred, and nineteen.
Ben Equals. So we need to think of a rule.
Awhi Its like double the number. Its nineteen, nineteen.
Ben What about 20? Oh you'll get 2020.

The ability to predict, form a conjecture, then test it, is indicative of a robust generalization process. In this case, and with others in the study, the children chose a particular path because they were using the spreadsheet, which determined the nature of their investigation: the visual feedback, showing a difference between their prediction and the actual output, evoked a tension influential in shaping the learning trajectory. The process shouldn't necessarily stop just there, however. An intervention, perhaps in the form of a teacher's scaffolding question, might initiate the investigation of why this visual pattern occurs.

The children were also able to quickly move beyond the constraints of the prescribed task, forming a fresh generalization.

Ben Oh try 1919.
Awhi One, nine, three, eight, one, nine.

Interestingly, they appeared to disregard this output and based their next prediction on their previous rule.

Awhi Now make it 1818 and see if its 1818.
Ben Look eighteen, 3, 6, eighteen.

This unexpected output caused them to reshape their emerging conjecture. After further exploration, they reconciled the output with their evolving theory.

Awhi What's the pattern for two digits? It puts the number down first, then doubles the number. This is four digits. It puts the number down first then doubles, and then repeats the number.

They were using a visual referent to the theory that was evolving; considering the actual visual sequence, rather than the procedure that is producing the number patterns. This indicates a form of visual reasoning. More specifically, the questions evoked, the path taken, and the informal conjectures they formed and tested, where fashioned by visual perturbations: the tension arising in their pervading discourse by the difference between the expected and actual output.

The next investigation relates to the traditional Grand Vizier problem, with the doubling of grains of rice for consecutive squares on a chessboard, and estimating how long the total amount of rice might feed the world's population. This investigation was initiated after the children had already had some experience using spreadsheets. Formation of a mental utilization schema (i.e. instrumental genesis, Artigue 2002) was evident.

Erin It goes 1, 2, 4, 8, 16 …., so its doubling
Kim =A1 times 2
Erin Is that fill down
Kim Go down to 64
Erin Right go to fill, then down

They interpreted the problem, framing their engagement with their fore-conceptions of both the mathematics and the digital technology, and used the spreadsheet to help them explore it. However, the output was in a visual form significantly different from what they were expecting, since it was in scientific notation, and they didn't recognize it.

Kim What the….
Erin Eh…
Kim What you…
Erin 9.22337 E + 18

This tension was later reconciled with teacher intervention and further exploration. A little later as they sought to estimate how long the rice might feed the world, there was no surprise with the scientific notation format:

Erin = sum (A1:A64)
Kim 1.84467E19
Erin How long will that feed?
Kim 1.84467E19 divided by 2000 [the number of grains of rice they had estimated would feed one person for a day]

The new understanding had been reconciled within their existing discourse. The visual tension when the actual output differed from the expected one was influential in the emergence of the new understanding. The learners shifted their perspectives (even if a minuscule degree) with each engagement with the pedagogical medium and activity. Each perceptual shift allowed them to re-engage from a fresh perspective, until either resolution was reached or teacher intervention was required. The learners' trajectory, in conjunction with other influences, was conditioned by the affordances and constraints of the digital, pedagogical media, and the technology gave opportunity for redirecting trajectories within rich, mathematical contexts (e.g. in this case, visual reasoning permeated their mathematical thinking). The propensities of digital media, in the educative sense, certainly promote mathematical dialogue, and re-interpretation of mathematics phenomena, which in turn fosters mathematical thinking. With both the HLT and the actual routes the learners take as they engage with the mathematical activity, the mathematical thinking can be described in two ways: a spontaneous response that is predominantly conditioned by the learner's preconceptions of the content or medium, and a reflective thinking that is shaped by broader learner discourses.

9.7 Shifts in Trajectories: Possibilities of Earlier Engagement with Powerful Ideas Afforded by DT, and the Development of Intuitions

This section is concerned with discussing the transformations or changes in didactic routes or learning trajectories, given that digital technologies may allow the possibility of early access to powerful mathematical ideas. Digital technologies have affordances for the development of learning trajectories that enrich intuitive representations, and can help in the transition to more analytical conceptions. For instance, the effective use of visualization and exploratory mathematics may give intuitions for formal proof by building up an overall picture of the relationships involved (Tall 1991). We thus begin by considering how DT can help develop intuitive thinking. We then present examples of early engagement with powerful ideas in some specific domains, not just focusing on the development of intuitions, but considering how learning trajectories can be transformed.

9.7.1 Using DT for Developing Intuitive Thinking

In this section we will reflect on how digital tools can help develop intuitions and mental representations (or *Vorstellungen*- see Meissner 2003, 2006) of objects, processes, relations and functions, and their potential relationship with individual learning trajectories. According to the *Dual Process Theory* (Kahneman and Frederick 2005; and Leron and Hazzan 2006), two ways[1] of internalizing our experiences from interacting with a problem, can be distinguished. On the one hand, we use our existing and not always conscious mental images or fore-conceptions (S1*Vorstellungen*) “spontaneously” and we do not change them if we do not see a need for it. On the other hand, if necessary, we develop conscious and reflective (S2) mental images on the base of our experiences. For a well-developed and powerful concept image both are essential: a sound and mainly intuitive “common-sense” and a conscious knowledge of rules and facts.

Most teachers or students or even researchers in mathematics often are unaware of their spontaneous and intuitive conceptions. In the traditional mathematics education classroom often we do not realize, or even ignore or suppress, intuitive or spontaneous ideas. But the teacher can adjust an intended learning trajectory to include tasks that stimulate creative, intuitive thinking, or alternatively allow space for imaginative exploration of the pedagogical medium or mathematical thinking as it emerges from engagement with the activities. As has been discussed in previous sections, the almost instantaneous nature of response (feedback) of digital technology has the potential to transform the intuitive thinking into analytical thinking while learning in mathematics.

In particular, guess-and-test behavior and guessing games can be valuable for developing intuitions. When we observe problem solving behavior from outside of mathematics education we realize that guess-and-test behavior is quite normal to build up mental representations (*Vorstellungen*) of the situation being confronted (Meissner 1982). When we observe children or adults working with a computer we also often see typical guess-and-test behavior. Students interact with the computer to discover properties and we see the repetition of similar keystroke sequences.

[1] In the *Dual Process Theory* cognition is seen as operating in two quite different modes called *System 1* and *System 2*.

Computer users tend to develop two attitudes that were unusual, until recently, in traditional mathematics education:

– They intensively use guess-and-test procedures (often unconsciously)
– They demonstrate a large and often unconscious knowledge without being able to communicate adequately about it: Being asked for rules they are quick in pressing diverse sequences of buttons, but very often they cannot give precise verbal descriptions or explanations. Their knowledge is sometimes fully situated in the digital context of the interface.

There are calculators which can work syntactically operating in the same manner as we speak in our daily life: For example, in order to input the expression "635 + 13% =..." we need to enter the following key stroke sequence:

[6], [3], [5], [+], [1], [3], [%], [=]

Meissner and his team (summary in Meissner 2003) taught percentages with the percent key, without using formulae or reverse functions or algebraic transformations of formulae. For all types of percentage problems, if necessary, a missing value had to be guessed and verified by pressing always the same key stroke sequence from above. The students became excellent in guessing each of the values needed and developed an astonishing "%-feeling" (getting a quite reasonable value already in the first guess).

The method used to teach percentages is an attempt to enrich the mathematical conceptions (*Vorstellungen*) of the users avoiding at the beginning an algebra-like symbolic language (no formulae or reverse functions or algebraic transformations of formulae). For Meissner and his team that systematic use of guess and test activities became a specific DT teaching method called One-Way-Principle or OWP (Meissner 2003). The OWP can be used to discover many functional relationships intuitively and/or consciously. The basic idea of the OWP is the following: First, learn the syntactical sequence of the buttons needed to get the output "Y" (when the input "X" and the operation "σ" are given). This is a relatively simple task when using a calculator or computer. The goal for the learner is to find a good first guess and then reach a given target with only a few more guesses. There is a big range of possibilities to apply the OWP method to develop S1 intuitive experiences. For example, σ may be a (+6)-operator or a (×5)-operator. With operators like these we develop number sense by exploring the four basic operations (including reverse operations). But "σ" also may be a symbol for trigonometric functions and "X" and "Y" are real numbers. Or "σ" is a symbol for a percentage function (see above). Or "X" symbolizes an algebraic term and "Y" the appropriate graph related to it via, for example, the *function plot software* σ. The OWP can also be a powerful method using spreadsheets, as described below in Sect. 9.7.2.

For a long time, many guessing games using DT have been developed by other researchers that are useful for developing intuitions and fostering analysis by students of a mathematical situation, such as classic "guess-my-function" or "guess my rule" activities. These have been carried out in many diverse DT environments such as Logo (Hoyles and Sutherland1992). More recently in the Weblabs project's (http://www.weblabs.eu.com) "Guess-my-robot" activity, children had to program sequences as behaviors of virtual "robots" in the visual programming environment ToonTalk and other children (in other countries) had to reproduce these robots (Mor et al. 2004). Other projects using DT environments aimed to create intuitions for difficult abstract concept such as those of limits of infinite sequences (Sacristán and Noss 2008; Kahn et al. 2005) and the cardinality of infinite sets (Kahn et al. in preparation), in order to make infinity-related ideas more accessible to younger learners via carefully designed didactic routes (learning trajectories); these are discussed further below. Before that we present an example illustrating the shifts in trajectories that can occur with DT in the specific domain of algebra.

9.7.1.1 Early Accessto Powerful Mathematical Ideas: An Example of Early Algebra

In this section we consider the issue of changes in learning trajectories and the development of "early algebra," for which very specialized work has been developed. The objective was to describe children's thinking and learning in this specific mathematical domain and develop a related route of tasks designed to promote mental processes in that domain. This sequence of tasks was intended to enhance children's developmental progression of thinking, enabling them to achieve specific goals in the mathematical domain in question.

The book of the 12th ICMI Study *The future of the teaching and learning of algebra* (Stacey et al. 2004) contains a chapter devoted entirely to the subject of early algebra. There, Lins and Kaput (2004) draw attention to the fact that introducing very young students to algebra does not mean bringing them closer to the study of more traditional forms of algebra, but rather to initiate them into a new algebraic culture. This, amongst other things, includes making drastic changes from the usual didactic routes to other routes that imply working with notations different from those of symbolic algebra. This is the case when using the *SimCalc Math Worlds* software, with which it is feasible for students with no prior background in algebraic symbolism to mathematically explore and analyze movement phenomena, with the support of a simulator and its graphic representation (Roschelle et al. 2000; see also Chap. 8 of this volume). The possibility of dynamically displaying the position, velocity and acceleration charts that correspond to one or several moving characters enables students to analyze the different relations existing between variables. Moreover, it introduces students into the notion of functional relations within a context of physical phenomena, and does so without symbolic representations.

The following are examples of two cases from a study (Perrusquía and Rojano 2006) carried out with 10 year-old pre-algebraic students who showed significant progress towards the notion of functional relationships through the use of the *SimCalc Math Worlds* (Fig. 9.11). The SimCalc activities involved students' analysis of position and speed graphics to help them gain a better understanding of the

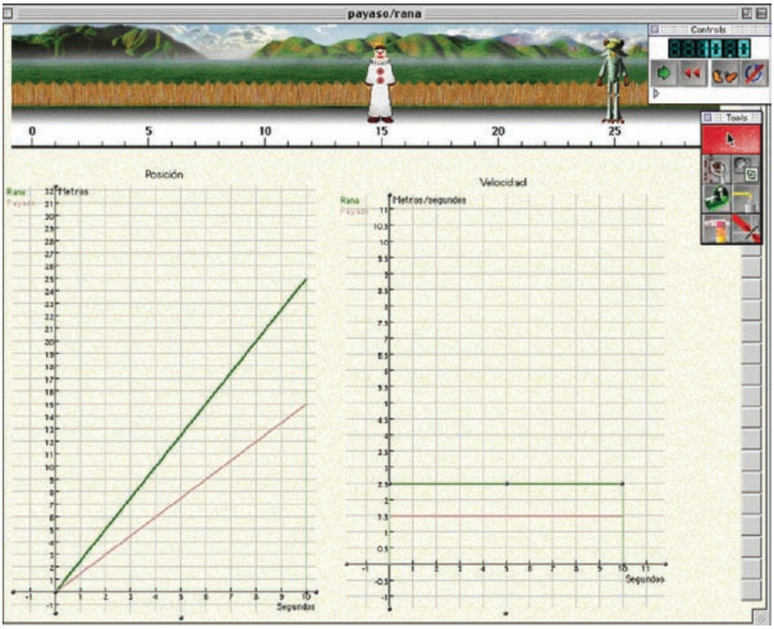

Fig. 9.11 Math worlds provides animated worlds, where animations move according to changes in graphics. Graphics are represented through rectangles meaning speed: the height of a rectangle means "how fast," and the width means "how long." Position, speed, and acceleration graphs are dynamically linked. If there is a change in speed, the corresponding changes in the position or acceleration graphs are instantly displayed

dependence relationship between two variables, as well as to include concepts such as "it goes faster" or "this is quicker" with a mathematical meaning.

The Case of Rodrigo

At the beginning of the study, Rodrigo had a low arithmetic proficiency level. His first explanations about motion took into account the physical features of characters, with definitions such as "*The clown is small, and that is why his legs go slower*" or "*The tires of the truck are bigger, and that is why it goes faster*." After completing the first SimCalc activities he included in his explanations elements related to the characters movement, such as "*the red one moves slow, and the green one moves fast*" until he finally took into consideration both variables: "*It moves one third every second*," or "*the red one goes up two floors every second*."

When first asked about his notion of speed, his answer was "*every lift goes up a number of floors per second*," making use of an example to generate an explanation. Once the activities sequence was concluded, his notion of speed evolved to include the two variables, distance and time, making it easier for him to calculate speeds from a position graph.

The Case of Ana Karen

Before the SimCalc activity sessions, Ana Karen showed to have a middle arithmetic proficiency level. At the beginning of the sessions she provided explanations such as "*The frog's steps are four meters long*," or "*The clown's steps are two meters long*." After comparing the speed of some characters she expressed the following: "*The truck goes faster, and the car slower*." After moving on through the activities, Ana managed to include a little more information: "*The slower clown moves five meters, and the faster clown moves eight meters*." In the middle of the activity sequence her definition of speed was "*the number of kilometers a car or anything else moves forward*." At this point she also perceived the possibility of using distance and time to calculate the speed from the information contained in the position graph. By the end of the sequence she employed a particular situation to explain the notion of speed, taking into consideration the time and distance variables, but on a particular example:

E: How would you explain speed?
Ak: Speed is the distance and time a car travels.
E: How do you read speed?
Ak: If we take meters and seconds, then it will be 81 meters per second.

The following questions emerge: How can those experiences be leveraged when the students have to learn proper algebraic language? Will that background conceivably make it possible to plot a route to another destination? This type of issue raises yet again previous debates regarding the importance of keeping the usual mathematical sign systems and what is known as transformational algebra, as part of the goals of compulsory teaching in mathematics. In this sense there needs to be a correspondence between the DT-based HLT and more traditional HLTs.

9.7.1.2 The Role of Spreadsheets in the Transition Towards the Algebraic Method for Solving Word Problems

Another example of a trajectory that aims to take advantage of representational systems in mathematics for developing algebraic thought at young ages, is one using spreadsheets. In this case students work with algebraic ideas such as generalization, the (functional) relation between variables and problem solving, using algebra-like notations, albeit notations that have strong numerical connotations. If the idea is to work with spreadsheets as scaffolds to build the usual algebraic knowledge, then the very limitations of the computer environment in terms of representing the variety of algebraic expressions will indicate the trajectory and moments in time when the leap to algebraic symbolization takes place. The following are examples from studies, where students engaged in spreadsheet activities designed to help them to cope with word problems involving two or more unknowns (Sutherland and Rojano1993; Rojano 2001). The first is an example from a study undertaken with 9-year old pre-algebraic pupils that illustrates how the use of a spreadsheet can assist in the analysis process of a problem's statement, such as that of the "Party Problem" below:

The Party Problem (a simple case)

420 people attended a cocktail party, the number of men was twice the number of women. How many women and how many men went to the party?

This problem can be solved using an algebraic method in the following way:

If x = no. of women, and y = no. of men
Then $y = 2x$
and $x + y = 420$

By solving this system of equations, it is found that: $x = 140$ and $y = 280$, which is the solution to the problem.

A spreadsheet method for solving this problem is as follows: Identify the unknown quantity (or quantities) as well as the problem data. Suppose that, that which is unknown, is known, and allocate an arbitrary value to one of the unknown quantities, for example the number of women (Fig. 9.12). This number is then introduced into one of the cells. In the neighboring cells, introduce the corresponding formulas for the number of men and the total amount of people who attended, as shown in the following diagram:

Note that these formulae shall include the name of the cell of one of the unknown quantities. The presupposed value is then changed until the number in the cell relating to the total number of people corresponds to the problem data (420). Figure 9.13 shows the moment in which this value is obtained and, as a result, the correct values for unknown quantities.

The spreadsheet method can help in the analysis of the problem's text by recording the steps of this analysis in a system of representations, in which natural language (column labels) is used along with numerical language and an algebra-like symbolic

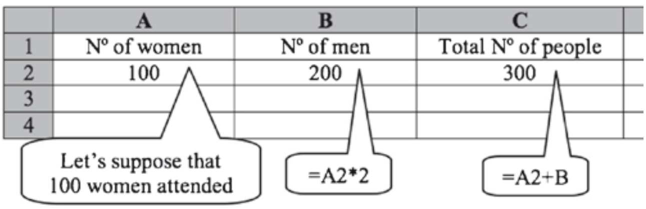

Fig. 9.12 Spreadsheet method to solve "The Party" problem

	A	B	C	D
1	N° of women	N° of men	Total N° of people	
2	140	280	420	
3				
4				

Fig. 9.13 The final correct values of the unknown quantities of "The Party" problem

language. Consequently, the analysis process, which consists of clearly stating the relationship between elements of the problem (data and unknown quantities), uses all of these languages:

- Natural language allows the presence of referents that provide the context of the problem
- Formulas allow relationships between data and unknown quantities to be expressed and, more importantly, allow functional relationships between unknown quantities to be expressed
- The supposition of a specific value for one of the unknown quantities allows the analysis and symbolization process to be undertaken, through the use of a known number instead of an unknown quantity
- The numerical variation of the assumed value for one of the unknown quantities incorporates one of the intuitive methods most frequently used by students, the method of trial and refinement (see Sect. 9.7.1)

In the case of the Party Problem, the spreadsheet method is illustrated using a simple case. However, this method can be used for the solution of problems of different levels of complexity. For example, problems can be addressed where the relationships between unknown quantities are more complex, or problems that include a larger number of unknown quantities (see Rojano 2001).

Problems similar to the Party Problem, as well as more complex problems, were used in the Anglo-Mexican Spreadsheets Algebra Project, carried out with groups of students in Mexico and England. In the framework of that project, case studies were carried out over a long period of time with children of pre-algebraic age (10–11 year olds) and with students resistant to algebra (14–15 year olds). In both studies, the children showed themselves to be capable of using a spreadsheet for solving different types of word problems, without having to wrestle with manipulative aspects of symbolic algebra and without having to reject their intuitive approaches to the solution of problems. More specifically, the results revealed that pre-algebraic children preferred to display the variation (of the unknown quantity to which an arbitrary value is allocated) in a numerical column generated by a formula such as =A2+1, whereas students between 14 and 15 years old were able to carry out this variation in a single cell.

Some algebra resistant pupils, who proved to be competent in the solving of equations, but who were unable to apply this knowledge to the solution of word problems, ended up combining their manipulative skills with the analysis of a problem statement with the help of a spreadsheet. Outcomes from this study show how a special use of a DT environment that incorporates intermediate sign systems can help students make transitions, from dealing with numerical language, to the use of algebraic code. It has been extensively reported that this particular transition is problematic when traditional algebra teaching approaches are used, but this study shows that using DT learning trajectories can be constructed that facilitate those transitions. On the other hand, differences showed by pre-algebraic and algebra-resistant pupils in their preferences of ways of representing variation in a spreadsheet suggest that actual learning trajectories can be strongly influenced by pupils' age and their previous experience with algebra.

9.7.2 Early Access to Powerful Mathematical Ideas: Exploring Infinity-Related Notions

Another area that has been researched in terms of developing DT environments and representational infrastructures for early access has been that of the mathematical infinity. Several projects have aimed to make difficult abstract concept such as those of limits of infinite sequences (Sacristán and Noss 2008) and the cardinality of infinite sets (Kahn et al. 2005, in preparation) accessible to younger learners.

9.7.2.1 A Logo Microworld for the Exploration of Infinite Processes

In the first study (Sacristán and Noss 2008), a computational microworld using the Logo programming language was designed to provide a means for students aged 14–17, to actively construct and explore different types of representations – symbolic, graphical and numerical - of infinite processes (infinite sequences and the construction of fractals, as explained below) via programming activities. In general, the computer setting provided an opportunity to analyze and discuss in conceptual (and concrete) terms the meaning of a mathematical situation. For example, drawing a geometric figure using the computer necessitated an analysis of the geometric structure under study and an analysis of the relationship between the visual and analytic representations.

The programming and explorative activities were part of a carefully designed didactic route included, and included:

- *Explorations of infinite sequences, such as* $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$, *and* $\{1/^n\}$, $\{1/n^{1.1}\}$,…, $\{1/n^2\}$, and their corresponding series, through geometric models such as spirals (Fig. 9.14), bar graphs (Fig. 9.15), staircases, and straight lines, and the corresponding Logo procedures, with a complementary analysis of the numerical values. These models constituted a straightforward way of translating arithmetic series into geometric form (e.g. in the "spiral" type of representation each term of the sequence is translated into a length, visually separated by a turn, so that the total length of the spiral corresponds to that of the sum of the terms, i.e. the corresponding series.) Through the observation of the visual (and numeric) behavior of the models, students were able to explore the convergence, and the type of convergence, or divergence, of a sequence and its corresponding series. The different geometric models for the same sequence, represented in Logo programming code, provided different perspectives of the same process.
- *Exploration of fractal figures* These included the Koch curve and snowflake (formed by putting together three Koch "segments") shown in Fig. 9.16, and the Sierpinski triangle. The explorations involved the study of their recursive structures (apparent both visually and in the programming code), and dealing with apparent paradoxes at infinity, such as a finite area bounded by an infinite perimeter.

One of the advantages of the microworld was that the *behavior* of the process could be observed, rather than the end result as is usually the case in traditional school

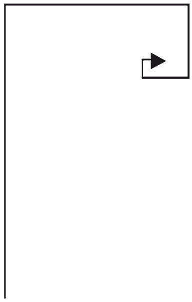

Fig. 9.14 Spiral model for the sequence $\{1/2^n\}$

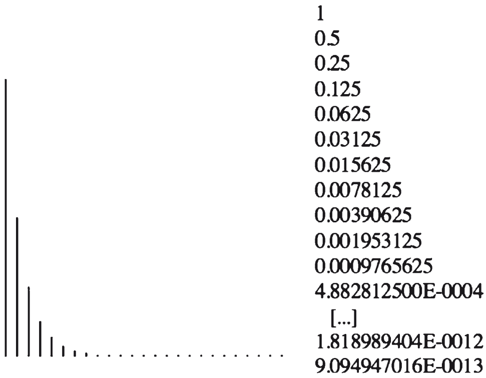

Fig. 9.15 Bar graph model for the sequence $\{1/2^n\}$ with numeric output

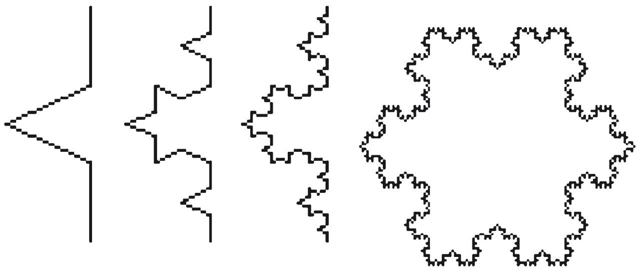

Fig. 9.16 Construction process of the Koch curve, and the Koch snowflake

mathematics. Observing the behavior, such as the rate of convergence, played a very important role for giving meaning and finding explanations as to why in a particular instance a process converged or diverged. The exploration of the behavior was done in several ways which included the observation of the process through its unfolding visual and numerical behavior, the possibility to compare different sequences and models, and in the case of series, coordinating the behavior of the series with that of the corresponding sequences.

Students discovered and explored limiting (or divergent) behaviors first through the graphical representations and then carrying out a back and forth process between these representations and numeric values. The graphical element played a role in indicating the existence of a limit when there was *visual invariance* through several stages. For example, in the fractal explorations of the Koch snowflake, the apparently invariant visual image conveyed the boundaries of the area, highlighting its independent behavior from the infinite perimeter that delineates it. At a second level, students would use numeric values, organized into tables, to complement and confirm the observed visual behavior and give an indication of the value of the limit or divergence of the sequences.

The microworld supported students in the coordination of hitherto unconnected or conflicting intuitions concerning infinity, based on a constructive articulation of different representational forms that was named *representational moderation* (Sacristán and Noss 2008). Representational moderation is sensitive to the direction of representations, or rather, to the trajectory between them. For example, in both the context of sequences and series, the approach was in the direction process to visual/numeric, which allowed a shift in intuitions: a common initial intuition is that if a process is infinite, its value will also be infinite[2]; through the microworld tasks, the intuitions were replaced by an understanding of the processes involved and the intuition could be dismissed. In the context of the fractal explorations, the approach began with the figure, and then moved to numerical analysis, which allowed some of the students to solve the apparent Koch curve "paradox" – the idea that the infinite perimeter of the curve could be formed by an infinite number of "zero-length" segments – by examining and coordinating the two processes involved: that is, by comparing the rate of increase of the segments, with the rate of decrease of the size of each segment.

The environment provided a language for asking questions, as well as tools for exploring these questions. In many cases students found what seemed like patterns and properties, which led them to formulate and test conjectures, as well as articulate relationships and build generalizations. In this way intuitions were developed before a formal proof in the way advocated by Tall (1991). Furthermore, the relationships uncovered constituted stepping-stones towards formal proofs; for this reason we called them *situated* or *pragmatic proofs*[3] (see Sacristán and Moreno 2003; Sacristán and Sánchez 2002).

9.7.2.2 Exploration of Infinite Sequences, Series and the Cardinality of Infinite Setswith ToonTalk

In the WebLabs project (see Sect. 9.3.2.6) there are further examples of the design of a DT-based representational infrastructure and learning trajectory to assist young learners in developing intuitions of the infinite that can be connected with more formal knowledge: by building on what they already know – or what they can "see" – and to engage with the computational structure in a quasi-formal way.

The attempt was to help children approach infinity-related ideas by providing them with an alternative formalism (the computational setting) with which to construct and then think and talk about these deep ideas. Children programmed infinite or non-terminating processes that produces infinite sequences and series, similar to

[2] Nuñez (1993) explains that this confusion arises when there are several competing components (processes) present; that is, when two types of iterations of perhaps different nature (cardinality vs. measure) are confused: the process itself and the divergent process of adding terms to a sequence.

[3] For example, during the explorations of sequences of the type $\{(1/k)^n\}$, some students discovered that the corresponding series: $\sum_{n=1}^{\infty} 1/k^n$, where the integer $k > 1$, converge to $1/(k - 1)$. They then tested the validity of their conjecture using all the available tools in the microworld, in order to "prove" it.

those in the previous example; in a second part of the project they also constructed infinite sequences such as the natural numbers, the even numbers, the integers, and the rational numbers, while simultaneously constructing one-to-one correspondences between the sets, in order to investigate their cardinality. As in the project described in Sect. 9.7.3.1, computer-programming activities had a central role. But in this case the ToonTalk environment was used. Here, also, the use of diverse tools and representations supported the dual view of a sequence as a process and an object. This was complemented with the interleaving of construction and argumentation activity - students built and discussed what they had built in a web-based system that was shared amongst students in several European countries.

In ToonTalk, programmers enter an animated world and build programs by training robots to manipulate boxes, perform arithmetic calculations, give birds messages to deliver, and more (see Fig. 9.17). They train these robots using specific example data and then recover generality by removing details (Kahn 2001). Two key characteristics of ToonTalk are that every computational process and its data are tangible and manipulable. In addition, the system employs exact arithmetic instead of floating point numbers (used by most other computer environments); this means that numerical investigations, such as those of infinite number sequences and series, can be as precise as desired.

The HLT that was designed aimed to encourage students to:

- Experience surprises arising out of the tension between intuitions of infinity and evidence revealed through activity
- Develop a non-algebraic language for describing, discussing and reasoning about infinity-related ideas: (a) infinite sequences, and in particular, the ideas of convergence, divergence and limits; (b) the cardinality of infinite sets, which was investigated by constructing one-to-one correspondences between sets

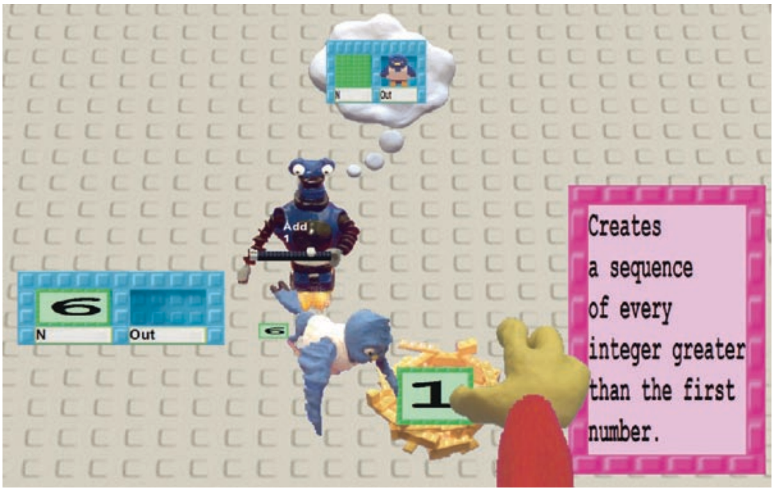

Fig. 9.17 The ToonTalk "Add 1" robot producing the natural numbers

These activities were tested in several groups, mainly in the UK and Bulgaria, of children aged 11–14. The results of this project illustrated how the curious child can learn some deep, interesting, and different mathematics without first having mastered the techniques that are normally only accessible to a few.

9.7.3 Early Access to Powerful Mathematical Ideas: Long-Term Impact

Although a good number of studies have shown that computer environments can play a role so that students, of very young ages, may work with sophisticated mathematical ideas such as the mathematics of variation, modeling and generalization (Rojano and Sutherland 2001; Kaput and Blanton 2001; Hoyles and Sutherland 1992) or infinity and infinite processes (Sacristán and Noss 2008; Kahn et al. 2005, in preparation) as described in previous sections, there has been, nonetheless, a high degree of uncertainty about the real long-term impact that early experience with such mathematical notions can have on students (Rojano 2008). This created the need to undertake longitudinal studies able to encompass several school grades with the same generation of students. In these cases, we take the perspective of hypothetical learning trajectory as applied to long-term learning processes. This not only represents a methodological challenge for research in mathematics education, but also means that researchers are faced with the arduous task of designing and developing a specific curriculum (well defined intended learning routes) that abides by the didactic approach and the technological tool(s) chosen.

It is noteworthy to refer to three studies bearing those traits: studies that were carried out in the last decade. One is the *Computer Intensive Algebra* project[4] developed by Heid (1996) at Penn State University; another is the *Visual Math* project led by Yerushalmy (2000)[5]; and a third one is the *Measure up Project*[6] spearheaded by Dougherty in Hawaii (Dougherty and Zilliox 2003).

[4] *Computer Intensive Algebra* is a beginning algebra curriculum that introduces students to algebra in the context of mathematical modeling computer explorations, that provide access to multiple representations and assist in reasoning about algebraic expressions (Heid 1996).

[5] In this study, a complete learning sequence (the *Visual Math* curriculum) is prepared in order to observe learning processes throughout a longitudinal period of 3 years (grades 7–9) using an alternate approach (a functional approach) to algebra teaching. One of the findings was that when using the alternate treatment, changes expected – for example in conception of functional variation and the rate of change – took a fair amount of time (Yerushalmy2000).

[6] In this project, algebra is introduced to pupils at the beginning of primary school. Its approach is based on a Russian framework created by the melding of multiple theories (e.g. theories by psychologists like Davidov and Vygotsky). Pupils begin with generalizations rather than with specific instances, so that they can see the concepts in action rather than trying to build the bigger picture from a variety of specific examples. Symbolism is naturally integrated to children's tasks as well as the notion of relationships between and among quantities (Dougherty 2001).

Strict longitudinal studies, such as these, enable researchers to follow the tracks and repercussions of knowledge acquired under a certain focus and within specific learning environments. In addition, they are also an open invitation to reflect upon possible learning trajectories that have those foci as a point of departure.

9.8 Concluding Remarks

In this chapter, we have identified and discussed themes that are relevant during the construction of both hypothetical and actual learning trajectories. We argued that the use of digital technologies to construct hypothetical learning trajectories offers the teachers the opportunity to examine or explore ways in which mathematical concepts and problem solving strategies can be developed. These trajectories become relevant to organize and structure potential paths or routes that can guide their students' actual development of mathematical concepts and problem solving approaches.

The different examples in this chapter also illustrate how the use of DT opens the possibility of conceiving new avenues for the learning of specific mathematical contents. Moreover, certain uses of DT give pupils access to advanced mathematical ideas, which are not currently considered in traditional curricula at the elementary and secondary school education level. These possibilities rely mainly on the potential of the DT environments to facilitate learners in making crucial transitions towards a mathematical way of thinking. In this way, DT can significantly alter how didactic and learning trajectories have been traditionally conceived.

Digital technologies seem to facilitate transitional processes that have previously been reported as being highly complex for the vast majority of students, such as transitions from the particular to the general; from what is concrete to what is abstract; from intuitive perception to formal thinking, from non-mathematical to mathematical representations (e.g. algebraic symbolism and graphics); etc.

We also recognize that there may be diverse ways or paths for students to construct or develop mathematical thinking and problem solving competencies. We contend that students' use of different digital – and representational – means or technologies, offers opportunities to represent and explore mathematical situations in terms of, or in accordance, to the facilities or potential associated with each tool. Thus, when students use more than one tool in their mathematical experiences they have an opportunity not only to think of mathematical situations or problems within multiple mathematical environments but also to use diverse resources and problem solving strategies.

How digital technologies shape learning and transform learning trajectories has a profound implication for teaching and for curriculum design. For instance, the examples in this chapter illustrate that hypothetical and students' actual learning trajectories when using digital media may involve contents and resources of various areas of mathematical domains. In this sense, the curriculum needs to be organized or structured in a way that facilitates the articulation of several areas or mathematical domains.

However, research into DT-based learning trajectories is still in its infancy. From a theoretical perspective, the idea of learning trajectories in the context of digital technologies still needs to be developed. From an empirical point of view there are many aspects that need to be researched. For example, one question that arises is what is the influence that can be had when hypothetical learning trajectories are made explicit; clearly when using DT actual trajectories can go in very unexpected ways, but making a HLT explicit for the teacher may have a significant effect that needs to be researched.

Finally, the ever more pervading presence of the Internet and of networking capabilities opens up many new scenarios of dynamics for classroom and learning environments (possibly beyond the school) that may lead to new forms of learning and shape learning trajectories in unforeseen ways. These of course, will also need to be researched.

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Chapter 10
Micro-level Automatic Assessment Supported by Digital Technologies

Chris Sangwin, Claire Cazes, Arthur Lee and Ka Lok Wong

Abstract This paper describes computer aided assessment of mathematics by focusing on the micro-level of automatically assessing students' answers. This is the moment at which a judgment takes place and so it forms the keystone the mathematical assessment process, so fundamental to the learning cycle. We describe the principle of automatic assessment at this micro-level and report some of the significant technical developments of the last two decades through examples of internet based systems.

Keywords Assessment • Computer aided assessment • Task design • Technology

10.1 Introduction

This paper describes contemporary computer aided assessment (CAA) of mathematics. In particular, we focus on the use of ICT for assessment activities and the micro-level of automatically assessing students' answers to individual questions. This is the moment at which a judgement takes place and so this micro-level forms the keystone of the majority of mathematical assessments. It is striking that in the first ICMI study (Churchhouse et al. 1986), no examples of such assessment systems at the tertiary level were described or exhibited. This situation has changed significantly in the last two decades.

Assessment is a fundamental part of the learning cycle, is central to learning and is also often a primary driver of students' activity.

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> In the early 1970 researchers on both sides of the Atlantic (Snyder 1971; Miller and Parlette, 1974) were engaged in studies of student learning at prestigious universities. What they found was that unexpectedly, what influenced students most was not the teaching but the assessment (Gibbs and Simpson, p. 4).

The outcome of assessments is feedback of various kinds and an item of assessment has a number of potential purposes. Assessment becomes formative when the information is used to adapt teaching and learning to meet students' needs. See Black and Wiliam (1998a, b) for a review of formative assessment. Sometimes termed "assessment for learning", feedback here could be qualitative, e.g. written comments tailored to the student's answer, or brief indications of where students' written work departs from model solutions. Summative assessment, or "assessment of learning", is to establish the achievement of the student. In mathematics, summative feedback is most often quantitative, either a mark or a percentage. Note that the nouns "mark", "grade" or "score" are used synonymously in the literature. In addition to these purposes, Wiliam and Black (1996), for example, also describe evaluative assessment that is to measure the effectiveness of the teaching or the assessment of students. Such assessments could have quality enhancement or quality audit functions.

It should be noted that rarely are the outcomes to a single assessment item considered in isolation. It is more common to aggregate data: for formative assessment a "profile" of the student's overall achievement is built. Indeed, a particular collection of mathematical assessment items might be constructed as a *diagnostic* instrument to provide students with a "skills audit" and the feedback consists of suggestions for further work. Similarly, summative assessment reduces the outcomes of a number of separate smaller items to a single numerical mark: either a percentage or grade. Computer aided assessment has a role in automating such aggregations and in the compilation of statistics for formative, summative and evaluative purposes. The focus of this paper is on the micro-level automated assessment of individual assessment items, not on theoretical perspectives on task *sequence* design – an important but separate topic.

For pragmatic reasons, an individual item of assessment may be used for a number of different purposes. For example, an "exercise sheet" of individual items may have a primarily formative function, with written feedback and a single numerical mark that is itself a crude formative measure. This mark could also contribute for summative purposes as "continuous assessment". Qualitative comments could be aggregated as an evaluative assessment to inform subsequent teaching (quality enhancement). The marks might also contribute as an evaluative assessment for quality audit. Strong messages are communicated to students by the choices made for assessment particularly when this is both formative and summative. A "reward for sustained achievement" needs to be balanced against an "opportunity to learn from mistakes". It is the use to which the outcomes of an assessment are put which has a greater bearing on the purpose of the assessment than the form of these outcomes or the nature of the task. Hence an online CAA system could be either a formative learning tool or a summative assessment system. It could be used to automate high-stakes public examinations. Furthermore, the ability to automatically generate data about an individual student or across a cohort is potentially very easy with CAA, allowing regular, detailed and accurate evaluative assessment.

The use to which a calculator, a computer algebra system (CAS) or dynamic geometry (DG) software has been put in the learning and teaching of mathematics is "*almost* *exclusively […] to model in an exploratory manner rather advanced mathematical ideas*", (IPC 2005, p. 3). Here we consider using these technologies to perform the assessment and generate outcomes, such as (a) a numerical mark, (b) written feedback or (c) statistics for the teacher concerning cohort achievement. These three outcomes approximately correspond with the summative, formative and evaluative purposes of assessment. An individual item may be designed and used with any of these purposes, and we shall consider all three. While the student is certainly using the software, the mode of use is, compared to the traditional use of an instrument, quite different both in terms of the specificity of the task and the richness of the feedback generated as a result. Hence, our focus here is on using technology to assess students' work, i.e. we include marking/grading answers to mathematical questions.

Assessment is a very broad field, and in this paper we concentrate only on one aspect. We acknowledge that we leave many issues unaddressed. For example, if a teacher encourages students to make extensive use of tools in a course but does not allow their use on the end-of-course test, are students being given the opportunity to show what they learned with the use of such tools? If the tools are to be used on the test, what kinds of test items can the teacher design to bring out the mathematical learning that may or may not have occurred? We make no assumptions about the use of technology by students in this paper. While this may seem strange, it is outside the scope of this paper. Our focus is exclusively on the micro-level of using digital technology to support the automatic assessment of student's work.

In Sect. 10.2 we shall describe the principle of automatic assessment at the micro-level. In Sect. 10.3 we briefly describe current practice through examples of internet based systems used for assessment of mathematics, including geometry, algebra and calculus. Section 10.4 discusses the potential future of such tools: at a technical level what can be assessed, what cannot?

10.2 Principle of CAA

To us the principle of mathematical computer aided assessment is the following: a student creates mathematical objects (e.g. an algebraic expression or geometric figure) using a computer; then the computer automatically establishes mathematical properties of these objects; on the basis of these properties it assigns outcomes, including feedback. After discussing what is suitable and/or possible to assess, we will examine the structure of mathematical objects created and then the variety of outcomes generated.

10.2.1 How to Implement an Assessment?

At the very fine-grained level we address here, automatic mathematical assessment seeks to establish various mathematical properties of a student's work on a specific mathematical question. For example, the teacher might ask "is the final answer algebraically equivalent to my answer?", "has an appropriate method been selected and correctly used?", "are the construction steps correct?", or "is this expression fully simplified"? We see first how to implement an assessment of the correctness of an answer both in an algebraic and a geometric situation.

As an illustration of the kind of CAA scenario which is the focus of this paper, consider the situation in which a student enters his or her response to a mathematical question, assumed to be an algebraic expression, into a computer aided assessment system. A CAS is then used to subtract the student's response from the teacher's response and to simplify the resulting expression algebraically. If the result is zero an algebraic equivalence between the student's answer and the teacher's answer has been established. That two expressions are equivalent in this way is a ubiquitous test for "correctness" and the ability to perform this test is often an important component of a mathematical CAA system. However as we shall see, there are many other properties that a teacher may seek to establish.

In the case of geometry, we highlight the difference between assessing declarative knowledge through written answers to traditional questions and assessing other forms of knowledge through manipulation, construction and experimentation. For example, a traditional item may require students to find unknown quantities or write a proof to indicate their understanding of geometrical theorems. When technology is involved, the task involves manipulating and constructing geometrical objects or formulating and testing conjectures with the given tools. Such a dynamic geometry (DG) environment allows students to demonstrate their geometrical knowledge in other modes. As an example, students might manipulate a DG worksheet to create a particular configuration. They submit electronically the final configuration of their dynamic figures in the browser when a task is finished. The final states of key variables are recorded, and from this the properties of the students' geometric figure established, with outcomes then automatically generated. Examples will be given of this in due course.

However, note the important pedagogic principle being implemented in both examples: the student interacts with a CAA system to create a mathematical object, either algebraic, geometric e.g. Cabri Geometry (http://www.cabri.net), or both, e.g. in the case of a GeoGebra worksheet: (http://www.geogebra.org). The student may use technology or traditional paper and pencil approaches for intermediate working depending on the circumstances. The CAA system then automatically assesses the student's answer that contains mathematical content, rather than a selection from a list of teacher provided answers, such as in multiple choice or multiple response questions. In evaluating the student's answer mathematical properties are established automatically, and based on these properties feedback can then be automatically generated to fulfil the purpose/purposes of the assessments, e.g. for summative assessment a numerical mark, for formative assessment textual feedback and statistics for evaluative assessment.

However CAAs' designers aspire to go further than to assess the correctness of an answer. In both the algebraic and geometric examples the central issue of assessment remains: what kinds of knowledge and understanding are we testing? Further, what action should we programme the system to take when we have enabled it to establish something relevant about the student's answer? The process with which a human teacher engages when assessing work at even this micro-level is both complex and subtle. Both for formative and summative purposes it involves them making many judgements rapidly.

In an algebra question, for example, in addition to the prototype of establishing algebraic equivalence this might include whether an expression is factored, "simplified" or perhaps a solution to a given equation. As a further illustration, we consider whether a student has found the general solution to a differential equation such as

$$y''(t) - 9y'(t) + 18y(\mathrm{t}) = 0. \tag{10.1}$$

Expressed as a CAS algorithm, we first substitute the response of the student into the left hand side of the equation and simplify, which includes performing the differentiation of the student's expression where necessary. If the result of this calculation is zero then the student's answer satisfies the differential equation. Other tests can be devised to ensure the expression is a non-trivial (i.e. $y(t) = 0$) general solution. In particular, that the answer consists of the superposition of two linearly independent solutions, and the presence of general constants can be established. However, the choice of which letters are used to express the general constants can be at the discretion of the student. The CAA system does *not* use a CAS to simply establish the algebraic equivalence of the student's answer with an expression such as

$$A\mathrm{e}^{3\mathrm{t}} + B\mathrm{e}^{6\mathrm{t}}.$$

While solving (10.1) is a relatively standard problem, the use of a CAS allows the teacher to set and assess questions that would require significant computation to establish the required properties, or have non-unique solutions. We examine such questions in more detail below.

The problem of recognizing that an expression entered by a student is factorized (over some field), is significantly more subtle than comparing the student's expression with the result of applying the CAS's "factor" command to the teacher's answer. For example, a CAA system may have to respond to any of the following expressions

$$(x-3)^2,\ (3-x)^2,\ (x-3)(x-3),\ (3-x)(3-x),\ 9(1-x/3)^2.$$

Only the first of these is returned by the "`factor`" command, while the others could all be argued to be correct factored forms even if they are not all fully simplified. Similar problems occur with other syntactic forms, such as partial fractions. To be useful as part of a CAA system, functions which establish such properties are needed. These functions ideally need to be able to generate feedback, for example "*a common factor can be taken out of the term … in the left hand side of your equation*". Whether or not the teacher opts to use such feedback depends on the circumstances, but it should be available.

The reader might consider all the different senses in which the word "simplify" is used in an average textbook on elementary algebra. Often "simplify" seems to be a synonym for "*do what I have just shown you*". Two different examples occur with what (Nicaud et al. 2004) terms *sorted* and *reduced* form, when a polynomial is represented as $x^2 + 2x + 1$ rather than $x + 1 + x + x^2$. If a CAA system is to provide useful feedback to students it must be capable of distinguishing between expressions that are not fully simplified in various senses, and respond. However, such functions are usually not present in a mainstream CAS designed for computation and subsequent *automatic simplification* to canonical forms.

In the educational context, the work of (Gray and Tall 1994) developed the notion of a *procept* to capture the duality between process and concept in mathematics. For example, basic arithmetic operations make use of the same symbolism to represent the product of the process: one half as 1/2, and as the process itself: divide one into two equal parts. They comment on the ambiguities in using the same symbol for both as follows.

> By using the notation ambiguously to represent either process or product, whichever is convenient at the time, the mathematician manages to encompass both – neatly side-stepping a possible object/process dichotomy. (Gray and Tall 1994, p. 120)

Unfortunately, while a mathematical expert (e.g. a teacher) might well use "*the notation ambiguously to represent either process or product, whichever is convenient at the time*", the teacher making use of CAA must be more explicit. Hence it is helpful to us if we think of the process as a *verb* and the concept as a *noun*. As a consequence, tests other than algebraic equivalence are needed which rely on the ability to switch off the automatic simplification of the CAS itself, something which is not possible with all mainstream CAS. For example, a very useful test is equivalence up to associativity and commutativity of elementary algebraic operations. In particular we do not wish to consider the addition symbol + as something to do, i.e. a process or verb, but rather as representing the concept, i.e. a noun. For example, $1 + 2x$ should be the "same" as $2x + 1$, but not the same as, e.g. $x + 1 + x$. Even if both are acceptable in a particular situation, a distinction should be drawn between them.

10.2.2 Structure in the Tasks or Students Generated Mathematical Objects

We discuss some principle features of using CAA in the following sections based on capabilities of CAA in creating or handling mathematical objects with respect to their properties. A set of questions randomly generated by the computer according to some parameters bears the structure of the mathematical content. Reciprocally, examining answers to open questions implies recognition of a structure in the space of possible answers. In either case, automating the generation of questions or evaluation of answers may allow the shift of attention from individual items (questions or answers) to the entire set with a structure. It therefore should have implications on what is being assessed. The formulation of feedback is also facilitated by the same capabilities of CAA.

10.2.2.1 Handling Algebraic Expressions

Many CAA systems make use of a CAS. The first system to make CAS a central feature was the AiM system (Klai et al. 2000), which uses Maple, as do a number of other systems including Maple's own proprietary MapleTA. Other systems have access to a different CAS, such as Mathematica or Derive. The STACK system (Sangwin and Grove 2006) uses the CAS Maxima. A common feature of these systems is their use of an *existing* CAS. There are significant and perhaps rather surprising differences between CAS implementations, although all mainstream CAS are designed for the research mathematician or student essentially using CAS as a "super calculator". As we have seen, the functionality required for the application of CAA is quite different and designers of CAA supplement the standard libraries with the appropriate extra functionality. It should be noted that a mainstream CAS is not required for CAA of algebraic or calculus questions. There are very many examples of highly mathematical CAA and computer based learning systems in which the authors replicate libraries of CAS-like functions, which represent and manipulate mathematical expressions. Hence, while they do not make use of a recognized mainstream CAS we would argue that they are in fact implementing computer algebra in its broadest sense.

Mathematically rich CAS functions are ideally suited to generating random versions of a particular problem within carefully structured *question spaces*. Worked solutions, with various steps, can similarly be constructed from templates. Such problems can be used for repeated practice or to reduce plagiarism and impersonation. Indeed, in the authors' experience, so far CAS supported CAA has predominantly been used to provide traditional practice of routine techniques. Since many of the systems cited above originated in higher education they have also seen application to questions from linear algebra, vector calculus and differential equations.

It might be argued that since the CAS can perform simplifications and other calculations, the students should not be required to do so fluently themselves. Even if fluency in the actual calculations is not a high priority, basic competence will always be necessary and so some practice and routine manipulations will remain a valid application. However, by harnessing CAA within a learning cycle group work could be encouraged to aid understanding of a topic, with each student evidencing their own learning by completing their unique set of tasks. Often no one cares about the actual answer itself, and the numbers used in a typical mathematics question are themselves unimportant. The ability to randomly generate questions within constrained variation may be used to help students perceive the structure of the problem underlying that which their version represents. This embeds the experience of the student. The quotes that follow are taken from students' course evaluation questionnaires conducted as part of routine quality audit of teaching.

> The questions are of the same style and want the same things but they are subtly different which means you can talk to a friend about a certain question but they cannot do it for you. You have to work it all out for yourself, which is good.

Given that CAS enabled CAA establishes properties, rather than simply checking for "the correct answer", more questions with many correct/acceptable answers can be set and assessed. As an example, consider the following question. "*Give an example of a function with a stationary point at $x = 1$*". To assess this, the CAS differentiates the student's answer with respect to x, substitutes $x = 1$, and simplifies. Hence, there is an infinite family of correct responses and as one student commented:

> Recognising […] the functions produced in question 2 was impressive, as there are a lot of functions […] and it would be difficult to simply input all possibilities to be recognized as answers.

More than one property can be requested, such as the following.

> Give an example of a function with a stationary point at $x = 2$ and which is continuous but not differentiable at $x = 0$.

The CAS functions are used to establish whether the student's answer (e.g. $x(x - 4)$) has each of the required properties. The screenshot shown in Fig. 10.1 is taken from the online system STACK, (http://www.stack.bham.ac.uk/).

In questions such as this the student must decide what properties are required, and then construct a mathematical object, such as a function, which satisfies them. The cognitive processes required are quite different from following or repeating a routine procedure given by the teacher. Of course, questions such as that shown in Fig. 10.1 could become routine: the context in which a question is set is crucial. However, the pedagogic potential for this style of question is well documented in the educational literature, for example Watson and Mason (2002) or Michener (1978). The work of Dahlberg and Housman (1997), suggests that it "*might be beneficial to introduce students to new concepts by having them generate their own examples or having them decide whether teacher-provided candidates are examples or non-examples, before providing students examples and explanations*" (p. 297).

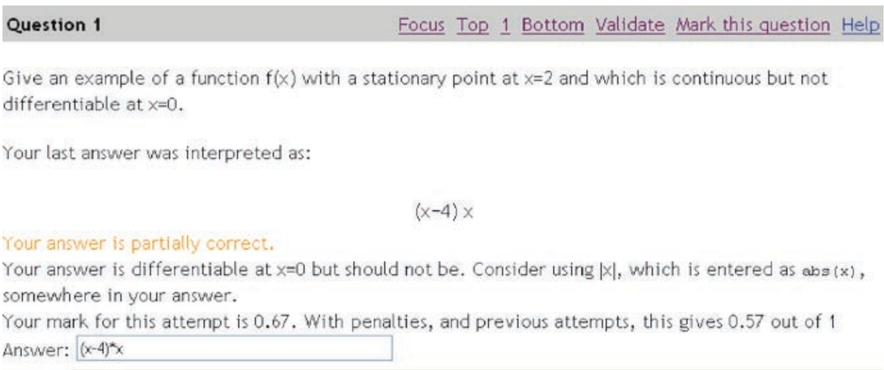

Fig. 10.1 An example of an open task

Such questions are usually absent from contemporary teaching, probably because of the practical constraints of time under which teachers operate. Using CAS-enabled CAA to assess such questions is considered in, for example, Sangwin (2005). It is readily acknowledged that the CAA described in this example only considers the final product and does not consider the solution process. Similarly, the pedagogic potential is related to generating and discussing examples and this requires integration within classroom practice.

10.2.2.2 Handling Geometric Figures

In a dynamic geometry environment, students can work directly on geometric figures. Some DG software (e.g. Cinderella and C.a.R.) allows teachers to set up assignments or exercises which can automatically check students' constructions against specific requirements. A student can create a figure using whatever tools are provided and the geometric properties of the final figure are automatically examined. Once again, the focus will be put on the specified properties resulting from a range of possible constructions.

Even without sophisticated constructions, a students' simple action of manipulating a dynamic figure can already be a meaningful mode to demonstrate their understanding of geometric concepts. This can be illustrated in a web based testing and learning platform, Geometry in Clicks and Drags (http://geometry.eclass.hk). It consists of simple tasks in which students are required to manipulate dynamic geometric figures in order to show their understanding of certain geometric properties and relationships. Students taking the tests are not required to have any experience in using DG software for constructions. By dragging movable parts in a dynamic figure, students can modify certain properties of the figure and submit the figure resulted from their manipulations.

Figure 10. 2 shows screen shots of two such tasks. Task 1 requires students to rotate a movable point P through a right angle anticlockwise about the origin in a

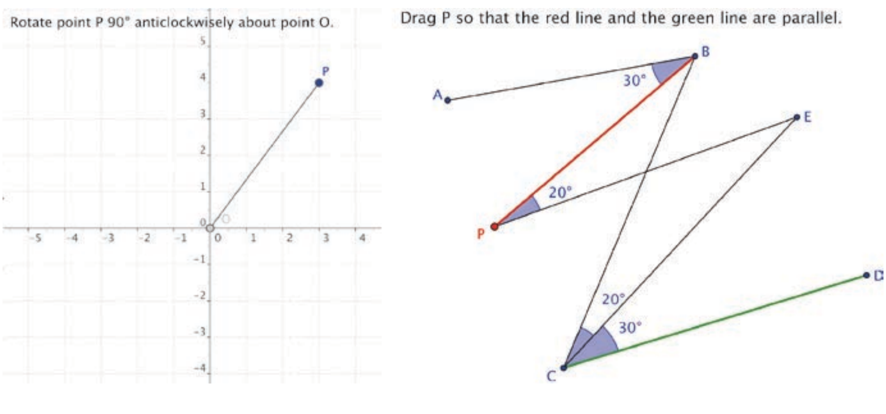

Fig. 10.2 Tasks 1 & 2

coordinate plane. In task 2, students are required to drag the only movable point P to make a segment PE parallel to another segment CD. Some angle measurements are shown on the screen and will be continuously updated as P varies. In both cases, the point P can be dragged freely on the entire screen.

Note that in task 2, an answer cannot be obtained *without* any dragging to explore the figure. This is contrary to some students' expectations in which an answer is obtained by calculation on paper and the dragging is just the final step to "input" the answer. In other words, for these students, the figure provided on the screen is not different from one shown on paper. One possible way to get an answer is shown in Fig. 10.3. It is much easier to make PE parallel to CD by putting P on the segment BC.

While performing this kind of manipulation, students are essentially providing a specific configuration out of a more general one to create certain properties, based on whatever they are allowed to vary. In other words, among the given constraints, there is freedom to vary certain geometric properties. In the case of task 1, for example, the direction and distance of P from O are the critical properties to be considered and could freely vary according to students' manipulations. While the CAA system can check the correctness of answers, it can also distinguish in what aspects and to what extent a wrong answer deviates from the correct one.

In task 2, in terms of the position of P, the answer is open, although in terms of inclination of PE, the answer is unique. In checking answers for this task, it is the angles involved, instead of the coordinates of P, which are being collected and processed by the CAA to determine whether PE is parallel to CD. The flexibility of DG behind the CAA in taking certain geometric properties in determining correctness of answers is more powerful than generic programmes in comparing students' answers against teachers' provided ones. Moreover, such power provides the basis for linking assessment to pedagogy.

For example, students' submissions to task 2 are summarized in a table shown in Fig. 10.4, in terms of a numerical value representing the inclination of the variable

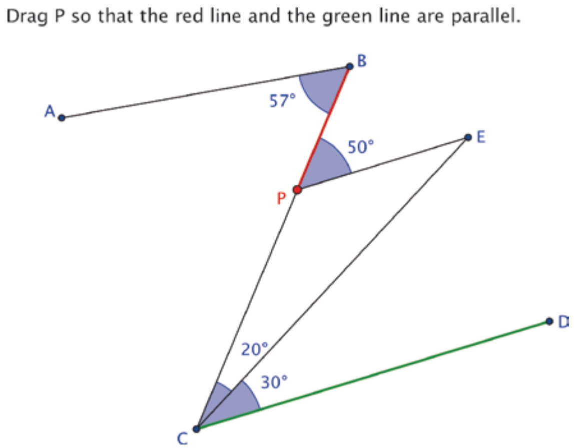

Fig. 10.3 One solution to Task 2

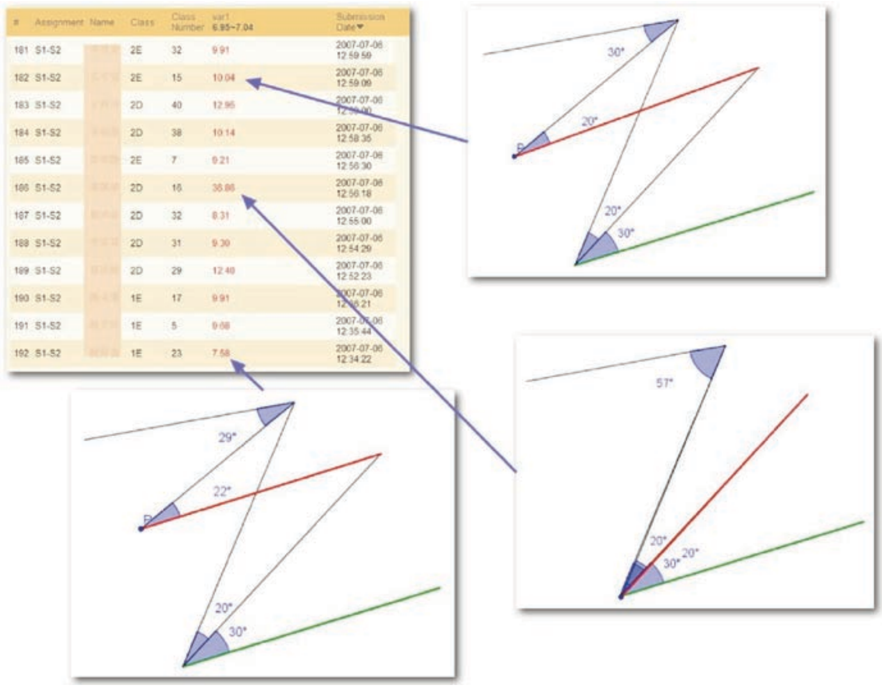

Fig. 10.4 Results of Task 2

line in that figure. Meanwhile, clicking at individual submissions in the table will retrieve that student's submitted figure in a pop-up window. Quickly browsing the answers in this way may help teachers to identify various approaches to this task.

In this CAA system, there are two distinct modes of assessments: evaluation and exploration. In the mode of evaluation, only one answer can be submitted by each student for each item; while in the mode of exploration, the number of submissions for each item is not limited. For example, task 2 can be set up in the exploration mode so that multiple submissions from the same student can be accepted. Note that the task is the same but the way the system stores and handles the answers will be altered. In doing so, students are encouraged to shift from considering individual answers to properties shared among a set of possible answers. This is in line with pedagogical strategies thoroughly explained and developed in Watson and Mason (2005), where the notion of moving from individual student's generated examples (with respect to a learning task) to a "space of examples" plays a crucial role. For this kind of simple task, with the outcomes easily captured in terms of some critical parameters (such as coordinates, lengths or angle sizes), the CAA system can support exploration and analysis of relations among answers, which becomes more than merely a collection of answers. In Sect. 10.3 there are further examples showing how these collections of students' answers can be represented and explored.

Randomly generated assessment items based on a set of parameters can also help to shift the attention from individual items to a class of items. Another web based CAA system, WIMS (Xiao 2000), described in the following section provides geometry exercises of this type. Examples given in this section can also be modified in this way. For example, in task 1, the given starting position of P can be randomly generated for each instance while keeping other aspects of the task unchanged. When gathering and analyzing the students' results, a new dimension that can be explored is how the difficulty of the task may vary according to the initial position of the point to be rotated.

10.2.3 Generation of Feedback for Students and for Teachers

One advantage of CAA is the ability to provide feedback to the users: students and teachers. In case of summative assessment the feedback is a mark. It is this automation that permits a saving time for teachers who are more available for other tasks such as helping students rather than assessing them. Nevertheless, assessment allows information to be collected for students and teachers; we examine this type of information.

10.2.3.1 Qualitative Feedback for Formative Assessment

Producing relevant and helpful qualitative feedback is one of the major challenges of automatic formative assessment. Such feedback is seen to be a major benefit of CAA, since it is provided almost immediately, i.e. it is synchronous with the student's work. If teachers wanted to provide such feedback it would be very time consuming and most probably would be asynchronous.

In CAA supported with either DG or a CAS the computer processes students' answers to establish geometric/algebraic properties. Here DG tests geometric properties based on certain geometric relationships or measurements inside the figures. In both cases such properties might include establishing "correctness", awarding partial credit if only a subset of the desired properties are satisfied, or examining whether the answer is the result of a common technical slip or known misconception. If the teacher knows, in advance, that certain misconceptions are common then the system can check if the answer appears to result from one of these. In this way, students' answers are not just compared with teachers' specified answers, but can be more meaningfully examined by the computer to find out what conditions are satisfied. That is a way to cope with partially correct answer. Let us imagine that a student is asked to attempt a classic integration problem. We shall assume for illustrative purposes that feedback is only to be generated at the end of their working process. If this student's final answer is incorrect then feedback of the following type might be given.

> The derivative of your answer should be equal to the function that you were asked to integrate. However, the derivative of your answer with respect to x is …, so you must have done something wrong!

Here, the … is automatically replaced by the derivative of the student's answer, as calculated by the CAS. Such feedback is designed to encourage the students to check the result for themselves by differentiating. Feedback must be given in the context of the particular version of the random question the student attempted. It is an emblematic example of formative feedback because it induces the student to a metacognitive behaviour: developing control strategies. Many classes' observations show that in classical exercises sessions, frequently teachers provide metacognitive information, and it is an important challenge to implement such appropriate hints in a CAA.

10.2.3.2 Cohort Achievement Data

In CAA it is usual for the system to automatically log all attempts and the associated outcomes. This generates a large dataset, both qualitative and quantitative. The advantages of being able to interrogate this data include the ability to easily see the numerical mark of each student on each exercise. In addition they may, depending on the system, know the length of time each student spent on each exercise and how many times it was attempted. In the case of randomly generated exercises they may search the results but group responses, not by student, but by information relevant to the random version. In this way, measures of question validity, a discrimination index and so on may be calculated automatically. For example, the system WIMS (Vandebrouck and Cazes 2005) builds two indices for a cohort of tertiary level students. The first concerns the *efficiency*. This is defined as the quotient of the total length of time spent on an exercise and the number of points achieved. The second index concerns the *difficulty*. This is the quotient of the number of times the exercise appears and the number of times students submit a result. Indeed, the more difficult the exercise the fewer students submit an answer. These indices allow the classification of exercises into three classes which approximate well those which teachers themselves anticipated in their a priori analysis. Yet, teachers seem to minimize the difficulties of each class, for instance, it appears that tasks of a low difficulty level, such as routine tasks or immediate applications, are not so immediate. In fact students need quite a long time to accomplish them. This phenomenon is emphasized because these tasks, due to their low difficulty level, are not often worked out in detail as examples by teachers.

When the CAA system records students' incorrect answers it provides quantitative data linked to one specific exercise. It is then informative for teachers to analyse the frequent errors and a geometry example will be given below. It may also be possible for teachers to provide asynchronous feedback or even to anticipate them by explaining the origin of the difficulty. Due to the exercise's format teachers see some errors that are invisible in traditional teaching. For example, Abboud-Blanchard et al, (2007), to solve an equation students have to choose between "multiply by…" or "add…". The equation was $3x = 14$. To some grade 9 pupils it was not obvious that divide by 3 was the same as multiply by 1/3 and hence it was not obvious to them that the multiplication tool was appropriate. In a subsequent interview, the teacher said she had not even anticipated this difficulty. She adds that in the classical lesson if a pupil said "I divide by 3" she only said "Ok, very good" and never asked, "so you multiply by what?" She said that she will repeat this lesson next year, but she will emphasize this point at the beginning of the digital sheet.

The use of cohort achievement data to complete the learning cycle, and the use of CAA as a tool through which to deploy research instruments are comparatively under exploited by the educational research community. More work needs to be done in this area to fully understand and exploit the potential it offers.

10.3 Results of Actual CAA Use

This section reports on effective use of CAA in different ordinary classes. We shall see examples of tasks, students' activity and their actual strategies. The end of the section discusses the limits of use of CAA and shows some cohort achievement data in geometry. We begin with examples from the WWW Interactive Multipurpose Server, WIMS (http://wims.auto.u-psud.fr). A collaborative project, available in six European languages, this was developed by French Professor Xiao Gang. It is a library of on-line interactive mathematics resources for all levels: from primary to tertiary education.

The exercise of Fig. 10.5 deals with knowledge about continuity and differentiability of functions. Students have to recognize that the given functions are of class C^1. Then it is enough to compute the limits and the derivatives of the two given restrictions and to equate the results. There are no specific suggestions for how students should complete this exercise. The expected answers are the numerical values a_1 and a_2. When students give a wrong answer, the computer provides the type of feedback shown in Fig. 10.6.

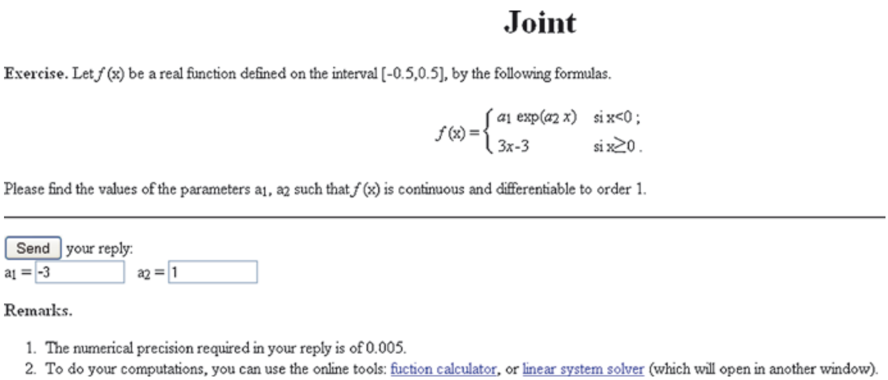

Fig. 10.5 Joint exercise in WIMS

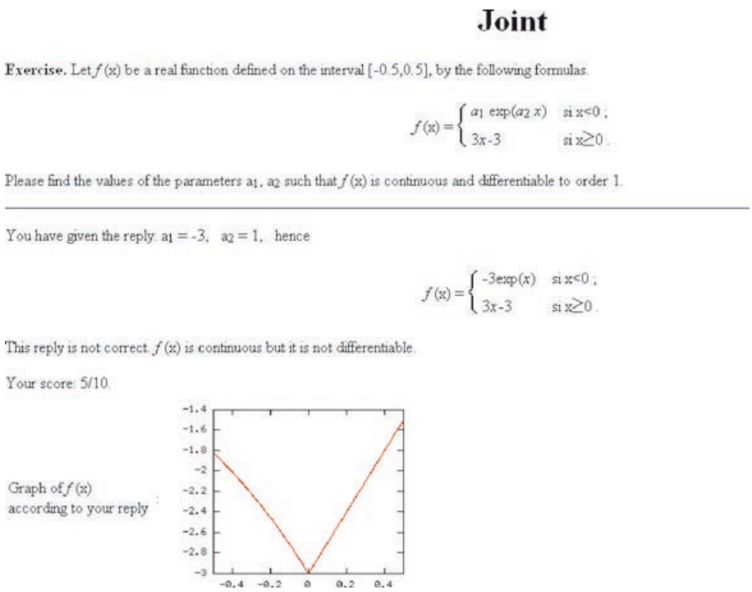

Fig. 10.6 WIMS' feedback for the joint exercise

The feedback is in the graphical register, while the text of the exercise is in the analytic register. Thus the level of the task seems not be modified by the feedback, but the "milieu" is enriched with that new register. This feedback does not necessarily help students to find the solution; it provides another viewpoint that helps only if students have sufficient knowledge to connect both registers (to think, for example, of the link between derivative and the tangent's slope). For this exercise, a random answer, or several successive attempts cannot lead to success without referring to the appropriate knowledge. Since most of these types of tasks are aimed at developing specific skills and concepts by working on randomly generated objects with a given set of parameters, students can practise repeatedly to improve their performance.

To solve the linear system shown in Fig. 10.7 students may choose between three options given in a panel: exchange the equations, add x times equation 1 to equation 2, or multiply equation 2 by a constant. Here the student's focus of attention is solely directed to choosing the operations to be performed and the computer actually performs them. Directing the focus of attention in this way would not be possible in a traditional environment. It is designed to encourage the student to learn step-by-step. The system is not only looking for the correctness of the student's final answer but also the rapidity with which the student can achieve this, as measured by the number of steps taken. In essence this device is much more directed by the formative aim of guiding the solution process and the efficiency of implementing this than by the summative aim of checking student's knowledge. Another important feature of WIMS, also found in many other systems, allows registered teachers to manage and customize activities by setting parameters of the equation such as integer, rational or irrational.

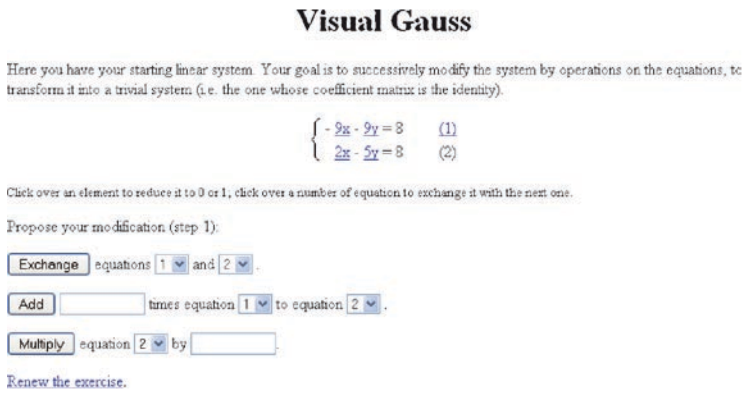

Fig. 10.7 Visual Gauss exercise

10.3.1 New Ways of Undertaking Mathematical Tasks

The examples of the previous section provide a small sample of the diversity of exercises currently available with mathematical CAA. So far we have concentrated on the type of task proposed. We emphasize the potential importance of open tasks (Fig. 10.1) and of new tasks especially in DG, such as shooting or plotting a point correctly (Figs. 10.2 and 10.9, below) or even partial tasks (Fig. 10.7). It would be either impossible to propose such exercises in a paper and pencil environment or so difficult that it is rarely done. Yet in context they have been found to be very useful as formative assessments to help students understand mathematics. And quantitative results obtain by tracking prove that students may work a long time to improve their performance on CAA exercises, (Cazes et al. 2006). A hypothetic explanation is that CAA modifies the didactic contract. In particular, for summative assessment students get immediate feedback that comes just in time, when students are really involved in the task. And for formative assessment students know that they cannot just wait until the solution is given. They must at least start to look for a result because they can never be sure that they will receive direct help from the teacher. Such qualitative results suggest a new relationship with mathematics is permitted by CAA. Indeed, ideally, students are immerged in a rich and reactive "milieu" in which they may form links between several registers. For example algebraic and graphic (e.g. joint exercises) or geometric and measure (see next exercises Figs. 10.9–10.12), or even specific and metacognitive (joint exercise or the exercise shown in Fig. 10.1). Students may also undertake experiments and make repeated attempts. Here, the CAA system may perhaps play the role of a partner providing helpful synchronous and customized feedback.

Moreover, CAA exercises do not ask for an academic proof but only for a correct result. What about the strategies used by student to solve them?

It is perhaps not surprising that new problem solving strategies appear. In the case of MCQ an eliminating strategy is possible, which is not specific to CAA or to mathematics but is inherent in the MCQ format. Dissatisfaction with the efficacy of this strategy was a driver for the development of more sophisticated CAA in which the student's answer is a mathematical object. Even if the student's answer contains more substantial mathematical content a new strategy, linked to the feedback, is possible: trial and error. In many CAA systems students are permitted to have several attempts. In many cases it is not possible to guess, and so the most effective strategy is to consider carefully and correctly interpret the feedback. This is not so easy. For example, in the joint exercise of Fig. 10.5 the explicit clause of the contract is to find the values of the two parameters a_1 and a_2. There is also an implicit demand: using the feedback in the case of a wrong answer to find a correct strategy. In the context in which the question of Fig. 10.5 was set, it is very difficult: students were not explicitly taught a routine method. In the open task in calculus (Fig. 10.1) the feedback is very helpful. By providing the hint "think of the absolute value" the difficulty level of the exercise is radically changed.

These observations are confirmed by others researchers in their survey of the use of technology in mathematics courses in England. Ruthven and Henessy (2002) observed that working in class with CAA facilitate "trial and improvement" strategies for the students and helps teachers to organize sessions where the students can work at their own pace. Is the new "trial and error" strategy encouraged by this CAA more genuinely mathematical than the elimination strategy of the MCQ questions it replaces? Is an elimination strategy to be discouraged? Does this provide a new relationship with geometry and is "answer almost correct" an interesting didactic concept? At this stage it seems impossible to make general statements on these issues. However, the precise circumstance of the use of a question needs to be considered carefully by the teacher. In particular, any feedback should be designed *by them accordingly* for their particular student group. That is why an off-the-shelf CAA package is unlikely to provide a particularly satisfactory solution.

10.3.2 Limits and Difficulties of Using CAA

The focus so far has been on assessment of student's answers. Randomly generating exercises has both benefits and pitfalls. Certainly students can train as long as they want on each exercise with new numerical values. However, students may develop some automatic strategy with no underlying mathematical reasoning. They may also become fatigued: practice needs to be effortful but of limited duration to be effective.

Currently the most significant drawback to CAA is that usually the only evidence on which to base the feedback is the final answer. It may be possible to find a correct answer, and hence accrue a good mark, but with no full mathematical understanding. As an example we consider Charles's work on joint exercise from Fig. 10.6, which is studied through the log file. Overall he works for 32 min during which time he makes four attempts. For his first attempt, 9 min, he scores 5 out of 10, i.e. he only found the value a_1. He quickly restarts the exercise and works for 13 min. This time he is fully correct (a mark of 10 out of 10). Next he restarts the exercise two additional

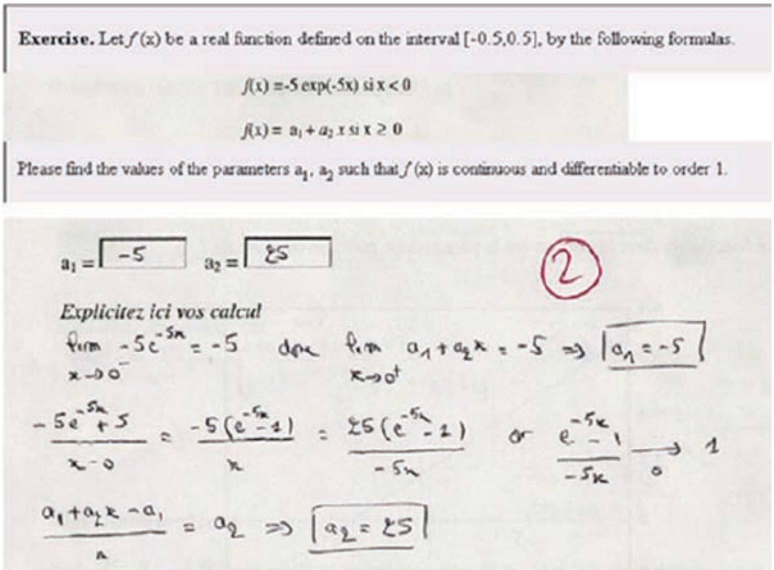

Fig. 10.8 Charles' sheet

times, both for 5 min, and obtains full marks each time. He always reads the feedback, but only for a few seconds.

Since Charles succeeded during his second attempt, i.e. 22 min after starting, it is really quite a long time. We consider the analysis of the time spent to be very important to the teacher: 9, 13 min, and then two times 5 min. A similar exercise was proposed in the assessment at the end of the semester. The extract of his sheet is shown in Fig. 10.8.

Charles develops the expected reasoning for a_1. He computes the right-hand and left-hand limits for each function and equates them. However, for a_2, he computes the increment ratio. There is a gap between what is expected and Charles' work. He did not use the most efficient technique, and this explains the length of time he spends on each exercise. The feedback and restarting the exercise are not sufficient to provide expert knowledge. Moreover, using the increment ratio is inefficient but actually works; and there is no limitation of time for that exercise. As anticipated in Sect. 10.3, the feedback does not change the task's level. Adapted to the student's answer, the feedback is likely to help connecting the graphical and the analytic registers. It can contribute to the elaboration by the student of a continuity and differentiability concept-image. However, in Charles' case, such an effect cannot be observed.

Since nobody examined Charles' personal strategy during the computer sessions he never received any advice on the most efficient method by the teacher. Hence, a correct answer does not mean either a correct or efficient method is used. These results are corroborated by other researchers. For example Gill and Greenhow (2008) report that "*students spent far more time on the feedback than expected, resulting* *in them being able to do only two or three questions in a 50-min test period rather than the five anticipated when writing the tests*".

When undertaking CAA, students need to interact with the machine, both to read mathematical text, on screen and to express themselves. In Dynamic Geometry the predominant mode of interaction is *dragging*. With other topics traditional written mathematical notation is used. This notation has evolved to be both *communicative* and as an *aid calculation and thought*. Regardless of their merits, these conventions have been embedded, probably irreversibly, by usage. On the other hand, when typing a mathematical expression into a computer keyboard the ability to take advantage of the features of traditional mathematical notation is severely limited. Essentially one has only a one-dimensional string of symbols taken from the limited alphabet found on, for example, Western computer keyboards. Translating mathematics into this limited format is a fundamental problem.

Computer scientists have addressed this problem by devising syntaxes to encode the meaning of an algebraic expression in a precise manner. These are used by a CAS, and indeed are used by students when using the CAS itself. However, there is a significant and surprising variety between implementations, even at the elementary level, see Ramsden and Sangwin (2007). Furthermore, few correspond closely to traditional notation. Whatever the benefits or drawbacks of notation in problem-solving and calculation, syntax has definite disadvantages if students taking tests are required to use it. Where that syntax uses conventions different from those underlying standard notation, the risk is that students' answers will be graded "incorrect" for purely syntactical reasons, leading to students failing, as it were, on a technicality. This has particularly important implications for high stakes assessment. This issue is far from trivial, and indeed the problems persist even when a student is provided with a "drag and drop" equation editor type interface such as the DragMath applet (http://www.dragmath.bham.ac.uk). Is, for example, the CAA system to interpret $x(t + 1)$ as a function application or an implied multiplication? While the experienced teacher may indeed be able to apply information from the context to disambiguate this, a CAA system is by its very nature strict and de-contextualized.

10.3.3 Interpreting Students' Solutions in Simple Geometry Tasks

In the geometry examples described here, we see the possibilities of extending the modes of assessment when enhanced with technology. This is not just replication of tests in conventional written format. Even if only simple responses are required, such as choosing from a list of options or the input of a numerical value, students have to manipulate the dynamic figures in order to obtain relevant information. Therefore, the response at least partly reflects their understanding and interpretation of their interaction with the dynamic geometric objects. In some cases it is the result of their manipulation (probably dragging of movable points), or even their creation of geometric objects, which directly serve as the output to be collected and processed by the computer.

We continue to elaborate on students' behaviour in this environment by referring to some DG based testing reported in Lee et al. (2006). We focus on the minimal DG specific support, which allows students to drag movable points on the screen and provides real time measurements. This means that students are not required to make constructions using DG tools, and in fact prior experience of using DG is not assumed. Two of the test items are described below.

Figure 10.9a shows the initial configuration of a dynamic figure. It contains a movable line passing through a fixed point outside the screen. This can be controlled by dragging a movable point P. Students are asked to drag P so that the unknown angle *a* equals 120°. Measurements indicate that the lines at the top and bottom of the screen are parallel and therefore Fig. 10.9b should give the results required.

Figure 10.10a shows the results collected from 169 students. It indicates the distribution of horizontal position of point P submitted by the students. Most of them can put P at the correct position. Meanwhile, about one eighth of the students put P at an unexpected position shown in Fig. 10.11b. Lee et al. (2006) conducted observations and subsequent interviews with another small group of students doing the same test, trying to understand their strategies that were not available merely from the results of the test. For example, one student did consider the configuration in 11b at the beginning, paying attention to the two changing angles above *a* which did maintain a constant difference of 2° as P varied. He eventually chose the correct position after focusing on the parallel lines but not clearly deduced from the given angles of 84°.

Another test item is shown in Fig. 10.11a. Students are asked to drag the only movable point D in the figure to make at least one pair of parallel sides. They are also asked to give more than one answer if possible. The results are better presented as a scatter plot indicating all positions chosen by students for the movable point D (Fig. 10.12a). Although the item did not require two pairs of parallel sides, the majority of the students started considering a parallelogram (Fig. 10.11b). Figure 10.12a shows how the points cluster at the intersection of imaginary lines parallel to AB and BC. Interestingly, about 28 students chose a position along a third imaginary

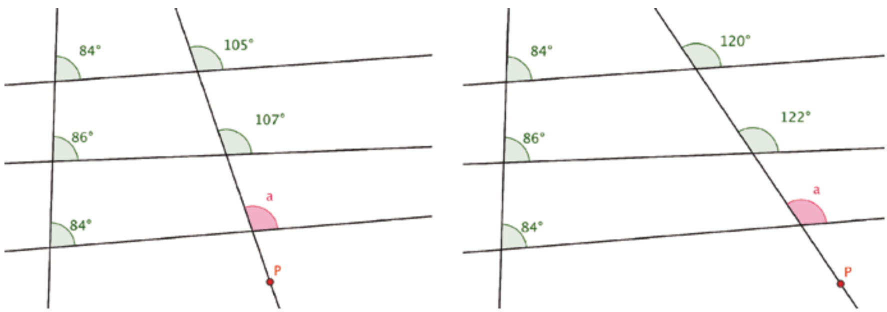

Fig. 10.9 Parallel lines

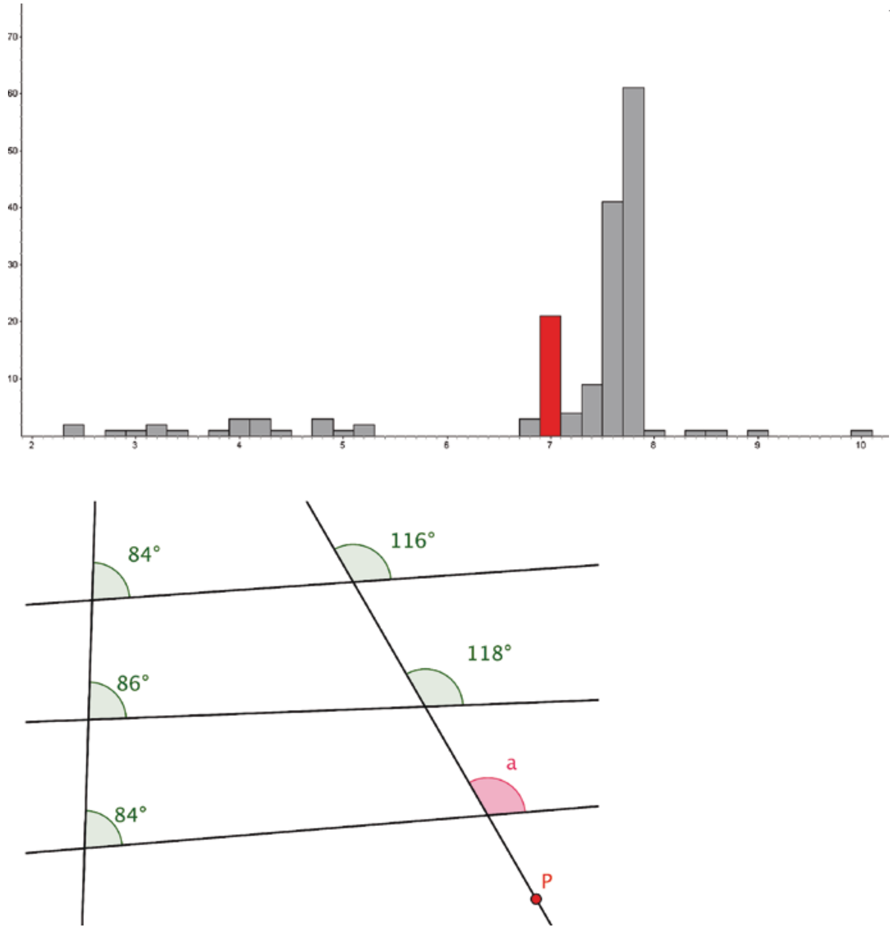

Fig. 10.10 Student's results for the parallel lines task

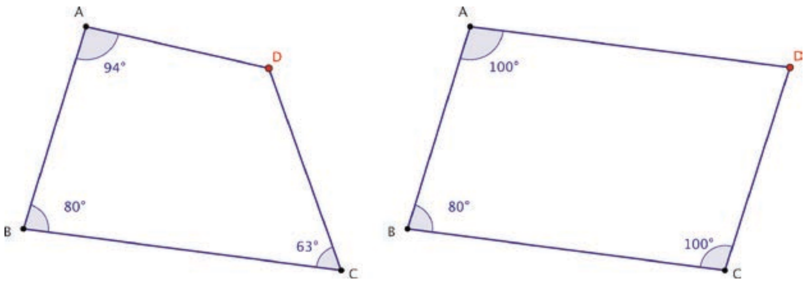

Fig. 10.11 Parallelogram task

line (dashed in Fig. 10.12a) that gave a pair of equal angles at A and C but no parallel sides. Correspondingly, in the later interviews, we observed students correctly making a parallelogram but explaining their choices based on the equal angles at A and C,

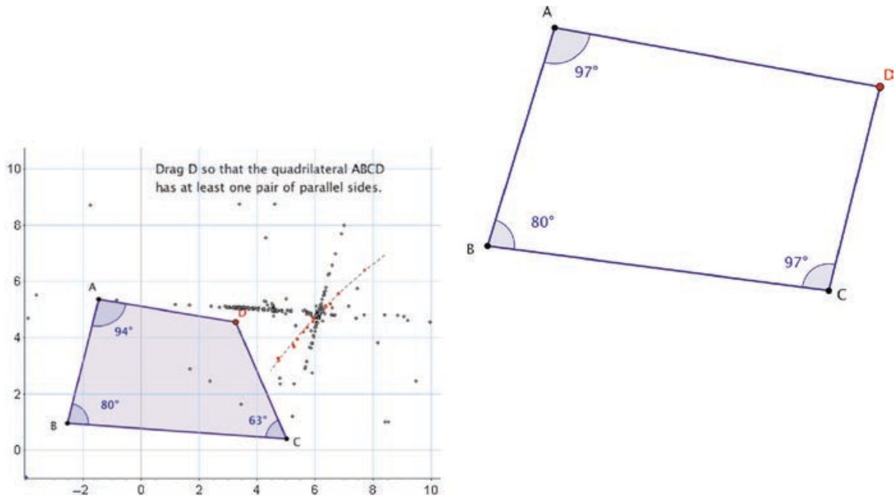

Fig. 10.12 Results of the parallelogram task

instead of any pair of supplementary interior angles. Their behaviour suggested that they were more attracted by the pair of changing angles in their reasoning than other information obtained in the figure.

These examples suggest some basic differences between dynamic and static figures when testing students' simple geometry knowledge. The kinds of feedback provided by this dynamic environment may provide support as well as new demands for students. Meanwhile, responses gathered and processed automatically allow researchers and teachers to explore students' conceptions at other levels.

10.4 Conclusions and the Future

The examples shown here rely on three specific CAA systems, but we believe these illustrate many of the current possibilities. We have seen very different types of tasks such as: open questions, new tasks like plotting a point correctly, or partial tasks such as solving a system by indicating the steps required. In the case of formative assessment, feedback may be adapted to the student's answer especially in open tasks and may guide the student to the correct answer, or it may feed the "milieu" of the exercise. In the case of summative assessment, we saw that CAA systems permit a quick evaluation and a saving of time for the teacher. Lastly, evaluative assessment has been illustrated, especially in dynamic geometry. Some results pointed out both how much time students spend working on CAA tasks and the variety of their solving strategies. However, one of the roles of the teacher is to consider false or incomplete strategies.

CAA is developing so rapidly that software development cycles overtake both annual teaching cycles and the ability to thoroughly evaluate particular initiatives and projects. This is deeply unsatisfactory but is an inescapable symptom of all contemporary technology. Furthermore, our students show immense sophistication with the use of technologies appealing to them, for example games and online communications. They reasonably expect institutions and their staff to keep pace with these changes. However not all CAA systems provide ready access to the teacher. In some the teacher has to be a system developer, writing computer code in obscure languages (e.g. Maple's programming language) to implement questions. Some systems, e.g. WIMS, allow teachers to choose parameters in otherwise fixed items. This is technically robust, simple for the teacher but lacks some flexibility. Others give the teacher full control through a form interface, e.g. STACK. Here teachers can modify existing items, or write their own from scratch giving those with sufficient technical expertise complete autonomy. A more serious problem is that of engaging colleagues in the theoretical aspects of CAA in this environment of such rapid change. For example, Sangwin and Grove (2006) refer to colleagues as "neglected learners". New CAA tools require new modes of thought and action on the part of institutions, teachers and students alike.

The strengths of CAA are the immediacy and mathematical sophistication of automatically generated feedback. Cohort data clearly has the potential to be better used by the teacher and for research. Currently only the final answer is available and future work needs to be done to combine existing "intelligent tutoring" systems with the best of the existing assessment technology. Being able to take account of steps in a calculation or ascertaining methods used both remain significant challenges. Unless the technical developments are guided by, and thoroughly investigated by independent educational research teams we risk missing the opportunity to embed significant improvements. Traditional approaches will be replicated in new formats.

It should be noted that the technologies we describe could be, and indeed often are, used simply to replicate in electronic form existing paper-based tasks. However this transposition from paper and pencil to digital environment is not transparent and the task is actually changed. The need to address emerging educational goals motivates the diversification of modes of assessment away from the traditional, dominant mode of timed paper-and-pencil tests. Technologies can support or even initiate such changes.

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Chapter 11
Technology, Communication, and Collaboration: Re-thinking Communities of Inquiry, Learning and Practice

Ruth Beatty and Vince Geiger

Abstract This chapter presents an overview of the role of technology in mathematics education within the framework of social learning theories. A review of past submissions to ICMI sponsored activities over the last 20 suggests that social perspectives on teaching and learning with technology have become increasingly prevalent. A review of recent literature, such as the proceedings of ICMI 17, as well as broader research sources, adds further support to the view that there is growing interest among the mathematics education community in how digital technologies can enhance mathematics teaching and learning through attention to social aspects of coming to know and understand. Four typologies of digital technologies and their role in collaborative practice are identified: technologies designed for both mathematics and collaboration; technologies designed for mathematics; technologies designed for collaboration; and technologies designed for neither mathematics nor collaboration. As new technologies continue to be developed and refined, they offer new ways to construe communication, collaboration, and social interaction and thus change the availability and feasibility of different kinds of communities of practice. This has implications for both research and practice.

Keywords Social learning theory • Socio-constructivism • Socio-culturalism • Communities of practice • Voice and discourse • Distributed cognition • Collaboration • Computers • Calculators • Graphing calculators • Graphics calculators • Digital technology • Information and communication technologies (ICTs) • Computer supported collaborative learning (CSCL) • Connective technology • On-line learning • Distance learning • Mathematics teaching and learning

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11.1 Introduction

At the Ninth International Congress on Mathematics Education, held in Makuhari, Japan, Stephen Lerman, in an address to the congress titled *The Socio-cultural Turn in Studying the Teaching and Learning of Mathematics* (Lerman 2000a), stated:

> It is taken for granted today that research on teaching and learning mathematics must take into account the social, historical and cultural milieu of schooling and pupils and of mathematics. (p.157)

and then further,

> The term social turn in my title is intended to signal something different, however, namely the emergence into the mathematics education research community of theories that see meaning, thinking and reasoning as products of social activity. (p.157)

Lerman presents his position in this presentation, and in other work (e.g., Lerman 2000b) and it is hard to argue, from a general point of view, that the influence of a social perspective on teaching and learning is not apparent in educational research, school curriculum reform movements or in current advice in relation to improving pedagogical practice. But is this position true of all branches of research in mathematics education? In particular, has there been a noticeable shift in interest by those involved in the study of how digital technologies can enhance the learning and teaching of mathematics towards social aspects of acquiring knowledge and of meaning making in mathematics classrooms?

This paper will address this question by first providing a brief description of theories of intellectual development that view social activity as central to the process of learning and teaching. In Sect. 11.2, we review the proceedings of a selection of ICMI sponsored activities over the last 20 years, and consider current conceptions of the role of technology in collaborative mathematical practice. Four distinct typologies of digital technologies and the role they play in mediating collaborative activity are discussed in Sect. 11.3. Finally, we outline anticipated future developments, and conclude with some implications for future research.

11.1.1 Social Perspectives on Learning

Of the three theories of intellectual development that have had greatest influence on school classrooms since the turn of the last century, behaviorism, constructivism, and socio-culturalism, only socio-culturalism was conceived with social activity as a foundation for intellectual development. Those who subscribe to constructivist theories, however, may now argue that a role for social interaction has been incorporated into the reconceived theory of socio-constructivism. While there has been considerable debate about the legitimacy of this claim (see for example Cobb 1994; Lerman 1989, 1996) it is not the purpose of this paper to engage in this discussion or to attempt to resolve the dispute, and so research that incorporates the notion that social life has a role to play in effective teaching and learning will not be excluded from consideration on the grounds of a particular theoretical position alone.

11.1.2 Socio-Constructivism

In response to the perceived shortcomings of behaviorism, a new class of theories was developed that collectively became known as cognitive theories of learning (Reynolds et al. 1996). These new theories sought to go beyond the behaviorists' simplistic stimulus and response paradigm to explain the complexity of human thinking and cognitive development. Of the many theoretical frameworks that sit under the umbrella of cognitivism (e.g., schema theory, connectionism), it was the range of psychological theories that became known as constructivism that had the greatest influence on mathematics education (Confrey and Kazak 2006). The constructivist position holds that learning is a process whereby the learner actively constructs symbolic representations of the world and uses interpretations of these representations to interact with the world (Noddings 1990). Fundamental to the constructivist understanding of intellectual growth is the Piagetian concept of disequilibrium or the cognitive conflict in which learners are engulfed when they encounter an idea that contradicts their current world view. From a constructivist perspective, intellectual growth takes place when the learner is able to rearrange cognitive structures in order to make sense of phenomena that conflict with their existing understanding of the world.

While Lerman (2006) rejects "piagetian research and especially constructivist and radical constructivist research" (p.350) as part of the socio-cultural paradigm, others (see for example Cobb 2000; Cobb and Bauersfeld 1995; Cobb et al. 1992) have argued that *social interaction* has an important role to play in constructivist theories of learning. In this view, interaction is fundamental to the process of disequilibrium as it is in social contexts that conflicting ideas between individuals may emerge (Palincsar 1998). Collaborative discussion also plays a role in the resolution of the conflict and its incorporation into new knowledge and meaning structures. While the role of tools per se receives less explicit attention in constructivist literature compared to writings in a socio-culturalist frame, cultural tools receive recognition as facilitators of cognitive conflict (Cobb 1995, 2002).

11.1.3 Socio-Culturalism

Unlike social constructivist perspectives, where cultural tools and social dynamics are seen as external supports to the construction of individual knowledge, socio-cultural perspectives of learning emphasize the socially and culturally situated nature of learning. While the history of this social perspective on mind is long (see for example Valsiner and Veer 2000) seminal work in this area is generally attributed to Vygotsky (e.g., Vygotsky and Cole 1978). Vygotsky emphasized the critical role of a student's own activity in learning and thinking while at the same time arguing that all learning takes place within a social context. Thus, socio-cultural theory shifts attention from individual to social modes of thinking, and emphasizes the role of language in learning, both as a tool for thinking and as a medium for communication.

Lerman (2006) states that socio-cultural theory is based around the following assumptions:

- Concepts appear first on the social plane and only subsequently on the individual plane
- The individual plane is formed through the process of internalization

He makes a clear distinction between socio-cultural theories and other theories that recognize the role of social interaction in learning on the basis of the alignment of theories (or not) with individualistic psychology.

From a Vygotskian perspective, as described by Luriia et al. (1979), there can be no strict separation of an individual from his or her social environment. In this view, cognitive development is the process of acquiring culture and so the individual and social must be regarded as complementary elements of a single interacting system.

Also central to socio-cultural theory is the principle that human action is mediated by cultural tools and is fundamentally transformed in the process (Wertsch 1985). These tools take the form of language, representations, and sign systems as well as physical artefacts. In the particular case of digital technologies, tools can be used to both amplify and reorganize cognitive processes through their integration into the practices of a community of learners. It is important to remember, however, that tool use must be incorporated into "structures of reasoning, and the forms of discourse that constrain and enable interactions within communities" (Resnick et al. 1997; p. 3) and so learning is not just the accompanying changes to mental structures that result from tool use, but also the appropriation of methods of reasoning and discourse that incorporate tool use as recognized by the community of practice. Thus, the introduction of digital technologies into a learning environment represents challenges to the learner that go beyond the mastery of a tool to new modes of reasoning and action.

11.1.4 Communities of Practice

Drawing from the socio-cultural perspective, new views of learning have emerged including those of apprenticeship (Rogoff 1990) and participation (Lave and Wenger 1991; Wenger 1998; Wenger et al. 2002). In *Cognition in Practice* (Lave 1988), Jean Lave challenged the notion that mathematical practices outside of schooling were merely the application of school mathematics. In a study of grocery shoppers and dieters, Lave observed that strategic decision making by the shoppers and dieters was heavily influenced by the contexts they were working in, that is, the knowing and processes for decision making were situated within a social milieu.

Consistent with that view Bishop (1988) argued that mathematics is a way of knowing that was culturally developed as a way of structuring a learner's experience.

Building on Lave's earlier work, Lave and Wenger (1991) described learning as a form of apprenticeship where novices are initiated into a learning community, or *community of practice*, through a process they termed "peripheral participation".

> A community of practice is a set of relations among persons, activity, and world, over time and in relation with other tangential and overlapping communities of practice. A community of practice is an intrinsic condition for the existence of knowledge, not least because it provides the interpretive support necessary for making sense of its heritage. Thus, participation in the cultural practice in which any knowledge exists is an epistemological principle of learning. The social structure of this practice, its power relations, and its conditions for legitimacy define possibilities for learning. (Lave and Wenger 1991, p.98)

In their view, learning is not associated with the individual internalization of knowledge, but rather can be conceptualized as the degree to which a learner participates in a particular community of practice. Experts within the community, for example, teachers or more knowledgeable peers, are responsible for the induction of learners new to the community into the culture of that community including beliefs, values, modes of discourse and means and methods of knowledge creation. Judgments about learning are therefore based on the increased range of participation of the learner within the community.

Participation in the community of practice is seen by Lave (1996) as the mechanism for learning or *becoming*:

> Rather than particular tools and techniques for learning as such, there are ways of becoming a participant, ways of participating, and ways in which participants and practices change. In any event, the learning of specific ways of participating differs in particular situated practices. The term "learning mechanism" diminishes in importance, in fact it may fall out altogether, as "mechanisms" disappear into practice. Mainly, people are becoming kinds of persons. (Lave 1996, p.157)

From this perspective mathematical activity is viewed as a process of enculturation into the modes and methods of knowledge creation, sharing, and validation, which characterize the practices of the community of that discipline (Goos et al. 1999). Students learning within a mathematical community of practice are expected to engage in debate about the validity of ideas and to defend positions or offer critique via explanations, justifications, and the provision of alternatives (Goos et al. 2000a).

Drawing on observations collected during a 2-year reform project of middle school mathematics classrooms aimed at fostering high-level thinking and problem-solving skills for students from economically disadvantaged backgrounds, Foreman (1996) developed a comparison between the range of activity setting in traditional and reform classrooms, the latter, she argues, conducted according to *community of practice* principals. She found that students in reform classrooms participated in a wider range of activity settings than in traditional classrooms, where students had less opportunity to initiate topics, redirect discussion, provide elaborate explanations, or debate issues. Foreman contrasts this learning environment with a classroom conducted by a teacher, Mrs. Hanes, where the following interactional scripts were observed: whole class recitation lead by the teacher, whole class presentations lead by one or more students, small group work lead by one or more students with the teacher's intermittent assistance, individual seat work and unofficial peer group activities. Foreman further argues that the increased range of activity associated with reform classrooms brings with it new task demands, values, and purposes. She views the appropriation of these new demands as the instantiation of a community of practice where students initially participate peripherally. As students appropriate new skills, norms, and idea of the community, they move to greater participation in the community and so demonstrate their learning. However, she notes that some students can be resistant to this participation even when other students in the classroom are working as a community. These students may demonstrate resistance to participating only within certain modes of interaction, for example, small group work, or may reject the collaborative norms of the classroom community altogether and remain passive and fail to contribute in any way.

11.1.5 Voice and Discourse

While Vygotsky provided new insight into the social aspects of learning, his description of the process of communication has been criticized for not reflecting the complexity of social discourse (Van Oers 2002), and in particular, the reciprocal nature of discursive negotiation as new ideas and meanings are explored. It is out of this concern that Bakhtin's theory of voice and literature from the field of discourse have emerged.

Bakhtin's theory of voice emphasizes the active, situated, and functional nature of speech as employed by various groups (Renshaw and Brown 1998). An act of communication, in this view, must always be constituted by a range of "voices" – the voice of the speaker but also traces of the voices of other members of the learning community who have previously used similar words or methods of argumentation acceptable to the community.

> …we would say that people's utterances in a communication process are not regulated by the processes that occur in direct interaction, but also by the historically developed style of communicating in that particular community of practice. (Van Oers 2002, p.68)

The development of such a voice allows members to recognize themselves as part of what Bakhtin called a sign community, in which a shared identity was manifest. The extent to which a speaker appropriates the style of communicating in the sign community can be used to make judgments about different levels of performance (e.g., explanation, justification, problem solution).

It is important to note the reciprocal nature of communication between the individual and the community. As the individual communicates in order to receive confirmation of their appropriation of the voice of the community, they may also progress the collective knowledge of the community and so change its voice. This constitutes development in the consciousness of the individual and also the collective.

This shift in focus away from views of learning as changes to the individual based on the acquisition of knowledge to a social view of learning that characterizes intellectual development as change in the way one communicates with others is what characterizes the studies concerned with learning discourse – the basis of the field of discursive psychology (Kieran et al. 2002). In Sfard's view (Sfard 2001, 2002; Sfard and Kieran 2001) learning mathematics is an initiation into a certain well-defined discourse and she uses the metaphor "thinking-as-communicating" to frame her research. This discourse is reliant on symbolic artifacts as communication-mediating tools and by meta-rules that regulate communication. Tools, and symbolic tools in particular (for example language, graphs, tables, algebraic formulae), are not viewed as simply the means of or media for communicating pre-existing knowledge, but rather tools are intertwined with the act of communicating and, therefore, of cognition itself. Thus tools can be conceptualized as cognitive intermediatories for communication within a community. Meta-discursive rules, on the other hand, guide the course of communicational activities within a community of learners.

Classroom studies based on the discursive field of collective argumentation (e.g., Brown and Renshaw 2000; Krummheuer 2007) have observed the greater range of communicative spaces available in classrooms conducted as a community of collaborative learners. In *collective argumentation* approaches, students are introduced to means of structuring classroom discourse aimed at the creation and sharing of knowledge – in the case of Brown and Renshaw to key words: represent, compare, explain, justify, agree, and validate – that facilitate students' co-construction of understanding. Students often work in small groups to initially represent a task, compare their representations with other group members, explain and justify competing representations to each other, before presenting their group findings to the whole group for validation. Krummheur (2007) emphasizes the importance of maintaining flexibility in the process of argumentation. The teacher's redirection of an argument away from what initially appears to be an unrelated path may result in a lost opportunity for the learning community to test and develop its capacity to self regulate, or to find an unexpected approach to solving a problem. Krummheur also cautions that too zealous an approach to directing the argumentation process brings with it the danger of lapsing into a transmissive mode of learning and teaching.

11.1.6 Distributed Cognition

Because the field of discourse focuses on the importance of language in the development of consciousness, it places greater emphasis on semiotic tools, such as language and specialized symbolic systems, than it does on physical artifacts as mediators for learning and thinking. Despite studies that consider physical artifacts, such as computers, as having a vital role to play in supporting discourse and as a result intellectual development (see for example Cobb 2002; Kirschner and Erkens 2007; Manouchehri 2004; McDonald et al. 2005; Pozzi et al. 1998), these tools do not appear to be given the same prominence as symbolic tools in theorizing the act of cognition.

An alternative theoretical perspective is that of distributed cognition in which cognition is not merely a social practice but an act distributed across individuals, collectives, symbolic and physical artifacts, as well as symbolic, virtual, and physical environments. Drawing on aspects of Vygotskian socio-cultural theory and recognizing the potential computer technology, Pea (1985, 1987, 1993) argues that humans are elements in a reasoning system that includes human minds, social contexts, and tools.

Hutchins' (1995) account of the process of navigation on a naval vessel (as described by Cobb 2007) considers the whole navigation team, including all physical and symbolic tools, as the reasoning system that provides for the safe piloting of the vessel into port. Further, this reasoning system is constituted by elements that exist in the moment of the act, for example, the navigator and the ships guidance system, as well as by elements that preceded the event which led to the development of the processes of navigation and the physical artifacts used to navigate. This is because traces of the intelligence of other minds that developed the procedures that guide navigation and of those that designed the maps or guidance system used by the navigator remain in those procedures and devices.

Further, the role of tools, in Pea's view, is more than an amplifier or extender of cognitive capability; tools can be used to reorganize mental processes, which in turn alter tasks as they were originally conceived.

> Computers are commonly believed to change how effectively we do traditional tasks, amplifying or extending our capabilities, with the assumption that these tasks stay fundamentally the same. The central point I wish to make is quite different, namely, that a primary role for computers is changing the tasks we do by reorganizing our mental functioning, not only by amplifying it. (Pea 1985, p.168)

Thus the nature of this intelligence is not always predictable as it emerges from the activity that is shaping intelligence in the process of engaging with the activity. Please see Chap. 7 of this book for a more elaborated description of the work of Pea.

11.2 The Growth in Social Perspectives on Teaching and Learning with Technology

In order to benchmark, in a short paper, the validity of Lerman's optimistic appraisal of a turn towards socially orientated theoretical frameworks in relation to the use of digital technologies, three ICMI sponsored events have been chosen across a span of some two decades. The first event, the initial event in ICMI's seventeen studies to date, *The Influence of Computers and Informatics on Mathematics and its Teaching*, was chosen because it was in the mid-1980s that micro-computers were having their first significant impact in educational contexts. The proceedings of this event will be reviewed in order to establish a baseline for gauging interest in the role of digital technologies in promoting social aspects of learning at this early stage. The ninth ICME in 2000 was included because it was at this congress that Lerman made the observation that frames this paper. Finally, because it is the most recent study in this area, the proceedings of the symposium associated with ICMI's seventeenth study, *Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain*, have been included in this review.

11.2.1 Early Accounts

The symposium, The Influence of Computers and Informatics on Mathematics and its Teaching (Churchhouse 1986), was organized under three themes, and the third, How can the use of computers help the teaching of mathematics?, is of relevance here.

The report on this theme opens with a discussion of what mathematics and mathematical activity might comprise in a future classroom. It was felt, in particular, that "the experimental aspects of mathematics assume greater prominence, and there is a corresponding wish to ensure that provision should be made for students to acquire skills in, and experience of, observing, exploring, forming insights and intuitions, making predictions, testing hypotheses, conducting trials, controlling variables, simulating, etc." (p. 24–25). Curiously, despite a description of what we would consider now to be activities students might engage in as a group, there is no commentary of how students might work with each other, or how such interaction could promote learning.

Later in this section there is acknowledgement that technology has the potential to influence classroom dynamics as "this creates new interactions and relationships between student, knowledge, computer and teacher" (p. 25). The use of the singular "student" is a further indication, however, that interactions among students were not a concern at that time. The advantage of the computer was seen as supporting the development of mental images that would assist in the acquisition of mathematical concepts and processes within individuals.

11.2.2 A New Millennium

Despite Lerman's optimism for the uptake of social perspective in education research articulated at the ninth ICME in 2000, the working group *The Use of Technology in Mathematics Education* provided only a modicum of support for his position. The reports of each subgroup of this theme reveal only one reference to the contribution of technology to the social aspects of learning. This appears within subgroup 4: *Conceptual and professional development of learners and teachers in technologically rich classrooms*, which notes "several informative empirical studies were presented that were routed in theoretical work in the socio-cultural perspective" (p. 277)

One such paper, *Classroom voices: Technology enriched interactions in a community of mathematical practice* (Goos et al. 2000b), theorized four roles for technology as a tool for amplifying students' cognitive processes and reorganizing interactions between human and technological agencies. This paper demonstrates a clear association with socio-cultural perspectives on teaching and learning but was only one of a very few of its type.

11.2.3 Current Climate

The 17th ICMI study, *Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain*, provides greater support for Lerman's optimism. The proceedings of this symposium were examined for indicators of a study's alignment with a social theme. These included references to socio-cultural theory, collaboration, learning communities and classroom discourse. Of the 77 papers included in the proceedings, 14 papers were framed around these ideas or made direct reference to them in their theoretical frameworks. This represents 18% of the studies included in the symposium. Further, an additional ten papers were framed around, or made reference to, the theoretical position of instrumentalization. While it is arguable that this is a social perspective, the concept of semiotic mediation through technological tools is often traced to Vygotskian theories of intellectual development and by association socio-culturalism. If these papers are included in this analysis, then 31% could be considered to exhibit traces of educational theory related to the social aspects of learning and teaching. Considering either figure, and acknowledging the broad brush nature of the analysis, 18–31% of papers represents a noteworthy shift in the interest of this branch of mathematics education towards the social and supports the claims of Lerman 7 years earlier.

11.2.4 The Role of Technology in Collaborative Mathematical Practice

Despite this recent interest in social perspectives of learning with technology, a decade or so ago research into the role of technology in mathematics learning and teaching was a relatively recent activity and studies in the area were limited (Kaput and Thompson 1994). There now exists a vast corpus of literature that draws upon a wide range of theoretical perspectives in an attempt to explain phenomena associated with learning and teaching mathematics within technology influenced environments (Hoyles and Noss 2003). Over this time a number of authors have attempted to define the territory.

Taylor, for example, (1981) suggested three ways in which technology, specifically computers, is used in education as a tutor, as a tutee, and as a tool. See Chap. 2 of this book section for a further elaboration of the tutor-tutee-tool dimension.

While Taylor's framework is a useful starting point to theorizing the role of computers in teaching and learning mathematics, in particular identifying the computer as a tool that can enhance the capabilities of humans, there is no attempt to discriminate between how technology might be used by individuals or groups of learners. Building on Taylor's commentary, Willis and Kissane (1989) have also added the category of *Computer as a Catalyst*. In this mode the computing environment is used as a means of provoking mathematical explorations and discussion or invoking the use of problem solving skills. This addition recognizes the potential of technology to support learning-focused interaction between students and suggests a meditative role for technology in learning.

The metaphor of *Computer as a Catalyst* is further extended by Goos and Cretchley (2004) in a review of the role of technology in mathematics education in the Australasian region. Their development of the metaphor further refines the view of the computer as a tool and catalyst for visualization, higher order thinking, and collaboration. While it is important to recognize that the three categories listed here are far from unrelated, this typography is useful to identify the primary focus of the research reviewed by these authors. Significantly, from the perspective of this chapter, the identification of the role of technology in fostering or in mediating collaboration is noteworthy. In particular, Goos and Cretchley observe that the role of technology in supporting students' knowledge building in a mathematical community of learners, such as in studies of Computer Supported Collaborative Learning (CSCL), has recently emerged as a significant theme for research.

Computer-supported collaborative learning (CSCL) is a fast-growing international field of research focusing on how technology can facilitate the sharing and creation of knowledge and expertise through peer interaction and group learning processes (Resta and Laferriere 2007). Typically, CSCL is built around a database to which members of a specific CSCL community share their developing understandings of an idea, concept, or topic through text, graphics, or other means. The paradigm of CSCL grew out of research on the demonstrated advantages for individual learning of working in groups (e.g., Johnson and Johnson 1989), and CSCL environments include web-based platforms, forums, and videoconferencing systems that allow both synchronous and asynchronous interaction, facilitating communication between temporally and/or geographically distributed participants.

Only recently have researchers begun to evaluate the particular affordances of this approach to teaching and learning, and to examine more closely what is uniquely feasible with new technologies that extend beyond supporting the development of peer-to-peer discourse. Studies have focused on the potential of technology to afford genuine *collaborative* efforts defined by Harasim et al. (1995) as "a learning process where two or more people work together to create meaning, explore a topic, or improve skills" by providing opportunities for exposure to multiple perspectives and interpretations (Koschmann 1994). Researchers have begun to identify the opportunities technology offers for the processes of sharing, critiquing, exchanging, and debating ideas as a way of developing both community knowledge and participating individual's understanding (e.g., Scardamalia and Bereiter 1996, 2003) and how technology can support qualitatively different learning environments (group cognition, collaborative knowledge building) (Resta and Laferrier 2007).

Stahl (2006) outlines some considerations with respect to technological tools necessary for the establishment of *collaborative* communities:

1. The use of cognitive tools by a collaborative community takes place through many-to-many interactions among people.
2. The cognition that the tools foster is inseparable from the collaboration that they support.
3. The relevant cognition is the group cognition; this is a linguistic phenomenon that takes place in discourse rather than a psychological phenomenon that takes place in an individuals mind.
4. The tools may be more like communication media than like hand calculators – they do not simply amplify individual cognitive abilities, they make possible specific forms of group interaction.
5. Rather than being relatively simple physical artifacts, tools for communities may be complex infrastructures.

This emerging field has underpinned studies specifically examining the support of computers for collaboration in mathematics learning.

11.3 Different Technological Typologies for Fostering Communication, Collaboration, and Communities of Inquiry

Taking socio-cultural perspectives of learning, we were particularly interested in identifying emergent roles of technology within the current range of pedagogical contexts. In particular, we were interested to identify research of technology as a support to communication and interaction, and the shift from classrooms as simply a collection of individuals towards classrooms as mathematical communities of inquiry – recognizing the importance of participation in communities of practice/learning (Lave and Wenger 1991).

In order to start to identify these roles, we began with a survey of papers from ICMI 17 to determine current practices. The goal of this section is to provide an overview of our findings. In particular, we outline four typologies of how different types of technology are designed and utilized to support the paradigm of *discourse communities – intellectual communities of practice* – where students engage in mathematics as a joint experience. These include technologies designed for:

1. Both learning mathematics and collaboration
2. Learning mathematics but not specifically for collaboration
3. Collaboration but not necessarily learning mathematics
4. Neither learning mathematics nor collaboration

We present each of the four typologies below with representative reports from the ICMI 17 Study, and selected articles from the broader research literature.

11.3.1 Technologies Designed for Both Mathematics and Collaboration

This typology includes tools with the facility for learners to work with mathematical concepts in a virtual environment that specifically includes a component designed for communication. In all the examples given below, the intention underlying the technology is the development of mathematical understanding through elicited student discussion. These studies examined the affordances of using computer-based tasks to mediate collaborative interactions, either in multi-site online interactions via the internet, or face-to-face in small group or whole group classroom-based settings.

11.3.1.1 Internet-Based Networks

From the ICMI 17 Study, technologies of the former type include the *Space Travel Games Construction Kit* developed by Kahn et al. (2006). In this simulation of computer game development, participants were provided with a construction kit comprised of small program fragments together with tools for customizing and composing them. This activity was designed to take place within the context of a metagame where learners are presented with a goal and need to interact with other members of their team in order to share components and acquire the knowledge to proceed. A component of this knowledge is mathematical in nature and so learners acquire this mathematical knowledge through interaction with peers.

Similar projects include the *WebLab*s (Simpson et al. 2006) and *Playground* Projects (Noss et al. 2002). In the *Playgrounds* Project, young students (ages 6–8) from various European Union (EU) countries worked in web-based programmable microworlds to collaborate on a variety of activities, including videogame construction, as a means of exploring domains such as numeric sequences, cardinality, and probabilistic thinking. Participants were able to communicate using two-way video, text messages, and the exchange of videogame program components within the programming environment *Toontalk*. For these younger children, the challenge of interactively designing computer games by engaging in both face-to-face and *distanced* collaboration allowed them to develop an understanding of the formalized rules that underpinned the components of the games, and these mathematical understandings were expressed as collaboratively built models. Time constraints on the project itself did not allow for the inclusion of an inter-site asynchronous communication platform. This was later integrated in the *WebLabs* Project, which involved older students (ages 13–15) who took part in the same kind of learning through program design/modification of mathematical models. In addition, this project included an asynchronous web-based communication/collaboration platform, WebReports, so that students from several EU countries could both communicate about their ideas and collaborate on the building of specific programs/models by sharing program modules through *Toontalks*. The findings suggest that on-line sharing of ideas, coupled with the ability to share models for collaborative construction/modification, not only increased students' motivation to work on mathematical problems, but also supported students' developing understanding of formalization and mathematical structure. "Mathematical ideas, expressed, and shared as models, can become the subject of reflection and discussion, and how this dialogue can begin to construct some rich understandings of mathematics that foreground its structure and the properties that follow from it" (Noss and Hoyles 2006; p. 21).

11.3.1.2 Classroom-Based Networks

In contrast to studies of multi-site on-line collaboration, a growing community of researchers (e.g., Stroup et al. 2002) has written about the potential of connective technology specifically developed to allow within-class exchanges of mathematical objects among students and teachers. These technologies integrate face-to-face interaction with the mediation afforded by a technology-based system as students work on a common task. Connective network components are made up of both a social component – face-to-face interaction, and a technological component – interaction between students and handhelds, or interaction between handhelds.

Investigators have looked at the affordance of combining dynamic computer-based representations and connectivity technology as students engage in teacher-led real-time comparisons of aggregated student solutions that served as the basis for rich discussions. One example of this kind of interactive learning environment is participatory simulations software such as *HubNet* (Wilensky and Stroup 1999, 2000), through which participants act as individual agents and observe how the behavior of dynamic systems as a whole emerge from individual elements. Another example of a different kind of public display of collective information is the *SimCalc Math Worlds* Project, (Roschelle et al. 2000; Roschelle et al. 2003; Hegedus and Kaput 2003, 2004). Students work with linked devices, allowing for aggregated representations of mathematical objects (e.g., linear functions) that are shaped and formed by multiple contributions and become the focus of shared attention for classroom discussion.

One consideration for supporting collaborative interaction is the topology of connective systems. In one topological model student messages travel only to and from the teacher hub, which supports teacher-led discussion and communication of artefacts, but which may not necessarily promote true collaborative inquiry given that the teacher leads the discussion, and that the group size (i.e., whole class) may be too large to allow for sustained collaborative discussion (White 2006). Another typology links small groups of students who share information between handhelds to collaboratively complete specific tasks. White's 2006 study demonstrated how adding a small-group networked component increased the opportunities students had to participate in classroom collaboration. In this study, middle school students engaged in collaborative problem solving about mathematical functions using connected devices linked through a local wireless network. Each student's device displayed different representations of linear functions (either tabular or graphic). Students were assigned specific roles within their group while working to integrate the multiple representations. White found that both the use of networked handhelds and the assignment of specific task roles increased student participation and their learning of functions.

In a study with younger students, (Zurita and Nussbaum 2007) 6 and 7 year olds worked in groups of three or five to transmit mathematical information between handhelds in order to carry out operations on whole numbers to reach certain "goal numbers". In this study, students were responsible for facilitating the successful outcome for all group members, and this collaborative disposition was underpinned by the self-coordination of the entire group through the wireless network. The results showed that students increased both their knowledge of basic mathematical operations with whole numbers, and their motivation to engage in collaborative mathematical activity.

11.3.1.3 Non-networked Software

This category includes technologies that have been designed to be used as the focal point for mathematical collaboration, but which do not necessarily include collaborative interface or connective technological components. In these studies, researchers included computer-based environments to act as the mediator of social interaction between student peers (with and without teachers) – the medium through which shared mathematical expression could be constructed and mathematical understandings enhanced (Pijls et al. 2007; Sinclair 2005; Lavy and Leron 2004; Vidakovic and Martin 2004; Manouchehri 2004). The software, therefore, serves a communicative function through which individual student meanings can be made explicit and open to negotiation.

An example of this is the computer Tools for Interactive Mathematical Activity (TIMA) project, designed to facilitate dyadic peer-to-peer and triadic teacher-student-student discussions as elementary students (K-8) engaged with a variety of computer-based activities (Olive 2000). These activities allowed participants the possibility of enacting mathematical operations on whole numbers and fractions, which were represented as on-screen manipulatives analogous to their real-world counterparts (beads, sticks, and fraction bars – see Chap. 3 of Theme C for a more detailed description of the TIMA). Olive found that the process of interacting with the on-screen manipulatives became the impetus for collaborative "games" underpinned by mathematical operations, and that the combination of interaction with and within the microworld and problem-solving communication brought forth students' mathematical reasoning. Other studies, such as Groupwork with Computers (Healy et al. 1995) assessed the potential of computer software to facilitate collaborative discussion in larger groups while engaged in computer-based mathematical tasks. In this study, 9–12-year-old students worked in groups of six on Logo-based math tasks designed to allow students to work together to "express and debug" their own conjectures.

Results from both these studies reflect the complexity of integrating PC-based software as a means of developing mathematical understanding. In both studies, the authors emphasize the need to consider the interaction between the pedagogical intentions underlying the design of computational environments, the children's responses to these environments, and the teacher's understanding of the epistemological and pedagogical goals of the software in order to successfully contribute to the development of children's mathematical thinking. However, the authors also conclude that discussion facilitated through joint engagement in computer-based activities led to robust conceptual mathematical understanding.

11.3.2 Technologies Designed for Mathematics but Not Specifically for Collaboration

Technologies in this category include those that were designed specifically for working with mathematical ideas but were not specifically designed for the promotion of social interaction. Studies include research incorporating devices such as graphics calculators and mathematically enabled software, for example, Maple, Derive, Cabri Géomètre and Geometer's Sketchpad, SPSS, and Excel in which authors have explicitly identified collaborative aspects present in:

- The co-construction, by students and teachers, of the use of a tool in mathematical meaning making
- Mathematical meaning making in which technology mediates interaction and learning in small group contexts
- The use of technology to stimulate conjecture and debate in whole class discussion

Working from a theoretical perspective in which psychological aspects of learning are coordinated with social aspects through students' interaction with tasks, each other and their teacher, Doerr and Zangor (2000) studied the co-construction, by the students and the teacher, of the graphics calculator as a tool for mathematical learning. As a critical aspect of the social context of the classroom in which this co-construction took place, the emergent norms for tool usage were also examined. In a case study of pre-calculus classrooms five modes of graphics calculator use were identified:

- Computational tool – where the calculator was routinely used by students to evaluate numerical expressions
- Transformational tool – where tedious computational tasks were transformed into interpretative tasks by focusing students' efforts on the interpretation of results rather than on any associate computation
- Data collection and analysis tool – here the calculator was used as a tool for data collection through the use of peripheral devices such as motion detectors, and the analysis of such data sets
- Visualization tool – the calculator here was used to develop visual parameter matching strategies to find equations that fit data sets, find appropriate views of

the graph and determine the nature of the underlying structure of the function, link the visual representation to the physical phenomena, solve equations
- Checking tool – where the calculator was used to check conjectures made by students as they engaged with the problem investigations

Interestingly, despite the collaborative nature of the co-construction of the tool by all classroom participants, Doerr and Zangor found that the tendency for the use of the graphics calculator as a private device regularly led to the breakdown of small group interactions. When this occurred, it also led to the disruption of the classes' collective endeavour to engage with a whole class task.

While this typography describes students' use of technology in a classroom environment it does not outline the role of technology as a mediating tool or make provision for the role of interaction and discourse in developing mathematical understanding as would be expected in collaborative environments.

In a series of papers written from the French curriculum context (for example, Artigue 2002; Guin and Trouche 1999; Trouche 2003) an *instrumental* approach to viewing students' activity in technology enhanced environments was developed (see Chap. 2 of this book for an extensive description of this approach). This approach describes the process of instrumental genesis in which the possibilities and constraints shape the conceptual development of the user, while at the same time the user's conceptualization of the artefact and thus its instrumentation lead, in some cases, to the user changing the instrument (Drijvers and Gravemeijer 2005). Social aspects of learning are recognised within the process of *instrumental orchestration* (Trouche 2003, 2005) and take the form of student activity where explicit schemas individuals have developed are shared with a small group or whole class. These schemas are thus available for appropriation by other class members. Thus utilisation schemas are essentially individual even though instrumental genesis may take place through a social process (Drijvers and Gravemeijer 2005).

An alternative construct for the way in which student learning is related to collaborative uses of technology is argued in a series of studies by Galbraith, Renshaw, Goos, and Geiger (Galbraith et al. 1999; Geiger 2005; Goos et al. 2000a, 2003). In their typography of students' use of technology, two levels, *technology as partner* and *technology as extension-of-self*, are related to student–student–teacher collaborative mathematic practice. This series of studies also provides evidence that technology can have a role in mediating classroom communities of inquiry in which the roles of teaches students and technology are integrated seamlessly in a classroom learning environment.

By contrast with the findings of Doerr and Zangor, Goos et al. observed that graphics calculators, as well as computers, could facilitate communication and sharing of knowledge in both private and public settings. In these cases students interacted both with and around the technology; for example, the calculator became a stimulus for, and partner in, face-to-face discussions when students worked together in small groups, or in the case of mediating whole class discussion, through the use of a view screen in which students were used to co-construct mathematical ideas and concepts with very little input from their teacher.

In a study involving undergraduate preservice teachers using NuCalc, an interactive algebra application, Manouchehri (2004) found an increase in the quality of mathematical explanations students offered during discussions, both in terms of complexity and mathematical content, while using NuCalc as compared to previous discussions that were not supported by any specific software. Four ways that the software supported discourse were identified:

1. Tool for assisting peers in constructing more sophisticated math explanations
2. Motivated engagement and increased participation in group inquiry
3. Mediated discourse resulting in a significant increase in the number of collaborative explanations constructed
4. Shifted the pattern of interaction from teacher to peer driven.

Further, Manouchehri concluded that because of the immediacy of feedback to students the software also supported a culture of conjecturing, testing, and verifying, formalizing mathematics and collaboration, and shifting the locus of power to the students.

11.3.3 Technologies Designed for Collaboration but Not Necessarily Mathematics

Technologies in this category include Information and Communication Technologies (ICTs), specifically, computer-mediated networked databases built to support social interaction, cooperation, and collaboration for learning and knowledge building. ICT systems can allow synchronous or asynchronous communication, can provide archival storage for the products of the connected group, and can enable learners to model, communicate, and document their shared understanding of new concepts. A central tenet of ICT-based epistemology is an understanding that it is not so much the individual student who learns and thinks as it is the collaborative group (Stahl 2005; p.79). Meaning making occurs in the context of joint activity (Koschmann 2002) – the inter-subjective construction of shared meanings achieved through group interaction (Stahl, p.82).

In the domain of mathematics education, there are two main areas of study that have included ICT technology. The first are those studies in which ICT is used in conjunction with classroom-based teaching in order to enhance the learning environment. These include studies of technology to support asynchronous and synchronous communication between students on-campus as well as those who are geographically distributed. In their 2007 review of ICT technology, Resta and Laferriere identify three primary aims of these kinds of studies including (1) to see how these environments support collaboration between students to enhance their learning processes (Kreijns et al. 2003); (2) facilitate collective learning (Pea 1994) or group cognition (Stahl 2006); and/or (3) to foster student engagement and keep track of student collaborative work and online written discourse (Stahl 2006).

The second area of study concerns the use of ICT in distance learning courses as the sole conduit for teaching (communication from teacher to learner) and communication (between teacher and learners, and between learners), to allow for flexibility in creating working groups that are not bound by proximity of time and/or space.

11.3.3.1 Enhancing the Learning Environment

Examples of Enhanced Learning ICT research reported at ICMI 17 include a study by Beatty and Moss (2006), of their research in the use of a web-based collaborative workspace, Knowledge Forum (KF) (Scardamalia and Bereiter 1994), to support Grade 4 students in generalizing with patterns as part of a teaching intervention in early algebra. In this study, KF was utilized after students had participated in a 12-lesson instructional sequence to determine how engaging in collaborative problem solving on the database would further develop their abilities to work with patterns and functions. The investigation revealed that the opportunity to work on a student-managed database benefited students by providing them with access to each other's theories and perspectives on the problems posed, and thus supported them in developing an understanding of algebraic rules for patterns. Because students were restricted to communicating their ideas to one another via a text-based discourse platform, the structure of this communication necessarily became precise and, subsequently, formalized. Students negotiated theories, questioned one another's theories, elaborated on their thinking and compared ideas. Students developed a community of practice in which the offering of evidence and justification for their conjectures became the norm, and this in turn supported students' deepening conceptual understanding both of mathematical functions, and the role of justifying (a precursor to proving) in mathematical discussions.

Another study by Kramarski (2006) evaluated the effects of software designed to display metacognitive prompts during problem solving. In the study, based on a study by Kramarski and Hirsch (2003) with Self-Regulated Learning within a CAS environment, a database was set up for 9th grade students with two kinds of scaffolds (1) scaffolds to assist students in self-regulating their cognition, motivation, and behavior while collaborating on problem solving; and (2) instructions of how to provide mathematical explanations. Both online supports led to an increase in students' abilities to successfully problem solve and provide mathematical explanations, which suggests that the explicit guidance given heightened the level of students' online discourse, and that this consequently increased their ability to engage in deeper mathematical reasoning.

Similar results from the mathematics literature indicate that students' need to clearly communicate their theories in asynchronous discussions results in enhanced levels of mathematical discussion (Jarvela and Hakkinen 2002) and mathematical understanding (Nason and Woodruff 2003; Hurme and Jarvela 2005). In these studies, whole group peer – peer discussions were conducted without the presence of a teacher voice on the database, so that students took on the responsibility for monitoring, critiquing, refining, and justifying their own and each other's problem solutions.

In all these studies, the activities that took place on the database provided an additional collaborative communication space that enhanced, but differed from, classroom-based collaborative work. Online, all participating students had an opportunity to enter solutions by contributing text (and in some cases, graphic) notes, and participate in discussions by responding to questions, providing, and building onto ideas and offering or critiquing findings. Through constructively commenting on offered solutions, students engaged in a process of revising and improving both their own and others' initial ideas, and so were effective in facilitating the construction of complex conceptual artifacts.

Generally in these studies the discourse space was student-managed – students learned with and from peers, which has been shown to be effective (Cohen 1994; Good et al. 1992). Historical problems linked to peer learning, such as the disparity regarding status and the associated quality and quantity of interactive contributions between low and high achieving students, were somewhat alleviated by having students take part in online asynchronous collaborative discussions – particularly when collaboration was between classrooms where participants had not met other than through notes posted in the database (Moss et al. 2008).

One limitation that has been noted with respect to using ICT technologies in the domain of mathematics learning is the need to find problems that elicit *sustained* meaningful knowledge building discourse, with the suggestion that non-conventional and ill-structured tasks should be used as they have the potential to allow students to engage in sustained problem solving within the ICT environment. This in turn enriches the authenticity of students' mathematical activity and their understanding about the nature of mathematical discourse (Nason and Woodruff 2003).

ICT and its abilities to archive discussions offers researchers a way of further studying distributed cognition. For example, Stahl utilized an online discourse platform, Math Forum, to further study the feasibility of studying "group knowledge", how group knowledge can be constructed in discourse, and how discourse analysis can make visible that knowledge to researchers (Stahl 2005; p.86). The focus of *Virtual Math Teams* Project (Stahl 2006), was to determine how groups of three to six middle- or high-school students build knowledge in a synchronous online community as they discuss mathematics in online chat rooms. In this study, students participated in synchronous discussions, and were also able to access and use a shared whiteboard for drawing geometric figures and for archiving and displaying contributed notes. The goal of the project was to generate empirical examples of concrete situations in which groups can be seen to have knowledge that is distinct from the knowledge of the individual group members.

11.3.3.2 Distance Learning

Distance learning is defined by the National Science Foundation (NSF) (2000) as education where learning occurs all or most of the time in a different place from teaching, and the principal means of communication between learners and teachers is through technology. One of the fastest growing modes of delivery is the use of internet-based ICT to deliver courses based on synchronous, asynchronous, or both synchronous and asynchronous communication where instructors and students are not present at the same location at the same time (Spiceland and Hawkins 2002).

Technologically supported distance learning represents a relatively new model for delivering courses to students. As this model continues to be more popular, researchers are now looking at the relationship between distance (time and space) and learning. In a study conducted by Simonsen and Banfield (2006) with 25 secondary mathematics teachers, the authors were interested in researching the quality, nature, and impact of asynchronous mathematical discussions that occur in distance learning courses, since in these courses students are virtually encouraged to take part in some level of mathematical discussion as a requirement of the course. They concluded that participating in asynchronous discussions supported high levels of sophisticated mathematical discourse. This in turn supported the construction of mathematical knowledge. The authors acknowledge that there is a need to examine the cognitive development of students involved in online learning, and the relationship between the quality of the discourse and students' success in the course.

Borba (2005) has utilized both synchronous and asynchronous forms of distance communication in online courses for mathematics teachers in Brazil, and in connecting Brazilian teachers with colleagues in other countries. Borba incorporates scheduled online chat sessions and asynchronous email discussions in order to research the kinds of communication that emerge in these environments, and how these differ from the kinds of knowledge that are developed from other media (such as pencil and paper). Mathematics that is done through synchronous chats on the internet differ from the mathematics that is conducted in face-to-face classroom settings as writing in non-mathematical language becomes an integral part of "doing" mathematics

It is important to note that distance learning is not just for students who elect to take online courses because of time/distance constraints. The impetus for developing internet-based courses in Borba's studies came from a desire to redress the social inequalities in Brazil, and to connect teachers living in remote parts of the country to research centers and universities. Sloan and Olive (2006) discuss how crucial distance learning is for rural schools in the US, particularly given that almost one third of US children are educated in rural schools. The authors equate the needs of rural US students to those of students attending schools in developing nations with respect to limited funds and/or remote geographic locations. Surveys show that for students in rural and economically stagnant areas, distance learning is important for gaining access to advanced mathematics courses, and that 46% of rural districts provide distance learning courses and utilize various modes of delivery including two-way interactive video and internet web-based courses. The authors outline the kinds of ongoing investment needed to continue to develop effective distance learning, including teacher professional development, and resources (money, hardware).

11.3.4 Technologies Designed for Neither Mathematics nor Collaboration

Finally, there were reports from other participants that noted the collaborative activity of learners which ensued from interaction with technologies that were not designed specifically for the learning of mathematics or to act as catalysts for social interaction.

The use of robots in a study by Fernandes et al. (2006) of K-8 level students investigated the potential for the use of robots to act as mediators between students and mathematics (see also Chap. 3 of Theme C). This paper documented the collaborative practice that followed when students were presented with problems that challenged them to program robots to follow a predetermined trajectory. Researchers reported that the mutual engagement in the enterprise lead to the co-definition between members of working teams of the meaning gained from the activity.

11.4 Future Developments

11.4.1 New Forms of Communities of Learners

If we consider learning as participation in a community of practice, what is clear from a review of these studies is that our conception of "communities of practice" needs to be continuously re-defined as new technologies change the kinds of communities of practice that are available and feasible. As these technologies continue to be developed and refined, they offer new ways to construe communication, collaboration, and social interaction. Ares and Stroup (2004) claim "Interactive devices make possible the layering of students' discursive communication with simultaneous transactions through the network, potentially allowing students to contribute to collective processes through channels that were unavailable in conventional classrooms." This is true for both classroom-based and online learning environments.

11.4.1.1 Amplifying, Enhancing, Broadening Classroom-Based Communities

There are three major ways in which connective technology can amplify and reshapes classroom-based discussions. These include (1) multiple modes of contribution (language, text, physical, and electronic gestures); (2) multiple representations (texts, graphs, visual displays of emergent systems, aggregated displays); and (3) inquiry-oriented discussion and analysis (Ayres and Stroup 2004; p.838). One of the most salient technologies is the use of *dynamic representations* to foster student collaboration. Students who work in pairs in front of the computer can engage in discussion mediated through a mathematical object that reflects changes in theories, which then allows for a back and forth of problem solving, theorizing, testing, and checking. In a PC sharing situation, a pair or group of students collaborate in front of a single machine and share a view of objects around which, and a representational context within which, to establish and negotiate collective meanings and convergent interpretations for the phenomena that structure a joint problem space (Roschelle and Teasley 1995; Goos 2004; Goos et al. 1999, 2003). In a networked environment, the relevant objects and representations of a problem are shared across devices of two or more students, and mathematical objects may appear in different views, configurations, or representational settings that require coordinating collective meanings and discursive negotiation (White 2006; p.361). Students developed a precision of discourse when working to calibrate their devices that showed different representations of the same object. This coordinating discourse, or *discursive alignment*, can lead to the development of a deeper understanding of the concept being considered.

Evolving classroom norms as connective technology becomes integrated into classroom practice can foster conditions for true collaboration. For instance, in the *SimCalc* Project, the identification of outliers in aggregate displays led to students collaboratively problem solving both the cause of the error and proposed solutions.

11.4.1.2 Online Communities: Virtual Communities of Learners

In web-based technologies, the communities of learners are housed within networked discussion spaces, and would not exist without the technology through which they are created. The result is entirely new forms of communities of practice linking members through space (synchronous discussions) and time/space (asynchronous). Researchers are now beginning to consider what is unique about the potential of collaborative study of mathematics while physically separated, and how this potential might be harnessed to support mathematics learning. Online communities also offer opportunities for groups to be national or international – bringing together different culturally bound paradigms of mathematical conceptions.

At the most basic level the attributes of these kinds of software, for instance the archiving of contributions, allow students, teachers, and researchers to trace contributions and review discussions, thus accessing a record of the development of increasingly sophisticated collective ideas. Bakhtin's "traces of the voices of other members of the learning community" become archived artifacts to document the development of collaborative communal understanding and, therefore, the intellectual development of the community, something that can get lost in real-time face-to-face classroom-based discussions. For instance, in the Knowledge Forum software, as students contribute and respond to one another, their notes are automatically linked, and the resulting webs provide an ongoing visual record of the development of increasingly sophisticated collective ideas. Stahl also highlights the importance of a "graphical referencing tool" that allows participants to reference existing items (or contributions) in the online environment by drawing a line from new message to the existing item, as a way of giving meaning to and structuring their online interactions.

However, it is not just the features of the software itself that promoted student learning, but also the pedagogical knowledge building or knowledge constructing or co-constructing principles that can be leveraged in virtual communities (e.g., Stahl 2006; Scardamalia and Bereiter 2003; Scardamalia 2002). Examples of these include the group responsibility for ownership of ideas that are given a public life in the database and ask for clarification or revisions of ideas with an eye towards moving the theorizing forward (Moss et al. 2008). A related concept is sustained idea improvement – purposefully revisiting initial theories to revise and improve, which provides students with an extended time to think (Moss and Beatty 2006). Finally, because all contributions are accessible by all participants, the collective knowledge of the group is "democratized" and all students contribute solutions, theories, critiques, comments, and ideas.

Research is still in the beginning stages of understanding how to design virtual communities that result in meaningful collaborative experiences. In their work within the *WebLabs* Project, (Matos et al. 2005) three components of online communities that seem to engender high quality mathematical and meta-mathematical discussion were identified – these include facilitation, reciprocation, and audience awareness. Facilitation refers to keeping the *WebLabs* game on track, setting new challenges to participants and shifting the conversation towards mathematical content. Reciprocation refers to the fact that participants tended to respond to more difficult math challenges with detailed explanations accompanying their solutions. Finally audience awareness suggests that participants seek out other participants who will engage in sustained interactions, leading to heightened levels of collaboration.

Researchers agree that sustained mathematical interaction is crucial for developing collaborative mathematical thinking through negotiating, formulating, revising, and critiquing solutions for complex problems. In *WebLabs*, participants needed time and experience to shift from the competitive and technical base level of the game to a collaborative effort of understanding the mathematical structure of their models and sharing of analytical tools. In the KF studies, students needed to engage in a certain amount of back and forth discussion before engaging in higher-level mathematical discussions (e.g., Moss and Beatty 2006; Nason and Woodruff 2003).

Consistent with Pea's notion of distributed cognition, the discussion above describes learning as taking place in physically proximate and in virtual communities in which participants and elements of their environment, including the technological mode employed, all contribute to changes in the learner and the community of learners.

11.4.2 Extending the Role of the Teacher

While there is a developing body of work associated with the role of technology in learning, less is known about how the availability of technology has affected teaching approaches (Penglase and Arnold 1996). There is now some evidence that technology can mediate more student-centred, exploratory, and discursive approaches (e.g., Simonsen and Dick 1997), but teachers' personal philosophies of mathematics and mathematics education are also influential (Tharp et al. 1997; Thomas et al. 1996). Thus, the nature of an available technology alone will not ensure the implementation of collaborative practices in any learning environment and so the classroom teacher or the designer of the virtual learning environment has a vital role to play in mediating the type of social interaction that is regarded as collaboration within a community of learners.

In classroom-based studies that included connective technology, researchers emphasized the teacher's deliberate calculated decisions about when and how to encourage students to use the software to facilitate collaborative problem solving (e.g., Manouchehri 2004, Goos et al. 2003). Similarly in reports of the SimCalc MathWorlds teachers orchestrated the learning experience, and utilized a deep knowledge of content and pedagogy to facilitate student learning along a trajectory from static, inert representations to dynamic personally indexed constructions, and that it was this deep knowledge that allowed teachers to take advantage of a connected classroom (Hegedus and Kaput 2004).

However, in studies of virtual communities, the teacher's voice in the community often seems peripheral or non-existent. For example, in a study by Simonsen and Banfield (2006) of the progression of mathematical discussion that evolved during an online graduate course, the researchers identified five roles for instructors/teachers when students are engaged in asynchronous discussion – resolve conflicts/confusion, validate responses, redirect tangential threads, expand ideas, and withhold any input. The researchers concluded that the majority of time, withholding of instructor input encouraged students to increase their participation in the discussion, and their ability to collaboratively problem solve. As these students were secondary school math teachers, it is possible they were already well-versed in the norms of mathematical discourse. If we consider studies of younger students, such as those involving high school or middle school (e.g., Stahl 2006; Hurme and Jarvela 2005; Jarvela and Hakkinen 2002) or elementary students (e.g., Beatty and Moss 2006; Nason and Woodruff 2003) there was minimal or no teacher or researcher voice in the database, and yet students were similarly able to successfully moderate their own discussions. It is important to remember, however, that in these latter studies, the online discussions took place as part of a broader instructional initiative. Online interaction does not evolve towards higher levels of discussion without proper grounding, monitoring, modeling, coaching, or contributing on the part of the instructor, particularly at the onset of instruction (Resta and Laferriere 2007). (For more on technology and teaching mathematics, please see Sect. 3 of this volume).

11.4.3 New Forms of Voice and Discourse

Technology facilitates new kinds of communication and there is a need to examine more closely this instrumented discourse, how different forms of discourse are engendered by different forms of technology, and how this in turn is linked to new kinds of mathematical understandings. Researchers also need to develop a way of communicating about instrumented discourse, and start to define a meta-discourse to look at its development across studies.

When considering face-to-face interaction mediated through technology, or studies in which technology is the catalyst around which discussion occurs, often the utterances of participants are incomprehensible in themselves, since they are bound with the on-screen images as an integral component of the discourse (e.g., Lavy and Leron 2004; Olive 2000).

Researchers have written about the differences in problem solving that occur when participants communicate online, and concluded that human knowledge is bound with the media of expression, and therefore new modalities of language that emerge from computer technology alter the kinds of participation and communication (and therefore cognitive) opportunities available (Borba 2005; Stahl 2006; Moss and Beatty, in press).

Beyond an examination of how technological communication modifies verbal or text-based discussion, researchers have also begun to examine the development of entirely new forms of communication. For instance in programmable microworlds, the programs themselves are designed to "allow users to express their own mathematical ideas" (Balacheff 1993) thus adding to the modes of communication traditionally available to students (text, symbols, tables, and graphs). Examples are the *Playground* and *WebLabs* Projects conducted by Noss et al (2002), where elements of the computer program, pieces of the models that students create, were the mode of communication.

11.4.4 Other Issues

11.4.4.1 The Case of Marginalised Members of a Community

Technology can foster peer interactions either face-to-face or online, by connecting learners. However, collaboration will not automatically occur simply because peer-to-peer interaction is supported and facilitated. *Connectivity* in itself does not necessarily entail *collaboration*. Some students will continue to opt out of the collaborative process. For instance, Geiger's (2006) paper describes a series of episodes in a secondary mathematics classroom in which a learner, who initially rejects his teacher's attempts to conduct his mathematics classroom according to socio-cultural principles, is eventually attracted to participate in the community of learners via experiences in which he is encouraged to pursue an interest in designing mathematics-based videos through the programming of his calculator. The technology in this instance is viewed as catalyzing the student's participation in collaborative practice during this incident and then into the future. Consistent with this finding, Goos (2004) also acknowledges that the nature of engagement and the extent of participation varied between students. While this is consistent with Lave's idea of *becoming* within a community of practice, Goos concedes that a small number of students remained resistant to the adoption of this specific community of learner's modes of knowledge creation and validation, and reminds us that inclusion does not necessarily guarantee participation or the appropriation of teacher's aspirations for the way in which the community of inquiry should manage itself.

11.4.4.2 Emergent Uses of Technology

Research has focused on studies in which the technologies used are designed to engender or enhance processes of collaboration, usually task-based and incorporating specific planning of how students will interact. Ramsden (1997) has argued that while it is not possible for a technology to be used for a purpose for which it is patently unsuited, emergent, or unexpected uses of technology are a territory that should be pursued, as it is often the divergent uses of technology by students and teachers that provide the most exciting outcomes. Thus, researchers must also accept and actively become aware of emergent uses of technology. For instance, Sinclair's (2005) study of high school students' use of Geometer's Sketchpad/Java Sketchpad to examine geometric concepts found that the pair interactions had greater impact on the collaborative learning environment than the researcher had expected. In another case, Lavy and Leron (2004) reported on how software that allows for dynamic interactions with mathematical artifacts can lead to unanticipated shared constructions of mathematical concepts. More documentation of uses of technology that are not necessarily part of the original design will be a necessary condition for progress of research in this area.

11.4.4.3 Unit of Analysis

The conceptualization of technology as an agent which is more that a simple tool that amplifies human capabilities brings into question the unit of analysis that should be employed when engaging in research based in technologically rich environments. Borba and Villarreal (2006) have referred to such a unit as humans-with-media, although his construct does not fully recognize the range of diversity of the relationships which develop in technologically enhanced communities of practice. Lavy and Leon (2004) refer to technology integrated with student discussion as a "super entity" that combines the effective attributes of each member – the individual learners and the computer environment act in synergy to contribute to a shared learning process. A review of current research with focus on collaboration reveals the need to expand this notion, as the unit of analysis becomes the interplay between "humans-with-media-with-many". There is a need to extend the discussion to include multiple media used in different contexts, and how these alter the way students collaborate. The unit of analysis ranges from individual + technology, small group + technology, whole class + technology and then to the communication of online virtual communities.

11.5 Conclusion and Final Remarks

Our central question was to examine increases in socially oriented theoretical frameworks in relation to the use of digital technologies, and how these technologies have been used to enhance the teaching and learning of mathematics by supporting the social aspects of knowledge building and meaning-making. As socio-cultural learning paradigms have evolved in the study of mathematics teaching and learning, there has been a corresponding increase in research into the ways in which didgital technologies can re-shape the way we conceptualize "communities of learners", and the kinds of mathematics understanding that these communities formulate.

Studies within the four typologies include a wide range of methodologies, perspectives, and parameters. Some look at the discourse between two or three learners in front of a desktop computer, others the interaction of many users contributing to an online database. In all studies, the underlying theoretical frameworks emphasized the importance of discourse and collaboration as essential to the process of learning mathematics. All used rich open-ended tasks, and all specified the affordances of the particular kind of technology used for engendering collaborative communities of practice – whether based around aggregated dynamic representations, or archiving threads of discussion in student-managed discussion platforms. And in all studies, technology was viewed as a means of mediating social interaction "not by constraining action, but by providing a medium through which shared mathematical expression can be constructed" (Healy et al. 1995).

The theories of intellectual development that have had greatest influence on teaching and learning since the beginning of the last century all were initially conceptualized before digital technologies (in the forms we understand them today) were available. There is a great deal of research that still needs to be carried out that considers the interplay between the developing role of technology and social theories of learning.

Finally, the authors would like to acknowledge that this paper could not have been written without internet based ICT – specifically email – which allowed us to collaborate while we were literally half a world away from one another.

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Section 3
Teachers and Technology

Chapter 12
Introduction to Section 3

Lulu Healy and Jean-Baptiste Lagrange

Abstract In this text, we present the aims that motivated the theme entitled "Teachers and Technology". The theme was organised to consider a variety of issues related to preparing teachers to teach mathematics in the digital age and to the challenges of appropriating and integrating technologies into pedagogical practices. Discussion during the conference gave rise to the structuring of the three chapters that compose this section of the book. The questions considered within these chapters and the relationships between them are briefly introduced.

Keywords Teachers • Pre-service and in-service teacher education • Classroom implementation

12.1 Introduction

Despite the fact that teachers have a central role in the mathematics classroom, they have been somewhat neglected players in research considering the relations between digital technologies and mathematics education. As noted by Healy (2008), in early research in this field the focus was largely directed towards the individual doing mathematics with software, and this is only gradually giving way to research which attempts to recognize and understand the role of the teacher and the challenge of teaching mathematics in the presence of digital technologies. Lagrange et al. (2003), in their review of research literature in this field, reached a similar conclusion, pointing to the relative paucity of systematic studies investigating the appropriation by mathematics teachers of digital technologies into their classroom practices. Those

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studies that do exist indicate that modifying teaching practices to include new tools is no mean feat for teachers. In addition to mastering the various possibilities for doing mathematics offered by different digital tools, they also are faced with the need to rethink a number of classroom management issues, adapt their teaching styles to include new forms of interactions – with students, between students and between students and mathematical ideas – take a more prominent role in designing learning activities for their students and confront a range of epistemic issues related to the acceptance and legitimization of unfamiliar or even completely new mathematical practices (see Laborde 2008). Not surprising then that the process of orchestrating technology-integrated mathematics learning is neither a spontaneous nor rapid one.

It was with this context in mind that the theme "Teachers and technology" was distinguished for inclusion within the study conference. The purposes of this theme were to document and to address issues related to the preparation of mathematics teachers in the face of ever-evolving resources and to consider complementarities and contrasts in the frameworks currently being employed in attempts to understand the role of the teacher. Prior to the conference, the following seven questions were formulated to serve as guides to those wishing to participate in the discussion related to this theme:

1. What theoretical frameworks and methodologies illuminate the teacher's role in technology-integrated environments for mathematics learning?
2. What kinds of pedagogical approaches and classroom organisations can be employed in technology-integrated environments including distance teaching and how can they be evaluated?
3. How can a focus on technological tools help us understand the ways in which mathematical practices and the roles of the teacher vary across settings?
4. How can teachers be supported in deciding why, when and how to implement technological resources into their teaching practices?
5. What kinds of pre-service education and professional development programmes are appropriate to prepare teachers to use technology in their mathematics classrooms and to help them to sustain ongoing use?
6. What can we learn from teachers who use, or who have tried to use, digital technologies for mathematics teaching?
7. How do teachers' beliefs, attitudes, mathematical and pedagogical knowledge shape and become shaped by their use of digital technologies in mathematics teaching and how are these issues influenced by access to resources and by differences in culture?

A total of seventeen papers were selected from those submitted for inclusion in this theme. These are available in the conference proceedings (Son et al. 2006). An initial analysis of the papers submitted suggested that in terms of the issues raised and the questions addressed, they could be grouped in three sub-themes: discussions of collaborations between researchers and teachers in in-service education scenarios, articles concerned with understanding issues related to classroom implementation and papers concerned with pre-service education of mathematics teachers in the digital age. These three sub-themes evolved into the three chapters which compose this section of the book.

The introduction of digital technologies into schools, and particularly mathematics classrooms, across the world, has been accompanied by a demand for in-service education courses to prepare practising teachers to make use of these new resources. Initial visions of the outcomes that might be expected of these courses were often extremely ambitious. In Brazil, for example, in the 1980s, the computer was to serve as a *catalyst for pedagogical change* (Valente and Almeida 1997), enabling innovative approaches to education and helping to form reflective citizens who would use exploit technology in the construction of knowledge which would empower them to better understand and transform their own socio-historical context. A huge, and essentially unsuccessful, challenge given that the dominant pedagogical approach of the time almost exclusively focused on teaching as transmission of ideas. As it turns out, the pedagogical aspects associated with the use of digital tool in mathematics teachers are rather more complex than originally imagined and the need to involve teachers as partners rather than students has become increasingly evident. This complexity stems, according to the authors of Chapter 13, from the fact that tools are a constituent part of culture, hence the introduction of new artefact necessarily involves the establishment of new cultural practices. The central argument permeating this chapter, which has its roots in the principles behind in-services courses such as those described by Hoyles et al. (1991), is the importance of forging partnerships with practicing mathematics teachers focused on the design of learning activities involving the use of digital tools and/or the design of the digital tools themselves.

The chapter revolves around cases studies from three different countries, Norway, Greece and Brazil, interpreted in terms of two different theoretical frameworks. According to the first, the process of communal design can be treated as a means through which to create boundary objects, objects intended to have utilities in the practices of different communities – in this case the teacher education community and the community of the mathematics classroom – and to permit the emergence of a common language, equally meaningful to participants on either side of the boundary. A goal of creating new artefacts is established as a central part of the in-service course in order to emphasis the role of all participants in the design process as agents of cultural change, which in turn necessitate the recognition and embracing of the complexity of including new artefacts into any community of practice. The second theoretical tool discussed in the chapter concerns the instrumental approach of Rabardel and Vérillon. This framework is in fact mentioned in all three of the chapters in this section, as well as in various other chapters in the book (see in particular Chap. 7 in Sect. 2). In this chapter, instrumentation is described as the shaping of thinking by the tool in the construction of mental schemes and instrumentalisation is considered as analogous to activities that involve the shaping of the tool by users (in the terms discussed by Noss and Hoyles 1996, for example). Given their strong emphasis on design, the authors take a rather particular stance in relation to instrumentalisation, arguing that while this seems usually to be perceived in terms of the way the tool is used by different individuals in different activities, involving teachers in the process of shaping the tool in ways that also influence their potential – built-in – uses may be particularly important for understanding the relationships between tools and mathematics learning.

Chapter 14 addresses the challenges associated with understanding the various facets that influence the use of technology (or not) by teachers in their own mathematics classrooms. As the authors point out, in contrast to the rather optimistic vision for the future of technology integration in mathematics education of the first ICMI study conference on technology (Churchhouse et al. 1986), the actual take up of technology within mathematics classrooms across the world has progressed rather slowly. Given this scenario, the chapter focuses on the search for theoretical frameworks that might illuminate the teacher's role in technology-integrated learning environments and clarify the factors that mediate teachers' use of digital technologies. In the first part of the chapter, various different theoretical frameworks for interpreting the teacher's role in technology-integrated learning environments are analysed and compared, by applying each in turn to data from two different research projects, one involving the use of Cabri-géomètre and the second a graphics calculator activity.

Of the frameworks considered, particular attention is centred around two. The first of these is the instrumental approach. In contrast to the previous chapter, where the process of instrumentalisation was given centre stage, here the material part of the artefact seems to be taken as given, and instrumentalisation is described as the learning of various uses of the tool – the example given in the text is that of learning to drag points. According to the authors, in this example, the complementary process of instrumentation involves aspects of the mental scheme concerned with why points are dragged, which will be related to the learner's conceptualisation of the geometrical properties associated with the tasks in hand. However, the authors are not mainly concerned with documenting the genesis of artefact into instrument, but how the teacher organises the conditions for instrumental genesis of the technology proposed to the students and the extent to which mathematics learning is fostered through instrumental genesis. They call this the process of 'instrumental integration'.

The second framework emphasised draws strongly from Valsiner's (1997) zone theory. This theory suggests that the developing structure of an individual's environment and his or her relationships with others in this environment can be described in terms of three zones: Vygotsky's Zone of Proximal Development (ZPD) is joined by the Zone of Free Movement (access to the various components of the environment and means or interacting with them) and the Zone of Promoted Action (activities promoted in the environment). In the development of the chapter, this framework is applied not only to interpreting the role of the teacher, but also to delineating the various factors that intervene in the process of implementing digital tools in the mathematics classroom. By applying different frameworks to the same classroom episodes, the authors are able to highlight the aspects most supported by each – for example, they suggest micro-level analysis emphasised in the instrumental approach might be complemented by zone theory where the lens is focused on the macro-level.

In the final chapter contained within this section, attention turns to teacher education. The authors of Chap. 15 are concerned that, despite the fact that teacher development courses increasingly consider the use of digital technology in the teaching and learning of mathematics, course developers do not yet have access to a robust corpus of literature documenting strategies already tried and tested by others.

In this chapter, they offer a number of dimensions by which teacher education courses, be they pre-service or in-service, might be characterised. Specifically, they focus on the implementation of technology in these courses, on attention to the changes in the teachers' role, activity and practices, and on the adaptation of teaching practices with regard to time and professional proficiency. In order to develop tools by which to classify these different aspects of teacher development programmes, five different courses, each of which was presented in some detail in the conference proceedings, are subjected to analysis.

In terms of classifying the beliefs underpinning the courses analysed, three areas are considered: views related to the implementation of technology into teaching (both teaching mathematics and teaching mathematics teachers); beliefs associated with the impact on technology on teaching practices, the teacher's role and teaching activities; and views on how to prepare teachers. In each case, the authors devise a graphical means of illustrating where each course is positioned in relation to each of the dimensions explored, allowing also a means of representing proximities and distances in the approaches adopted in each course. In relation to the question of identifying the practical decisions regarding course design, the authors discuss two areas: how technology is addressed in the course curricula and the teaching strategies utilized. Radar charts are used to contrast the treatment in the courses of six possible issues related to technology use that might be discussed. Finally, four strategies used by teacher educators across the five courses are identified and described. It is perhaps worth noting that the only strategy common in all five courses was that of demonstrating good practice, the other three strategies being that of involving future teachers in role playing activities where they are invited to assume initially the role of students, reflection on action in the sense of Schön (1983) and learning in communities – a strategy perhaps motivated by theories such as those described in Chap. 11.

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Chapter 13
Working with Teachers: Context and Culture

Anne Berit Fuglestad, Lulu Healy, Chronis Kynigos and John Monaghan

Abstract This chapter concerns collaborations between teacher educators and teachers in activities involving digital technologies in the teaching and learning of mathematics. In light of the complexity involved in introducing new artefacts into existing cultures of practices, we focus on our attempts to develop ways of working with teachers so that they can become active participants in designing practices and routines appropriate for the particularities of their own classrooms. Three case studies are presented, from three different countries, Norway, Greece and Brazil, each of which describes the participation of teachers in a process of communal design of mathematical tools and activities. Two theoretical notions, *boundary objects* and *instrumental genesis*, are employed in order interpret the case studies and to illuminate the challenges associated with involving teachers in considering when, how and why digital technologies might be used fruitfully in the teaching of mathematics.

Keywords Digital technologies • Culture and context • Communal design • Teacher education • Boundary objects • Instrumental genesis

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13.1 Introduction

In this chapter, we look at ways that mathematics educators who are not school teachers can work with teachers on the integration of digital technologies into their teaching. Our starting points are threefold: that working with school teachers is not telling school teachers what to do but requires that a dialogue be established in which different views are respected; that considerations of the context and culture of school teaching are paramount to establishing such a dialogue; that teaching is a complex undertaking and introducing digital technologies into teachers' classrooms adds to this complexity regardless of whether teachers find this an easy or a difficult thing to do. We restrict attention to in-service secondary school teachers because of a belief that issues with the integration of digital technologies into primary, secondary and university teaching, and of novice and established teachers, whilst having some commonalities, vary and the case studies we present focus on serving secondary teachers. This chapter is organized in five sections. The next section considers issues of context, culture and teachers' practices and sets the scene for three case studies of working with teachers in Brazil, Greece and Norway. The final section draws lessons from these case studies.

13.2 Context, Culture and Teachers' Practices

"Context" and "culture" appear unproblematic in their everyday use. As teacher educators, we can go into schools in our respective countries and speak of the "context of teaching and learning" and the "classroom culture". But when we attempt to transcend specific schools in specific countries the terms become problematic. Cole (1996) describes context as "perhaps the most prevalent term used to index the circumstances of behavior" (p. 132) and presents two metaphors for context: that which surrounds and that which weaves together. With regard to secondary teaching, context is that which surrounds and situates a teacher in a class, with resources, in a school, in an educational sector and in a country. Cole views the relationship between activity in surrounding layers dialectically. The metaphor, however useful, has its limitations. At the heart of what teaching and learning are about is what teachers ask students to do but "the boundaries between "task and its context" are not clear-cut and static but ambiguous and dynamic" (p. 135). It really just depends on what we, as analysts, focus on – our unit of analysis – and trying to focus on everything at once leads to empty generalities. Viewed in this way, context weaves together rather than surrounds and teaching can be seen as a thread winding between other peopled and institutional threads.

It is common to think of culture in terms of the customs of people of other nations or ethnicities but this can be somewhat vague and relativistic. We turn again to Cole for clarification, "artifacts are the fundamental constituents of culture" (1996, p. 144). Cole's argument, very briefly, is that the things people do in their everyday settings involve a multitude of coordinated artefacts which mediate their attitudes and beliefs, their social interactions and their actions on the nonhuman world.

In teaching, if new artefacts such as digital technologies are to enter the classroom in other than a peripheral form, then a new culture needs to be established in which digital practices are coordinated with established artefacts and routines.

Pedagogy is a central consideration in working with teachers using digital technologies. Pedagogy, "the fundamental social context through which cultural reproduction-production takes place" (Daniels 2001, p. 69), concerns why and how teachers do things with artefacts around them. There are many pedagogies, many things a teacher can do and many things a teacher can use, but what they do, what resources they use and how they use them are intrinsically bound up with context and culture. The coupling of culture and context allows us to retain a focus on pedagogies in practice, to retain a focus on commonalities across education systems and across countries, whilst recognizing the dangers of imposing approaches that may be inappropriate or unviable given the specificities of the system in question (Nkhoma 2002). It is also critical that the complexity of introducing digital technologies into teachers' practice and how it affects all levels of classroom activity is recognized. Monaghan's (2004) study of the activities of teachers attempting to make sustained use of digital technologies in their mathematics classroom shows how a stated goal of, for example, providing a rich spreadsheet activity for one's students can become overshadowed in practice by emergent goals (in the sense of Saxe 1991) such as managing the printer queue and the behaviour of students waiting for work to be printed.

It is our aim to keep in mind, as we present the case studies in this chapter, that the material conditions that teachers work with and under, their established practice, their classroom routines, prior to the introduction of digital technologies must be appreciated if we are to work with them to develop appropriate practices which exploit the potential of digital technologies. As we hope the case studies will show, we do believe that digital technologies can allow teachers to reflect critically upon existing practices, but this can only occur if we are sensitive to the fact that digital technologies do not enter a teaching void, but enter into existing cultures of practice. Teachers need to make sense of the entrance of this new artefact into their practice. This sense making is unlikely to happen by others telling teachers what to do, it needs to occur in situations within which teachers themselves are active participants in designing the practices and routines appropriate for the particularities of their own classrooms.

In this vein, Korthagen and Kessels (1999) discuss Aristotle's notion of *episteme* versus *phronesis*, i.e. theoretical de-contextualized knowledge applicable to a wide spectrum of situations versus situation-specific knowledge derived directly from experience within that situation and aimed at meeting a problem within the situation itself. In arguing for a realistic teacher education pedagogy, they suggest that building on teachers' phronesis is critical to their understanding a theory and most importantly in building a constructively reflective habit of mind in the teachers, helping them perceive their profession as a developing one.

In seeking frameworks which help illuminate the challenges associated with working with teachers and involving them in considering when, how and why digital technologies might be employed fruitfully in the teaching of mathematics, we identify two in particular that we will use when interpreting the case studies. The first theoretical notion to which we turn is that of “boundary objects”. Boundary objects “inhabit several intersecting social worlds… They are weakly structured in common use, and become strongly structured in individual use.” (Star and Griesemer 1989, p. 393). Their importance has been noted by researchers concerned with interaction between communities (Wenger 1998; Hoyles et al. 2004; Kynigos 2002, 2007a). Certainly an issue in working with teachers is to develop practices in which artefacts introduced within teacher education courses, softwares, microworlds, suggested activities, worksheets, etc., come to have utility beyond this context, traversing the boundary between practices associated with the community of teacher educators and entering into those of the teachers and their students. Boundary objects do not carry meaning with them, instead meaning is recreated, in action, that is, when we present, say, a microworld to a teacher. So we cannot assume that the meanings that we build in to the microworld (or any other artefact) are transparent to the teacher. Teachers will construct their own meanings, which will be influenced by their past experiences and beliefs as well as their interactions with these objects. As Hoyles et al. (2004) point out, this implies mutual negotiation and meaning-construction should be established “as the norm for both sides of the boundary, rather than the preserve of one protagonist.” (Hoyles et al., p. 321). An activity conducive to such negotiation and that may place the artefacts in the role of boundary objects is that of *communal design of artefacts* by researchers and teachers (Kynigos 2007b; Healy 2006a, b). Communal design can generate the need to be explicit, to reflect and to express meanings through argumentation and result in objects that are both the centre of the activity and also function as communicational tools, shaping a common language within (and between) communities.

A second theoretical notion that informs the analyses presented in this chapter is the notion of instrumental genesis, the process by which artefacts become transformed into instruments. Vérillon and Rabardel (1995) use the term *artefact* to describe a given human-made object. Its appropriation as an instrument proceeds in two directions: towards the self and towards the context in which it is employed. The first direction, *instrumentation*, involves the shaping of thinking by the tool and its integration, through the construction of schemes of instrumented actions, into the individual’s own cognitive structure. The second, *instrumentalisation*, refers to the shaping of the tool, of how its functionalities and affordances are adjusted and transformed for specific uses. Hence, for any individual person, the artefact becomes an *instrument* as he or she develops a set of schemes associated with its use, allowing the artefact to be appropriated and integrated into his or her practices. Like the debate surrounding boundary objects, this perspective too emphasises how the potential role of digital tools cannot be expected to be transparent – neither to teachers nor to learners – and if they are to be integrated in a significant form into mathematics classrooms, an understanding of how to engender the process of instrumental genesis is crucial. In the case of working with teachers, the instrumental genesis process is particularly complex since artefacts become instruments not only in the mathematical practices of teachers but also in their didactical practices.

We now turn more specifically to examples of work involving collaborations between teachers and teacher educators in three different parts of the world, Norway, Greece and Brazil, and to how the notions of boundary objects and the reciprocal shaping of tools and thinking might contribute to building an understanding of developing technology-integrated teaching practices. Each example focuses on the activities of participants in teacher education courses or developmental projects, themselves components of research projects aimed at investigating the use and implementation of digital tools in mathematics learning. Describing the context in which projects like these are situated is a complex thing to do, since one can select a large variety of contextual characteristics, from the macro-level of the educational system and the cultural-historical and political time in which the course took place, to the specifics of the institutional dynamics and the roles of the actors engaged in the course. Although the specific aims of the three projects varied, as did the mathematical ideas under study, all three courses can be considered at least to some extent innovations within the education systems of the countries in which they concerned – more precise details as to how are included in the examples themselves.

13.3 Case 1: Using Inquiry Cycles in Activity Design

The first example is situated within the Norwegian project *Information and Communication Technologies and Mathematics Learning* (ICTML). The founding ideas for the project are *learning community* and *inquiry*, or the idea of building *communities of inquiry* in which teacher educators and teachers work together to develop teaching with digital tools as a support for students' mathematical learning. The ICTML project is run in close collaboration with the project *Learning Communities in Mathematics* (LCM) with the same fundamental ideas and mode of working with teachers (Jaworski 2007). Inquiry is a central concept in the work, as an approach to the work on mathematics, digital tools and teaching, and for the teachers' collaboration. This implies the building of a culture of asking questions, conjecturing, investigating, experimenting and seeking answers both in school-based teams of teachers and in their classrooms. Furthermore, inquiry is seen as an attitude, a willingness to wonder and seek to understand and developing into "inquiry as a way of being" (Jaworski 2004).

The activities in the project encompass workshops in which teachers and teacher educators work together on mathematics, inquiring into the mathematics involved, the possible ways of using the actual tasks or problems and the implementation in the classroom. Alongside these workshops, in each school that takes part in the project, a team of teachers meet to discuss their work on implementing digital tools with mathematical tasks or problems in their classes.

In working with the implementation of inquiry approaches related to the use of digital tools, one aim was to follow what in the project is named *an inquiry cycle*. This is a cycle similar to other models for development of teaching, an action research cycle, design cycle or cycle for learning studies or lesson studies (Jaworski 2004). The cycle is seen as consisting of the main steps: plan, act, observe, reflect and feedback. It starts with a teaching plan, which is then acted out as it is implemented in practice. The teaching and the pupils' work in class are observed by teacher educators and/or teacher colleagues and later all those involved reflect upon and discuss the work. The discussions include feedback and further planning for a new cycle. In the following, an example from a lower secondary school with students in grade eight will be described.

The teachers in the school, Richard, Victor and Otto, had set their own goal for their participation in the ICTML project. They aimed to develop Excel activities which would support pupils' learning of specific topics, such as fractions, percentages, area and volume and the like, by using inquiry approaches, experimenting and investigating connections for the topic. They wanted to build up an electronic library of Excel activities connected to the textbook they used and involving various mathematical topics. Two teacher educators had been visiting their classes and had also attended some of the school team meetings. The impending visit of one of the teacher educators, Aud, to Richard's class motivated him to develop a new Excel activity for the topic he was just about to teach. His aim was that this would serve as a new item for the school's electronic library. Richard carried through his planning just in time, the night before the first lesson on fractions, percentages and decimal numbers. The first task was about comparing and finding connections between fractions and corresponding percentages and decimal numbers (see Fig. 13.1).

In the phase of implementing the plan in his class, Richard gave an introduction to the task using a computer with a projector and explaining how to locate and load the file and open the first task, on Sheet 1 in the Excel-file. The activity involved choosing four numbers that sum to 100 and entering these into the first column. The numbers in the second and third columns were generated automatically and the task for the students was to find connections between the three columns and to record their observations and possible explanations in a textbox in Excel. After the introduction, the class moved over to the computer lab next door. The lesson was observed by Aud, but also by Richards' two teacher colleagues, Victor and Otto,

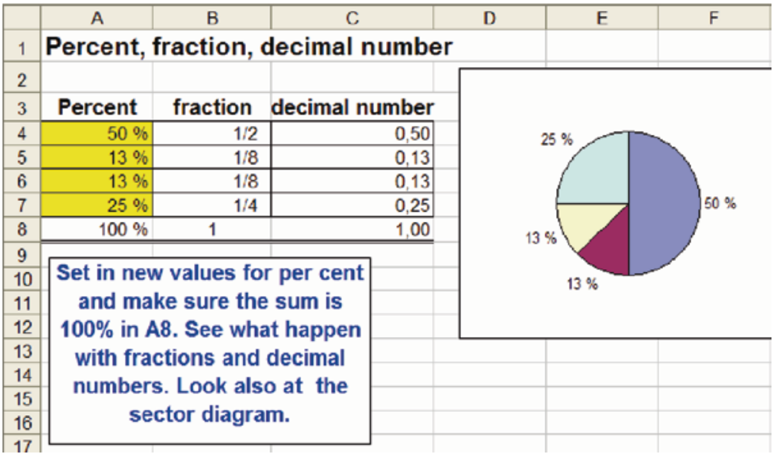

Fig. 13.1 Excel activity on percentages, fractions and decimals

who came in during the lesson, observed and to some extent discussed with and supported the students in their work.

After the lesson, the school team including the teacher educators, held a summary and reflection meeting to discuss observations and experiences from the class and to decide how to proceed with the topic. It had become evident during the observations that several students had problems in expressing themselves in written form and a first reaction from Otto was that the task was too complicated. It appeared too difficult to compare many items at one time. Other ways of setting the task were discussed, like, for example, exploring what percentage leads to four parts, five parts or eight parts in the fraction (denominator) and so on. Affordances and constraints of Excel concerning use the of fractions, presentations of equal valued fractions, use of diagrams and other features were discussed together with possible solutions to facilitate the kind of tasks and illustrations wanted. Hence the school team inquired into both pedagogical and mathematical issues alongside of technical aspects associated with Excel. Richard was also able to capitalise on some of the issues raised in the team discussion in planning his next meeting with the students. Following the work in the computer lab, Richard conducted a class discussion to sum up and reflect on the students' experiences. He challenged them to describe their thinking, using as a starting point the observations and explanations they had written in their Excel file. This gave an opportunity to address some of the difficulties that had been observed in the session in the computer lab.

As a result of his participation in the various loops of the inquiry cycle, Richard became motivated to prepare a new collection of Excel tasks to be used in the lessons which followed shortly after the team meeting, his increasing enthusiasm evident:

> This is great fun. I will enjoy working more on it. I will make something; it is always difficult in the start.

Preparing the new tasks involved him in successfully negotiating, with the help of one of his colleagues, the technical challenges of constructing Excel files that behaved as he wanted and enabled, in turn, the next cycle of the inquiry cycle, as Aud, along with Richard's school colleagues, observed this new lesson (Fuglestad 2008). In this way, the school team meeting, which represents both the reflection and the feedback in the inquiry cycle, contributed both to the process of designing the Excel activity and to the process of instrumental genesis by which Excel was being appropriated as an instrument in Richard's teaching practices. Both aspects of the genesis process can be identified: Richard makes modifications and adjustments to specific aspects of the Excel tools presented in the original activity (instrumentalisation) while, at the same time building an extended repertoire of schemes of instrumented action (instrumentation).

It is important to stress how these developments in Richard's practices were associated with his participation in the inquiry cycle. The opportunities for reflection and feedback provided motivation for development and the mutual help between participants also indicates the development of a learning community within the school involving the teacher educators and the school team. Possibilities to extend this learning community occurred later as experiences from this cycle of work were shared amongst the other participants in the ICTML project during a workshop held at the university.

From the point of view of the teacher educators, the cycle brought evidence of the successful building of an inquiry community focussed on the teaching and learning of mathematics with digital tools. The problems and questions raised in the reflection and planning steps of the cycles themselves had some characteristics of inquiry: asking new questions, conjecturing, willingness to modify approaches to investigate new tasks. The students' difficulties with mathematics revealed in the work posed challenges for the team to work on in further planning. The school team discussed how to meet the challenges by providing suitable investigative tasks for exploring the concepts further. The software became a tool for inquiry into mathematics, providing opportunities to experiment and investigate and the school team, together with the teacher educator, developed a culture for inquiry approaches.

13.4 Case 2: Half-Baked Microworlds as Catalysts for Instrumentalisation

This second case study focuses on the ways in which teachers working within the Greek education system used mathematical exploratory software during an in-service professional development course aiming to prepare them to engage in school-based teacher education themselves on the subject of using technology in the mathematics classroom (Kynigos 2007b). The context of the Greek system is characterized by a centralized nationwide administration coupled with a single national curriculum. In this sense, the teacher is placed in the role of the technical implementer of this curriculum rather than in the role of a professional implementing a developing personal pedagogy. With such constraints, it is very hard to distinguish innovation from systemic reform, and it is difficult to imagine individual teachers involved in curriculum design and in trying out alternative teaching methods, both of which were central to the aims of the course. In this kind of context, it was unlikely that teachers would start constructing things with a piece of educational technology unless this was done through starting up a program for teacher education that aimed to institutionalize not only the use of technology in schools, but also the idea of teacher professional development supported by the system.

The course was a constituent element of a middle-scale initiative from the Ministry of Education involving the installation and use of digital technologies in 10% of secondary schools ("Odysseia" project[1]). The objective of the teacher education course described here was to train experienced teachers, selected by the Ministry of Education, to become teacher educators in the use of digital technologies for teaching and learning in their respective subjects. During and after completion of the course, these teacher educators were relieved from their school duties and given the task of engaging in in-service teacher education programs in 3-5 schools neighbouring their own. The aim of the course was (a) to provide the teachers with methods, knowledge and experience in in-service school-based

[1] http://odysseia.cti.gr/English/ODYSSEIANEW.

teacher education, and (b) to educate them in the pedagogical characteristics and uses of exploratory software and communication technologies.

The following snapshot from the course (Kynigos 2007b) illustrates how what is termed a *half baked microworld* was used as a boundary object for negotiations between the teacher educator and the teachers. The microworld consisted of a procedure in *Turtleworlds*,[2] a Logo-based turtle geometry program with variation tools allowing the user to dynamically manipulate the values of procedure variables and to observe continuous changes to the corresponding figures in the same style as dynamic geometry software (Kynigos et al. 1997). The procedure was designed so that a right-angled triangle would be formed only when the relationship between the value of two variables (*:x* for the angle and *:a* for one of the perpendicular sides) corresponded to the *sine* function, i.e. $\sin a = x$ when the hypotenuse is equal to 1. The students would need to investigate many values and make a conjecture about the kind of relationship between the variables. The microworld thus consisted of a buggy procedure in the sense that a mathematical relationship necessary to create a right-angled triangle was missing.

Five teacher educators-to-be were given the microworld and its design was explained to them. They were then requested to design another one using the same design principles. The result was a procedure for constructing an arc and a chord with one input, the length of the arc in degrees. The teachers seemed to have missed the point, creating a program containing all the necessary mathematics to create the arc and chord. Their instructor then initiated a debate questioning the teachers on what kind of mathematics they thought that their students could do if they were given this model. The teachers were challenged to reflect on the kind of mathematics they themselves engaged in while constructing the model and to think of re-designing it so that the students would be engaged in some similar kind of mathematics. Reference was made again to the design idea of the half-baked "right-triangle" microworld. The teachers subsequently changed from a mode of working which had focused primarily on the construction of a working model without really reflecting on how their students could use what they would create. Their new activity involved an explicit design agenda to transform their construction from that of a model serving demonstration purposes, to that of a mathematically incomplete model inviting students to engage with experiential mathematics of conjecturing, measuring, observing and forming theorems. As in the Norwegian case, the iterative process of design, and the teachers' reflection on the first version of their microworld tool, provided opportunities to reflect upon the possible uses of the artefact they were creating and to extend their repertoire of instrumented action schemes.

During the episode, this change of perspective on what people could do with the software and on how they might use it to generate some mathematical meaning became apparent. During a subsequent interview, George made a distinction between the use of computers as teacher demonstration tools and tools for student learning through personal use.

> George (interview): Proper use (of software) is not as a means for demonstration of the teacher's abilities but how the student will use it as a tool for learning.

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[2] "Turtleworlds" is an E-slate Logo microworld combining turtle graphics with dynamic manipulation of variable values (Kynigos, 2002). It can be downloaded from http://etl.ppp.uoa.gr.

Furthermore, he provided some insightful comments about learning and pedagogy, showing some rather deeply thought-out constructivist-oriented ideas, which, again in common with the Norwegian case, associate mathematics learning with an inquiry process.

> Questioner (interview): With respect to what can this tool overturn traditional education?
> George: In that student and teacher are both stakeholders in a research process.

> George (interview): Now when I work with students I will operate totally differently. I will let the student create and participate in his/her difficulties.

Nick referred to his experience of co-teaching in the classroom, pointing out that "weaker students become more responsible and active" something corroborated by Kostas. The views expressed by all three of these teachers, rather than resulting from theoretical knowledge and beliefs, seemed instead to emerge from experience with the course. It cannot be said that the teachers directly projected their own experiences on their students. However, activities such as the one described in this section, where the teachers were immersed in explicitly constructing tools and microworlds for student engagement were crucial.

As the process of design continued, the teachers subsequently engaged in discussing the mathematics behind its construction in order to design a tool for others to experiment with. Their new focus was on which mathematics to "take away" from the model, in order to give the students the opportunity to carry out an experiment and complete the building of the model by inserting a mathematical relation they worked out for themselves. Within this situation, the way in which they seemed to address the role of the teacher, and the use of the software reflected the views expressed in the interviews: attention was given to Nick and Kostas's concern about the under participation of struggling students and to George's desire to involve students in working through their own difficulties. Rather than trying to create a working model for the user to manipulate, they began to think about designing an incomplete model so that the user would join in the construction process. The user-student's role would be to experiment with the incomplete model in the attempt to work out the type of relation required for it to work properly. The task then turned into trying to establish what mathematics to remove and what to leave in so that the experiment would be interesting and encourage focus on the mathematics intended by the designers. The decisions taken are shown through comparison of the two sets of code for the complete and the incomplete model (Fig. 13.2).

In the procedure mystery, the length of the chord has been substituted by an independent variable. Execution of the program creates an arc and a segment

```
to arc :t                            to mystery  :t :x
ht pu                                ht pu
repeat (360-:t) [fd 1 rt 1 ]         repeat (360-:t) [fd 1 rt 1 ]
pd                                   pd
rt :t                                rt :t
fd (2*(180/pi))*(sin :t)             fd :x
bk (2*(180/pi))*(sin :t)             bk :x
lt :t                                lt :t
repeat 2*:t [fd 1 rt 1 ]             repeat 2*:t [fd 1 rt 1 ]
end                                  end
```

Fig. 13.2 Logo code for the arc procedures

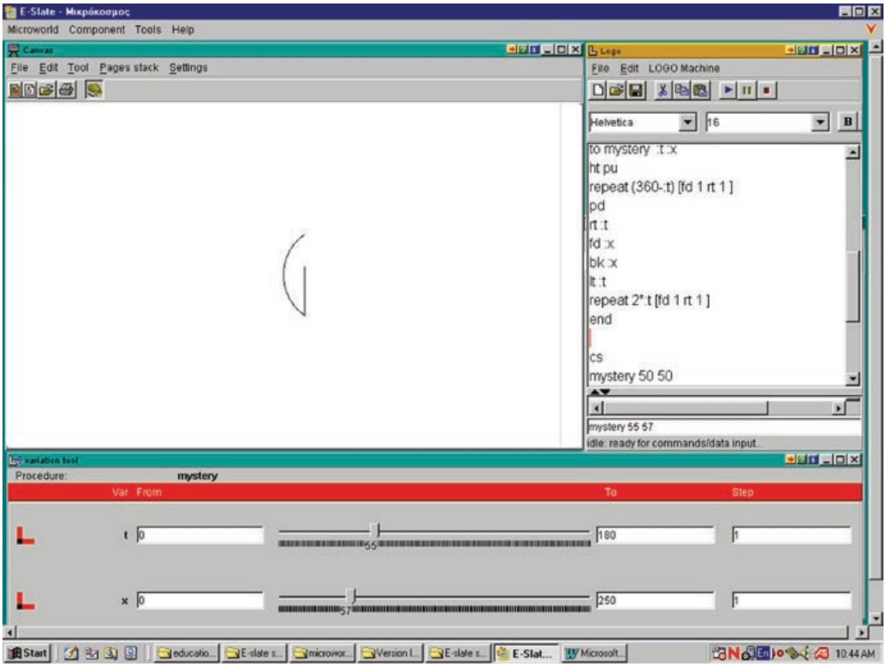

Fig. 13.3 Inputting values to the mystery procedure

which, although has the right orientation and one end on the edge of the arc, it is not necessarily the right size to be the corresponding chord (Figs. 13.3 and 13.4). So the mathematics which was taken away from the correct arc and chord model was the relationship between number of degrees and length of corresponding chord, which was exactly the idea which the students were intended to work with. The students can thus begin by trying out different values and writing them down to look for relations between degrees and segment size (an example is presented in Fig. 13.3).

They can then use the two-dimensional variation tool to find the locus of points for which the segment becomes the chord (Fig. 13.4) and hence discuss the curve resulting from the locus of points and think about what type of relationship it might represent between the two parameters.

This example shows how the generation of these instruments was a process which challenged teachers' knowing with respect to teaching and learning mathematics, but also regarding their view of the nature of mathematics itself. In a sense, it was the instrumentalisation process which seemed to play a critical role in bringing genuine mathematical discussion and activity into the context of professional practice. In parallel, the same process encouraged reflection on mathematical teaching and learning issues. Up till now, instrumentalisation has mainly been perceived as a process to study the kinds of uses of technological artefacts within educational practice. What this study suggests is that it might be worthwhile to consider this perspective as a design factor for teacher educator development contexts.

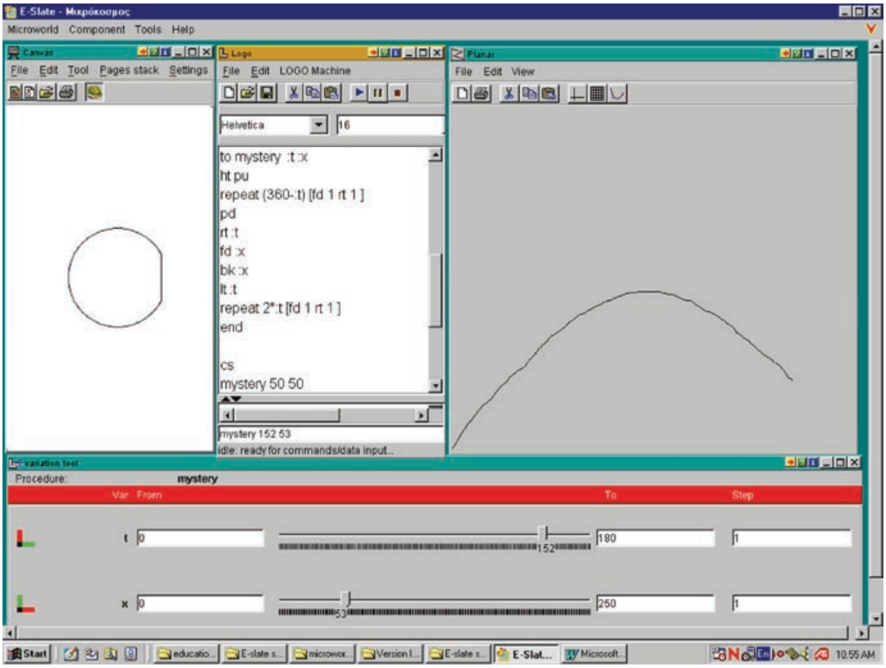

Fig. 13.4 Locus of points for which the segment becomes the chord

13.5 Case 3: Communal Design of a Tool for Statistical Explorations

The third case study presents snapshots from a project in which teachers working within the Brazilian educational system were joined by teacher educators and computer programmers from a research group of a post-graduate research program for mathematics education. The activities of this research group are aimed at investigating the processes by which mathematical knowledge is constructed in the presence of digital technologies, and at supporting teachers in integrating learning activities that support engagement in these processes into their teaching practices (Healy 2006b).

This case study presents a brief synopsis of the mathematical strategies and the pedagogical reflections that emerged as some of the members of this group attempted to develop, collaboratively, one such environment: a computational microworld in which ideas related to random processes in the context of average and spread could be explored and expressed. This choice in itself brought an added complication to the scenario: for most of the teachers involved in the project, the arithmetic mean is a familiar statistical measure (perhaps even the only familiar measure for some), however, as Stella (2003) reports, the dominant view in

Brazilian mathematics classrooms is that of mean as algorithm, its meaning synonymous with the mathematical operations used for its computation. To encourage reflections on this initial point of view, the challenge became to design a digital tool which, instead of calculating the mean for a given data set, given the size of a data set and its mean, could calculate possible data-sets.[3] The data-sets were to consist only of positive whole numbers. Eleven members of the research group expressed an interest in working on this challenge, along with one teacher educator. Eight of the eleven were school mathematics teachers, four working in the public school system only, two who taught in schools from both the public and private sectors and two who worked exclusively in private schools. Only one of the teachers had extensive experience with the use of computers (and this included computer programming) prior to participating in the project. The remaining three members all worked in areas involving computer programming. The group of eleven split into four smaller groups, each with an assigned programmer whose role was to coordinate the formalisation of the ideas expressed.

It is important to be clear that the ostensive aim for those working on the challenge was not about the learning of particular mathematical content, nor was it to come up with some activity that could be directly transported into the mathematics classroom. The immediate aim was the construction of a tool that worked, making this study a little different from the previous two cases where the emphasis on building teaching activities was explicit throughout. However, in all four groups, the strategies developed involved the designers as mathematics learners in reflecting upon statistical concepts incorporated in the tools under development and especially on random processes and, at the same time, participating in the design process also involved them in reflecting on possible pedagogical practices that might be incorporated into their own classroom work. Episodes from one of the group's design attempts show some of the issues related to the building of meanings for random which emerged (for a more details of the development of strategies, see Healy 2006a).

13.5.1 Designing as Learning: From Means and Spread to Distribution as a Space of Possible Values

At first, the four members of this group were very unsure as to how to start. They experimented with the random tool of *Imagine Logo* and while Lise, the group's programmer, seemed prepared to think about the problem completely in abstract terms, for the others it was important to ground the challenge in a particular situation. The problem was hence rephrased: the challenge became to generate possible sets of eight families with a mean size of 4. This problem (also in Mokros and Russell 1995) became the reference for much of the subsequent discussions. The first tool created by this group drew, not surprising, on their knowledge of the algorithm for calculating the arithmetic mean. It was summarised by Lise as follows:

[3] A digital version of a problem originally proposed by Mokros and Russell (1995).

> For the first seven families, use random to choose a value between one and eight.[4] Add the seven values and take away from thirty-two for the last family.

There are two problems with this strategy. The first has to do with the random tool. There was an initial tendency among participants to forget that the command random 8 generates the numbers 0, 1, 2, 3, 4, 5, 6, 7. The number 8 is not a member of the list of numbers produced, while 0 does appear. This problem was relatively easy to debug, perhaps particularly because of the family context (*you can't have a family with no people*). The second problem, which led the group to baptise the strategy "*too random*", is that when formalised into a procedure, this strategy frequently results in a last value for the data set that is "illegal". For example, if the first seven numbers were, 4, 7, 2, 8, 7, 3, 5, the last number in the set would be −4. Explorations of this "broken tool" opened a window onto the functioning of the random tool used in its construction, allowing its designers to explain that, since each of the numbers within the defined range had an equiprobable chance of being selected, the mean value of the $n - 1$ randomly chosen values tended to the midpoint of the range. The larger the data set, the greater this tendency and the more "equal" (uniform) the spread of numbers selected. Furthermore, if the mean value was less than the midpoint, the last value tended to be negative and vice versa.

The discussion of the equiprobable nature of the random tool's output, led the group to express another concern: they began to think that the distributions that they were seeking to construct were not uniform. This problem seemed especially evident to them because of the way the challenge was originally expressed in terms of distributions of family sizes. While at the moment of programming this context may have been left completely aside, at the point of assessing the procedures that were constructed, it came back to the forefront creating a conflict between distributions that did not correspond to their expectations, but were mathematically valid in terms of the constraints of the challenge. Elza expressed her problem with the random tool thus:

> I don't know if this is right, but with random, the more we picked the more equal the spread of numbers and I don't think we want an equal spread. It should depend, depend on … on the situation, yes, but on the mean as well. If we find all the right lists and then choose from them, not like at random, well maybe a different random, the set might be more realistic.

She conjectured that the "ideal" shape of a distribution could be found by constructing a list which contains all the possible data sets for a given mean and range, then plotting all the values in this list – setting off a new cycle in the design process as Gustavo (both teacher and programmer), a member of another group, eagerly took on the challenge of designing this new tool. It is interesting to note that this groups' broken tool, functioned in a similar manner as the half-baked microworld tool in the example from Greece – one difference being was that in this case the tool was not intentionally buggy.

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[4]Use of the random tool necessitated the explicit choice of maximum and minimum values.

13.5.2 Distributing the Instrumental Genesis Process?

The challenge of simulating data sets given the mean and the range of possible values certainly appeared to involve the designers as mathematics learners in deep thinking about random processes and permutations and in making various generalisations about how the shape of a distribution is related to these properties. As they created this new instrument, they also embarked on a shared process during which the random tool, originally only an artefact (in the sense used by Vérrilon and Rabardel) for the majority of the school teachers, also began the process of transformation into an instrument, with attention given to both technical aspects associated with its use (subject-instrument relations), as well as the relations between the instrument and the statistical concepts that were the object of study for the participants (instrument-object and subject-instrument-object relations).

A particular feature of this case study is the bringing of participants from quite different communities to a new practice. The presence of computer programmers alongside the teacher educator and teachers distinguishes this example from the previous two cases. Communication between those with and without programming experience varied, some of those without were keen to make sense of the formalisms produced by their more fluent working partners, others were more interested in seeing their strategy played out on screen than in appropriating the details of the language. But did the fact that the group was composed of intersecting communities mean that the objects constructed had legitimacy beyond the particular challenge for which they were created?

In one sense, the answer will always be yes – the nature of tools, generally speaking at least, is that they are reusable. All those who participated in this challenge, even those who talked more or less exclusively about the family problem, accepted that their tool should work regardless of the number of elements in the set, or the values of the mean and the interval. This is only part of the answer, however. A bigger question is whether the tool, or the experience of designing a tool, was associated with reflections about practices outside of those associated with the project itself – whether it could be legitimately considered as a boundary object. Following the participants into their other worlds was not part of this project, however, during the final session, as the four groups were reporting back on the strategies they had devised, there was talk about the pedagogical opportunities that might cross the frontier between university and school.

Feelings were mixed, especially perhaps of because the different realities and possibilities in the schools within which the teachers worked. All of the mathematics teachers judged the experience as valid, but some, like Leo, felt that the constraints associated with teaching school mathematics mean that experiences such as this are luxuries associated with university life that would be difficult to imagine in their own classes.

> The problem that I see is that I just don't have enough time to spend exploring the concept like this… I am not sure that, well it's important to know how to calculate means, that is coming up more frequently now, but we don't have much time to spend on "the why's"..

Other teachers were keen to use the tool they had created with their own students, but in a rather different manner: to aid exploration but not expression of the mathematical ideas.

> With my students, they could use the tool that we built to see that different data sets can have the same mean. They could generate 'legal' data-sets quickly and observe what happens when you change the mean, with data-sets of different sizes also and …. help them see mean as more than just a calculation.

The third opinion expressed in the discussion suggests that for a small group of the teachers the design experience led to a more radical reflection on teaching mathematics. For these teachers it was not the tool as product that was important, it was the process of tool design. Maria expressed this thus:

> What was most important for me was not the final tool, it was thinking about how to create the tool. I, because of this, I feel I really understood the different explanations, the mathematics in the tools… I'm thinking, maybe not exactly this problem, but this process. It's important. It's this that… that… I would like to do something like this with my students too…

13.6 Reflections on the Case Studies

Although conducted under different conditions, with somewhat different aims, in different locations, with different kinds of participants and in different countries, there are a number of striking similarities between the case studies we have presented in this chapter. First and foremost, the emphasis on design is central to each story. In attempting to establish a dialogue to which all participants of the respective projects could contribute, the strategy chosen was to centre activities around the process of elaborating and experimenting with new instruments aimed to support new mediations of mathematical and/or teaching practices. This strategy differs from one in which different communities take on different aspects of the integration of technology, with computer programmers largely responsible for tool design, university-based researchers, curriculum developers and teacher educators for task design and school-based teachers for subsequently delivering the products in their mathematics classroom. In such an approach, teachers bypass completely the design phase, with the result that they may not feel ready to come up with new activities of their own or even to adapt existing activities according to the particular needs of their students. Where the distance between the affordances and pedagogies incorporated in the tools and tasks of others and the existing routines and culture of the teachers' is very large, it may be that for the teachers epistemic issues are so far detached from their phronesis, that there is little motivation to negotiate the challenge of instrumental genesis.

This problem can be understood in terms of the notion of boundary objects introduced above. When tasks and tools are presented wholesale to teachers, the meanings built into them by the designers are not transparent on at least two levels: the presence of technology brings epistemological changes to the mathematical content involved and the pedagogical assumptions made by the designers may be quite different to those underlying the teacher's usual practices. With the aim of creating learning activities that serve our intersecting worlds, crossing the boundary between university-based research and school-based teaching, a strategy that we argue is worth investigating is to involve all group members not only in the design of activities to support mathematical reasoning, but also in the design of the computational environments which form the context in which this reasoning is to take place.

Another factor common to the three stories is that, while design is central, the products of the design process were not intended to be well polished "finished" softwares – indeed the second two studies stress the potential of working with broken or *half-baked* tools. Rather, all three projects engaged their participants in what the Norwegian example terms inquiry cycles during which tools and activities were created that represented the participants' tinkering and which could be subsequently tinkered with by others – an approach highly reminiscent of the constructionist agenda of Papert (1980, 1991). In such a strategy, the process of instrumental genesis begins with the genesis of an artefact itself – since it could be said that, for the designer, the artefact comes into the world already as an instrument – moreover, the emphasis on design moves this process from the individual plane to the social. The evidence suggested in the three examples indicated that the building of artefacts that might serve as boundary objects will involve participants (from both sides of the boundary) in making explicit their own knowledge about the mathematical issues concerned, their beliefs about the learning trajectories that students follow and their thinking about how best to mediate between their students' (and perhaps their own) personal knowledge and the mathematics they are aiming to teach. This brings us full circle to considerations of culture and context. Since we view artefacts as fundamental constituents of culture, their successful insertion into any new practice necessarily implies cultural change. Perhaps by involving teacher in all stages of the design process, the full extent of the repercussions involved in using digital tools in the classroom – their impact on not only students' learning and teachers' didactic approaches but also on classroom management, on teaching time, and on mathematical knowledge itself – becomes more apparent. And by increasing the sense of ownership that teachers feel for the tools and tasks to be implemented, perhaps it also becomes more natural for them to accept the challenges of becoming active agents in the process of creating new cultures of practices which capitalise on the possibilities of digital tools.

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Chapter 14
Teachers and Teaching: Theoretical Perspectives and Issues Concerning Classroom Implementation

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Abstract This chapter analyses and compares various theoretical frameworks that illuminate the teacher's role in technology-integrated learning environments and the inter-relationship between factors influencing teachers' use of digital technologies. The first section of the chapter considers three frameworks drawing on instrumental genesis, zone theory, and complexity theory, and examines their relevance by interpreting lesson excerpts from alternative theoretical perspectives. This section also outlines research on relationships between teachers' beliefs, attitudes, mathematical and pedagogical knowledge, and institutional contexts and their use of digital technologies in school and university mathematics education. The second section considers classroom implementation issues by asking what we can learn from teachers who use, or have tried to use, digital technologies for mathematics teaching. Issues arising here concern criteria for effective use and the nature of what counts as "progress" in technology integration. The final section of the chapter identifies work that needs to be done to further develop, test, and apply useful theoretical frameworks and methodologies.

Keywords Theoretical frameworks • Teachers and teaching • Instrumental genesis • Zone theory • Complexity theory • Affordances • Teacher beliefs • Teacher knowledge • Institutional context • Technology integration

In 1985 the first ever ICMI Study undertook a critical review of the influence, potential, and constraints in using computers in mathematics teaching and learning, presenting an optimistic vision for the future of technology integration in mathematics education (Churchhouse et al. 1986). Only a few years later, Kaput (1992) predicted that technology would become rapidly integrated into every level of education. He also claimed that the challenges of describing "the roles of technology in mathematics

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education [are] akin to describing a newly active volcano – the mathematical mountain is changing before our eyes" (p. 515). While researchers in this field would probably agree that Kaput's analogy captures the fast pace of technological change, evidence accumulated over the last 15 years indicates that the predicted integration of digital technologies into mathematics teaching and learning has proceeded much more slowly (e.g., Cuban et al. 2001; Ruthven and Hennessy 2002). Many of the authors who contributed to this part of the ICMI-17 Study referred to possible reasons why technology still plays a marginal role in mathematics classrooms, noting that access to technology resources, institutional support, and educational policies are insufficient conditions for ensuring effective integration of technology into teachers' everyday practice (Son et al. 2006). Taken together, these findings suggest that more sophisticated theoretical frameworks are needed to understand the teacher's role in technology-integrated learning environments and the inter-relationship between factors influencing teachers' use of digital technologies, as well as what counts as "effective" use and how progress in technology integration might be identified.

This chapter draws on contributions to Theme B of the ICMI-17 Study to address the following questions:

1. What theoretical frameworks illuminate the teacher's role in technology-integrated environments for mathematics learning?
2. How do teachers' beliefs, attitudes, mathematical and pedagogical knowledge shape (and how are they shaped by) their use of digital technologies in mathematics teaching and how are these issues influenced by access to resources and by differences in culture?
3. What can we learn from teachers who use, or have tried to use, digital technologies for mathematics teaching?

The chapter is organised in three main sections. The first section considers the origin and relevance of various theoretical frameworks for analysing the role of the teacher and the influence on technology integration of a range of personal, contextual and professional factors (questions 1 and 2 above). Research on classroom implementation issues is presented in the second section of the chapter (question 3). In the final section we ask how these theories might help us to develop visions of the future use of digital technologies and new approaches to pre-service and in-service teacher education.

14.1 Theoretical Perspectives

What theoretical frameworks illuminate the teacher's role in technology-integrated environments for mathematics learning?

This section draws on papers contributed by Assude, Grugeon, Laborde and Soury-Lavergne, Brown, and Sinclair[1] to consider three theoretical frameworks for examining the role of the teacher in technology integration.

[1]The contributions cited in this chapter are all available in Son et al. (2006)

14.1.1 Instrumental Genesis

Artigue (2002) points out the unexpected complexity of the instrumental genesis regarding the introduction of technology into mathematic teaching and learning.

According to the instrumental approach developed by Vérillon and Rabardel (1995), the individual must learn how to use a tool for carrying out a task by means of the tool. When the tool is complex and offers the possibility of performing operations referring to theoretical domains, this process of instrumental genesis may be long and may need the help or intervention of a more expert person. As technology involved in mathematics education embodies mathematics, the technical and the conceptual parts are intrinsically intertwined (Artigue 2002): the use of technology shapes the knowledge constructed by students (Hoyles et al. 2004).

The main idea of this approach is to consider that an instrument is a complex entity combining a material or symbolic object with structures that organise the subject's actions. The part of the instrument which is external to the subject is called the artefact. The internal part is constituted by the schemes of use and results from both a subject's personal construction about the way to use the artefact and an appropriation of social pre-existing schemes. The process of constructing schemes, the instrumental genesis, is a two sided process. On the one side, the construction of schemes is oriented toward the use of the artefact: the instrumentalisation. On the other side, the construction of schemes is oriented toward the task to be achieved: the instrumentation. For instance, in a dynamic geometry environment, the drag mode can be seen as an instrument to identify geometrical properties of a figure. The pupil must learn how to drag points (instrumentalisation), which is rather easy, but also why to drag points (instrumentation), which is strongly related to his/her conceptualisation of geometrical properties. Classroom observations have revealed that several weeks are needed until pupils decide to drag points on their own with a mathematical intention and not only to see objects moving. This gives evidence of the instrumental genesis and the need for the teacher to support it.

The contribution of Assude, Grugeon, Laborde and Soury-Lavergne to the ICMI-17 Study draws on the instrumental approach to propose the idea of instrumental integration. Instrumental integration is a means to describe how the teacher organizes the conditions for instrumental genesis of the technology proposed to the students and to what extent (s)he fosters mathematics learning through instrumental genesis. It rests on two main characteristics of the teaching situation. The first is the know-how of the pupils regarding the artefact. The second is the didactical aim of the tasks given to the pupils. It is drawn from indicators like: the focus of the task, the solving techniques, the content of the teacher interaction with the pupils, the links with paper and pencil activities. The combination of the two characteristics produces four different modes of technology integration into mathematical teaching, described below in order from lowest to highest level of instrumental integration.

If pupils are beginners:

- *Instrumental initiation* occurs when the teacher's aim is mainly that the pupils learn how to use the technology. The given tasks focus on the way to use the technology. The relation between know-how and mathematical knowledge, thus the level of instrumental integration, is minimal.

- *Instrumental exploration* occurs when the teacher aims at improving both some know-how and some mathematical knowledge. Pupils explore the technology through mathematical tasks. The relationship between know-how and mathematical knowledge can vary according to the mathematical task and to the content of the teacher interventions: the teacher may just give information on how to use a specific facility of the artefact or (s)he may express links with mathematical knowledge.

If pupils are already introduced to handling the artefact:

- *Instrumental reinforcement* occurs when pupils are faced with instrumental difficulties while solving a mathematical task. The teacher gives them information about how to use a specific item of the artefact to allow them to overcome the technical difficulties. But the teacher's aim is improving mathematical knowledge. The relationship between know-how and mathematical knowledge varies according to the way the teacher formulates his/her help for using the artefact.
- *Instrumental symbiosis* occurs when pupils are faced with mathematical tasks that allow them to improve both their know-how and mathematical knowledge because these are connected. The relation between know-how and mathematical knowledge is therefore maximal, thus so is the instrumental integration.

The example below shows the use of modes of integration to design a task with dynamic geometry. The mathematical learning objective is the counter-example principle. Four diagrams are displayed in Cabri. They have not been obtained by the same construction process but they all look like a triangle and an inscribed quadrilateral in the triangle (Fig. 14.1). The pupils are asked to answer questions about parallel or perpendicular lines: are lines BC and GF perpendicular? Are lines GF and DE parallel? Are lines EG and DF parallel? To answer, the pupils are supposed to move every free point of the diagram and to observe whether the property is preserved or not (Figs. 14.2 and 14.3). This kind of task is possible only after significant work about moving objects and interpreting mathematically what happens on the screen.

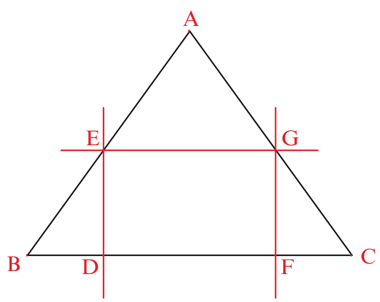

Fig. 14.1 The initial state of all four diagrams

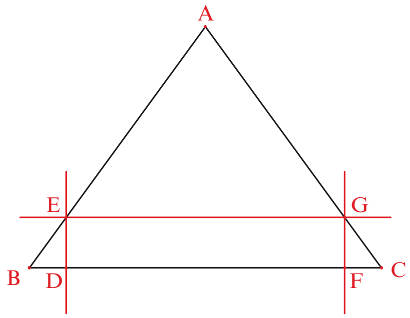

Fig. 14.2 Dragging D

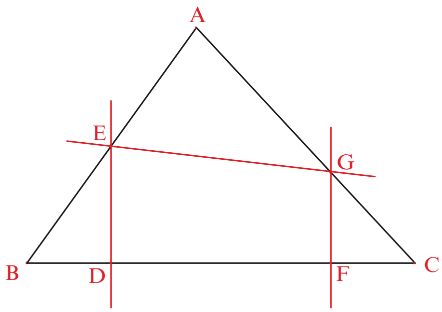

Fig. 14.3 Dragging A, B or C

With pupils already introduced to Cabri, this task can be considered as coming within instrumental symbiosis. The task is of instrumental and mathematical nature: pupils need to decide (1) to drag elements (2) to drag enough points elements in order to decide about the validity of mathematical properties.

Because the lines ED and GF always look parallel when A, B or C are dragged, pupils must find a mathematical reason in the construction program (obtained in Cabri with the facility Replay construction): ED and GF were both constructed as perpendicular to line BC and dragging D has no influence on the quadrilateral because F is the reflected point of D with respect to the midpoint of segment BC. However the relationship between dragging and mathematics may strongly vary according to the prior knowledge of pupils. Pupils may be attracted by only the fact that "it moves" once they drag points but do not pay attention in a more precise way to what happens while dragging or misinterpret the phenomenon because they do not relate it to geometrical properties. In this case, the expected instrumental symbiosis turns into instrumental exploration since the pupils may be considered as beginners with respect to the interpretation of drag mode and the aim of the teacher becomes to reintroduce dragging as a tool for checking properties. In the same vein, the activity may turn into instrumental reinforcement: if pupils do not know how to get information on the construction program of the diagram, the teacher may give information about the existence and use of the tool "Replay construction" and extend the instrumental abilities of the pupils.

Beyond this example, modes of integration are also planned to describe actual practices. When the gap between the planned mode of integration and its actualisation in the classroom is important, it reveals the incomplete instrumental genesis of the pupils and thus may be a research tool for analysing the integration of technology.

Conceptualising these modes of integration allows us to point out two more things. First, the integration of technology depends not only on the teacher but also on pupils' knowledge, regarding the technology as well as the mathematics. Second, there is a possible evolution of the instrumental integration by the teacher, which characterizes an increasing interdependency between the know-how and the mathematical knowledge into the management of the teaching.

14.1.2 Zones and Affordances

The work of Assude and colleagues is representative of a theoretical tradition that has developed specific tools for studying the process through which a material or symbolic object, or "artefact", becomes an "instrument" through construction of personal schemes of use. The theoretical focus is on this process of instrumental genesis and on studying the interplay between the pragmatic and epistemic value of instrumented techniques – between technical work and construction of conceptual understanding of the mathematics. Other researchers have found it useful to adapt more general educational theories in order to study the role of the teacher in technology-rich classroom environments.

Brown's contribution to the ICMI-17 Study draws on Valsiner's (1997) zone theory from the field of developmental psychology to elucidate the teacher's role in technology-rich teaching and learning environments (which she labels as TRTLEs). Valsiner expanded on Vygotsky's Zone of Proximal Development (ZPD) and proposed two additional zones that describe the structure of the developing child's environment and relationships between the child and other people in the environment. He describes the Zone of Free Movement as structuring "the child's access to different areas of the environment, to different objects within these areas, and to different ways of acting upon these objects" (pp. 67–68), noting that internalised ZFMs regulate relationships between person and environment. The counterpart of the ZFM is the Zone of Promoted Action – the "set of activities, objects, or areas in the environment, in which the person's actions are promoted" (p. 192). Valsiner argues that these two zone concepts should not be separated; rather they should be considered as a ZFM/ZPA complex that "canalises" the child's development.

In conjunction with zone theory, Brown uses the construct of *affordances* (Gibson 1979; see also Scarantino 2003) to describe the offerings of the TRTLE – the potential relationships between the teacher and/or students and the environment that facilitate or impede teaching and learning. To take advantage of opportunities

Table 14.1 Affordances of TRTLEs allowing particular views of functions to be observed

Manifestations of the affordance	Conditions enabling perception	Conditions promoting enactment (ZPA)
Using current settings	Serendipity	Task: find graph of data/function Task: identify model of physical curve
Setting viewing window to given values	Lesson element, window settings given	Quadratic function test, sketch function over a specified domain
Edit viewing window to *include* key feature, get a better/global view	Lesson element focused on setting of a 'good' window Functions task requiring exploration of graphs of families of functions	Teacher promotion: can you show me a bit more of your graph? Contextualised task requiring a suitable WINDOW Functions task requiring students to explore graphs of families of functions
Editing viewing window to allow key features to be clearly visible	Teacher scaffolding – understanding of the effect of c in the function $f(x) = x^2 + c$	Lesson element, adjusting WINDOW settings to view key features Contextualised task requiring a suitable WINDOW Functions task requiring exploration of graphs of families of functions

arising, teachers and students need to *perceive* affordances and *act* on them (Drijvers 2003). *Affordance bearers* are defined as specific objects within the environment, such as forms of technology, that enable an affordance to be enacted. Table 14.1 shows an example for the affordance that Brown calls "function view-ability" when students are using graphics calculators. The second column describes conditions *enabling perception*, or circumstances where a teacher or student action allows a particular affordance to be perceived. Conditions enabling perception include those where a learning experience is provided during which the student experiences a particular affordance, for example, where students are expected to follow instructions and experience the particular affordance, thus facilitating future enactment. The third column describes conditions *promoting enactment*, that is, circumstances where a teacher or student action promotes enactment of a particular affordance. For example, through the wording of a task the teacher promotes the direct setting of the viewing domain of the graphics calculator. Affordances are thus linked with zone theory in two ways: (1) they help define the Zone of Free Movement, and (2) teachers can organise a ZPA that promotes their enactment.

Brown provides an example of a teacher (James), who observed that his students experienced difficulties in determining settings for the graphics calculator allowing particular views of a function to be observed:

James: Interestingly, a lot of kids find the notion of setting a WINDOW to a particular graph [difficult], especially if you are doing real, in inverted commas, applications where you do some linear modelling and you might have so many books sold for so many dollars which … is a problem that kids can relate to. And inevitably [you] see them with a graph with the four quadrants. And when you say to them, 'Now is it

> realistic to have a negative number of books?' 'No', or 'A negative amount of dollars?' and, 'No'. 'Well then, are those values realistic to have on your graph?' 'No'. 'Well, you would have more efficient use of your graph if you deleted those bits and use your WINDOWs'. 'But I don't know how to use WINDOWs, I don't understand.

This teacher sees *function view-ability* – focusing on the section of the function that is relevant to the modelling context – as an essential affordance for his students to enact. Brown describes an occasion where he was observed promoting use of context (finding the biggest box volume) to select an appropriate viewing domain:

James: When you do those cut-outs, of *x*, what is the biggest value of *x* that you can cut-out? If you have a look at your picture, what did you put for your diagram? If you started to make those corner cut-outs bigger, what would be the biggest cut-out that you could make?
Cam: Five.
James: Five. That is right. So for your WINDOW, you would set *X*min to be zero. And *X*max to [pause]?
Cam: So that, is that [wrong]?
James: No, you are right up to there.
Cam: So we didn't have to do that much?
James: But, beyond here [$x = 5$] it is not a realistic part of the problem. Because the biggest value of *x* you could ever cut out is 5. Okay?
Cam: Yeah.
James: So you would set your WINDOW to?

Here James had deliberately organised the learning experience so as to promote this particular manifestation of the affordance *function view-ability*. This ZFM/ZPA complex is also intended to orient students' ZPDs towards possible futures where they will be able to independently *perceive* and *enact* this affordance where appropriate.

We have so far presented two contrasting theoretical frameworks for analysing the teacher's role in technology-integrated learning environments: one based on the technology-specific theory of instrumental genesis and another that has applied the more general concepts of zones and affordances to technology contexts. To examine the relevance of these frameworks it is instructive to re-interpret some of the lesson excerpts within each study from the perspective of the alternative theory.

14.1.3 Instrumentation Theory Applied to a Zone/Affordances Excerpt

In the teacher–student dialogue from the function view-ability excerpt above we observe instrumental *reinforcement* from the teacher when he provides information about how to use the WINDOW to view the graph appropriately: his aim is to improve the students' ability to relate mathematical knowledge to the problem context. Instrumental *initiation* is suggested in the segment of Table 14.1 that refers to "Setting viewing window to given values" for the purpose of sketching a function over a specified domain, as this implies that the aim is to teach students how to use an aspect of the technology. On the other hand, a lesson element that focuses on setting a "good" window that includes key features of the graph, together with a task requiring exploration of graphs of families of functions, is consistent with instrumental *exploration* because students explore the technology through a mathematical task, or with instrumental *symbiosis* when they explore specific window settings to visualize properties they conjecture about a function.

14.1.4 Zone Theory and Affordances Applied to an Instrumental Genesis Excerpt

Now let us return to the Cabri example involving triangles and inscribed quadrilaterals. Whether this activity becomes one of instrumental exploration, reinforcement, or symbiosis depends on the conditions enabling perception and promoting enactment of the *affordances* carried by "dragging points" and "Replay construction"; that is, by the *ZFM/ZPA complex* organised by the teacher and learning environment. Conditions enabling perception might be established through previous lessons in which students experienced the particular affordance; conditions promoting enactment would include the wording of the task (whether specific instructions were given regarding use of Cabri features) and the teacher's promotion via questioning about the mathematical or technical aspects of the task.

While both these theoretical frameworks can be applied to lesson events to analyse the teacher's role in integrating technology into the mathematical practices of the classroom, looking through alternative lenses brings different issues to the foreground. We might say that the theory of instrumental genesis proceeds from a micro-level analysis of interactions between mathematical tasks and instrumented techniques to pose questions about pedagogy; while zone theory is primarily concerned with macro-level issues of learning through interaction with other people and the material and representational tools offered by the learning environment, where technology is one such tool.

14.1.5 Complexity Theory

The third theoretical framework is illustrated by Sinclair's reflections on her own practice as a teacher educator, looking through the lens of complexity theory to analyse how teachers can nurture development of a learning system in a technology-supported environment. Here the focus is on the classroom as a whole rather than on the teacher, students, and technology. Sinclair argues that complexity theory – the study of adaptive and self-organising systems – "challenges us to see the whole system in a new way".

Davis and Simmt (2003) were the first to apply complexity theory to the teaching of mathematics, evoking the conceptual shift, described by Cobb (1999), "away from *mathematics as content* and toward *emergent terms*" (Davis and Simmt 2003, p. 144, original emphasis). They propose five conditions for emergence of a mathematical community as a learning system:

1. Internal diversity: but not the kind of diversity achieved by structured group work or other formal classroom organization strategies, because "diversity cannot be assigned or legislated, it must be assumed – and it must be flexible" (Davis and Simmt 2003, p. 149)
2. Redundancy: which provides the necessary degree of sameness to allow people to interact while compensating for each other's weaknesses
3. Distributed control: acknowledging that the locus of learning is in the collective rather than the individual
4. Organised randomness: establishing the enabling constraints necessary for generative activity
5. Neighbour interactions: providing sufficient density of interactions between agents to open up new conceptual possibilities

Sinclair first examined three mathematics activities she had used with high school students with respect to the five conditions listed above: a linear transformations project using a spreadsheet; a set of proof tasks with JavaSketchpad; and an independent study that made use of a variety of technological applications. In only the latter activity was she satisfied that all the conditions were met for development of a learning system.

To bring Sinclair's use of complexity theory into our broader discussion of theoretical frameworks for studying technology integration, we take two insights arising from her analysis and consider these from alternative theoretical perspectives. The first insight is related to Davis and Simmt's (2003) argument that "emergent events [such as the emergence of mathematical ideas in the collective practices of the classroom] cannot be caused, but they might be occasioned" (p. 147). The difference here is between tasks that are *prescriptive* (specifying what is *permitted*; everything else is forbidden) versus *proscriptive* (specifying what is *forbidden*; everything else is allowed); in other words, emergence requires enabling constraints. One reason for the success of Sinclair's independent study task was that it was proscriptive, providing "a structure for sharing, play, and individual choice". The notion of enabling constraints is thus reminiscent of zone theory's *ZFM/ZPA complex*, which provides freedom to explore within limits set by the teacher (although Brown would argue that, in a TRTLE, secondary students can often move beyond these limits).

A second insight came from Sinclair's reflection on her practice as a teacher educator and the realisation that she was not using technology with her pre-service students in a natural, spontaneous way as she did in her own mathematical work. "Instead, I was teaching applications of technology" – an approach to task design that Assude and colleagues might describe as *instrumental initiation*. Sinclair therefore set out to make technology an integral part of her teaching, shifting the focus from how to use the technology towards how technology can be used to explore mathematics – that is, towards *instrumental exploration* or even *symbiosis*, improving both instrumental abilities and mathematical knowledge.

14.1.6 Theoretical Perspectives: What Is the Teacher's Role in Technology Integration?

Table 14.2 summarises three complementary interpretations of the teacher's role in technology integration, each corresponding to a different view of technology itself.

Table 14.2 Theoretical interpretations of the teacher's role in technology integration

Author	Theory	View of technology	Teacher's role
Assude et al.	Instrumental genesis	From artefact to instrument	Organise conditions for instrumental genesis
Brown	Zone theory Theory of affordances	Offering affordances in relationship with the user	Organise conditions (ZFM/ZPA complex) for student perception and enactment of affordance
Sinclair	Complexity theory	Element of a learning system	Organise conditions for emergence of a mathematical community

How do teachers' beliefs, attitudes, mathematical and pedagogical knowledge shape (and how are they shaped by) their use of digital technologies in mathematics teaching and how are these issues influenced by access to resources and by differences in culture?

Each of the papers discussed in the previous section implicitly drew attention to the mathematical and pedagogical content knowledge (Shulman 1987) that teachers require in order to integrate technology into their classroom practice. However, it is well known that many other factors influence the extent and manner of classroom integration in school mathematics.

14.1.7 Factors Influencing Technology Integration in Schools

Goos's contribution to the ICMI-17 Study adapted Valsiner's concepts of the Zone of Proximal Development, Zone of Free Movement and Zone of Promoted Action to devise a theoretical framework for analysing relationships between factors influencing secondary school mathematics teachers' use of technology. In this research, the zone concepts are used to theorise *teachers*', rather than *students*', learning (cf Brown's research discussed above). Previous research on technology use by mathematics teachers has identified a range of influences related to teacher knowledge and beliefs, school structures and institutional constraints, and professional learning opportunities (e.g., Fine and Fleener 1994; Manoucherhri 1999; Simonsen

Table 14.3 Factors affecting technolo\gy usage

Valsiner's zones	Elements of the zones
Zone of Proximal Development	Skill/experience in working with technology Pedagogical knowledge (technology integration) General pedagogical beliefs
Zone of Free Movement	Access to hardware, software, teaching materials Support from colleagues (including technical support) Curriculum and assessment requirements Students (perceived abilities, motivation, behaviour)
Zone of Promoted Action	Pre-service education (university program) Practicum and beginning teaching experience Professional development

and Dick 1997; Walen et al. 2003). Goos proposes that these influences represent elements of a teacher's ZPD, ZFM and ZPA, as shown in Table 14.3.

Goos illustrated how the zone framework may be used by analysing case studies of a novice teacher and an experienced teacher in different school settings.

Vignette #1: Novice (Pre-service) Teacher

In her university pre-service course Sandra gained experience in integrating technology (computer software, Internet, graphics calculators) into mathematics teaching and learning and she developed a strong commitment to using technology in ways consistent with a student-centred teaching approach. Her practicum school had many computer laboratories but had only recently purchased its first class set of graphics calculators. Sandra had not observed other mathematics teachers use any kind of technology with their classes. Because none of these teachers knew how to use the graphics calculators, it was easy for Sandra to borrow the sole class set for her own teaching. She decided to use the graphics calculators with her senior secondary class for solving linear programming problems. The students had never used graphics calculators before, so Sandra devised a worksheet with keystroke instructions and encouraged students to work and help each other in groups. She was surprised to encounter strong resistance from the students, which seemed to stem from their previous experiences of mathematics lessons. Other teachers focused on covering the content in preparation for pen and paper tests and did not allow the students to work in groups. The students were not interested in helping each other or in learning how to use technology if this would be disallowed in assessment situations. Sandra was not discouraged by this experience and continued to seek ways of integrating technology into her teaching of mathematics.

Sandra experienced tensions and contradictions within her ZFM/ZPA complex and its relationship with her own ZPD. Some elements of her ZFM, such as her easy access to calculators that no other teacher wanted to use, presented favourable opportunities to use technology; however, her students' negative attitudes and lack of motivation, together with an assessment regime that excluded technology, represented potential constraints. Further tensions arose from inconsistency between the pedagogical actions promoted by teachers in the school (school ZPA) and the technology emphasis in her pre-service course (pre-service ZPA). Sandra's willingness to persist with technology integration suggests that she was able to re-interpret her professional environment in the light of her own goals and beliefs regarding technology – that is, she attended to only those elements of her ZFM/ZPA complex that engaged productively with her ZPD and would thus "canalise" her development as a teacher committed to student-centred learning with technology.

Vignette #2: Experienced Teacher

Lisa gained little benefit from her initial experiences of professional development involving graphics calculators. These sessions emphasised procedural aspects of operating the calculators and the mathematics presented was too difficult for participants to engage meaningfully with the technology. After several workshops she felt confident enough to use graphics calculators in her teaching, but only as a tool for graphing and statistical calculations. Lisa later volunteered to participate in a research-based professional development program that demonstrated the impact of technology in developing students' understanding of mathematical concepts and in facilitating classroom discussion. She began to see different ways of using graphics calculators that she hadn't thought of before, commenting that "It really enhanced group work, we were really starting to think when we were fitting functions to the data, we had to really understand what the intercept and gradient mean: we weren't just *doing*, we were really *understanding* at a higher level".

In contrast with the case of Sandra, institutional constraints (ZFM) seemed to play little part in Lisa's learning, possibly because she was Head of her school's Mathematics Department and therefore in charge of obtaining resources and writing curriculum and assessment programs. In this case, the research-based professional development program created a ZFM/ZPA complex that met Lisa's need to focus on pedagogical, rather than procedural, aspects of using technology.

14.1.8 Factors Influencing Technology Integration in University Mathematics Departments

Compared with the wealth of research on technology integration in school mathematics, much less is known about factors influencing use of technology for mathematics teaching and learning in universities. Drawing on school-level research in this area, Lavicza has designed a study investigating the extent to which Computer Algebra Systems (CAS) are used in university mathematics departments in Hungary, the UK and the USA and factors that influence the integration of CAS into university mathematics education. In the first, qualitative, phase of the study, he interviewed 22 mathematicians, observed classes, and collected course materials from these countries as a prelude to a larger scale quantitative study. Analysis of this qualitative data identified three clusters of issues related to the participants' personal characteristics, institutional and technology factors, and conceptions of mathematics, mathematics teaching and learning, and the role of CAS. Academics' conceptions appeared to be a crucial factor influencing technology integration, more so than for school teachers, possibly because the greater academic freedoms of university life tend to lessen the impact of curriculum and other policy pressures.

In the quantitative phase of Lavicza's study, 1,103 questionnaires were completed by mathematicians in the three participating countries (24.6% response rate). Although the responses are yet to be analysed in detail, preliminary findings indicate that current use of CAS is much higher in universities than in schools and that CAS technology, support and computer laboratories are readily available to university mathematicians for teaching purposes. Further analysis is expected to identify relationships between mathematicians' personal characteristics and institutional settings, their CAS use in teaching and research, their conceptions of mathematics, and their CAS-related conceptions. This work has the potential to highlight similarities and differences between technology integration in schools and universities, especially if a common theoretical framework is used to compare research at the different levels of education. For example, the zone theory framework developed by Goos may provide a means of analysing technology integration across diverse curricular organizations and educational levels.

14.2 Classroom Implementation

What can we learn from teachers who use, or have tried to use, digital technologies for mathematics teaching?

Asking what we can learn from teachers who use digital technologies raises questions concerning criteria for effective use and the nature of "progress" in technology integration. These are the classroom implementation issues upon which we focus in this section.

14.2.1 Defining Criteria for Effective Use

What counts as "success" in technology integration can be evaluated either theoretically or empirically. Assude et al.'s notion of instrumental integration, explained earlier in this chapter, is a tool for theoretical evaluation that can identify different modes of integration by teachers. Using this tool, "progress" would be represented by increasingly sophisticated modes of integration, such as when teachers design tasks that increase the relationship between students' instrumental abilities and their mathematical knowledge.

However, Ruthven and Hennessy (2002) have argued that there are limitations in using preconceived theoretical models if the goal is to capture teachers' perspectives on effective practice. Instead, they collected empirical evidence to develop a model "of what practitioners conceive as the successful use of computer tools and resources to support mathematics teaching and learning" (p. 51). From group interviews conducted with the mathematics departments in seven UK schools they identified well developed themes that ran across transcripts. *Success themes* explicit or implicit in teachers' accounts reflected teaching aspirations concerned with gaining students' *participation* in classroom work, the *pace and productivity* of this work, and the resultant *progression* in learning that occurred. Further analysis yielded ten *operational themes* describing the affordances and mediating processes that teachers associated with success. These in turn were linked to broader *pedagogical themes* concerned with promoting investigation and supporting consolidation. While investigation and consolidation were viewed as complementary rather than oppositional aspects of teaching and learning, computer use was particularly important in making investigative activities accessible to students and viable in the classroom.

It may be that neither theoretical nor empirical criteria on their own are sufficient for defining what counts as effective use of digital technologies if we are genuinely interested in learning from teachers who are incorporating such technologies into their practice. Bringing together theoretical and empirical accounts may give rise to richer interpretations of teachers' practice and highlight similarities and differences between teacher and researcher views on the nature of "progress".

14.2.2 Identifying Change

Research that investigates the impact of digital technologies on classroom practice is moving towards more mature analysis of teacher learning by asking *how* technology is used and how this changes the teacher's role rather than by simply contrasting the teaching of particular mathematical topics with and without technology. Some examples of such research already exist in the literature. For example, Farrell (1996) studied classroom interactions in technology-integrated pre-calculus classrooms and observed a shift in both teachers' and students' roles towards that of consultant and fellow investigator, accompanied by a similar movement away from teacher exposition towards planned or informal group work. Goos et al. (2003) developed four metaphors in order to theorise the varying degrees of sophistication with which teachers and students work with technology: technology as "master", "servant", "partner", and "extension of self". These metaphors can be used to characterise particular classroom episodes as well as to analyse changes in technology use over time.

Contributions to the ICMI-17 Study raised several issues around classroom implementation and identifying change in teachers' practices. Chow reported on his plans to conduct a study of Singapore's junior college teachers at a time of transition to a new mathematics curriculum that expects use of graphics calculators in all advanced level courses. This study provides a rare opportunity to investigate how the concerns of teachers, their teaching strategies and their roles change when they attempt to integrate graphics calculators into teaching and learning. Chow also aims to identify common features among teachers who are successful in integrating graphics calculators into the curriculum, thus raising again the need for clear criteria – whether theoretically or empirically determined – that define "success".

Miller and Glover described how they have studied UK teachers' changing pedagogical strategies as they become more confident in using interactive whiteboards (IWBs) to teach mathematics. Along with other contributors to the ICMI-17 Study, they noted that availability of digital resources alone does not guarantee enhancement of teaching and learning, and they emphasised the importance of professional development that fosters both technological competence and pedagogic flexibility. Drawing on analysis of over 100 video-recorded mathematics lessons in 25 schools (pupils aged 11–14 years), Miller and Glover identified three stages of development in teachers' use of IWBs, outlined below.

- *Supported didactic*: the teacher makes some use of the IWB but only as a visual support to the lesson and not as an integral tool to conceptual development; there is little interactivity, pupil involvement or discussion.
- *Interactive*: the teacher makes some use of the potential of the IWB to stimulate pupil's responses from time to time in the lesson and to demonstrate some concepts; elements of lessons challenge pupils to think by using a variety of verbal, visual and aesthetic stimuli.
- *Enhanced interactive*: this approach is a progression from the previous stage marked by a change of thinking on the part of the teacher who seeks to use the technology as an integral part of most teaching in most lessons and who looks to integrate concept and cognitive development in a way that exploits the interactive capacity of the technology.

To sum up, studying "progress" in technology integration requires researchers to keep in mind the following questions:

- How is "progress" defined? (theoretically or empirically defined criteria)
- What changes, or gets better? (teacher roles, mathematical practices, task design, teaching strategies, classroom interactions, teacher knowledge and beliefs)
- What do we mean by "better", and for whom? (teacher, student)
- How do we know (methodology), and what by what means do we "measure" change?
- How can we explain it? (theory)

14.3 Future Visions

We began this chapter by looking back at the first ICMI Study on technology in mathematics education and reflecting on the gap between aspirations and reality regarding the pace and scope of integration of digital technologies into mathematics classroom practice. To conclude it we now look forward into the future to identify work that needs to be done to further advance the field in terms of theoretical perspectives and classroom implementation.

Taken together, the contributions to the ICMI-17 Study reviewed in this chapter present a consistent argument that (1) teacher characteristics (their mathematical and pedagogical knowledge, beliefs and attitudes, skill and comfort in using digital technologies), (2) institutional contexts (access to resources, policy pressures, curriculum change), and (3) professional learning and development influence the integration of digital technologies into mathematics teaching. To create future-oriented visions of technology integration, we propose the following lines of inquiry:

- Continue to develop theories that target each of these three factors separately
- Further investigate theories that seek to explain relationships between these factors on a broader scale
- Develop theories of practice to inform pre-service and in-service teacher education, in order to strengthen recommendations about what counts as "progress" or "success" in technology integration
- Conduct research that makes use of these theories to test their operational value and domain of validity

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Chapter 15
Teacher Education Courses in Mathematics and Technology: Analyzing Views and Options

Brigitte Grugeon, Jean-Baptiste Lagrange and Daniel Jarvis, with Mara Alagic, Mili Das and Diana Hunscheidt

Abstract Research in the field of teacher development courses in mathematics and technology is still in its infancy. In order to offer reference marks for this field, this chapter explores the variety of views and options that underpin such courses. The investigation considers the views and options provided by five contributions to the ICMI study conference from three different continents, each based on a specific teacher education course. The authors propose to characterize the views with regard to three aspects: implementation of technology in the classroom and in teacher education, changes in teachers' role, activity and practices, and adaptation of teaching practices with regard to time and professional proficiency. They also propose to classify the options first with regard to the content, and secondly with regard to teaching strategies. Six types of content – curriculum, potential of software, instrumental genesis, new and old tasks, new teaching abilities, professional context – and four main strategies – demonstration, role playing, "in practice", and learning communities – are identified.

Keywords Mathematics • Technology • Teaching • Teacher education • Professional development • Pre-service • In-service • Course design

15.1 Introduction

The integration of a new artefact into a teaching situation necessarily alters existing stability and requires teachers to undergo a complex process of adaptation. In the case of digital technologies, the modifications required in routine practices are likely to be particularly pronounced. The teacher also needs to consider how the new

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artefacts, the new representations and alternative learning strategies made available by technology depend on the various factors of a teaching situation: content to be taught, curriculum, specificities of students and more generally of the learning context. These complex issues have huge implications for pre-service and in-service teacher professional development. As Musley et al. (2003, p. 396) have argued in their international study into technology use within teacher education programs, there are a variety of different ways in which technology may be used. They distinguish between three different types of uses:

1. Creating and using videotape, videodisc and multimedia resources in order to make a wide range of pedagogical interactions available for analysis
2. Using the Internet and communication software packages to enable and facilitate information and communication in professional development
3. Using computers, calculators and other electronic resources for doing mathematics

In this chapter, we focus on the third of these different purposes as an important aspect of preparing teachers for work in technology-rich classrooms. Research studies in this domain are in short supply. The literature consists mainly of reports on how specific tools were used, or on how specific programs were designed, rather than on what learning took place, or on the broader question of how teachers learn (Musley et al. 2003, p. 425). There is, in particular, a need to compare and share experiences from different countries and contexts and to seek frameworks which allow us to pinpoint the commonalities and differences in the approaches currently being adopted. This arose as a major concern in the study conference working group that dealt with teacher education. Five examples of teacher development courses, each developed in a different country were presented and discussed, and the working group participants felt the need for guidelines and reference marks to make sense of their commonalities and differences. It appeared first that each course was based on assumptions or beliefs about technology in mathematics education related to classroom implementation of technology and teacher preparation. This chapter uses the word ‘views’ to denote these assumptions or beliefs. Then, practical choices and decisions in the implementation of courses were identified as possible reference marks. This chapter uses the word ‘options’ to denote these choices and decisions.

The aim of this chapter is then twofold: to investigate the various views which underpin approaches adopted in courses aiming to support teachers taking up the challenge of using technology for mathematic education; and, to examine the various options that inform the practical decisions adopted by those designing, organising and implementing such courses. To this end, we focus our analysis on five examples of teacher development programmes, each developed in a different country that were presented and discussed during the study conference. The first part of the chapter consists of a very brief description of the aims and the organization of each of the development programmes (for more details, the reader is referred to the respective papers in the conference proceedings). In the second part, we offer a series of frameworks by which to characterize the different views that underpin the five examples. These views deal with the relationship between technology and teaching/learning, with related changes in teaching practices, and with the teaching strategies the five examples aim to promote. Finally, drawing again from these five examples, we identify core components related to the practical decisions concerning the content proposed in the courses and the strategies by which the teachers participating in the courses are to access this content.

15.2 Examples of Teacher Development Courses

The papers presented at the study conference offered an interesting diversity of teacher development courses. Within the five courses, three are aimed at pre-service teachers, and two others at in-service teachers. Two are prepared for work especially in primary school and two for the secondary sector. Two of the courses include substantial online activity; the others were more concerned with the integration of technological tools in the ordinary classroom. The period of time dedicated to teacher preparation also varies. This diversity is also reflected in the geographical location of the courses in three different continents.

We refer in the texts and tables to each course using short acronyms, and name authors when referring to their specific views as expressed in their papers.

15.2.1 In Service Teacher Development Course "Mathematics Investigations" (MathInquiry) (Based on Alagic 2006)

The course "Mathematics Investigations" is designed for pre-service teachers in the USA and aims to address current demands to integrate both digital technologies and inquiry-based approaches into the teaching and learning of mathematics. The course has five components, each weighted differently: problem sets, reflections, self-evaluations, readings, and a final presentation. At the end of a problem set, the students are required to produce a metacognitive reflection report, which describes their thinking during the process of problem set design. Weekly discussions are carried out through the use of online courseware, as the classroom learning network includes discussion groups. As a final product, each student compiles a Digital Resource File that consists of five problem sets, a final presentation, and additional resources relevant to their future work. Students are guided and encouraged to develop their fluency in dynamic geometry, spreadsheets, selection and use of virtual manipulatives, and other Web resources. Furthermore, they are required to design Problem sets so that these digital tools are implemented in a meaningful way for different grade levels of mathematics learning and teaching. University-wide available courseware is used to support complementary online activities, group discussions, and the virtual classroom.

15.2.2 *A Bachelor of Education Course at the Institute of Education for Women in India (BecIEW)* (Based on Das 2006)

In this course, trainee teachers practice teaching skills in a classroom environment in which peers play the role of students or observers. After practicing five or six teaching skills, trainee teachers take part in actual practice teaching sessions for 3–4 weeks in secondary schools. Regarding technology use, the course specifically targets the following abilities:

- Using technology comfortably in the classroom
- Integrating technology as a complementary tool in the classroom
- Providing students with a variety of experiences through dynamic approaches
- Motivating students towards learning mathematics

15.2.3 *Mieux Apprendre la Géométrie avec l'Informatique*[1] *(MAGI)* (Based on Assude et al. 2006)

This paper is about in-service primary teacher professional development courses carried out in France within the national project MAGI. This research and development project involves twenty researchers, teacher educators and teachers divided into groups located in different places in France. The project specifically focuses on the integration of the dynamic geometry environment, Cabri-Geometry, into ordinary classrooms at the primary level. There are two main parts, or stages, of this project: a 3-week long primary school teacher professional development course, and, a documentation of teachers' practices involving the evolution of their *instrumental integration modes* (see Chap. 14) as observed 1 or 2 years after the 3-week course is completed.

15.2.4 *A Teacher Development Course for Prospective Primary Mathematics Teachers (TdcPt)* (Based on Hunscheidt and Peter-Koop 2006)

This teacher preparation program offered at the University of Oldenburg in Germany is aimed at future primary mathematics teachers and involves a 4-h compulsory module on the use of digital and electronic technologies in the mathematics classroom. Two learning environments have been developed in

[1] English translation: improving geometry learning with computers.

the context of a university methods course, and these have then been replicated in the form of teaching experiments in Grade 4 classrooms by pre-service teachers. The two learning environments go beyond the use of special software designed for (primary) mathematics classrooms and involve a robot and a monitoring device. The development of the learning environments is guided by the paradigm that ICT-related objects serve as "tools, not toys" within a classroom that fosters the extension of mathematical understanding and the introduction of new content areas.

15.2.5 The Bachelor of Education "ITeach Laptop Learning Program" (BEdITeach) (Based on Jarvis 2006)

The Faculty of Education at Nipissing University in North Bay, Ontario is among the first in Canada to adopt a laptop program for all teacher candidates enrolled in the Bachelor of Education program. Since 2001, teacher candidates in the "ITeach Laptop Learning Program" have been required to purchase and use a portable laptop computer for all courses offered in the Bachelor of Education (BEd) program. Course instructors are encouraged to make use of this technology by way of preloaded, discipline-based software applications, Internet-based resources, and information/links housed on their own instructor websites.

15.3 Characterizing the Varied Views that Underpin Teacher Development Programs in Technology

15.3.1 Views Concerning the Implementation of Technology in the Classroom and in Teacher Education

All of the programs mentioned above have in common the aim of motivating and supporting teachers for using technology, but they differ in terms of the related views on technology use and the modes of use supported within the implementation. We have classified these views and modes of use along two axes.

The first axis (horizontal in Fig. 15.1) refers to classroom use of technology. It separates those courses underpinned by the idea that technology will necessarily improve learning from those courses that consider a more problematic contribution to mathematics learning. Courses assuming improvements, like Das' BecIEW, are characterized by the view that technology will make teaching more dynamic, interesting, and effective as compared with a chalk-and-talk method. Teachers are portrayed as being able to explain, interact, and illustrate based on student learning needs, and the appropriate use of technology enables the visualization of mathematical concepts and applications, minimizing the abstractness of the topic and thereby reducing instances of math-phobia. The goal of these courses is then to acquaint

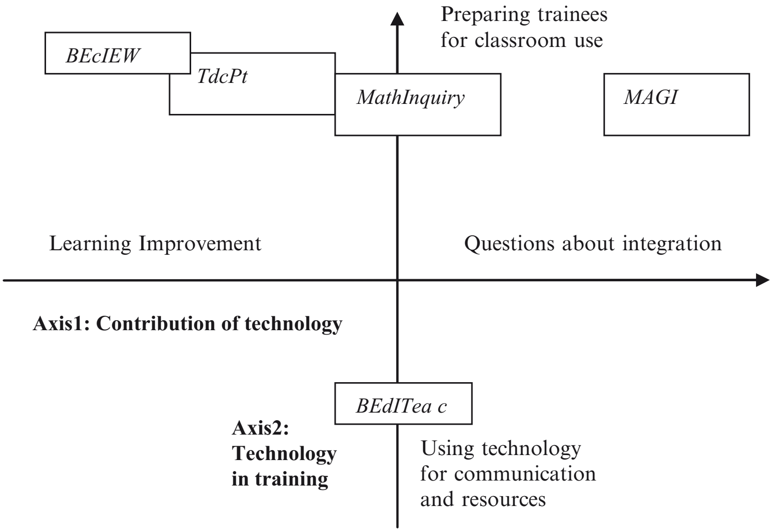

Fig. 15.1 Views concerning the implementation of technology

teachers with, and inspire teachers to adopt, the implementation of technology in their classes on the basis of these improvements. Courses that consider a more problematic contribution, like in Grugeon's MAGI, stress the complexity of the classroom integration of technology, and the necessity of introducing a reflection on the instrumental dimension whose importance with regard to teacher issues is stressed by Chap. 13. Courses like MAGI also considers an anthropological didactic perspective as explained in Chap. 23, especially questioning the way technology changes mathematical techniques and the interactions between new digital techniques and old paper/pencil techniques.

Courses positioned more towards the middle of this axis – the BEdITeach and MathInquiry, for example – assume a relatively neutral position. While aware of technological challenges in implementation, the instructors remain optimistic about technology advantages for teaching. In these courses, technology is treated as a creative and complementary teaching resource but not necessarily as a definite improvement in itself.

The second axis (vertical in Fig. 15.1) deals with the relationship with technology within the training courses. It distinguishes between courses taking advantage of technology for communicating, collaborating, accessing and sharing resources, and courses aiming more explicitly to prepare trainees for classroom implementation. Along this axis, the question of the transfer from training to classroom is critical.

Courses based on the potential of technology for communicating and collaborating, like Jarvis' BEdITeach, require extensive use of technological means by both the teacher educator and the student teachers. They are based on the assumption that expertise in the use of digital technology in education is better acquired through ongoing, "hands-on" exposure than through specific computer sessions. Within this perspective, the transfer to the classroom is seen as a kind of by-product, that is, as technology use becomes an integral part of their practices, it will make sense for them to extend these practices from the university to the school classroom. In Jarvis's BEdITeach, the student teachers were required to have a laptop and course instructors were encouraged to make use of this technology by way of pre-loaded, discipline-based software applications and Internet-based resources. He observed that student teachers often used their laptops in their BEd classes, along with a digital projector, to demonstrate software applications and slideshow presentations. On some occasions, these presentations took the form of the teacher introducing a new mathematics problem or concept. At other times, the presentations served as a precursor to a classroom exploration with technology.

At the other end of this axis, the other four courses all aimed at preparing teachers to specifically use technology within their classrooms. Here, the practices emphasized within the courses aimed at enabling trainees to transfer their learning to new settings and events: Alagic's MathInquiry provides a strong case in point. This course featured built-in requirements for sense-making, self-assessment, and reflection as to what worked and what still needed improvement, as teachers employed teaching practices congruent with metacognitive approaches to learning.

15.3.2 Views About Changes in Teachers' Role, Activity and Practices Underpinning a Course

Technology impacts upon classroom teaching practices and this impact is central in teacher development courses. Our five example courses are all based on the assumption that technology has a deep impact, but they tend to differ regarding the aspects of this impact which they each arguably privilege. We have distinguished three poles which help to characterize these differences (Fig. 15.2).

The first pole is about the new role of the teacher, when using technology. For instance, Das (BecIEW) considers that the function of a teacher is to facilitate, communicate and mediate among students. She expects that technology will help him/her and that he/she will therefore become friendlier, more dynamic, and interactive. Alagic (MathInquiry) also believes that a major change associated with the use of technology in the mathematics classroom is that teachers come to adopt the role of facilitators of learning. She adds that this change should result in classrooms becoming more learner-centred and less knowledge-centred.

The second pole is about the new kinds of activities that a teacher will develop in the presence of digital technologies. Jarvis (BEdITeach) developed the use of web resources for those who participated in his course, intending that they would transfer this new activity into their classes. He maintained a Course Schedule webpage on which the required readings for each session were listed, along with topics for discussion, and related websites. He also used the Course Schedule webpage to support the trainee teachers in software explorations during the actual workshops

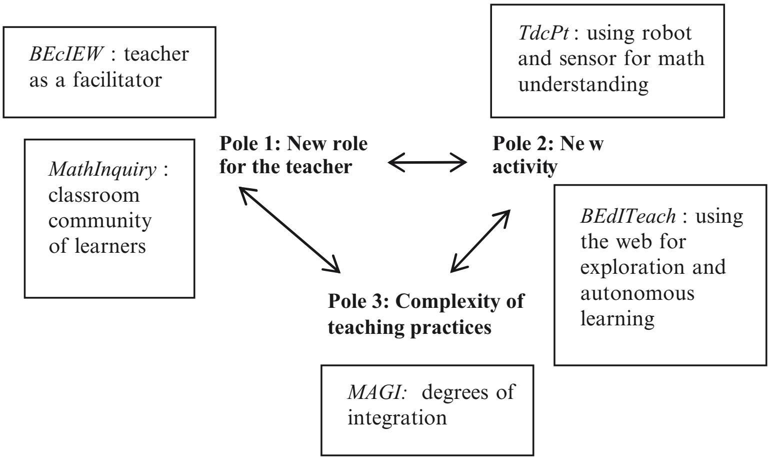

Fig. 15.2 Views about changes in teaching practices

and assembled a website featuring mathematics-related links organized by categories. Obviously this new kind of activity is conceived to help the learner to learn more autonomously, yet also collaboratively, as well as through exploration. Hence, this conception relates well to the role which the teacher is expected to adopt within the technology-integrated classroom. Indeed, Jarvis (BEdITeach) notes that managing this new activity is not obvious for the teacher. He/she must be diligent in enforcing some kind of accountability mechanism to assist students in staying on-task during classroom workshops because there are so many interesting/distracting web spaces available that students are often prone to distraction.

This places BEdITeach close also to the third pole, which is based on the idea that technology brings a new complexity into teaching practices that may not be captured by a focus that privileges just one aspect of the teaching situation. MAGI (Grugeon) particularly exemplifies this pole, characterizing the process of changing practices by several degrees of instrumental integration.

15.3.3 Views About How to Prepare Teachers

Courses about the use of technology aim to prepare teachers to adapt their practices, while taking into account the actual circumstances and contexts of the trainees' practices. In relation to the set of views associated with this challenge, we organize the five example courses along two axes (Fig. 15.3). A temporal axis distinguishes short and long-term integration while the other axis deals with teachers' professional proficiency.

The temporal axis (horizontal in the figure) separates courses that privilege trainees' short-term entry into the new practice of using technology and courses that

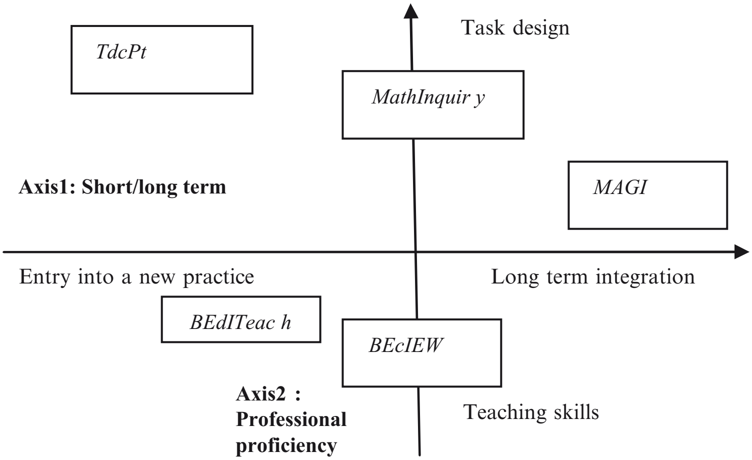

Fig. 15.3 Views about how to prepare teachers

try to take long-term integration into account. For instance, Jarvis (BEdITeach) stresses that of the brevity of practice teaching sessions for his teacher candidates often makes it difficult for them to implement an ambitious software-based activity in the classroom, and concludes that the most common and competent use by the trainees of the laptops during mathematics classes was in presenting mathematics problems or new mathematics learning using digital slideshow software. For him, therefore, while using slideshow software brings perhaps not the best contribution to math teaching/learning, it is nevertheless an effective use for his trainees in the concrete context of their teaching.

At the other end of this axis, courses considered by MAGI try to go further and to create conditions for more ambitious uses of technology. It is worth noting that BEdITeach is for pre-service training while MAGI deals with in-service. It is also important to stress the interdependence of this view with the second axis in Fig. 15.1. The ubiquitous use of technology in BEdITeach suggests a view in which technology is expected to permeate practice; those who begin to use technology in relation to short-term learning aims are seen as beginning on the path that will gradually lead into an approach which more fully integrates technology. By means of a contrast, the approach adopted by Grugeon and her colleagues (MAGI) stresses the challenge of long-term integration from the very start. This said, in the analysis of teachers' practice, the levels of what they describe as instrumental implementation also seem to pass from a more peripheral to a more central use: from instrumental initiation, where the emphasis is more on learning to use technological than on new approaches to mathematical knowledge, through instrumental exploration and reinforcement, to the level of instrumental symbiosis, an approach which takes advantage of specific affordances of the technology to enable new relationships with mathematical objects.

The professional proficiency axis (vertical in the figure) distinguishes between classroom teaching skills and professional content-related knowledge. Technology impacts upon both of these aspects of the organization of teaching practices.

Das (BecIEW) chooses in her course to privilege the first aspect, aiming to support her student teaching in managing the classroom, playing the roles of facilitator, communicator, and mediator. She specifically trained teachers to minimize their use of blackboard work while explaining critical steps, thereby reducing the abstract nature of the topic by repeating explanations and giving examples. This served to provide quality teaching for every learner so that no one in the class felt neglected and so that teachers were able to interact, gather feedback, and evaluate pupils' level of achievement through supplied worksheets. She also saw the management of time as an important factor. This choice is consistent with constraints within actual teaching: size of the class is sometimes very big; prescribed textbooks are not compatible with technology; the number of computers is insufficient; and 40-min classes are considered too short.

At the other end of the axis, following Ball (2000, p. 244), Alagic (MathInquiry) aims "to prepare teachers to know and to be able to use subject matter knowledge effectively in their work as teachers." She chose to concentrate on the design of relevant tasks for students. In her course, each trainee had to compile a Digital Resource File that consisted of five problem sets, a final presentation, and possible additional resources relevant for their future work. Each problem set utilized technology tools in an essential way and demonstrated a gradual development of selected concepts through a sequence of rich problems. Reflection is crucial in this approach and at the end of the problem set the trainees are requested to report about their thinking during the process of design.

15.4 Identifying the Various Practical Decisions Related to Course Organisation

This section concentrates on the various options that can be chosen in relation to the organisation of teacher professional development courses incorporating digital technologies. We categorize these options by referring to two core components of teacher education courses: the contents proposed in the course, and the strategies that teacher educators develop in order to help teachers to access this content.

15.4.1 Contents Proposed in the Courses

Drawing from the five cases of teacher development courses, this section identifies and presents six types of content that can be taught in pre- or in-service courses. The five cases do not privilege the same content. Figure 15.4 displays

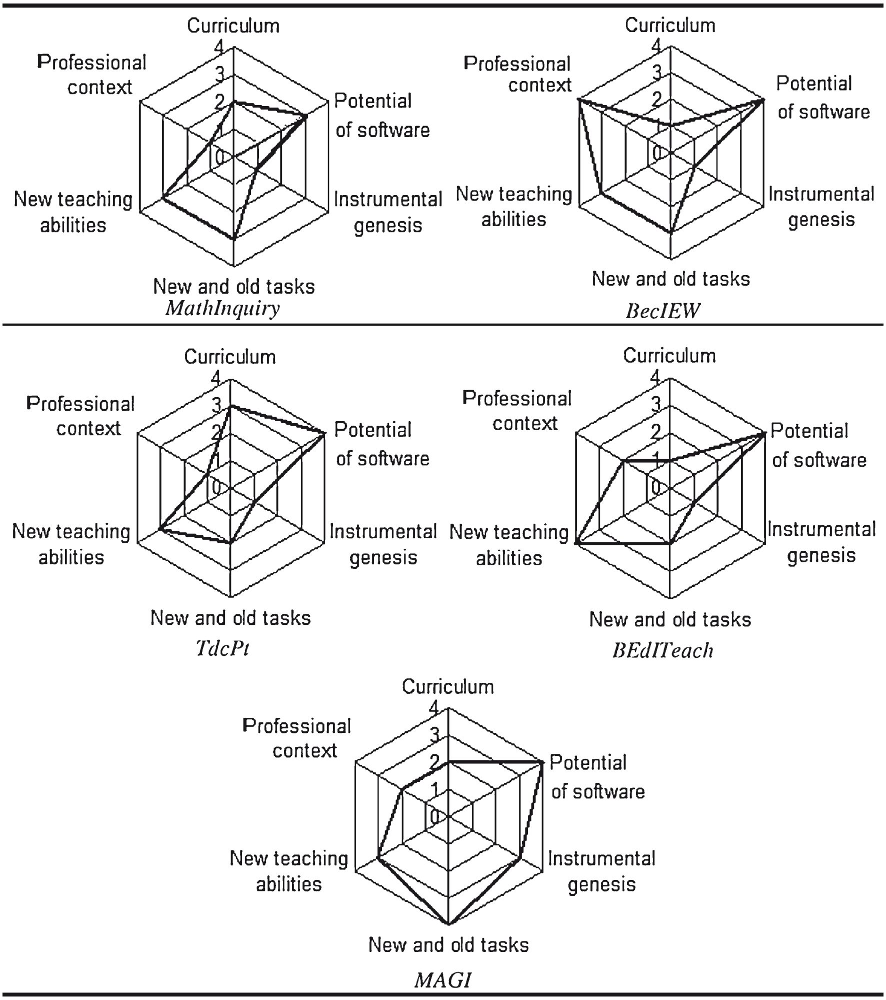

Fig. 15.4 Contents in teacher development courses

radar charts showing how each one can be located on a scale for each of the six aspects of learning content presented above.

15.4.1.1 Content 1: The Impact of Technology on Mathematics and the Resulting Evolution of the Curriculum

This type of content appears more or less in all five courses. This basic information can certainly not be left out in a teacher development course.

15.4.1.2 Content 2: The Potential of Computer Applications for New Alternatives in Mathematics Learning

There is a wide range of potentialities from research and innovation that can be presented to teachers. For instance, in both Grugeon's MAGI and Das' BecIEW, the teacher educators demonstrate and explain the potentialities of the integration and the use of dynamic geometry, e.g., the "drag" mode that provides a means of distinguishing between properties that are valid only for a specific drawing and geometrical properties of a figure (Laborde and Capponi 1994). The potentialities of technology for changing lessons are also highlighted: conjecture and inquiry become major aspects of the lesson, theorems are illustrated, and meaningful proofs can be constructed.

15.4.1.3 Content 3: The Ideas of Instrumental Genesis and Intertwined Mathematical and Instrumental Knowledge

These ideas are receiving increasing attention in research into teaching and learning mathematics with technology (Guin et al. 2004). In teacher development courses, discussion of these notions with teachers represents an important means by which to help pre-service and in-service teachers consider how to organize a series of sessions so as to take into account students' instrumental genesis. Perhaps because the notion of instrumental approach initially appeared in the French literature, from among the five courses it is Grugeon's MAGI which includes explicit opportunities for this discussion while introducing students to dynamic geometry use. In lessons involving dynamic geometry, the school students might learn, for example, to create and drag a point, to create lines, etc. Teachers have to identify the underlying instrumental abilities. They can also become aware that such abilities can be institutionalized as a new instrumental knowledge. The teacher educator offers then teachers a chance to reflect on tasks intertwining mathematical *and* instrumental knowledge as a support for students' conceptualization.

15.4.1.4 Content 4: Creating New Tasks and Making Them Work Together with Older Tasks

In all the teacher development courses described herein, new tasks integrating technological artifacts are introduced. If a teacher wants that students' conceptualization following these tasks really contribute to their mathematical understanding, he (she) has to highlight explicitly the links with existing paper/pencil tasks. Making student teachers reflect on new tasks and how to make them work together with older tasks may then represent one strategy useful in preparing to this important aspect of classroom implementation of technology.

For instance, Grugeon (MAGI) proposes to base courses on reflections upon dynamic geometry tasks for students, by analyzing figures and conjecturing geometrical properties. She also asks the teachers to build paper/pencil tasks for which students could use the properties conjectured within a dynamic geometry environment. Das (BecIEW) adopted a similar approach in order to help the trainee teachers taking her course to reflect on tasks involving the invariance of a variable triangle area with fixed base and elevation. In the MathInquiry course (Alagic), the problem set devoted to geometrical thinking challenges pre-service students to compare and contrast, via metacognitive reflection, what they can learn with dynamic geometry that was not available in traditional approaches, providing some hints about the dialectic between new and old tasks.

15.4.1.5 Content 5: New Teaching Abilities

The integration of a new artefact into a teaching situation modifies the professional practices and requires teachers to undergo a complex process of adaptation. In some stages of a lesson, students work alone or in groups to solve a problem with technology or paper-and-pencil. In others, the teacher orchestrates collective discussions. Teachers have to learn to identify new teaching roles brought about by technology at different stages. Most courses described in this chapter take into account this dimension. For instance, a particular ability is stressed in MAGI, based on the fact that the instrumental language can play an important role in students' verbalization, lessening the use of mathematical language. Teachers have to learn to organize the connection between these languages and to take greater advantage of this connection for students' learning.

15.4.1.6 Content 6: Introducing Technology into a Professional Context

Teachers have to learn to make the best of the context in which they will have to use technology. For instance, BecIEW takes into account:

1. Differences between schools' equipment that make things possible in some schools but not in others
2. Differences between topics – some are more easily taught with technology than others
3. Differences between institutional contexts – in some contexts technology will be more easily accepted

15.4.2 Teacher Educator Strategies

Besides deciding upon the contents they aim to make their student teachers access, teacher educators have to choose the strategies by which they will undertake to reach these aims. We identified four possible strategies used within the five courses

	Demonstration	Role Playing	In practice	Communities
MathInquiry	X	X		X
BecIEW	X	X		
TdcPt	X		X	
BEdITeach	X			X
MAGI	X	X	X	X

Fig. 15.5 Strategies used in the cases of teacher development courses

described in this chapter. The table in Fig. 15.5 summarizes the different strategies used in the cases of teacher development courses. The ideas underlying the first two strategies ('demonstration' and 'role playing') are inspired by the notions of 'monstration' and 'homology' (Houdement and Kuzniak 1996).

15.4.2.1 Strategy 1: Demonstration (Showing How to Achieve a Specific Goal)

With this strategy, the educator shows how to use the digital artefact and illustrates particular activities with it. This strategy doesn't emphasise discussion about conditions and constraints of technology integration. Every course uses this training strategy to some extent. The main aspect here is immediate efficiency: in a short time, student teachers are introduced to a practical classroom implementation of technology. However the effectiveness is far from guaranteed when this strategy is used alone because student teachers have little opportunity for reflecting and discussion on the rationales for the implementation and the adaptation to their teaching context.

15.4.2.2 Strategy 2: Role Playing (Teacher as a Student)

In this strategy, the educator organises a course session in two steps. In the first step, he/she asks the course participants to resolve a task as if they were students; in a second step, he/she orchestrates a discussion, highlighting the aims of the activities and the decisions that a teacher has to take. This strategy is also used in many courses. Student teachers can reflect on cognitive issues related to the task from their own experience. The main advantage is then that student teachers can enter into a reflection about technology use, even when they had little previous experience in this field. The limitation is that the reflection is introspection into the relationship of an adult already educated in mathematics and technology, more than really an opportunity for thinking about classroom situations involving students with little mathematical knowledge.

15.4.2.3 Strategy 3: In Practice (Teacher as Reflective Practitioners)

In courses based on this strategy, participants design a teaching situation with the help of the educator at the university. then they put it into operation during classroom sessions. Finally, they analyze the sessions with the educator back the university. As an example, in TdcPt (Hunscheidt) prospective primary teachers experiment with selected situations in their classrooms involving a robot and a monitoring device. Following this, they analyze selected tasks using classroom observations and extracts of pupils' work. Since Schön (1983) the idea of reflection in action turned out to be a central feature in professional education and it was implemented in a variety of strategies, involving various teaching contexts and means of observation. It seems that when classroom technology use is concerned this idea is relatively new and that few strategies based on this idea have been thus far conceived and experimented.

15.4.2.4 Strategy 4: Learning in Communities

Courses like BEdITeach (Jarvis) are based on virtual *networks* as means to build on-line communities sharing information, resources and expertise. BEdITeach is for prospective teachers and it focuses on mathematical knowledge rather than directly on professional development. It nevertheless assumes that the experience of learning in a community will impact on future teaching practices: course participants regularly share their new findings and ideas relating to online resources and mathematics computer software with the instructor who functions as a co-learner. This instructor/student relationship reinforces the problem-based approach in accordance with the international reform movement in mathematics education. In MathInquiry (Alagic) course, learning in communities is seen as a means to share knowledge and reflection within a group of peers. It should help to initiate communities of teachers, exchanging useful information on different systems and cultural practices, thinking about specific learning experiences, sharing and discussing content as well as views about students' learning.

15.5 Conclusion

As we noted at the beginning of this chapter, research about teacher development courses in technology and mathematics is still in its infancy. We felt that investigating the variety of views and options that underpin such courses could represent a useful contribution to the development of this research field. We classified the views with regard to three aspects: implementation of technology, changes in teachers' role, and adaptation of teaching practices. We classified the options first with regard to the content, and secondly with regard to teaching strategies. We draw these classifications from the exploration of five cases of teacher development courses provided by

ICMI study conference participants. Although they come from three continents and involve very different contexts, they certainly cannot be considered as representing the whole diversity of possible options and views. Nevertheless we notice that the classifications presented within this chapter provide informative and helpful insight. In each of Figs. 15.1–15.5, the courses are positioned at various places along the given continua, with the proximity among, or distance between, the courses made visually apparent. The figures also give a clear account of each of the five course's ambitions, content/strategy privileging, and options.

- MathInquiry focuses on mathematical knowledge and on the teacher as a task designer. Its aim is to prepare teachers to know and to use knowledge effectively in their work. This "metacognitive" approach is consistent with the potentialities of technology and its impact on mathematics. In this course, communities of learning favour human interaction as well as virtual communication.
- In BecIEW, the context in which technology can be implemented is important. Assuming that technology will improve teaching, the author chooses to focus on teaching skills and to develop these by way of "in-practice" strategies.
- TdcPt considers the use of a specific technology (robot and sensor) and its potentialities. It assumes the creation and evaluation of classroom situations taking advantage of these potentialities as a good strategy for teacher education.
- BEdITeach does not directly aim at professional development. It nevertheless assumes effects of learning "in communities" with technology on future teaching practices. This strategy heavily relies on virtual communication and resources.
- For MAGI, the potential of technology to improve learning is balanced by the necessities like students' instrumental genesis. A course should help teachers to achieve a better degree of integration of technology in their classrooms by considering this genesis and the dialectic between old and new tasks. This can be done only on a long-term basis and through the use of varied strategies.

Each course, then, has its own idiosyncratic consistency, yet the reasons behind specific options are not always easy to determine. This could be characteristic of teacher development courses, especially within a new domain such as technology for teaching/learning. While it is widely acknowledged that teacher development is crucial for the successful integration of technology in the mathematics classroom, there is very little presently available to guide policy makers, researchers and teacher educators regarding the relevance of different viewpoints, content selection, or the actual effectiveness of various teaching and learning strategies involving technology. We hope that the classifications that this chapter provides will assist in the posing of appropriate questions for future development and research studies.

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Section 4
Implementation of Curricula: Issues of Access and Equity

Chapter 16
Introduction to Section 4

Colleen Vale and Cyril Julie, with Chantal Buteau and Jim Ridgway

Abstract The nature and extent of the implementation of digital technology in mathematics curricula along with issues of access and equity were the issues considered by one working group of this ICMI study. A summary of the discussion conducted by the working group and the findings reported in the following chapters on this theme are presented. It is clear that widespread and sustained use of digital technology is not common and that where digital technology is used there are complex and confounding equity issues.

Keywords Intended curricula • Implemented curricula • Access • Equity • Curriculum reform

16.1 Introduction

Since the first ICMI Study, "The Influence of Computers and Informatics on Mathematics and its Teaching" (Churchhouse et al. 1986) more than 20 years ago, developments in digital technology have resulted in the emergence of a range of applications for mathematics and mathematics teaching and learning. Furthermore, governments have developed policies to promote the learning and use of digital technologies throughout education systems in general as well as for mathematics learning in particular. Thus there has been some systemic implementation of digital technologies in mathematics education as a result of policy initiatives, alongside more scattered implementation as a result of specific innovations and initiatives.

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In designing the current ICMI study the members of the International Planning Committee (IPC) were aware that neither centralised nor local initiatives have tended to result in widespread and sustained use of digital technologies in mathematics curricula and in mathematics teaching (Artigue 2000, cited in Hoyles et al. 2004; Wong 2003). One of the main objectives of this study was to understand the nature and extent of the integration of digital technologies in mathematics curricula and to consider the factors influencing integration in different countries and education sectors and the implications for reform and change. Moreover, "cultural diversity and how this diversity impinges on the use of digital technology, particularly in developing countries" was also a major focus of the study (Hoyles and Lagrange 2005, p. 1). Likewise we were aware that access to, and use of, digital technologies for mathematics learning differ within countries according to socio-economic, gender and cultural factors (Hoyles 1998). Again we were interested in understanding the extent and implications of these differences and to discern the factors influencing equity with respect to technology and mathematics.

A set of key questions framed the inquiry into the nature, extent and equity of the technology-rich mathematics curricula (IPC 2005). In this chapter we provide some background regarding the impact of digital technologies on mathematics curricula and introduce the chapters included in this section. We conclude this chapter with a discussion of the emerging issues and consider their implication for reform and change in mathematics curricula.

16.2 Mathematics Curricula

16.2.1 Intended, Implemented and Attained Curricula

We agreed that in order to investigate the integration of digital technology in mathematics curricula, the outcomes regarding access and equity and the implications for reform and change we needed to consider three aspects of curriculum: the intended curriculum, the implemented curriculum and the attained curriculum (Robitaille and Garden 1989).

By intended curriculum we mean the planned curriculum that is normally documented and considered the formal curriculum for mathematics learning for a level or sector of education. The intended curriculum may be a national or local document and records the intended learning for students in the educational setting or sector. Hence it is a statement of policy regarding mathematics learning regardless of whether it has been prepared by a government or 'authorized' educational body, or the teachers who will implement it. The integration of digital technologies in intended mathematics curricula is therefore subject to political, social and cultural forces concerning the place of digital technology in education, in mathematics and in mathematics education. Wong (2003) pointed out in his analysis of the influence of information and communication technology on mathematics curricula in countries from the 'west' and 'far east' that the intended curriculum can take many forms. It can be a checklist of attainment targets or standards, a program of studies, a framework for curriculum writers, "codes of practice", general principles and directions for teachers or an examination syllabi. In some countries the intended curriculum is mandatory, in others not. Wong also found that in some countries the role of information and communication technology is located elsewhere in education or curricula and not explicitly stated in the mathematics curriculum.

The implemented curriculum, on the other hand, is the curriculum of the mathematics classroom. It is mathematics curriculum that actually happens and may therefore vary significantly from the intended curriculum. It is constituted in the practices of teachers and learners in classrooms or learning environments. Since classrooms or learning environments are communities, 'orchestration' or integration of digital technology into classroom practice is also subject to political, social and cultural factors related to mathematics, digital technologies and to teaching and learning. Furthermore different students in the system or classroom may experience the intended or implemented curriculum differently giving rise to questions regarding access and equity in mathematics learning.

The attained curriculum is the mathematics achieved by the learners and is identified through the assessment of mathematics learning and may be substantially different from both the intended curriculum and the implemented curriculum.

16.2.2 The Mathematics Curricula of Different Countries

Two questions formed the basis of the inquiry with respect to the nature and extent of integration of digital technology in mathematics curricula:

> How have mathematics curricula and values changed to reflect developments in mathematical knowledge and practices afforded by digital technologies?
>
> How have countries with different economic capacity or with different cultural heritage and practices implemented digital technologies in mathematics education?

In the period since the 1980s, Wong (2003) found that information technology received increasing attention in the intended school mathematics curriculum of countries in the 'west' and the 'east'. He observed that mathematics curricula allowed students to use calculators and computers, made provision for the necessary facilities, encouraged student use of information technology and called on teachers to actively incorporate information technology in mathematics teaching. Moreover the changes with respect to the integration of information technology in mathematics curricula over the past 20 years represented a shift "from students (in the passive sense: permission) to teachers and probably back to students (teachers facilitating active use among students in learning mathematics)" (Wong 2003, p. 297). Wong cautioned against making assumptions about the more explicit reference to information technology in the intended curricula of western nations compared to nations in the far east, and argued that the way in which information technology is positioned in the mathematics curriculum is more important than the amount.

Wong observed three ways in which information technology was positioned in mathematics curricula. In some curricula information technology was regarded as integral to the effective teaching and learning of mathematics. In other curricula information technology was positioned as an enhancement to learning which could enrich the learning experience of students. Thirdly, in some curricula mathematics appeared to carry the responsibility of developing information technology literacies among students. With regard to this third position, he found that in some curricula the development of information technology skills in a mathematics context were evident, while in other curricula there was caution not to confuse the teaching of computer skills with the teaching of mathematics. According to Wong, the kinds of digital technology that may be effective or enhance mathematics learning, and the relative emphasis on content and processes in mathematics curricula underpinned the positioning of information technology in mathematics curricula. These issues, Wong argued, are questions about mathematics itself and the value and purpose of learning mathematics. The ways in which information technology is positioned and questions concerning mathematics itself and the value and purpose of learning mathematics are illuminated in Chaps. 17 and 19 regarding the intended and implemented curriculum in this section.

Posadas (2006) explained the ways in which the integration of digital technologies in mathematics education may be related to the policies of the United Nations Educational, Scientific and Cultural Organisation (UNESCO), in particular "Education for Sustainable Development" (UNESCO n.d.) and "Education for All" (UNESCO 1990). The relationship between the use of digital technologies in mathematics education and UNESCO's goals for the enhancement of the human condition in the interest of sustainable development, especially as they apply to late developing countries, are discussed in Chap. 17.

In Chap. 17 Julie, Leung, Nguyen, Posadas, Sacristán and Semenov describe the intended and implemented curricula of culturally diverse nations including Russia, South Africa, China, Vietnam and several Latin-American nations. The influence of social, economic, political and cultural factors for these countries of differing economic capacity and cultural heritage are analysed. Similar to Wong (2003) the positioning of digital technology in the intended mathematics curricula in these countries reflected an emphasis on enhancement of mathematics learning and/or developing information literacies. The Russian mathematics curriculum is a contrasting example wherein the nature of mathematics in the digital age appears to have been more explicitly identified resulting in a "new mathematics" curriculum.

Julie and co-authors also discuss the significance of integrating digital technology in mathematics curricula for the enhancement of the human condition and sustainable development. Not surprisingly, the authors of Chap. 17 show how economic incapacity has severely hampered the implementation of information and digital technologies in late developing countries and especially in low socio-economic communities within these countries that are often located in rural or geographically isolated regions with inadequate infrastructure.

In Chap. 19, Assude, Buteau and Forgasz include examples of intended mathematics curricula and discuss the factors affecting the implementation of technology-rich mathematics curricula in western, eastern and late developing countries. They too show that the political, social and cultural factors influencing intended curricula are concerned with either the development of ICT literacies and/or the capacity of digital technologies to support or enhance mathematics teaching and learning. Their typology of factors influencing both intended and implemented curricula, as Wong (2003) argued, also concerns the nature of mathematics and mathematics epistemology.

> Economic, political and social factors have resulted in the integration of digital technologies in the intended mathematics curricula of countries irrespective of the cultural background or economic capacity. However as we explain in the following sections of this chapter, economic capacity, cultural heritage and a number of other factors have impeded the wide spread integration of digital technologies in implemented mathematics curricula.

16.2.3 The Mathematics Curricula of Different Sectors

In contrast to the first ICMI study which mainly reported developments in the western world and the potential to use technology to model mathematical ideas, the papers prepared for the current ICMI study were largely concerned with curriculum, teaching, learning and design at the school level, especially secondary. The nature and extent of the integration of digital technology in mathematics curricula indicated above and discussed in more detail in the following chapters, result from an analysis primarily of school mathematics curricula.

A small number of papers prepared for the study conference of ICMI Study 17 reported on the integration of digital technologies in tertiary mathematics education (for example, Andresen 2006; Buteau and Muller 2006; Dana-Picard and Kidron 2006; Lavicza 2006; Makar and Confrey 2006). Lavicza (2006) noted the absence of evidence of widespread and imbedded use of digital technologies in tertiary mathematics curricula reported by ICMI Study 11: "The Teaching and Learning of Mathematics at University Level" (Holton 2001). However a substantial number of mathematicians use CAS for teaching at some level (Lavicza 2007). The mathematical and epistemological factors influencing the implementation of Computer Algebra Systems (CAS) in tertiary mathematics education emerging from Lavicza's (2006) international comparative study contribute to the typology of factors discussed by Assude, Buteau and Forgasz in Chap. 19. Interestingly, researchers examining curriculum, teaching and learning of tertiary students studying to become teachers of mathematics authored most of the ICMI Study 17 conference papers about tertiary mathematics. Perhaps this indicates that pedagogical and didactic factors as well as epistemological factors are critical for implementation in tertiary settings.

The Mathematics Integrated with Computers and Applications undergraduate program (MICA) at Brock University in Canada (Buteau and Muller 2006) provides a case in point. MICA grew out of an evolution of technological use in mathematics courses at Brock University (Muller 2001) that started in the 1970s. Pead and Ralph with Muller (2007) reported that

> During the two years intensive development of MICA, faculty sought to create a modern mathematics program that would foster creativity, and the mastery of mathematical concepts and their applications, while making the best possible use of modern technologies. (p. 135)

In addition to a revision of traditional courses, innovative core project-based courses, also called MICA, were introduced in which, among other things, students select a mathematics topic of their choice, research it, develop an interactive computer environment to further research their topic, and repeat this development until they are ready to communicate their findings and understandings through their own interactive computer environment; see MICA Student Projects web-site (n.d.) for examples of student work. Buteau and Muller (2006) reported,

> We have found that the approaches, activities, and experiences in the MICA courses are able to harness the students' motivations thereby empowering them to become their own mediators in the development of their mathematical knowledge and understanding. (p. 8)

In other words, by involving tertiary students early in a rich technology environment, student abilities, interests, engagement in mathematics may be different from those shown in traditional programs.

The use of digital technology to engage very young, and even pre-school learners in mathematics experiences were not reported in the current ICMI study, though Yelland (2007), for example, shows how young children play and use technology to develop mathematical ideas. Clearly more research about the integration of digital technology in the learning environments of pre-school children and children in the early years of schooling is required.

Also absent from the study were reports on the place of digital technologies in mathematics curricula within vocational education settings. Yet it would appear that the implementation of digital technologies in vocational mathematics education is consistent with political reforms in vocational education that advocate greater flexibility in delivery of these courses and the economic imperatives of information literacies and the vocational application of mathematics. Research about the use of digital technologies in this sector of education is available, see for example Javed and Vale (2006) and Noss et al. (2002), but clearly more studies of this sector are needed.

16.2.4 Factors Influencing Implemented Curricula

Two further questions regarding the implementation of digital technology in mathematics curricula provided a focus for chapters in this section:

> What approaches, strategies or factors foster or impede the implementation of technology-rich mathematics education? What issues are involved for policy-makers, administrators and teachers in the organisation of technology resources in educational settings?

> What have we learned about the process of change and reform in mathematics education through our successful and unsuccessful experiences of implementing digital technologies in mathematics education?

As discussed in the following chapters the implementation of technology-rich mathematics curricula has occurred through the system-wide initiatives or evolved through local innovation or research activity. In most countries governments have provided infrastructure for schools and teachers, ranging from hardware such as computers and electronic interactive whiteboards, to computer network access and to network supported teaching materials or software. In some countries or jurisdictions, the implementation of technology-rich mathematics curriculum has been supported by large-scale professional development projects as in Mexico (Chap. 17; Ursini and Sacristán 2006), and TELMA in Europe (Artigue et al. 2006). Elsewhere, smaller-scale innovation or professional development projects are intended to provide exemplars that others will follow through the provision of appropriate teaching and curricula materials. Through the description and analysis of the various initiatives and projects of the vastly different countries reported on in Chap. 17, Julie and co-authors show that implementation of digital technology in mathematics classrooms is dependent upon well-funded large or system-wide projects that provide not only infrastructure and digital technology resources but also incorporate well planned and structured professional development and training for teachers and the provision of ready-to-use digital tools, learning objects and teaching materials.

According to Artigue (2000) the slow progress in the integration of digital technologies for mathematics learning in school classrooms is due to an invariance of values and norms in mathematics education, an underestimation of both the complex process of transforming mathematics with technology in the classroom and the mathematical demands placed on learners, and a dissonance between technical and conceptual aspects of mathematical activity (cited in Hoyles et al. 2004).

In Chap. 19, Assude, Buteau and Forgasz provide a typology to analyse the factors influencing implementation and consider resistances and change factors. The typology and factors influencing implementation embrace primary, secondary and tertiary mathematics, though the strength and influence of particular factors may vary between the sectors due to cultural differences (see for example Assude et al. 2006; Trigueros et al. 2006). At the tertiary level the factors influencing implementation are different in character from those in the school sector because they arise in educational systems that have radically different responsibility structures; see for example Kozma (1985). Assude, Buteau and Forgasz argue that the resistances to the implementation of digital technology in mathematics curricula are personal, epistemological, ethical, economic, symbolic and institutional. Julie and co-authors agree with Wong (2003) since they report that the dissonance between the intended and implemented curriculum in various countries arises because the integration of technology is positioned according to whether it enhances mathematics learning and whether mathematics should carry responsibility for developing information literacy. Though cultural heritage may appear to explain resistance and some differences in implemented curricula between countries, the factors concerning pedagogy, didactics and epistemology as argued by Assude et al. (Chap. 19) and Artigue (2000) is more likely to explain the resistance of teachers.

We observed that system level curricula change that involved the integration of digital technologies in high-stakes assessment, that is the attained curriculum, was more likely to result in widespread implementation for particular school-level mathematics courses, for example the Victorian Certificate of Education, Australia (Forgasz et al. 2006) the International Baccalaureate (Leng 2006), and the two major standard tests, the Scholastic Assessment Test (College Board n.d.) and American College Testing (ACT n.d.), for admission in most American universities. In Chap. 10, Cazes, Lee, Perrusquia, Rojano, Sangwin and Wong discuss the relationship between assessment practices and the implementation of digital technology in mathematics curriculum.

16.3 Access and Equity

UNESCO's policy of "Education for All" (UNESCO 1990) guided our consideration of implementation of digital technologies for this ICMI study. Furthermore the IPC recognized that the implementation of digital technologies in mathematics curricula did not affect the learning of students equally. Hence two key questions concerning access and equity guided the inquiry reported in this section:

> How and to what extent has the use of digital technologies in mathematics education enabled, or eroded, equity and agency in mathematics education?
>
> How and to what extent has technology-integrated mathematics contributed to, or reduced, differences between countries in participation and achievement in mathematics?

Findings emerging from these inquiries are reported in Chap. 18 by Forgasz, Vale and Ursini. Prior to discussing the findings of a range of studies concerned with equity issues, the authors begin the chapter by defining equity and agency in the context of digital technology and mathematics education. They discuss the meaning of equity as access to digital technology for mathematics learning and to technology-rich mathematics curricula, equitable distribution of resources, equitable pedagogies and equitable learning outcomes. This framework of analysis draws upon the literature and the categories used by UNESCO in their evaluation of educational policy regarding "Education for All" (Sherman and Poirier 2007). For the studies discussed in this chapter, the findings with respect to gender differences are not conclusive. In some countries the use of digital technologies in mathematics has enabled a gender gap in achievement or affect to close while in other countries the gender gap has widened. Socio-economic differences in access to, and learning outcomes of technology-rich mathematics within and between countries are also complex.

The authors of Chap. 18 also considered:

> How can digital technologies be used in mathematics learning to respond to the diverse needs of all learners, regardless of mathematics achievement, sex, class, ethnicity or cultural background?
>
> What can students and teachers with limited access to digital technologies, or access to modest technologies for mathematics learning do with technology that is empowering for students?
>
> How can the use of digital technologies in mathematics education support the learning of students with special needs?

The authors present some principles regarding equitable pedagogies with respect to the use of digital technologies in mathematics in Chap. 18 but they note a scarcity of studies that focus on the development of agency for mathematical learners with technology. Furthermore, while we know that schools servicing students with special needs are using technology there appears to be a gap in the literature in this field. One study conference paper by Healy (2006) described a research program aiming to design and evaluate digitally based learning environments for deaf students and blind students in Brazil. Multi-media involving video and audio will provide opportunities to explore mathematical expressions in diverse ways to engage these learners. Clearly more studies involving students with special needs are needed if we are to understand how digital technologies may support the diverse needs of all learners of mathematics.

16.4 Conclusion

Finally we comment on the visions and possibilities of mathematics curricula both idealistic and realistic about how mathematics curricula should be changed in response to technology mediated knowledge, the diverse needs of all learners and for countries of different economic capacity and cultural heritage.

Mathematics, a human construction and practice with a variety of purposes, is in constant change and dialectically entwined with technology. Education systems and curriculum designers and mathematics teachers make decisions about the nature of the "new mathematics" and how it is integrated and positioned in the intended mathematics curricula and implemented and attained in classrooms. While we may imagine the possibilities that rapidly emerging digital and communication technologies affords for mathematics, its application and for learning, the evidence suggests that there will always be a lag between the development of "new mathematics" and its implementation in education systems. The lag is not just because it takes time and resources to change education systems but because teaching and learning are cultural practices with embedded assumptions and values. Implementation of mathematics afforded by digital technologies is more likely to occur when and where there is a shared vision among political leaders, education authorities, mathematicians and mathematics teachers.

The potential exists for late but fast developing countries to by-pass the curriculum experiments and out-dated technologies (PCs, land-lines, computer laboratories) of the early but slow developing countries in the manner envisaged by Papert (2006) where there is the political will and the drive for sustainable development, equity and improvement of the human condition. To do otherwise runs the risk of widening the gap between countries of differing economic capacities and threatening the place of mathematics in our cultures. As Conway (1997) implored when reflecting on the integration of digital technology in undergraduate mathematics:

> We have to embrace technology, I don't mean just tolerate it; embrace it and celebrate it. The professional mathematics community must adapt and learn how to best incorporate technology into instruction. With the existence of powerful, inexpensive computers,

> I see mathematics departments rethinking their entire curriculum. Otherwise, we are out of business.

The same applies to school mathematics. The chapters that follow in this section and others in this volume enable us to feel optimistic about mathematics and the possibilities to enhance the mathematics learning of students with diverse needs and cultural heritage.

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Chapter 17
Some Regional Developments in Access and Implementation of Digital Technologies and ICT

Cyril Julie, Allen Leung, Nguyen Chi Thanh, Linda S. Posadas, Ana Isabel Sacristán and Alexei Semenov

Abstract Access to and implementation of digital technologies for mathematics teaching and learning across and within countries and regions display similarities and differences. This chapter is derived from regional presentations made at the ICMI Study 17 Conference held in Vietnam in December 2006. The descriptions of the situations in four countries (Russia, Hong Kong, Vietnam, South Africa) and one region (Latin-America) give a sense of the similarities against the general background of a global goal for schooling in the twenty-first century. The complex issue of universal access to digital technologies for meaningful mathematics learning, it is suggested, requires concerted efforts to address a host of mitigating factors.

Keywords Digital technologies • Mathematics education • Access and implementation • Regional perspectives from Russia and developing countries in Asia • Africa and Latin-America

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17.1 Introduction

Access to and use of technology for mathematics teaching and learning is now more or less universally accepted. However, this ideal manifests itself differentially across different regions in the world. Additionally, the use of digital technologies needs to take cognizance of larger global imperatives regarding the aims and goals of education. This chapter draws together reports – presented at the ICMI 17 Study Conference – on policies, directions and initiatives regarding access and implementation from various countries and regions: Russia, Hong Kong (A Special Administrative Region of China), Vietnam, South Africa, and several countries in Latin-America. The regional reports are set against the background of UNESCO's imperative for sustainable development to which technology-driven mathematical education should contribute. The descriptions in the reports show that there has been definite progress since the first ICMI study on technology use in mathematics teaching (Churchhouse et al. 1986) regarding late developing countries but that much needs still to be done for equitable universal access to be realized.

17.2 Macro Perspective on Education for the Twenty-First Century by Linda S. Posadas

Macro perspectives refer to the desires expressed by countries across the world regarding access. Here we present these perspectives from the point of view of the United Nations Educational, Scientific and Cultural Organisation (UNESCO).

UNESCO's perspectives on access are underpinned by the concept of sustainable development seen as "Development that meets the needs of the present without compromising the ability of future generations to meet their own needs and a process of change in which exploitation of resources, the direction of investments, the orientation of technology development, and institutional change are all in harmony and enhance both current and future potential to meet human needs and aspirations." A further complementary definition is "improving the quality of human life while living within the carrying capacity of supporting ecosystems" (UNEP 1991). Thus the emphases are on the inter-generational responsibility in meeting human needs and the protection of the Earth's capacity for regeneration.

Sustainable development is concerned with benefiting both people and ecosystems and this requires people who are able to think in terms of systems, having the skills of understanding complexity, finding and identifying linkages and relationships, doing quantitative and qualitative analysis, and presenting data in formats that are comprehensible and useful to non-specialists who have policy and decision-making authority.

Agenda 21 was adopted in 1992 as the guide to implementation of the sustainable development agenda. Chapter 36 of Agenda 21 emphasizes the critical role of education in improving the capacity of the people to address environment and development issues. It called for the key action of re-orienting education towards sustainable development, by integrating environment and development as a cross-cutting issue into education at all levels, introducing new courses, revising and upgrading curricula, introducing innovative teaching methods, re-training teachers, as may be needed. To further bolster the sustainable development agenda, the United Nations General Assembly at its 57th session in December 2002 declared the period 2005–2014 as the Decade of Education for Sustainable Development, with UNESCO as the lead agency for its implementation. This is accompanied by the Education for All (EFA) Goals that, amongst other issues, call for universal primary education by 2015 and improving the quality of education. Within these parameters it is conceivable to think about "Mathematics education for all" with "Mathematics Literacy for Sustainable Development" and to ponder the use of technology in mathematics teaching in such an agenda. In the context of providing mathematics education for all, in a manner that would develop citizens who can thrive and contribute to sustainable development, information and communications technologies have become a useful tool. In this regard technology

- Increases the educational options for the marginalized sectors of society (e.g. distance learning);
- Introduces new approaches to pedagogy/course materials development;
- Introduces new social/cultural opportunities for interaction among peers;
- Enhances the efficiency and effectiveness of educational administration (at classroom, school and system levels).

But the realization of the above will require that the challenges below be faced and addressed.

- Reformulation of current mathematics education approaches (e.g. through the exercises and homework problems, examples) to bring in the sustainable development concept and build awareness;
- Developing the ability to apply mathematical concepts and processes to the life, work and culture of one's own society, an appreciation of the contributions of mathematics, and awareness of its limits;
- Developing the attitude, value and skill to distinguish between use and misuse of mathematics (e.g. misuse of statistics, probability concepts).

Thus at the macro level, access to technology is not merely regarded as some technical solution to problems facing mathematical teaching and learning, nor is it a mere "shift in parameters" due to its availability and power. Rather mathematical technologies and their use should, in a considered way, take cognizance of the global thrust to enhance the human condition in the interest of sustainable development.

The wide goal for provision of access plays itself out within specific local contexts as can be seen in the five cases from different regions of the world which are now discussed.

17.3 Regional Reports

17.3.1 Case 1: Russia by Alexei Semenov

One of the issues that has been deliberated upon for some time is what the nature of mathematical content should be, given the availability of information communication technologies. There seems to be consensus that this availability calls for a change of content. In the Soviet Union attempts were made to bring about a change in content by considering informatics as a new mathematics. In offering informatics as new mathematics, consideration was given to the nature of current content. This was viewed as ad hoc comprising different kinds of content at an epistemic level. Some of these kinds of content are meta-content (discovery, collaboration, project, etc.), technology-content, application-content, generalization- and transfer-content, mathematics-content (activity-content), and mathematics-in-other-school-subjects. In line with informatics as content, it was proposed that "programming is the second literacy." Textbooks dealing with informatics appeared in the mid-1980s. One such textbook was used by 2.5 million students in the final years of schooling. The introduction of informatics was accompanied by appropriate software, teacher training and the design of personal computers focusing on school applications.

A major motivation for propagating the introduction of informatics was the virtual disappearance of the primary goals – development of the child's communication and reasoning abilities, understanding of the world and being an independent learner and a creative artist. Informatics afforded the opportunity for the re-insertion of the primary goals around the rubric of a "New School for the society of *information age*." Furthermore, technology can substitute for the technical mathematical skills that are the focus of so much time in mathematics. It is not the case that technical skills are not needed. Rather, the issue is that a new set of technical skills should be introduced and this is now happening in Russia. Fundamentals of Informatics deals with discrete mathematical objects and processes. The relevant constructs are explored by students in visual and playable ways in both real and computer-based microworlds. As such the basis is laid for modern computer mathematics and classical continuous mathematics.

Thus for the local intended curriculum in Russia, access and implementation are linked to Informatics strongly supported by technology with the intention to change mathematics to serve both some neglected goals of schooling and affording the opportunity for engaging students in mathematics relevant for the information age.

17.3.2 Case 2: Hong Kong (A Special Administrative Region of China) by Allen Leung

When Hong Kong returned to the People's Republic of China in 1997, the newly established Hong Kong SAR Government made a decision to transform the school educational environment into a technology-rich setting to meet the need of a fast changing society. A 5-year ICT education strategy was announced in the first policy address of HKSAR in 1998 and was evaluated in 2004 (Education and Manpower Bureau, Hong Kong SAR 1998, 2005). One of the targets of the policy was to have teaching in at least 25% of the curriculum supported through IT within 5 years. These policy and evaluation initiatives were further reviewed in 2004 in the hope of mapping out future strategies (Education and Manpower Bureau Hong Kong SAR 2004a, b).

Within all these initiatives to change to an ICT-rich curriculum, mathematics was regarded as a "Key Learning Area" and the most natural for ICT implementation. The core of the major curriculum reform was a set of generic skills and ICT skills. Regarding the latter, the capacity to seek, absorb, analyse, manage and present information critically and intelligently in an information age and a digitised world was the major objective. Information Technology for Interactive Learning was a vehicle for realizing the policy intention.

The policy intention was supported by various mechanisms via teacher enablement, curriculum and software, hardware provision and network infrastructure. In this regard most schools became well-equipped and installed Intranet and Internet access, with a computer and data projector in each classroom. Each school has at least one multi-media laboratory and some schools provided every teacher with a laptop computer. To further bolster and promote the use of ICT, exemplary teaching materials and ready-made ICT applets were available through dedicated websites. Detailed instructions for using specific tools (e.g. *Sketchpad*, *Excel*) to complete a teaching or learning task were also provided. Other resources included individual mathematics teachers' websites; schools and professional bodies sharing their self-developed teaching materials through the Internet; projects funded through the government; and ready-made spreadsheets, PowerPoint presentation slides and dynamic geometry files provided by mathematics textbook publishers.

Teachers were inducted into the use of ICT through continuous development programmes. These include seminars and workshops (half or 1 day) and longer (4–5 weekly 3-hour session) programmes. A typical 3-hour seminar usually consisted of sharing first-hand experience from a frontline teacher's perspective on the effective use of IT resources. Domains covered included dynamic geometry, algebraic graphing and data analysis, e-Learning platform and assessment. The emphasis was usually on techniques and presentation. A longer 15-hour workshop would be more comprehensive in introducing ways to use tools like *Geometer's Sketchpad*, *Geogebra*, *Cabri-Géomètre* (2D and 3D), *Excel*, *Fathom*, and Web-based inquiry-oriented learning approaches to design teaching and learning materials. Resource materials in these types of courses were usually ready-made for teachers to adopt as they designed their own materials. In some workshops, there was a Web-based forum for teachers to share their experiences in implementing ICT in their mathematics teaching. At the level of initial professional education of teachers, ICT is integrated in mathematics teacher training programmes at tertiary institutes like the University of Hong Kong where prospective teachers were motivated to use different types of ICT environments in their mathematics teaching.

Regarding implementation there seems to be a disparity between common usage of ICT in local mathematics classrooms and the pedagogical strategies proposed in the policy, such as learner-centered approaches and exploratory or investigative work as suggested in the curriculum documents. An evaluation study indicated that actual classroom use of ICT tended to be more teacher-centered than student-centered learning, involving predominantly didactic expository teaching such as explanation and demonstration. Some teachers also expressed a preference for working out ways of using ICT in their own classrooms rather than importing the exploratory examples from overseas. They preferred a more teacher-centered approach since they believe that this would work better in the local context. This approach, however, does not necessarily imply a purely transmission mode of teaching, as skillful uses of ICT can encourage students' active learning. An example of this is a lesson where a teacher used ICT in a demonstration manner and led the discussion. While the teacher was in control of the computer during a mathematical investigation in a classroom where students had no other computer access, he also saw evidence of students' hypothesis making and testing. His earlier attempts of explorations in the computer room with less teacher guidance, by contrast, did not lead to such fruitful discussion and investigation (Lee et al. 2003). Despite some successful examples of using ICT in lessons, most school teachers were willing to consider using ICT in their teaching only if they were given ready-made ICT resources that fit their teaching – an indication of the very realistic fact that teachers did not have the time or the motivation to prepare their own ICT teaching materials.

The skill- and teacher-oriented use of ICT in classroom probably reflects the fact that East-Asian teaching is deeply ingrained in the Confucian Heritage Culture (CHC) (Wong 2000). Amongst other things in this tradition, the teacher is the respected master and the source of knowledge, and students are apprentices and reflective practitioners. Teaching is viewed as the effective performance of well-structured and well-implemented lessons. Teaching and learning stress both fundamental techniques (via repeated practice) and fundamental knowledge (established as a strong foundation for further conceptual development). This is embedded in a cultural belief that skillfulness (ability to do certain things well) can bring about cleverness and creativity. Hong Kong is a place where East meets West. The constant tension and fusion among different cultural artifacts often brings about innovative perspectives of seeing and doing things. Hence using the Hong Kong context to do comparative studies on ICT in teaching and learning mathematics should be a fruitful research area in exploring the not yet known pedagogical potential of ICT.

17.3.3 Case 3: Vietnam by Nguyen Chi Thanh

Vietnam has an extensive state-controlled network of schools, colleges and universities but the number of privately-run and mixed public and private institutions is also growing. There has been an increase in the number of students completing general education and the limited number of universities makes it difficult for them, after completing school, to continue further education. The secondary school ends with a national examination for which the success rate was 80.38% in 2007. Students also need to pass competitive examinations to enter universities and the success rate is about 25%. These examinations are content-driven and have a high level of difficulty. In order to prepare students for these examinations, supplementary courses

BỘ GIÁO DỤC VÀ ĐÀO TẠO **ĐỀ THI TUYỂN SINH ĐẠI HỌC, CAO ĐẲNG NĂM 2006**

ĐỀ CHÍNH THỨC **Môn thi: TOÁN, khối A**
Thời gian làm bài: 180 phút, không kể thời gian phát đề

PHẦN CHUNG CHO TẤT CẢ CÁC THÍ SINH

Câu I (2 điểm)

1. Khảo sát sự biến thiên và vẽ đồ thị của hàm số $y = 2x^3 - 9x^2 + 12x - 4$.
2. Tìm m để phương trình sau có 6 nghiệm phân biệt: $2|x|^3 - 9x^2 + 12|x| = m$.

Câu II (2 điểm)

1. Giải phương trình: $\dfrac{2\left(\cos^6 x + \sin^6 x\right) - \sin x \cos x}{\sqrt{2} - 2\sin x} = 0$.
2. Giải hệ phương trình: $\begin{cases} x + y - \sqrt{xy} = 3 \\ \sqrt{x+1} + \sqrt{y+1} = 4 \end{cases}$ $(x, y \in \mathbb{R})$.

Câu III (2 điểm)

Trong không gian với hệ tọa độ Oxyz, cho hình lập phương $ABCD.A'B'C'D'$ với $A(0;0;0)$, $B(1;0;0)$, $D(0;1;0)$, $A'(0;0;1)$. Gọi M và N lần lượt là trung điểm của AB và CD.

1. Tính khoảng cách giữa hai đường thẳng $A'C$ và MN.
2. Viết phương trình mặt phẳng chứa $A'C$ và tạo với mặt phẳng Oxy một góc α biết $\cos\alpha = \dfrac{1}{\sqrt{6}}$.

Câu IV (2 điểm)

1. Tính tích phân: $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{\cos^2 x + 4\sin^2 x}}\,dx$.
2. Cho hai số thực $x \neq 0$, $y \neq 0$ thay đổi và thỏa mãn điều kiện: $(x+y)xy = x^2 + y^2 - xy$. Tìm giá trị lớn nhất của biểu thức $A = \dfrac{1}{x^3} + \dfrac{1}{y^3}$.

Fig. 17.1 Sample items from the 2006 Vietnamese university-entry mathematics examination

are given by teachers at universities. Figure 17.1 shows an extract of items from the mathematics examination for entry into universities in 2006.

The competitive nature of these examinations and the emphasis on manipulative skills of complexly composed problems requiring "exact" rather than approximate answers, mitigate against the use of calculators.

The system of education in Vietnam underwent reforms in 1990, in 2000 and in 2005. "These reforms [however] seem to focus more on teaching content and organisation than on the transformations of teaching practices" (Bessot and Comiti 2006). Content such as vectors in geometry, computational science, basics of combinatorics, integral calculus and mathematical statistics were introduced. One of the aims of theses reforms was to reduce the highly theoretical nature of the curriculum and introduce aspects of applications of mathematics to practical real life problems. This is captured in the teacher's guide book for Mathematics at Grade 10 as "Learning mathematics at secondary schools should help pupils in training skills regarding solving mathematical problems and applying mathematics in the life." The number of problems to the application of mathematics to real life situations is, however, still very small. For example, an analysis of the mathematics text book at Grade 10 reveals that in algebra there are only 5.4% and in geometry 2.5% of the problems are of an applications nature.

The Ministry of Education and Training of Vietnam (MOET) pays a lot of attention on using new technologies for teaching and learning. At the end of 2003, about 96% of the upper secondary schools in Vietnam were connected to the Internet, large budgets were available for equipment and national conferences about using technology in education were regularly held.

In mathematics, new technologies used are essentially non-graphic calculator and geometry software. For example, there is the introduction of using calculators such as the Casio Fx 500MS in the mathematics text book at Grade 10. Since 2005, there are three sessions of 45 min each at the end of the school year on using calculators. The general aims of using the calculator are to verify results and to aid calculations (Nguyen Thi Nhu 2004). In co-operation with the Casio calculator company,[1] the MOET organizes a national competition on using the Casio calculator for "talented" students.

Geometry software has been introduced in classrooms in some urban regions of Vietnam since around 2000. This is encouraged and in 2005, a mathematics teacher has been awarded a national prize ("knight in information and communication technology") for using *Cabri-Géomètre* in his teaching. The geometry content of the university entrance examination deals with analytical geometry so the use of geometry software remains optional during mathematical activities. Geometry software packages are essentially used by teachers to illustrate properties and demonstrations in geometry (Nguyen 2006).

The above narrative indicates that much needs to be done to realise some of the possibilities information communication technologies of mathematics teaching and learning. Some of the issues needing attention are: the incorporation of digital technologies for mathematics teaching and learning in pre-service teacher education courses at universities; mathematics curriculum changes which take due cognizance of digital technologies, change in assessment practices, awareness-building and sensitization of practising teachers and educational decision-makers of the benefits of digital technologies for teaching and learning mathematics and researchers to connect with researchers and research activities across the world. Last, but not least, much effort should be expanded to make both hardware and software more affordable.

17.3.4 Case 4: South Africa (and Some Developments in Sub-Saharan Africa) by Cyril Julie

Within most Sub-Saharan countries there are explicit statements in curriculum and accompanying documents favouring the use of calculators and computers in

[1] Binh Tay Company at Ho Chi Minh City.

Table 17.1 Availability of computers in South African schools (Department of Education 2004, p. 12)

Schools with computers	Schools with computers for teaching and learning
39.2%	26.5%

school-going mathematics. This is the case for Botswana; Namibia; Ghana; South Africa; Uganda and Zimbabwe. For the South African situation it is, for example, stated that:

> Every South African learner in the general and further education and training bands will be ICT capable (that is, use ICT's confidently and creatively to help develop the skills and knowledge they need to achieve personal goals and to be full participants in the global community) by 2013. (Department of Education 2004, p. 17)

This is a major task given that physical access to ICT is available to less than 50% of schools in South Africa as can be seen from Table 17.1 below.

The envisaged use of ICT for teaching and learning is in line with those proffered in most countries. For example statements such as "Create knowledge and new information by adapting, applying, designing, inventing and authoring information" and the creation of "A learning environment that advances creativity, communication, collaboration and engagement" (Department of Education 2004, p. 14) are found in these policy documents.

With respect to provisioning it is stated that "The [South African] Department of Education supports the development of refurbished facilities for second hand computers." (Department of Education 2004, p. 30). There is thus no aversion to refurbishment given the overall budgetary demands and constraints in late developing countries.

For school mathematics there is a deep realization that the implementation of an ICT-driven curriculum will be hampered by resource constraints. Thus some of the stated goals are:

- Proper conceptual understanding will be required to enable learners to use calculators appropriately and effectively.
- Where possible, learners should get opportunities to use spreadsheets and other computer tools. (Department of Education 2003, pp. 11–12)

Clearly these are guarded statements with riders indicative of a mindfulness of the exacerbation of disparities between schools populated by learners from high and low socio-economic backgrounds. The guardedness is also linked to the fact that the state can be legally challenged to concretely support her policies. This guarded position is also evident in the regulations for school examinations where only non-programmable and non-graphic calculators are permissible. As an aside in a country such as Uganda logarithmic tables are still used (Opolot-Okurut 2004).

The TIMSS report (Mullis et al. 2004) indicates that availability of computers in some Sub-Saharan African countries is still severely limited both at school and at home as reported by the Grade 8 sample for the countries listed in Table 17.2 below.

Table 17.2 Availability and use of computers in some Sub-Saharan countries (%) (Mullis et al. 2004, pp. 140–142)

	Availability of computers	
	Have computer	Do not have computer
Botswana	16	84
Ghana	24	76
South Africa	37	63

	Use computer				
	Home and school	Home only	School only	Other than home and school	Not at all
Botswana	5	6	23	5	61
Ghana	9	9	21	26	34
South Africa	16	11	18	27	28

Table 17.3 Nature of use of computers in some Sub-Saharan countries (% of students) (Mullis et al. 2004, p. 296)

	Computers are not available	Computer used for half or more of the lessons			
		Discovering principles and concepts	Practicing skills and procedures	Looking up ideas and information	Processing and analyzing data
Botswana	93	0	0	0	1
Ghana	85	0	1	1	2
South Africa	83	3	3	3	2

Regarding the nature of use, it is also evident that the usage of computers in the curriculum is still extremely limited (see Table 17.3). Furthermore, with those who use computers in more than 50% of their lessons, the usage tends to be in the direction of high-order cognitive skills.

Despite the situation described above, there are large scale and state-supported projects to enhance the use of computers in mathematics teaching and learning. In South Africa, for example, the Dinaledi project is such a project. Its major goal is to increase the supply of learners to pursue science and technology-related careers requiring mathematics for access into these fields of study at higher education institutions through the establishment of specialist mathematics and science secondary schools across the country. Schools in low socio-economic status (LSES) environments are targeted and these schools have fully-equipped computer laboratories with access to the Internet. Support is provided to schools through specialist teacher-advisors. They generally run school-based workshops on the use of computers in mathematics teaching and are general resource persons on which teachers in these specialist schools can draw. In addition they search for appropriate software to present to teachers. An example of such a search is making teachers aware of the availability of open-source software such as *GeoGebra* and its possible use in the teaching of transformation geometry. The expected outcome of computer integration in mathematics teaching and learning is, however, aimed at the improvement of

Computer-generated problem	It costs R1.75 to send an SMS for a competition. Zinele sends 8 SMS's. How much will she have to pay?			
	Student		Computer response	
	1st response	8 × 1.75	1st feedback	Type a number
	2nd response	12.8	2nd feedback	Try 8 × 1.75

Fig. 17.2 Sample computer drill-and-practice sequence

mathematics achievement results and thus drill-and-practice programmes underpinned by the corrective feedback paradigm is still in vogue as is evident from Fig. 17.2.

Given the interest in improvement, research revolves basically around the question "Does the use of computers improve mathematics achievement scores?" and modalities of student use of computers. As is the case generally with the use of computers and improvement of achievement scores, the tentative results point in the direction of the use of computers not negatively impacting on achievement. Regarding use and research at the tertiary level the emphasis is on the use of mathematics-dedicated package such as *Mathematica* (Engelbrecht and Harding 2005). De Villiers (2004) reports on a similar use of *Sketchpad* for geometry teaching and learning.

Generally access to appropriate software is influenced by a time lag between frontier development of this software and release for experimentation in late developing countries. This has the effect that for such countries there is a lack of sustained developmental research to work out optimal ways to implement ICT in mathematics teaching and learning in their context. Thus the "problem" manifested starkly in ICT as the "technology gap" is not only an issue of research, discussion and deliberation in mathematics education. These are necessary but not sufficient. The "problem" is within the broader political realm and starting points should also be sought within this realm; this was the case with the issue regarding the adoption of the Hindu-Arabic calculation technology as a replacement for the counting-table technology, which was resolved politically and economically.

17.3.5 Case 5: Latin-America by Ana Isabel Sacristán

Latin-America is one of the most homogeneous regions of the world, in that most of its countries have very similar cultures (even language: except for Brazil, all of the countries are Spanish-speaking) and they also share similar problems. It is therefore pertinent to discuss the issues of this region as a whole.

The Latin-American countries are developing ones with problems of strong socio-economic inequalities and often very diverse geographical territories (that are often vast with hard-to-reach and isolated areas). Within each one, there tend to be also many regional disparities.

All of these issues imply that access to digital technologies is very inconsistent (Tedesco 2005). For example, there is much difference between urban and rural areas;

and the strong socio-economic disparities, entail profound differences between privileged private schools and the public school systems; the quality of teacher preparation is also generally very unequal.

There are three types of integration of digital technologies into the school systems of this sub-continent; these are due to:

- The initiative of individual teachers and/or schools,
- Privately-funded projects (e.g. by IBM, Microsoft, Intel, etc.), and
- Government-sponsored projects.

General Latin-American overviews of some projects involving the use of digital technologies for education in general, are given by Fonseca (2005) and in the report by Universidad de La Frontera (2005), Chile, entitled *Experiencias Innovadoras en Informática Educativa*.

Besides small- or large-scale endeavors for incorporating digital technologies to schools, in most countries there are research projects being carried out in universities into the use (and development) of a variety of digital and ICT tools for mathematics education, and their integration into schools. In the next sections, a summary of the efforts that have been done in selected countries is given.

17.3.5.1 Latin-American Countries with Mainly Small-Scale Efforts of Integration of Digital Technologies Due to Individual Initiatives by Teachers or Schools

There are many Latin-American countries where there are few, if any, large-scale programmes for incorporating digital technologies into the educational area; in those countries only projects by individual teachers or smaller institutions are carried out, although sometimes there are regional or local efforts as well. Some of the countries (at least until 2006) in this category include Argentina, Ecuador, Guatemala, Honduras, Panama, and Uruguay.

In those countries, most efforts generally take place at the tertiary education level (i.e. university level), where the use is mainly of Computer Algebra Systems (CAS) such as *Derive, Matlab* or *Mathematica* (although this one to a lesser extent due to its cost) – and sometimes also of Spreadsheets (*Excel*) – in selected mathematics courses or programmes; or for research purposes. Statistics software and equation plotters (e.g. *Graphmatica*) are also common, although not in classrooms. Dynamic Geometry systems like the *Geometer's Sketchpad*,*Cabri-Géomètre* or *Cinderella* are used by some university teachers and researchers, but their use is very limited and they are not well-known in many of these countries. More recently in some countries (e.g. Panama) there is some interest in *Descartes* as a means to create interactive web activities, but its use is very restricted, if at all. But even at this tertiary level, there generally seems to be a lack of investment in the use of digital technologies in these countries, so interested individuals do as best they can.

At primary and secondary levels, the lack of hardware has hindered widespread integration (in many countries, such as Guatemala, the only schools that use ICT are private ones). Despite a rising consciousness of the potential of digital tools for mathematics teaching and learning, there seems to be little action. There are also very big differences between regions, and in some places teacher-training institutions do not provide courses on the use of digital technologies until the very end. In some of these countries (e.g. Uruguay) it also seems that mathematics teachers are still very resistant to change and to the inclusion of digital tools into their practice. In Argentina, Giuliano et al. (2006) observe that teachers have little knowledge of the possibilities offered by new technologies, and when they do use digital tools, they select their activities, contents and teaching strategies according to traditional teaching stances.

The use of the Internet by primary and secondary teachers also remains limited; in Uruguay, an attempt in 2001–2002 of a distance virtual education mathematics course financed by private foreign investment, failed due to lack of users (Bermúdez 2006, personal communication). Nevertheless, there are efforts by private institutions (e.g. Centro Babbage, http://www.centrobabbage.com, in Argentina) to train teachers in the use of new technologies for mathematics and present them with tools (e.g. Dynamic Geometry) that still lack widespread use in that particular country; other groups, many of them in Argentina, promote the use of digital technologies and constructionist approaches like Logo (e.g. FUNDAUSTRAL, http://www.fundaustral.com.ar; Rosa Kaufman's *Laboratorio de Computación*, etc.).

Interestingly, in many countries across the sub-continent, presentation and word-processing tools like *Powerpoint*,*Word* (and its equation editor) – as well as *LaTex* and *PDF* files – are frequently cited as among the most-used tools for mathematics teaching and course management at all levels.

17.3.5.2 Latin-American Countries with Large-Scale, Either Government-, or Privately-Sponsored, Projects

There are countries in Latin-America where there have been large-scale efforts for integrating ICT and digital technologies into mathematics teaching and learning, at primary and secondary levels. These include Brazil, Costa Rica, Chile, Mexico, Colombia, Cuba, and Venezuela. We will summarize the information of each of these (except Cuba) in turn. It is also worth noting that both Brazil and Mexico have had, for several decades, important government distance education programmes, via television (de Moura Castro 2005) that can be thought of as precedents for the incorporation of more modern digital technologies into the school systems.

Brazil

The case of Brazil is summarized by Healy (2004, 2006). She recounts how digital technologies began to be introduced in Brazil in the late 1970s and early 1980s when university researchers developed studies on the use of the computer inspired by the constructionist perspective of Papert (1980, cited by Healy 2006) and the "Logo methodology" (with emphasis on programming languages and the Logo experimental approach). By the late 1980s, insertion of digital technologies into the public education system, began through government investment with five centers in universities[2] throughout Brazil (e.g. through projects such as EDUCOM and FORMAR). The computer was then seen as a catalyst for and instrument of didactic and pedagogic change; this was “a huge challenge given that the dominant pedagogical approach of the time almost exclusively focused on teaching as transmission of ideas” (Healy 2006).

The use of technologies was seen as an innovative and motivating pedagogic approach. However, more than 20 years later, none of the government-funded programs seem to have yet resulted in the intended transformations to the educational system. There has been also no impact on official forms of assessment, access to technological resources remains sporadic and unevenly distributed (with calculators being more prevalent than computers), and teachers are more resistant to change than originally anticipated, having insecurities about their own mathematical practices. In that sense, Healy (2004) explains that CAS-type software is more easily inserted (though not integrated) into classroom practice than programmable tools; there seems to be the perception that the more “recognisable” the mathematics, the more “legitimate” the software (and perhaps also less challenging to existing practices).

One important result, however, has been the recognition of the importance of the role of the teacher at each step of the integration process. Until now, for example in teacher training programmes on the use of digital technologies, there has been emphasis on how learning can be supported but much less on what can be learnt either in terms of the mathematical topics involved or the kinds of meanings constructed by learners. Furthermore the reciprocal relationship between tools and thinking is not taken into account. Teacher-training is now to be carried out as distance-learning; the impact this will have, remains to be seen.

Costa Rica

The case of Costa Rica is unique among Latin-American countries. In Costa Rica there have been massive continuous efforts to incorporate digital tools and ICT into schools since 1987. The first efforts (focused on the teaching and learning of the Spanish language and mathematics) began with the introduction of Logo programming (first using *Logowriter* and later *Microworlds* Logo) as well as robotics. This was done through agreements between the government (the Ministry of Education) and the “Omar Dengo” Foundation, with funding by Intel (and IBM) and support from the MIT media lab and other national and international institutions and universities (FOD 2004). The collaboration between the Ministry of Education and the Omar Dengo Foundation continues to this day. It is estimated that 50% of primary level school children has had access to these programmes, although priority is given to low-income or rural children and/or schools in order to promote equality.

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[2] It is worth noting that although research in mathematics education has been a part of the insertion process in Brazil, its role has not been central.

This programme has been considered innovative, has increased the motivation of teachers and students and has received favourable and positive reviews despite its lack of measurable impact (e.g. through traditional assessment). It is also a programme that provides multimedia facilities, access to Internet and local networks (de Faria Campos 2006).

Despite criticism of discontinuity between the primary and secondary school level programmes, there is a smaller similar project (PRIES) for secondary schools, which incorporates *Microworlds* Logo.

However, nowadays there are other projects in place in Costa Rica, that support the use of digital technologies, such as PROMECE (the Program for the Improvement of Education in Costa Rica), also promoted by the Ministry of Education, which is in place in approximately 20% of secondary schools (1,500 teachers and over 25,000 students). This programme equips schools with all types of multimedia hardware and software (including *Microworlds*, *Geometer's Sketchpad*, virtual encyclopaedias, music software, and other video, web and office tools), as well as Internet access and promotes a holistic approach to education aimed to develop creativity and collaborative projects.

In general, as stated in the Ministry's official documents, digital technologies are seen as tools, that can promote collaboration and assist in computations so that attention can be placed on the reasoning processes involved in problem-solving activities (MEP 2005a, cited by de Faria Campos 2006).

Another project, begun in 2002, promoted by the Omar Dengo Foundation in Costa Rica, and also implemented in the Universidad de San Pablo University in Peru, and the ESPOL Centre for Information Technology in Ecuador, is the *Ciberaprendiz* project. This project, funded by the Inter-American Development Bank, and in collaboration with several US institutions, promotes the use of the Internet for activities of communication and collaboration with students all over the world, in order to improve the learning of science and mathematics (de Faria Campos 2006).

Costa Rica also has strong teacher training programmes – as well as research projects – in the use of digital technologies for mathematics teaching in learning, that are in place in several universities.

Mexico

In Mexico, one of the first large scale projects to incorporate digital technologies into classroom was a badly thought-out government initiative which, in 1989, shipped custom-made computers to schools without giving proper training to teachers on their use or developing an integration programme. This initiative was thus a total failure, and the Logo programming language, which came in the ROM memory of those computers, was partly blamed. This failure hindered many future initiatives for almost a decade.

In 1997, however, the Mexican Ministry of Education took the initiative to incorporate computational technologies into the primary and secondary (middle-school) levels. For secondary schools (children aged 12–15 years old) the initiative began with the EMAT (Teaching Mathematics with Technology) programme, and a parallel one for Physics (EFIT); in a later phase, the ECAMM project (Teaching Science through Mathematical Modelling) was added, and more recently one for Sciences (ECIT).

The conception and design of EMAT, as well as the choice of tools, was led by a group of national and international researchers in mathematics education and took into account results of previous studies in computer-based mathematics education around the world. EMAT provides activities and a pedagogical model for incorporating the use of technological tools in mathematics classrooms, in a constructionist way, aimed to enrich the teaching and improve learning (Ursini and Rojano 2000). The pedagogical model emphasizes changes in the classroom structure such as the requirement of a different teaching approach and the way the classroom needs to be set up. In particular, the pedagogical model emphasizes a collaborative model of learning, and a role of the teacher as guide, mediator and promoter of the exchange of ideas and collective discussion. A study carried out in Mexico and England (Rojano et al. 1996) revealed that in Mexico few students were able to close the gap between the formal treatment of the curricular topics and their possible applications. This suggested that it was necessary to replace the formal approach of the official curriculum of 1997, with a “down-up” approach capable of fostering the students’ explorative, manipulative, and communication skills. EMAT was seen as a catalyst for changing classroom practices.

In its first phase (1997–2000), the project researched the use of Spreadsheets (*Excel*), *Cabri-Géomètre*, *SimCalc*, *Stella* and CAS activities with the TI-92 calculator. These tools were piloted with nearly 90 teachers and 10,000 students at the secondary school level, which allowed for changes before massive implementation. In the second phase (2001–2007) the use of some of the tools[3] used in the first phase (Spreadsheets, *Cabri-Géomètre*, and the TI-92 calculator) continued and was expanded gradually in the national public school system, with local regional authorities assuming responsibility. In that phase, the Logo programming language was also added as another one of the tools; this decision was taken at the suggestion of both national and international advisors who evaluated the first phase and pointed out that there was still the need for more expressive activities (such as programming), on the part of the students.

In the academic year 2002–2003, the EMAT project had been implemented in 731 schools in 17 states, with 2,283 participating teachers and close to 200,000 students (out of a population of over 5.7 million lower secondary school registered students); since then, many more states have joined the programme, although the incorporation remains very uneven from one region or school district to another.

During the pilot stages, teacher training was done directly by the national and international experts, and there was continuous support of the teachers. In the expansion phase, however, due to the immensity of the scale of the programme, this was no longer possible. A cascading model was implemented: the experts trained trainers who in turn trained teachers and/or head-teachers, who in turn were sup-

[3] Simcalc and Stella were dropped because it was hard to fit these tools into the curriculum without more extensive teacher-training that was hard to achieve.

posed to support other teachers. The problem is that this has a "faulty line effect" and the quality of the training has been diluted (particularly in reference to the understanding of the pedagogical model). But this is an unavoidable problem.

Diverse evaluations of the EMAT programme (e.g. Trigueros and Sacristán 2008) have highlighted the complexity of the implementation of projects such as this one: there are many difficulties that were not foreseen (particularly technical and administrative). There is still a lack of resources and time for a widespread use of digital tools in classrooms; and teacher preparation, both technical and mathematical, as well as continuous support, remain insufficient, which leads to insecurities. All of the above issues imply that the use of digital technologies is inconsistent and often sporadic. Teachers also have difficulties in integrating the pedagogical model, as well as to technology-based activities, to their mathematics teaching practice. Not surprisingly, results showing the impact of the use of digital tools on students' learning are also inconsistent. Thus, this project put into evidence the importance of the role of the teacher in the use and integration of digital technologies to the mathematics classroom.

On the other hand, after a decade, there are many teachers who have successfully changed their practice to incorporate the proposed pedagogical model not only when using the digital tools, but also in their regular practice. In these cases, positive benefits of the use of digital technologies on students' learning and attitudes have been observed (Trigueros and Sacristán 2008), which show successful implementations of the digital tools. However the changes are gradual and do take time.

Nevertheless, the EMAT project was groundbreaking in changing the role of the teacher and the traditional passive attitude of children (as well as their attitudes towards mathematics) and opened the door for richer ways of incorporating technologies in schools. In fact, the use of digital technologies and of the EMAT materials is now suggested in the new national curriculum of 2006.

In other projects, ways to incorporate EMAT activities into the established distance education (via television) *Telesecundaria* programme that exists in Mexico since the 1960s, has been researched since 2001. More recently, electronic whiteboard activities for this *Telesecundaria* programme have also been developed, primarily as teaching aids.

Another important project in Mexico that began in 2004 is the*Enciclomedia* programme for primary schools. Enciclomedia, has been massively implemented in all primary schools in Mexico; it aims to help teachers by providing resources, computer interactive activities and strategies (mainly designed to be used on electronic whiteboards), through links in an enhanced electronic version of the mandatory textbooks (Lozano et al. 2006).

Colombia

In Colombia, the project "Implementing new technologies in the secondary school mathematics curriculum" was sponsored by the government from 1998 to 2004, for the country's public schools. International experts from the UK, Mexico, and Chile acted as consultants, and the model was very similar to the Mexican model EMAT. It had as main tools *Cabri- Géomètre* and CAS (often using calculators). The project began in 1998, with an exploratory phase. In the year 2000, a pilot phase was carried out in collaboration with 17 universities, where 60 schools were provided with graphic calculators, and there was a programme of continuous teacher training. In 2002 the project expanded to include 121 public schools and the support of 24 universities. From 2004 onwards, the responsibility of the project was delegated to local governments. Several regions have purchased *Cabri-Géomètre*, and some also *Derive*, to complement the use of the calculators with computers.

Castiblanco Paiba (2002) mentions that at a local level there has been progress in the development of a pedagogical model for the teaching and learning of mathematics with computational tools that integrates insights onto the nature of mathematics and the cognitive aspects involved in learning. There has also been progress in identifying which processes are involved when students work with calculators in the classrooms that could lead to new strategies and mathematical content. But she also recognizes that the development of a project such as this is a slow and complex process that can only have a noticeable impact for education if it involves a permanent plan of teacher-training, the cooperation and collective work of institutions, the compromise and motivation of teachers and authorities; and provides support materials and the continuing equipment in DT for school infrastructures.

Chile

In Chile, a large government-sponsored project, the *Enlaces* Project, began in 1995. This project promotes the use of the Internet and office tools (word processing, *Excel*, *Powerpoint*) for primary and secondary schools. It is meant to be "a support for learning" (Mineduc 2002a) – and is used mainly for Mathematics, as well as History and Geography, teaching – and is also meant to be associated with university projects. By 2002, Enlaces was incorporated in more than 500 schools in the country. Over 75% of teachers in Chile have been trained in the *technical* use of the technological tools, and 85% of them use the Internet regularly (Mineduc/DESUC 2005). This places Chile as the top Latin-American country in terms of the number of teachers trained and also in the use of the Internet (Mineduc 2002b).

However, there seems to be a lack of an accompanying pedagogical model or theoretical framework. The uses of Enlaces are mainly for information-seeking in the Internet, and many users use the system for instant messaging and email. For mathematics teaching, the use of the electronic whiteboard is promoted (Villarreal and Marinkovic 2005). There have also been some efforts (Galaz 2005) to use dynamic geometry (with *Cabri-Géomètre*). Some institutions, like the Centro Comenius of the University of Santiago, have made attempts to develop a mathematics curriculum with Enlaces (Oteiza and Villarreal 2005), and some positive results have been observed, particularly in terms of promoting a cultural change for the acceptance and integration of digital tools in the mathematics classroom, and improved motivation. In spite of these, however, there doesn't seem to be a general integration of the program for mathematical learning. Critics claim that there is still a lack of generalised use (or even knowledge by teachers) of useful software or DT tools for mathematics lessons (even the use of spreadsheets doesn't seem to be commonplace). In fact, the knowledge that teachers tend to have is technical and superficial, so they still lack confidence in the use of digital technologies, and there are few links made with classroom work.

Venezuela

The Venezuelan experience is not as important as those in other countries with large-scale implementations but it is still worth noting. During the early 1980s, the Ministry of Education of Venezuela started to develop programmes for improving education in schools (Cabanzo et al. 1997). One of its strategies was to introduce computers into schools; this was begun in 1985 with an agreement with Epson who aimed to develop software and created a computer laboratory in one primary school where teachers and students were trained in the use of commercial software for mathematics (and language). But it is not until 1992 that the first large-scale project began, the "Proyecto Simón". This project funded through agreements with IBM and Epson, aimed, not only to provide computers for schools but also to incorporate a pedagogical approach. The part funded by IBM, following the company's commitment in Costa Rica, promoted Papert's constructionist approach through Logo programming activities (using *Logowriter*); whereas the part sponsored by Epson fostered a behaviourist approach and Computer-Assisted Teaching. This led to the creation of two types of laboratories, but the Epson part was dropped in 1993. Almost simultaneously (beginning in 1989) the government launched the "Computer in the school" program through which it created links with university and other research groups for the creation of pilot centers. However, apparently this initiative was more political, lacking clear aims and funding (Hernández 1993, cited by Cabanzo et al. 1997). In 1997, the Simon Project was abandoned (despite increasing interest and demands from children) when IBM closed its office of Educational Informatics that coordinated the project; there had also been problems with teachers who needed broader training in order to cope with children's demands but the pyramidal structure of teacher training that was in place did not work very well. Nevertheless, individual teachers and schools continued the project at much smaller scales – until even today – such as through the regional project "*Francisco de Miranda*", or the *Fe y Alegría* schools, both of which focus on a constructionist approach and used *Microworlds* Logo.

In a separate line of efforts and pedagogic approach, the National Centre for the Improvement of the Teaching of Science (CENAMEC) has been involved in the development of educational software, tutorial systems, as well as pedagogic materials, such as those for the use of *Cabri-Géomètre* in primary schools. However, these efforts are limited. Presently, the current government of Venezuela wants to promote the use of freeware tools.

17.4 Conclusions

The above narratives indicate that there are similarities and differences with regard to access and implementation of digital technologies across and within the respective countries and regions. The outstanding similarity is the acceptance at political and bureaucratic level of the use of digital technologies for mathematics teaching and learning in all the countries. However, the translation of policy into practice is a much more daunting task. Both human and physical, for example, resource constraints, given the differential economic realities of the respective countries, partly account for this phenomenon. This differential realization of access is nothing new but in terms of quality mathematics education for and by all, it is brought to the fore much more starkly. The experiences from Latin-American countries, for example, show that initiatives for implementation are carried out, at different times and places, in different levels: from individual teachers and/or schools, to privately-funded projects, to government-sponsored projects; and they highlight the difficulties of large-scale implementations. In general, even under massive government implementation, there remain unequal access, unequal resources, and sporadic use of the digital technologies in schools.

Political decisions and administrative issues also affect the implementations, the quality of the training of teachers as well as its continuity and that of the projects themselves.

A second issue that is illuminative, is the expression, in most of the above countries, for a form of access to the epistemic machinery of mathematics through the use of digital technologies. Although the use that is made of these technologies tends to vary at different levels of schooling (e.g. with a predominance of CAS-style tools used at tertiary level compared with more “exploratory” tools at lower school-levels) and there is still a dominance in the use of these technologies as ICT (i.e. for purposes of information and communication), an emphasis on problem-solving, exploration and inquiry, is being placed and prioritised and mere drill-and-practice uses are de-emphasised.

A further common theme is the use of open source software and packages. This has opened accessibility to resources with which to do mathematics but how things are played out in spaces of teaching and learning is still in need of in-depth investigation.

Also, as is made evident from several implementation projects of digital technologies, the role of the teacher is very important, and his/her beliefs, insecurities and lack of mathematical and technical preparation affect the possible impact that the use in the classroom of these technologies can have on students’ learning and even attitudes. The need for careful, considered and continuous work with teachers is thus extremely important. A priority in this kind of work should be the integration of digital technologies with the work that teachers are required to do, to take them into account at all steps of the implementation process, and to assist them in developing pedagogical strategies. The most successful implementations of digital technologies for mathematical teaching and learning seem to be those where teachers are able not only to add, but to fully integrate and articulate the use of digital technologies into their wider lesson plans and teaching (Trigueros and Sacristán 2008). A lesson from all the reviewed projects is that changes are gradual, take time and effort, and that there is still much that needs to be done.

In the last ICMI study related to computers and informatics in teaching and learning (Churchhouse et al. 1986) there was one chapter dealing with the then-called Third World situation. The thrust of the chapter was on the inequitable distribution of computers for mathematics teaching and learning between economically differentially positioned nations. This situation has not changed in a substantive way. Neither did it stay the same. There are developments such as the lowering costs of computers and programs such as the "One Laptop per Child" project, which could ease physical access. Advances in communication technologies as well as in international networking make it possible for the lag time between frontier developments and local implementations to be diminished. However, conditions of schooling, social stratification, even within early developing countries, and political conditions mitigate strongly against all students benefiting from the possibilities for meaningful learning of mathematics which digital technologies have to offer. Despite the possibilities offered by technology, as perceived by the UNESCO agenda described in Sect. 2, such as increasing the educational options for the marginalized sectors of society, many of these are not yet widely seen, and there is still a need to reformulate the educational approaches when using new technologies. To realise this ideal will require thoughtful navigation, action and support, in the first instance, from the global mathematics education community.

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Chapter 18
Technology for Mathematics Education: Equity, Access and Agency

Helen J. Forgasz, Colleen Vale and Sonia Ursini

Abstract In this chapter, issues of equity – including gender, access, and agency – with respect to the learning of mathematics with technology are examined. Research findings are not equivocal. Compared to late developing countries, where issues of access to technology can be complicated by educational and cultural values and beliefs, there seems to be greater access to technology to be used for the learning of mathematics in developed nations. There also appears to be some disparity in findings on the relationship between technology use and gender differences in mathematics achievement; in some countries the gender gap favoring males may be closing, while in other countries, where there have been little or no gender differences in the past, the gap may be widening. Areas in which more research is needed have been identified.

Keywords Technology • Gender • Access • Equity • Agency

18.1 Introduction

Skovsmose and Valero (2002) argued that the rhetoric of western countries is that mathematics prepares students for active citizenship while the reality is that mathematics maintains the social order. They observed that some students gain access to the power that knowledge of mathematics affords while others do not, and that the demarcation between these two groups matches the differences in economic and political power of different social groups. International studies of mathematical performance, such as the

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Third International Mathematics and Science Study (TIMSS) and Program for International Student Assessment (PISA), provide evidence of social inequity and gender inequity in mathematics achievement within, as well as between, countries around the world (Mullins et al. 2000; OECD 2004). While studies such as these clearly identify differences in achievement that provide evidence of inequity in mathematical outcomes, defining equity is a more complex task (Bishop and Forgasz 2007).

Apple (1989) stressed that in spite of experts' efforts, mathematics and science curricula very often contribute to the reproduction of gender differences. He argued that the situation might get worse with the introduction of technology to support mathematics and science teaching. This concern was put forward again in the ICMI Study "Gender and Mathematics Education" where it was reported that the use of technology in mathematics might erode the advances made towards gender equity in mathematics (Hanna and Nyhof-Young 1995); this view was re-iterated by Hoyles (1998) at the ICMI Study, "The teaching and learning of mathematics at university level".

Issues of equity, access, and agency, as they pertain to mathematics teaching and learning with technology, are discussed in this chapter. Commencing with definitions of the terms, a range of research findings from a number of countries are then presented. To close the chapter, summaries of the findings and conclusions based on them are accompanied by a discussion of the implications and perspectives on future directions in the field.

18.2 Definitions of Equity Including Access and Agency

Many theoretical perspectives have been used to define equity. At the ICMI study conference on gender and mathematics education in Sweden in 1993, Fenemma (1995) defined three aspects of equity which had been the subject of research in the field: equal opportunity, equal treatment, and equal outcomes. However, she observed that in many studies it had been shown that equal opportunity and equal treatment in mathematics classrooms were insufficient to achieve equal outcomes. In the literature and research concerning other socio-cultural factors and education it has been argued that equity is also about fairness and justice (e.g., Gewitz 1998), and that a commitment to "closing the gap" between the achievement of people of different socio-economic and cultural groups, including females and males, is at the heart of equity (Secada et al. 1995). Definitions of equity that include fairness and achieving justice reveal that equity is different from equality, and that there is the need to include notions of equitable pedagogies (Hart 2003). Furthermore, enhancing the human condition and building capacity so that individuals become agents in their own lives and make a difference in society are central to UNESCO's goals of "Education for All" and the Decade of Education for Sustainable Development (UNESCO 1990, n.d.).

In considering equity, four student characteristics are of interest: gender, socio-economic status, race or ethnicity, and disability. Sherman and Poirier (2007) also felt that educational equity within countries should be considered by regional characteristics such as urbanity and wealth. In their evaluation of educational policy of sixteen countries concerning the agreed international objective of "education for all" (UNESCO 1990), they used three categories of indicators of educational equity: access and progression, resources, and outcomes. These categories are helpful in reporting on equity with respect to digital technology and mathematics education. Consideration is also needed of how fairness and justice can be identified, described, and defined when students and teachers use digital technologies in mathematics classrooms. Definitions are now presented of the various dimensions of equity for mathematics education with digital technologies that are examined in this chapter, as well as the ways in which these dimensions can be measured or identified.

18.2.1 Access

Access to mathematics education, and hence opportunity to learn, is fundamental to equity. Enrolment is an indicator of access to, and progression in, education. But, in which mathematics courses are students from the various equity categories enrolled? Policy makers concerned with equity have described equitable mathematics curricula as "mathematics for all" (Bishop and Forgasz 2007). Notions of "mathematical literacy" that prescribe the mathematical requirements for active citizenship, or agency, have underpinned these mathematics curricula. In the digital age, mathematical literacy involves the facility to use digital technology in a range of contexts in which mathematical thinking and problem solving skills are needed to solve problems and interpret information (Keitel et al. 1993). Skovsmose and Valero (2002) argued that students need access to the powerful ideas of mathematics. In their view, knowledge of mathematics is not only needed for active citizenship and for professional careers dependent on mathematics, but also enables people to have a sense of control over their own lives within society. Students in the digital age therefore need access to digital technology for mathematics learning and problem solving.

It cannot be assumed that all students are enrolled in the same intended mathematics curriculum. There is a lot of evidence to suggest that the mathematics curriculum is defined differently for different groups of students within countries, and that these curricula do not result in equitable outcomes (Bishop and Forgasz 2007). In Chap. 16, Vale and Julie with Buteau and Ridgway discussed the extent to which the facility to use technology in a range of mathematical contexts is included in the intended and the implemented curriculum. The enrolment and progression of different categories of students in mathematics subjects in which students have the opportunity to develop facility with digital technologies are therefore equity-related issues.

18.2.2 Resources for Equity

The provision of educational resources including facilities, equipment, learning materials, and teachers is also fundamental to equity. Sherman and Poirier (2007) used government expenditure per pupil and pupil-teacher ratios as measures of equity with respect to resources. However, these measures are insufficient as the quality of resources is also important and includes, for example, class sizes, the quality of materials and equipment, and the quality of teachers that can be measured by their qualifications, knowledge, and experience.

Also of interest are measures associated with the digital resources for mathematics education (both hardware and software), as well as the infrastructure (buildings, electricity, and telecommunications) needed to run these tools. Computer to student ratios, and student ownership of computers and hand-held technologies are therefore pertinent to a study of equity. The term, “digital divide”, is used to describe the marked difference in access to digital technologies between developed and late developing countries, and between high and low socio-economic communities. However, Setati (2003) warned that these measures of resources might not be adequate indicators of equity with respect to access. In late developing countries that have poor schools, the availability of computer technology does not necessarily translate into classroom use. Setati (2003) argued that in these schools administrators' and teachers' conceptions of resources as possessions needed to be considered. In these contexts, consideration for the care, protection, and security of resources often limited teacher and student access to them for learning purposes.

In many studies it has been shown that the quality of teaching is critical for student learning and hence equity, and that inexperienced teachers and teachers with limited qualifications are often employed in schools located in socially and culturally disadvantaged communities (Bishop and Forgasz 2007). Teachers' qualifications, training, knowledge, beliefs, and experience with respect to using digital technologies for mathematics learning and teaching are therefore important indicators of equity with respect to the allocation of resources. Further discussion of teachers and teaching as critical factors in mathematics learning with digital technology is included in Sect. 3 of this book.

18.2.3 Equitable Pedagogies

Equitable access to mathematical learning with digital technology is non-trivial. However, the classroom learning environment and the pedagogical approaches adopted by teachers also need to be considered. Equity for students in schools that are poorly resourced with digital technologies, or who are from poor family backgrounds with limited access to digital technologies, means that access to mathematical learning with technology extends beyond the inclusion of learning activities using digital technologies and beyond ensuring that particular groups of students have equal “hands on” time with materials and digital technologies in classrooms. The quality of these students' mathematical learning experiences needs to include: high expectations of the students; that the content and purpose of the mathematical learning activities connect socially, culturally, and politically with the students' lives; and that teachers support the students in the development of mathematical thinking and practices with digital technologies (Boaler 2002; Gutstein 2003; Vale 2003). Equity involves paying attention to diversity in the classroom, and to providing for different needs arising from the different positions and identities of the students in the classroom. Hence equitable pedagogical practices are situated, that is, they relate to the learners in given settings and are respectful of diverse cultural realities (Anthony and Walshaw 2007; Cobb and Hodge 2002; Quiroz and Secada 2003).

Digital tools are cultural artifacts, and teachers need to be aware of the different needs, positions, and identities that students may have with respect to these tools (Vale 2002). Moreover, teachers and students may find themselves on different sides of another digital divide, the different experiences of technology arising from the different ages and exposure to digital technology of teachers and students, and hence the different expectations and beliefs about using technology for mathematics learning. The cultural and pedagogical practices with regard to the various digital tools are therefore important considerations for equity. There is also the need to reflect on the ways in which digital technology can be used in mathematics teaching and learning to disrupt the social reproduction that Skovsmose and Valero (2002) observed, and to "close the (equity) gap".

18.2.4 Equitable Outcomes

As indicated above, achievement and performance data are used to measure equity with respect to mathematical learning outcomes. The Program for International Student Assessment (PISA), conducted by the Organisation for Economic Cooperation and Development (OECD), is designed to find out how well 15-year-old students are prepared for the "challenges of today's knowledge societies" (OECD 2004, p. 20). Literacies are the focus of the PISA study and mathematical literacy is defined as students' ability to "apply mathematical knowledge and skills and to analyse, reason and communicate their ideas effectively as they pose, solve and interpret problems in a variety of situations" (OECD 2004, p. 23). The PISA study assesses students' familiarity with mathematical concepts and processes, and their capacity to make decisions related to their lives or understanding of world affairs.

Not surprisingly, the findings from the PISA study conducted in 2003 showed that countries with higher levels of income and expenditure on education recorded higher levels of mathematical literacy. However, variation within countries was greater than the variation between countries (OECD 2004). These variations occurred between education systems and programs, between schools, and between groups of students within schools. Mathematical literacy was positively correlated with socio-economic status, with the relationship stronger in some countries (e.g., Australia) than in others (e.g., Germany) (McGaw 2004). Gender differences favoring boys were not as great as differences by other factors, but they were most clearly observed among the highest achievers. In 27 of the 40 countries for which gender analyses for PISA 2003 were reported (OECD 2004) there were statistically significant gender differences favoring males; in 12 countries there were no statistically significant gender differences, and only in Iceland was there evidence of girls outperforming boys in mathematical literacy (OECD 2004). The instrument used by PISA to measure mathematical literacy was a pen and paper test that was conducted without the aid or use of digital technologies. There is therefore a need to consider equity of outcomes with respect to students' technical knowledge of digital tools in a mathematics context, and students' mathematical knowledge in a digital context.

Attitudes towards mathematics are indicators of students' dispositions to use mathematics, and are outcomes of mathematics learning. Hence students' attitudes towards the use of technologies for learning mathematics and for solving mathematical problems are important outcomes of mathematics education in the digital age. Further, a number of affective factors have consistently been strongly associated with participation and progress in mathematics (see for example, Watt et al. 2006) and hence are important indicators of equity in mathematics education. Researchers have also explored the relationship between attitudes to technology and the use of digital technologies in mathematics. In studies of the attitudes of tertiary mathematics students and of secondary students, stronger correlations have been reported between attitudes to the use of digital technologies and attitude to computers than between attitudes to digital technologies and attitudes to mathematics (Forgasz 2004; Galbraith et al. 2001; Pierce et al. 2007). Due consideration of equity issues with respect to attitudes towards the use of digital technologies for mathematics and the relationships to other outcomes is warranted.

18.2.5 Agency

Mathematical literacy also involves the disposition to use mathematics and digital technologies to meet social demands and to participate actively in society. Teaching mathematics well gives students access to mathematical knowledge and skills, and hence power in society. The idea of empowering students to act in, and on, their world is central to the notion of agency. Gutstein (2003), drawing on the work of Friere (1992), described agency as being able to "read the world" with mathematics, and for people to believe that they themselves can make a difference in the world. Agency therefore is concerned with both the capacity and the disposition to use mathematics to effect change in one's personal life or in the lives of others. A definition of mathematical literacy for which the objective is citizenship (e.g., Jablonka 2003) could be interpreted as contributing to social reproduction. Mathematical agency, on the other hand, encompasses a disposition for social action. Gutstein (2003) and others (e.g., Burton 1996) have argued that mathematics is not context-free, and that pedagogy for social justice needs to engage students in mathematical inquiry of the social and political phenomena in their communities and societies. Through these activities students develop skills in critical mathematical literacy, that is, the capacity to critique the mathematical models used in the political process and for solving social problems (Jablonka 2003; Skovsmose 1994). In this sense, mathematical agency is an equity and social justice issue. Of interest, therefore, is how teachers and students have used digital technologies to understand mathematics and the natural world, as well as their social and political worlds, and to develop the belief that their knowledge of mathematics equips them with the power to provide justice in their own communities.

In the following sections research findings from studies focusing on the various aspects of equity with respect to digital technologies in mathematics education are presented and discussed. The research findings are drawn from studies conducted in different countries, regions, schools, and classrooms.

18.3 Research Studies

18.3.1 Equity and Mathematics Learning with Technology

In 1995, Fennema (1995) argued that there had been little progress towards gender equity for lower achieving girls. With a focus on preferred learning styles and the needs of the most successful and socially advantaged mathematics students, Hoyles (1998) signalled that the use of technology for mathematics learning could be a factor promoting increased gender inequity. There is recent evidence that technology may actually accentuate gender inequities in the mathematics classroom. For example, in Victoria, Australia, where the great majority of people are familiar with technology, Vale (2002, 2003) studied grade 8 and 9 students using technology in the mathematics classroom. The findings were summarised by Vale (2006) as follows:

> While the behaviours and attitudes of girls and boys were similar in many respects, the classrooms were masculine domains since the behaviours and interests of the boys defined the cultural norms of the classroom. The boys were louder… more demonstrative and public about their computer knowledge and competitive about their achievements in mathematics and with computers… Boys benefited… because they took control of their own learning to learn more about computers… Girls and their needs and interests were on the periphery…; they did not participate in general classroom discussions, males denigrated their achievement and the teachers were generally ignorant of their computer skills, especially girls with lower math achievement. Some high achieving girls worked individually as silent participants. (p. 2)

Vale (2006) added that "without adequate support from their peers or teacher, students who were not computer literate were excluded from the mathematical learning" (p. 4).

The teachers in Forgasz' (2006a) study believed that boys and girls worked differently with computers in the classroom. Forgasz (2006a) summarised as follows:

> …boys' competence, confidence, and interest in computers generally, appear to advantage them over girls when computers are used in the mathematics classroom. It seems that teachers feel the need to focus boys' attention to the task at hand and encourage and support girls to engage with the technology. It would appear that without positive intervention with girls, it is more likely that boys will gain more from their interactions with computers in the mathematics classroom. (pp. 459–460)

In Mexico where, in contrast to Australia, the great majority of people are not familiar with technology and the great majority of students only infrequently use computers at school, Ursini et al. (2004b) found that when technology was introduced into mathematics classrooms it seemed to lead to improved gender equity with respect to students' behaviors. In the study, 24 teachers were asked about the behavioral changes of 1,113 12–15 year old students when using technology for mathematics. The teachers believed that the use of technology modified the girls' and the boys' behaviors, but in different ways. For example, boys with 3 years experience in using computers at school were less keen to work individually than their peers with less experience in using technology, while computers did not influence the girls' preference for almost always wanting to work in collaborative teams. The use of technology did not influence the boys' reticence to ask for help, but pushed the girls into being more participative and less inhibited in asking for help than in classes without technology. After using technology in the mathematics classroom for 3 years, the teachers felt that there were no longer major differences between boys' and girls' behaviors. These findings suggest that the use of technology was promoting gender equity in these dimensions of behavior. The teachers, however, stressed that the more the boys used technology, the more they tended to abandon the mathematics task and focus on learning more about computers, while girls always tried to complete the assigned tasks.

The Mexican findings described above might be strongly culturally dependent. In fact, the Mexican teachers explained that boys rarely asked for help in order to avoid others making fun of them, and that boys usually participate when they are confident that they have the correct answer. In contrast, girls are not expected to know much, and it is very acceptable for them to participate less and ask for help. When technology was introduced into the classroom, however, the vast majority of girls and boys had no experience using them; this seemed to have inhibited the boys and led the girls to participate more. The teachers explained that in order to be equitable in the mathematics classroom, they always try to help girls more than boys, arguably exhibiting their gendered perceptions of girls being less capable than boys. Teachers' gendered perceptions of girls and boys were also reported in another Mexican study (Ramirez Mercado 2006) in which primary teachers clearly expressed beliefs that boys were "naturally" talented in mathematics, although their views on what denoted intelligence varied widely. Girls were considered obedient, following rules, and achieving good marks because they worked hard.

Forgasz and Griffith (2006) compared the Victorian (Australia) Certificate of Education (VCE) results of students in two parallel (Mathematical Methods) courses, one in which graphics calculators were used and the other in which CAS calculators were used. (It should be noted that the Mathematical Methods CAS program was a pilot study and the enrolment numbers were relatively small.) At the highest levels of achievement in both courses there was an achievement gap favoring males. However, the gap was wider in the course in which CAS calculators were used and the pattern was consistent over the 3 years (2002–2004) for which the data were examined. In the next few years, all students in Victoria taking Mathematical Methods will be using CAS calculators. Forgasz and Griffith (2006) noted that while teachers were generally optimistic about the introduction of the CAS calculators into the Mathematical Methods course, there was the potential that males may be advantaged over females in using CAS calculators in these high stakes examinations (results are used for university entry), and that this warranted careful future monitoring.

In an earlier study, Forster (2002) explored gender-related effects in the Western Australian Calculus Examinations from 1995 to 2000, 3 years before and 3 years after graphics calculators were introduced. There were many more boys than girls enrolled in this subject, and no clear-cut patterns were evident with respect to gender differences in achievement along a number of dimensions. Forster (2002) was cautious in her conclusions and recommended further study. However, there were some data on actual calculator use from sample data for a rectilinear motion application. These data indicated that most boys and girls “chose to use the technology when it was an option and scored better than students not choosing it; and a greater percentage of boys than girls chose it and scored better than the girls” (Forster 2002, p. 816).

In a recent study, Ursini et al. (2007) found that using technology to support mathematics learning benefited only a small group of high achievers, and girls benefited more than did boys. The mathematics topics that were learnt better with the support of digital software (spreadsheets and Cabri Géomètre) were: ratio and proportion, perimeter and area of simple geometric figures, calculating percentages of numbers, pre-algebra, very simple linear equations, and linear graphs. However, the researchers believed that further research was needed to identify the areas of mathematics in which the use of technology would improve students’ learning.

Yerushalmy (2006) discussed the need to study changes in cognitive hierarchies when learning with technology from the perspective that “computational technologies allow us to improve the design of mathematical learning environments” (Yerushalmy 2006, p. 6). Findings from various studies were described, including some involving a Visual Math curriculum, described by Yerushalmy (2006) to be:

> an algebra, pre-calculus and calculus curriculum where technology is being used to help learners develop knowledge from their perceptions of the world and to develop conceptual understanding of symbols. (p. 2)

From one study, it was reported that compared to traditional algebra students adopting an equations-based approach, those using the Visual Math curriculum provided a wider range of solutions and were more likely to get the correct answer. An implication of these findings is that using this technology might widen the achievement gap between students using the technology and those, including students from late developing countries, who do not have access to it.

The research findings discussed in this section provide evidence that the introduction of technology into the mathematics classroom has had mixed outcomes with respect to some equity dimensions. The findings from Australia and from Mexico were not consistent; in Australia there was evidence that there was a widening of the gender gap when computers and sophisticated hand-held technologies were brought into the mathematics classroom, supporting the predictions of Hoyles (1998) and Hanna and Nyhof-Young (1995). In Mexico, however, classroom behaviors in response to the novelty of technology introduced into the classroom appeared to challenge the strong gender stereotyped behavioral expectations that teachers had of their students.

18.3.2 Equity and Attitudes, Beliefs, and Values Associated with Technology Use for Mathematics Learning

In general, the use of computational tools has had a positive impact on children's attitudes towards mathematics. However, in studies of computer use in education it has been commonly found that males, compared to females, hold more positive attitudes to computers e.g., grade 8 students' mathematics learning with Geometer's Sketchpad (Dix 1999).

Forgasz (2002a) examined various equity groups' gender-stereotyping of computer use for mathematics learning. Traditionally, mathematics and computers have been considered male domains. Forgasz (2002a) found that socio-economic status mediated gender differences in attitudes to computers for mathematics learning. Students of high socio-economic status, that is those most likely to have greatest access to computers, were found to hold the strongest traditional gender-stereotyped views; and Australian Aboriginal students, considered to be at the lowest socio-economic level in the country, held the least stereotyped views.

In a study in which three groups of grade 7 students (64 students in all) were taught mathematics differently – two groups used technology and the third did not – Isiksal and Askar (2005) found no significant gender differences with respect to mathematics achievement and mathematics self-efficacy, but that the boys had significantly higher mean scores than girls for computer self–efficacy. Significant correlations were found between the self-efficacy scores and achievement.

Forgasz (2002b) compared a large sample of Australian grade 7–10 students' gender-stereotyped views of mathematics, of computers, and of computer use for mathematics learning. It was found that students no longer appeared to stereotype mathematics as a male domain but clearly considered computing to be a male enclave. With respect to computer use for learning of mathematics, they were more ambivalent, with their views sitting between those for mathematics and those for computing. The gendered directions of the views of male and female students were remarkably similar, and although the strengths of males' and females' views varied, there was no clear pattern evident. Vale and Leder (2004) found that attitudes to the use of computers for learning mathematics were more strongly correlated with attitudes to computers than to mathematics, and that the relationship was stronger for boys than for girls.

Pierce et al. (2007) described the development of a new scale entitled *Mathematics and Technology Attitudes Scale* (MTAS) that had five subscales – (affective engagement AE, behavioral engagement BE, confidence with technology TC, mathematics confidence MC, and attitude to learning mathematics with technology MT). Findings from the administration of the scale to a sample of 350 grade 8–10 students in six schools revealed that boys had significantly higher scores than girls on all scales except BE. Most bi-variate pairs had significant positive correlations. Interestingly, MT was positively correlated with TC for boys and negatively with MC for girls. It was also found that most students in each school agreed that it was better to learn mathematics with technology.

In Mexico Ursini et al. (2004a) developed the AMMEC (Actitudes hacia las Matemáticas y las Matemáticas Enseñada con Computadora) scale. The results of administering the scale revealed that boys tended to be more positive than girls towards mathematics and towards mathematics taught with computers. Comparing the attitudes towards mathematics of two groups (458 using spreadsheets or Cabri, and 221 not using technology), Ursini et al. (2007) found attitudes to be more positive among those using technology, although this was not true of all students. After 2 years using computers, a sizable percentage of students retained a negative or neutral attitude toward mathematics, with more girls than boys feeling this way. It was also found that the great majority of students, whether they used technology or not, were positive towards technology use for mathematics learning (more boys than girls), although a huge percentage remained neutral (more girls than boys).

Alajääski (2006) used the differences in pre- and post-test scores to determine 53 (43F, 10M) Finnish polytechnic students' attitudes towards a Web-based approach to the learning of mathematics/statistics. The attitudes of females, students with higher ICT-orientations, and students with stronger mathematical backgrounds were found to be less favorable at the end of the course, while males, students with lower ICT-orientations, and students with weaker mathematical backgrounds had more positive attitudes. Alajääski (2006) concluded that it "seems that the students with better overall motivation to study mathematics/statistics are most critical of the Web technology based studying platform." (p. 78)

There are some research findings in which the anticipated benefits to all students of using computers for mathematics learning have been challenged and which revealed gender differences in related beliefs. Forgasz et al. (2006) discussed findings from a 3-year study in which data were gathered on teachers' and their students' views on whether computers assisted students' understanding of mathematics. About 60% of the teachers believed that computers aided students' mathematical understandings, with a higher proportion of male than female teachers believing this to be the case. With only about 30% agreeing, the students were less convinced than their teachers of the positive impact of computers on their learning; there was also a higher proportion of male than female students in support. The findings of the reported study appear to provide further evidence of a "digital divide" between teachers and students in regard to their perceptions of the effects of technology use on mathematics learning and also, perhaps, on the potential the technology might have to promote the mathematics learning of some students.

The research studies reported in this section reveal mixed findings on students' perceptions of the benefits of computer use on their mathematics learning. With the exception of the findings from Mexico, it would appear that male students and male teachers are more positive than their female counterparts about the benefits of technology use for mathematics learning. Further research is needed to explain the reported findings that students from high socio-economic backgrounds held the most gender-stereotyped views of the effects of computer use for mathematics learning.

18.3.3 Resources for Mathematics Learning with Technology

When looking at differences in ownership of computers, Instituto Nacional de Estadïstica Geografia e Informatics (INEGI) (2007) reported that in 2004 there were between 40.8 and 138.7 per 1,000 inhabitants in Latin America, China and Russia who owned computers, while in Europe (except Spain and Italy), the United States, Canada, Asia (except China and Malaysia), and Oceania there were between 486.6 and 762.2 per 1,0002007 inhabitants owning computers. (It should be noted that Africa and India were not mentioned.) These figures are strong indicators of the differences in access to technology for people around the world. In Mexico in 2005, for example, INEGI (2007) data revealed that only 19.4% of the population had a computer at home. The main reason for not having computers was economic, but there was also nearly 16% who had no idea about the possible advantages that a computer could offer. Only about 30.4% of the Mexican population aged over six had used a computer, and the great majority used them about once a week. Computers are used at home (43.3%), at school (31.6%), in public places (27.6%), and at work (27.9%). There were no significant gender differences in access to computers. Mexicans use computers mainly as word processors but, interestingly, they are also being used for educational purposes, with 58.9% using computers to support school teaching or learning. In 2006, most users (63.1%) were younger than 24 years of age; only 34.3% were between 25 and 55 years of age. These official data (INEGI 2007) suggest a generational gap in access to computers which may partly explain teachers' resistance to incorporate the technology into their daily mathematics classes, as found by Ursini et al. (2005) who worked with 12 teachers participating in a pilot study aimed at incorporating technology in rural and semi-rural schools.

In Australia in 2004/2005, computers were found in 67% of all homes, and the Internet was connected in 56% of all homes (Australian Bureau of Statistics 2006). Forgasz (2006a) reported findings from repeated surveys about the technology use for mathematics teaching of secondary school mathematics teachers from 29 schools (2001: 95 respondents; 2003: 75 respondents). In both years well over 90% of the teachers rated their computer skills as average or better and were at least willing to have a go at using them in their mathematics teaching. All of the schools they worked in had computers available, with all, except one, having computer laboratories. In 2001 and 2003, about 2/3 of the teachers said they had used computers in their mathematics teaching. In both years, 39% of the teachers said they had used computers for "just one topic", about 10% said they used them at least once a week, with the rest indicating they had used them less frequently.

In order to enrich and improve the current teaching and learning of the standard secondary level mathematics curriculum for children aged 12–15, the Mexican Ministry of Education has, since 1997, promoted the EMAT (Mathematics Teaching with Technology) project. The aim of the project is to integrate computer technology into public secondary schools throughout the country using specially designed materials (worksheets with curriculum based activities to be solved using computer software, mainly Excel or Cabri Géomètre) based on a constructivist approach (Ursini and Sacristán 2006). Taking into account teachers' lack of familiarity with technology and to help them to gradually adopt the tool for teaching mathematics, the teachers were offered a 1-week workshop in which they were taught basic computer skills and how to use the EMAT materials. Although they were free to modify the worksheets, their general lack of experience did not allow them to do so. At the end of 2002, there were nearly 730 schools equipped with 15 or more computers using EMAT, that is, 2,280 teachers representing about 200,000 students. Most of these students are from low economic backgrounds, they do not have computers at home, and their access to technology is limited to a few hours a month at school. The vast majority of teachers cannot afford their own computers, so they have no easy way to familiarise themselves with the technology out of school, or to explore and prepare their lessons to incorporate technology. Moreover, computers are usually in the media-room and a technician is in charge of determining when teachers or student groups can use the computers.

18.3.4 Access to Mathematical Learning with Technology

Even in well-resourced schools, however, access to technology for mathematics learning is not guaranteed. Thomas (2006) revealed that while in New Zealand the number of computers in schools had increased, as too had the frequency of their use, access to them was still a major obstacle for use in mathematics learning. Forgasz (2006b) reported similar findings for Victorian (Australia) secondary schools. As discussed in more detail in another chapter in this book, access to computer hardware has been found to be a major obstacle for some mathematics teachers, while simultaneously for other teachers it is a major facilitator to their use of technology for mathematics teaching.

Exploring the potential for technology to enhance access to mathematics for students living in rural and economically stagnant areas, Sloan and Olive (2006) highlighted the opportunities afforded through Distance Learning and web-based instruction. However, they signaled that it is not enough to have technological tools, but that it is also necessary for a conscientious effort to be made by educators to embrace the technology, and that a multitude of issues needed to be addressed including: professional development, equitable access to appropriate hardware and software, Internet access and controls, and out-of-class availability of computers.

Focusing on late developing countries, Gray (2006) signaled that computers were not as rare as might be thought, and that there has been rapid, superficial growth in the Internet. However, she maintained that educational use of the Internet was very limited outside developed countries. She argued that while most universities in late developing countries have Internet access, a single terminal may serve as many as a thousand students, and that while secondary schools may have computers, they seldom had Internet access. She added that at the primary level, the use of computers for educational or administrative purposes is rare in most late developing countries.

On the other hand, business use, including commercial cyber-cafes, had expanded greatly, revealing the potential for broader application of the technology. Gray (2006) claimed that the data reflected inequities between countries, but also revealed within country inequities with city-dwelling, prosperous citizens having access to the information and skills available via the Internet. Even in developed countries, rural areas or depressed urban areas there may be difficulty in securing access to qualified teachers and material. Gray (2006), as Sloan and Olive (2006), considered distance learning applications via the Internet to have great potential, especially in reaching rural areas and in maximizing the use of scarce teaching resources. In particular, Gray (2006) stressed that girls clearly did not have equal access to education in many late developing countries. For example, in sub-Sahara Africa only six of every ten girls attended primary school (compared to eight of every ten boys), with the situation becoming even more disparate beyond the primary school level. She maintained that vast distances, lack of sanitary facilities, and sexual harassment problems can be overcome through distance learning. Setting up Internet access points, particularly in rural areas, had the potential to transform girls' educational prospects.

In summary, access to technology for mathematics learning varies, with late developing countries clearly having fewer resources than in the developed nations. Several researchers, however, hold high hopes that the Internet has the potential, when coupled with appropriate pedagogical developments, to enhance the learning opportunities for those in more remote regions and, in particular, to girls who are denied educational opportunities because of their sex.

18.3.5 Agency as an Outcome of Mathematical Learning with Technology

Research findings and examples of approaches for achieving agency as an outcome of mathematics learning have been documented in the literature (see for example, Gutstein 2003; Gutstein and Peterson 2006). However explicit reference to the involvement of digital technologies in that mathematics learning are rarely reported or analysed. Three examples were identified in the ICMI study – one at the tertiary level, and two others at the secondary level. It was suggested that agency ought to be a goal for mathematical learning with technology.

At the tertiary level, Muller (2001) differentiates between flexibility of student access to mathematics and flexibility of student action in mathematics. In mathematics service courses (designed for students who are not majoring in mathematics), technology enhances flexibility of student access to mathematics. In these courses, the emphasis is on developing an understanding of the role of mathematics and its uses in other disciplines. Students are expected to understand the mathematical methods and models that are applied in their disciplines. However, they are not expected to know the mathematics necessary to develop the models themselves. For example students in Data Analysis courses should develop a conceptual understanding of the linear regression model and they can do this by experimenting with a graphical model without being expected to be able to derive the formulae for the coefficients of the line. Through the MICA (Mathematics Integrating Computers and Applications) program in a tertiary setting, Buteau and Muller (2006) have found that:

> The approaches, activities and experiences in the MICA courses are able to harness the students' motivations thereby empowering them to become their own mediators in the development of mathematical knowledge and understanding. (p. 8).

In a study of secondary teachers' equitable practices with digital technologies (Vale 2006), integrated projects that were socially and culturally relevant to students were used. The projects were usually open-ended and aspects of the tasks were negotiated with students. Using mathematical or statistical software applications, the students explored mathematical concepts, or a problem or issue of social or cultural relevance (such as the status of women). They presented their findings using a range of digital technologies and media. Interviews and workshop presentations were used to explore the teachers' equity objectives, beliefs, and practices associated with the use of digital technologies. While making mathematics relevant was clearly a goal for the teachers, the mathematical concepts and skills that were empowering for students were not examined and remain to be established.

Using examples of their work with teachers of mathematics, citizenship and geography, Ridgway et al. (2006) illustrated the potential for the development of reasoning from evidence through the use of appropriate computer interfaces for multivariate data analysis. These kinds of mathematical activities provide opportunities for students to understand important social issues and to make informed decisions about their own well being and suggest that the use of appropriate digital technologies may enable students to be agents for their own learning and social action.

18.4 Conclusion

Not surprisingly the availability of resources for mathematical learning with digital technology has been shown to vary according to the economic status of countries and regions within countries. The extent of access to mathematics learning with digital technologies is more complex, involving cultural and educational values and beliefs, however the evidence from some studies shows that even when schools are resourced with digital technologies some students are denied access. There are no reports from large international studies on students' facility to use digital technologies in contexts involving mathematical thinking and problem solving. Such studies would have the potential to identify the implications of limited resources and restricted access to resources with respect to equitable outcomes, and may provide some clarification of the conflicting results with respect to gender equity reported in this chapter. International studies of classroom practices would also have the potential to resolve the disparity of results showing that in some countries the use of digital technologies may be closing the gender gap, while in other countries, where there have been little or no gender differences in the past, the gap may be widening.

More studies are needed to provide information about the gap between low and high achievers in relation to the use of digital technologies. Further research on the relationship between mathematics learning with digital technologies and agency is also needed, since the development of agency as a consequence of mathematics learning is an important dimension of national and international human capacity building.

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Chapter 19
Factors Influencing Implementation of Technology-Rich Mathematics Curriculum and Practices

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Abstract Using different levels of analysis, we identify some factors influencing the integration of digital technology in mathematics and we try to explain the contradiction between the strong political will for this integration and the weak implementation in mathematics classrooms. When one wants to change something, resistances often arise; we have identified some of them, for example, personal, institutional, symbolic and didactical resistances.

Keywords Typology of factors • Changes and resistances • Levels of analysis • Mathematics curriculum • Mathematical practices • Digital technology

19.1 Introduction

In this chapter, we will answer the question: What approaches, strategies or factors foster or impede the implementation of technology-rich mathematics education? We have organized our chapter in two sections. In the first section, we are interested in identifying a typology of factors that foster or impede this implementation, and in the second section we try to explain why we can observe a strong political will about digital technology integration and a weak implementation in mathematics classrooms at all levels of schooling and education.

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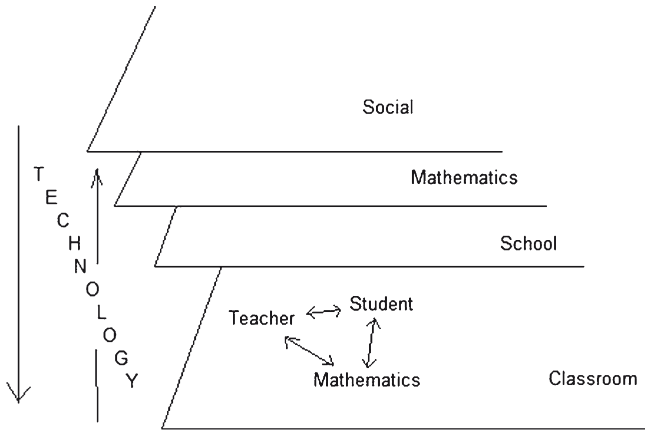

Fig. 19.1 Different levels of analysis

19.2 Typology of Factors

In our attempt to identify some factors influencing the integration of digital technologies in the mathematics curriculum we will choose different levels of analysis:

- The social, political, economical and cultural level
- The mathematical and epistemological level
- The school and institutional level
- The classroom and didactical level

These levels are not independent and there are many intersections. An element belonging to the school level can also be relevant on a social level and an element belonging to the mathematical sphere may also be part of the social and cultural sphere, but our aim in separating these levels is to draw attention to some specific functions of each one. For example, with respect to the epistemological level we want to point out the role of the specific mathematical knowledge for teaching and learning mathematics in the classroom (Fig. 19.1).

19.2.1 The Social, Political, Economical and Cultural Level

There is a growing expectation that mathematical education uses digital technology. There is a strong political will for integrating digital technologies in the official, that is planned, curriculum for secondary and primary schools. We can observe these political, social, economic and cultural factors in different national curricula or in local curricula, and in the financing and development of research projects. In the USA, for example, technology is one of the six principles underpinning the *Principles and Standards for School Mathematics* (NCTM 2000). In the French curriculum, *The Common Base of Knowledge and Skills* (MENESR 2006), one of the seven competencies is “mastering common information and communication technologies” and this competency is on the same level of importance as “mastering the French language”. In Mexico, the Mexican Ministry of Education sponsored a national project called EMAT (Teaching Mathematics with Technology) and “there is a large government-sponsored campaign, that includes many advertisements on radio and television, claiming that computers (without any reference to the way they are used) improve children’s learning” (Ursini and Sacristán 2006, p. 2).

In most countries, the official curriculum presents rhetoric about different factors fostering the integration of information and communication technology (ICT) in the teaching and learning of mathematics. For example, in the UK’s *National Numeracy Strategy, Framework for teaching mathematics from Reception to year 6* (NNS), we can read:

> ICT includes the calculator (…) and extends to the whole range of audiovisual aids, including audio tape, video film and educational broadcasts. You can use ICT in various ways to support your teaching and motivate children’s learning. (DfEE 1999, p. 31)

ICT is presented here as a tool for the teachers to support their teaching, and a tool to motivate children’s learning. Calculators are presented in the UK NNS as powerful and efficient tools that have a role to play in subjects including geography, history and science, allowing primary aged children to use real data. Here the rhetoric is based essentially on the usefulness and the power of calculators; they are tools for working with real data and for making relations with other domains. Other elements of the rhetoric include dealing with pupils’ activities including: “explore, describe and explain number patterns”; “practice and consolidate their number skills”; “explore and explain patterns in data”; “estimate and compare measures of length or distance, angle, time and so on”; “experiment with and discuss properties in shape and space”; “develop their mathematical vocabulary, logical thinking and problem-solving skills”.

In the French curriculum, *The Common Base of Knowledge and Skills*, social and cultural factors are pointed out:

> Digital culture involves the safe and critical use of technology of the information society. This include IT, multimedia and the Internet, which now permeate economic and social fields. This technology is often learnt by experimenting outside of school. Nevertheless, schools must allow each pupil to acquire a set of skills that will allow him/her to use technology in a more thought-out and effective manner. (MENESR 2006, p. 35)

The numerical culture is an international culture and schools must prepare pupils for it. The use of ICT is social and the integration of ICT in school is initially an answer to these social needs. Other arguments can appear in the official curriculum, for example the benefits of ICT for learning, but these rationales are not the primary motivation for integrating ICT. Such educational justifications are a way to implicate the education community even if some researchers have raised doubts

> a) whether computers have any real value in learning (Cuban 2001), and b) whether current teacher use is qualitatively and quantitatively sufficient to promote any benefits that might exist. (Thomas 2006, p. 1)

In most countries, school and tertiary educational systems have evolved with diverse traditions, are faced with different constraints and challenges, and operate differently. In the section on “Effecting Curriculum Change” in their paper, Hodgson and Muller (1992) pointed out that school mathematics curricula are normally developed by a small group of experts in Ministries or Boards and are then implemented by a very large population of teachers. At the tertiary level the mathematics curriculum is developed by the professors in the departments of mathematics who will be teaching it; the decisions and implementations in classrooms are taken by the same actors. At this level of education it is easier to see the "transnational" aspects of the numerical culture. In a study involving universities from Hungary, UK and USA, Lavicza (2006) mentioned that:

> no distinctive teaching traditions of technology use at the university-level were identified…. This result accords with Atweh, Clarkson, and Nebres' (2003) idea that mathematics research and mathematics education have become an international enterprise, particularly at the university level. (p. 4)

Compared to the school level, there are no social or cultural factors to impede, or foster, the integration of digital technology in university mathematics education.

These social and cultural factors are very important in fostering the implementation of digital technologies because they legitimate what we do in mathematics classrooms. In spite of any political support, the implementation of digital technologies in mathematics classrooms in different countries is different (see Sect. 1, Chap. 1), but everywhere it is difficult. Why is there such a distance between the planned curriculum and the implemented curriculum? We must examine other levels to identify factors that may explain this difference.

19.2.2 The Mathematical and Epistemological Level

Starting in the 1960s more and more mathematicians have made use of digital technology in their research. New research areas such as Discrete Mathematics, Simulation, Theoretical Computer Science, and others have been developed. Some areas such as Applied Mathematics naturally embraced technology. Even some mathematicians whose research focus is Pure Mathematics have introduced an experimental approach to their research using technology to explore conjectures but always with the aim of developing a proof. According to Lavicza (2007) a substantial number of mathematicians use CAS (computer algebra systems) for teaching at some level. Based on the questionnaire responses of 1,103 mathematicians (24.62% response rate), 67% of them use CAS for their own research and 55% integrate CAS into their teaching at least on an occasional basis.

How does the teaching of mathematics take into account these changes in mathematicians' practices? These epistemological factors are important in the consideration of the "reference" for the teaching of mathematics. What is the mathematics reference for teaching? We have a new historical and cultural situation because the "reference" of mathematical knowledge with digital technologies is very young.

At the university level, the department is not only the environment of the implemented curriculum, but also the decision maker for curriculum development.

One may think that this could easily foster technology innovation in curricula, yet the reality is that evolution and innovation in university mathematics education are slow processes. There is in fact a strong internationally uniform “mathematics university department culture”. Traditionally, mathematicians view doing mathematics as an individual activity. There is a strong focus on proofs. Teaching is usually valued as secondary and way behind research, which may reinforce a common attitude towards teaching to copy one’s own personal, traditionally abstract oriented, experience since one’s success supports it, although the vast majority of undergraduate mathematics students do not become mathematics academics. Lavicza (2006) argued that, due to academic freedom:

> Mathematicians have better opportunities than school teachers to experiment with technology integration in their teaching. However, academics are frequently more concerned with research than teaching and so experiments with technology in their teaching may be seen counterproductive. (p. 4)

There is a need for a radical change in teaching approach by faculty and teaching assistants in a technology rich undergraduate mathematics curriculum of the *Mathematics Integrated with Computers and Applications* (MICA) type (Ben-El-Mechaiekh et al. 2007) described in the introduction to this section. Anguelov et al. (2001) wrote:

> Courses which focus on exploration within the topical content, practical experience, self-discovery, exposure to the problems which led to the development of the theory, and applications which are the reasons for its continued existence, can immediately make use of computer-based mathematical tools. (p. 154)

In our view the traditional role of the faculty as exhibiter of knowledge needs to shift to become more of a facilitator role.

At the secondary or primary school level, mathematics knowledge takes as reference “old” mathematics, and the integration of digital technologies is a professional problem for the teachers (see Sect. 2). Secondary or primary teachers have less knowledge of new mathematics, of how mathematical knowledge is changing and how the mathematicians’ practices are changing. For example, some teachers are not aware of the role of multiple representations of mathematical concepts (Laborde 2007; Hoyles and Noss 2003) or the development of an experimental approach to mathematics. Laborde (2007) said:

> These two features of technologies, their embodiment of mathematical knowledge and the range of computing and graphical capabilities they offer, contribute to their usefulness for experimenting and change the very nature of mathematical activity by shifting the balance in favor of an experimental approach in a broad sense, including activities such as modeling, simulation, and trials on a large scale. (p. 72–73)

Our question is: How can we reconcile the old and the new if we don’t know very well the practices of reference? We can investigate factors at other levels to reflect on this problem. For example, the existence of resources or teachers’ professional development appear to be conditions for fostering the use of digital technologies in classrooms.

19.2.3 A School or an Institutional Level

Integration of digital technologies in the teaching of mathematics is legitimated at the social and epistemological levels and some mathematical practices of reference are indicated at the epistemological level. The school or institutional level is the "environment" of the implemented curriculum. We can find different factors in this level because educational organizations vary in function by country or region or institutional level. Monaghan (2006; see Sect. 3 in this book) focused on cultural factors and provided a particularly useful theoretical framework. Let us examine some of them.

The material factors in a school are an essential condition for using digital technologies: computers, accessibility to the computer lab (if it exists), money to buy software, technical assistance, encouragement of the school and so on. Most schools and tertiary institutions are constrained by funding from governments (or communities) and by the social and economic infrastructure (e.g., availability of electricity, digital networks). The political will in some countries is not followed by economic possibilities and realities or economic decision making. Even in the USA, some rural schools do not have good material conditions but according to Sloan and Olive (2006):

> There is one technological resource that can provide students in almost any location with the very best learning opportunities available anywhere in the world. This technology is known as Distance Learning, and even at its most basic level, any school with Internet access can open new doors of opportunity for its students… Some schools have found that distance learning can provide a virtual schooling alternative across the curriculum. (p. 1)

There are also cultural factors in schools or institutions concerning the different ways of working together in schools or in communities of practices or in-service training. The relationships with technologies are fostered if the teacher is not isolated but a member of a community in technology.

The availability or the conception of resources (pedagogical material) is one factor for using technologies but it is not a sufficient condition. In some countries (such as France) the official curriculum requires use of technology but there is (in general) no material prepared for teachers, but in other countries there is pedagogical material ready for teachers but still there is no use of technology (Thomas 2006).

The assessment practices and requirements in schools or educational systems are important factors in fostering the uses of technology. Forgasz et al. (2006), for example, reported on two studies in which teachers' views of graphics calculator use were examined. In one study, Victorian (Australia) and Singaporean teachers' views were compared. The findings suggested that:

> Mandating technology tool use in an assessment program, as was the case in Victoria, plays an important part in explaining the extent of their use by teachers, and may also account for the Victorian teachers' preference for graphics calculators over computers. (p. 4)

Findings from the second study revealed that, in general, teachers in Victoria "believed that graphics calculators have had a positive impact on their teaching and on students' learning outcomes, and that the curriculum has been enriched"

(p. 5. See Sect. 3 of this book for a more detailed discussion of assessment issues related to technology use).

The development and implementation of the university mathematics program (MICA) described in Chap. 16 had to overcome many institutional factors. Because faculty members have the final say on the mathematics curriculum they are teaching, most of the effort was directed at exposing the faculty to situations that involved the teaching and learning of mathematics using technology. This was achieved through many years of sustained technology development and implementation in traditional mathematics courses. In the early days the Department had to work hard to get the necessary laboratory and software facilities and had to provide different training to its Teaching Assistants. Muller (2001) presented a brief history of these developments. It may be surprising that although this paper was written only 2 years before the introduction of the MICA program, the author did not predict the Department's radical curriculum change.

19.2.4 The Classroom and Didactical Level

At this level, we consider a classroom as a didactical system constituted by the teacher, the students and the mathematical knowledge. We distinguish some personal or human factors and some didactical factors that can foster or impede the use of digital technologies in the classroom. Some teacher factors can be identified, such as teachers' conceptions about the technology itself or its place or role in the classroom, professional development in the use of digital technology, familiarity with software, professional identity and so one. These teachers factors are discussed in the chapters of Sect. 3.

In the section "Technology and Modeling in a University Mathematics Program", Pead et al. (2007) raised a number of human factors arising for both faculty and students in the undergraduate MICA program.

From a faculty point of view, the MICA courses require a very different teaching approach, which cannot be communicated by simply passing teaching notes from one faculty member to another, but require extensive discussions on how to develop an open teaching environment in which students are constantly urged to raise questions and to propose their own conjectures. The MICA course environment is very different from that of a traditional mathematics lecture course, and faculty must be willing to move from the role of "knowledge provider" to a less secure role of facilitator. Pesonen and Malvela (2000) suggested that:

> As the students become more familiar with computers and the programming environments get more flexible, we could give the students more freedom to do the learning wherever and whenever they want. This would require very different kinds of guidance facilities from the teacher's side, but it would also mean very different but exciting experiences for the students. (p. 122)

According to Ben-el-Mechaiekh et al. (2007), most high school graduates see mathematics as a set of rules and procedures and have little experience using technology to support and enhance their own learning of mathematics. Many of the students in the MICA program were reluctant to get involved with computer programming. However, results of an internal survey showed that when students were asked how beneficial technology was in their learning of mathematics, 91% indicated technology to be positively beneficial in MICA courses (Ben-El-Mechaiekh et al. 2007) and, in all first year courses, 76% indicated technology to be positively beneficial in all mathematics courses.

Some didactical factors that foster digital technology integration can be identified. One of these factors concerns the didactical transposition: what kind of transformation is required to adapt mathematical knowledge when using technology in a classroom? The problem of designing these transformations is discussed in Sect. 1. One other factor concerns the problem of management in the classroom: how does the teacher orchestrate the work in the classroom? How does the teacher organize the mathematical work for pupils? What kind of tasks does he/she propose? What are the available techniques? What is the relationship between the paper and pencil work and the instrumented work? What about the assessment? The answers to these questions are not evident, and the role of research is very important to disseminate some well-worked solutions (see contributions in Sect. 3).

The instrumental factors concerned with learning about the use of the tool and the relationship between technical and conceptual mathematics must be taken into account (see contributions in Sect. 2). For example, the first year university students in the MICA program were progressively introduced to computer programming as a means to explore their own mathematical conjectures and to communicate interactively their results (Ben-El-Mechaiekh et al. 2007). Initially, for these students computer programming is a means to an end, but in the upper years it evolves to be a natural component of their learning of mathematics.

Assude and Gélis (2002) pointed out the role of the dialectic between old and new practices for fostering the implementation of digital technologies in mathematics: if the distance between old and new is significant teachers change less than if this distance is not significant. In the following section we discuss the degree of change required and it appears necessary to define some indicators for clarifying this degree.

19.2.5 Multi-level Factors

Many teachers indicate that time is a problem for integrating digital technology. For example, Thomas (2006) cites a typical teacher's comment about this problem: "Access to computers at required time (of year and within school timetable blocks)". He also noted as problematic

> … the time and effort needed by both students and teachers in order to become familiar with the technology. It appears that some teachers are concerned that this instrumentation phase would impact on time available for learning mathematics. (p. 7)

Assude (2005) identified different kinds of time: didactical time, capital-time, the pace of a session, and some economical temporal strategies fostering ICT integration. For example,

> One of the conditions of integration is the teacher's command of the didactic time, which allows the teacher to have a global view of how the teaching of certain content is progressing, and to have an idea of what has to come after an activity. This condition allows teachers to know where they are and where they are going. This condition cannot necessarily be satisfied in the first year of integration of new technologies. We think that even experienced teachers (the teachers participating in our research had each been teaching for more than 15 years) are not necessarily ready to face time management difficulties when the way of working with the class changes and when ready-made outlines are not available. (p. 200)

19.3 Explaining the Problem

In this section we point out that this typology is a way to explain why we can observe a strong political will about digital technology integration and a weak implementation in the classroom.

Foundations and legitimacy are a "problematique" in the integration of digital technologies: some factors can explain why it is important to implement technology-rich mathematics in the classroom. Why is it worth changing? What's the economic and symbolic value of changes?

One set of factors shows that the political decisions are necessary but they are not sufficient because the problem of implementation in classrooms is complex. Some factors can foster this implementation in some situations yet impede implementation in others. Are some factors more determinant than others? What about resistances? What kinds of resistance exist in these different levels? Investigating the issues about the use of calculators in primary school, Assude (2007) identified the role of resistances in relation to changes. When we want to change something, resistances always arise, so we must insert these resistances in our model just as we must insert an electrical resistance of a substance (passing through an electric current). We can identify several types of resistances: personal, institutional, social, economic, symbolic, ethical, epistemological and temporal, as shown in Fig. 19.2.

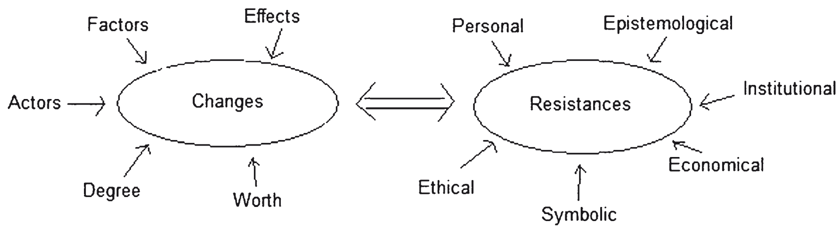

Fig. 19.2 Changes and resistances (Assude 2007)

In this section we present some examples of the institutional and personal resistances. The worth of change is a very important factor in the teachers' representations.

> There are a number of possible reasons for a low level of computer use in mathematics teaching and learning, including teacher inability to focus on the mathematics and its implications rather than the computer and many teachers not believing that the computer has a real value in student learning. (Thomas 2006, p. 1)

Computers can challenge teachers' technical knowledge and place demands on their time and energy; teaching strategies also need to be modified (Goodson and Mangan 1995). Barriers to the implementation of computers in classrooms that have been reported include: lack of time release to learn how to use computers and insufficient class time available for students to use computers (Smerdon et al. 2000); not enough computers (Hadley and Sheingold 1993; Smerdon et al. 2000); insufficient appropriate software and related information (Hadley and Sheingold 1993); teachers' self-doubt, lack of interest, and lack of knowledge about computers (Hadley and Sheingold 1993); lack of technical support, maintenance, and advice (Finger et al. 1999; Hadley and Sheingold 1993); and the high costs associated with hardware and software (Finger et al. 1999).

Many of the obstacles to computer use for the teaching of mathematics are similar to those for computer use in classrooms more generally. Manoucherhri (1999) reported that U.S secondary mathematics teachers who did not use computers said they lacked experience and access to educational software, lacked knowledge about how to use computers to improve learning, and had not been trained to use computers in their mathematics teaching. Andrews (1999) found that school mathematics departments in the UK did not have well-developed policies on computer use, that strategies for professional development were inadequate, information technology coordinators were unable to help other colleagues, and there was a lack of technical support. In an Australian study, Norton (1999) found that computer coordinators believed that mathematics teachers under used available computer resources, claiming difficulty of access as an excuse to mask other reasons such as computer phobia, a lack of software knowledge, concerns about changing teaching roles and not covering the curriculum, and lack of planning time for computer based mathematics learning. In the early years of schooling, Travers (2001) claimed, there had been relatively little impact of ICT. Some of the reasons for this included: the provision of only one computer per room; limited high quality *open-ended* software; and that many early childhood teachers were negative about computers, seeing them as providing only passive experiences and believing that students should be actively engaged in discovery.

In a survey of 485 secondary mathematics teachers in Queensland, Goos and Bennison (2004) found that there appeared to be access to software and graphics calculators in schools but there were access problems that did not guarantee that teachers and students were able to use them when needed or appropriate. Teachers reported difficulty in getting classes into computer laboratories and said that the lack of time and relevant professional development were obstacles. While generally supportive of the potential offered by technology, doubt was expressed about its use in helping students' mathematical learning – a finding opposite to that reported by

Forgasz (2006a) but similar to that reported by Thomas (2006) – and that those lacking professional development on graphics calculators were more likely to hold this view. Goos and Bennison (2004) reported that teachers identified professional development on the integration of digital technology into classroom teaching to improve students' mathematical understandings as a pressing need.

In a study of 12 teachers teaching 300 students in the educational subsystem known as "Telesecundaria" (Distance High School Program using TV) used in rural and suburban areas in Mexico, Ursini et al. (2005) explored teachers' resistance to the of use computers in the classroom for the teaching of mathematics as part of the EMAT Project (Mathematics Teaching with Technology). They found that teachers' resistance may be due to technical problems when using technology and the digital divide between teachers and students when students know more about computers than the teacher. Further, in accordance with Thompson (1992), they identified that teachers resist change if they question the relevance and benefit of computer use for themselves or their students. Lack of preparation time to explore EMAT activities due to teachers' administrative workload was another contributing factor. They concluded that:

> The results demonstrate a certain level of resistance by teachers to change and, on the other hand, the enthusiasm of students in relation to a different pedagogical focus in which they play a more participatory role and can use their knowledge. The huge shortcomings surrounding this educational system are also clear, ranging from teachers' lack of preparation to the difficulties of many students to attend school on a regular basis, to understand the content of materials and to carry out activities. (p. 193)

As early as 1992, Zammit (1992) identified factors that served to encourage or discourage computer use by mathematics teachers in Victoria, Australia. Using a limited, but appropriate for the period, definition of classroom computer *users* as those who used them at least once a term, 102 teachers were identified as *users* and 250 as *non-users*. The mathematics teachers were asked to rank a set of factors that had encouraged or hindered their use of computers. Access to computers and the availability of software were the strongest encouragers for *users*, followed by self-motivation to keep up to date, the need for students to learn to use technology, and a supportive computer coordinator. *Users* and *non-users* also ranked seven factors that discouraged them from using computers. For *users*, difficulty accessing the computer room and too few computers were the major obstacles; for non-users lack of confidence and skill with computers, insufficient time to review software adequately and computers not being a priority in the subject were identified as the key hindrances.

Forgasz (2006b) more recently explored the factors that 96 Victorian (Australia) secondary mathematics teachers in 2001 and 75 in 2003 from the same schools identified as encouraging or inhibiting their computer use in classrooms. Teachers were asked to list factors that had encouraged and discouraged their use of computers in their mathematics classrooms. The three most frequently cited encouraging factors were the same in 2001 and 2003, only the order changed. In 2001, they were appropriate software (mentioned by 41% of respondents), access to computers (37%), and personal confidence and relevant skills (32%); in 2003 the order was access to computers (40%), personal confidence and relevant skills (37%) and lastly appropriate software (29%), the most common response in 2001. Access to computers was the most commonly cited discouraging factor in 2001 (60%) and 2003 (67%). The perceived need for professional development, and technical issues were equally the next most frequently cited (31%) obstacles. In 2003, professional development issues, and time related issues were equally the next most commonly mentioned inhibitors (22%). Thomas (2006) surveyed teachers in New Zealand about access to and use of computers for mathematics learning. While the number of computers in schools had increased over time, access to them was a key obstacle to their use as mathematical learning tools.

19.4 Conclusion

It is interesting to note that across the many studies reported above from different national and international settings spanning more than a decade, similar factors were identified as encouraging or inhibiting mathematics teachers' use of technology for mathematics learning. Access to technology was a notable factor that served to encourage many mathematics teachers and also appeared to act as a barrier to others. It seems, too, that institutional or didactical factors – access to hardware, software issues, professional development needs, technical support and resources – appear to outweigh personal factors, such as confidence, in preventing teachers from using technology in their mathematics teaching. But some teachers are not persuaded of the value of technological changes for mathematics learning.

Many changes are necessary to integrate digital technologies into the teaching of mathematics, and we may not see their effects for some years. The scale of time for social change is not the same as may be desired. We now see some changes (personal and institutional) but they are only "little changes" when practices are compared to the aims and political imperatives. Perhaps in the next 30 years we will see more substantial change in the nature of mathematics and mathematics learning in digital technology learning environments. We can hypothesize that changes are at first general (such as the motivation for learning) and that specific changes in mathematics knowledge appear in a second phase. For example, when discussing the outcomes of the Mexican national project called EMAT (Teaching Mathematics with Technology), Ursini and Sacristán (2006) argued that:

> We do know that the use of technological tools does develop motivation, a more positive attitude towards mathematics, an increase in student participation, in student abilities to defend their ideas; that the technology-based environments allow students to generate and test conjectures and to go from particular to the general. (p. 6)

The changes in mathematics knowledge and mathematical practices that are emerging in the digital age are more difficult to implement in school classrooms because this mathematical "reference" is not yet clearly established or understood by teachers. More research is needed on the emerging mathematical reference and its didactical transposition. We conclude this chapter by stressing the role of the epistemological dimension in furthering the implementation of technology-rich mathematics.

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Section 5
Future Directions

Chapter 20
Introduction to Section 5

Celia Hoyles and Jean-Baptiste Lagrange

Abstract This last section consists of three chapters reflecting the plenary panels and lecture at the Study conference. The authors look at the overall landscape concerning the potential and impact of digital technologies on mathematics teaching and learning, and consider future prospects and challenges.

Chapter 21 comes back to the issue of design already addressed in Sect. 1, stressing that software design is a crucial dimension in the educational use of technology and the key to transformative practices. While the authors of Sect. 1 were primarily researchers, the choice for this chapter has been to ask creators and designers of well established and widely used environments to contribute from their own unique expertise. They participated in a plenary panel on this topic at the Study Conference. This chapter helps readers to understand better what design decisions consist of, how they are connected with visions of teaching and learning, and how they can give rise to evolutions in practice and future designs.

Chapter 22 takes up the challenge of the development of networks and of the World Wide Web. Although it is hypothesized that connectivity will strongly impact mathematics education, at the time of the Study conference, research on this topic was in its infancy. A group of researchers in this field was therefore invited to participate in a plenary panel at the Study Conference - either at the meeting itself or through a video link, and to write their contributions in this chapter. Some contributions describe experiments that take advantage of connectivity within one classroom, while others focus on between-classroom interactivity. In both scenarios, teachers' actions in supporting new communities of practice are recognised as crucial, and new roles for the teacher noted, although it is acknowledged that these roles have as yet been undertheorised.

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Chapter 23 is written by M. Artigue from her plenary closing address. She first takes advantage of her personal experience for analyzing the evolution of the relationship between digital technologies and mathematics education over the last two decades, and for situating the reflection about the future into a historical dynamic. Then, she focuses on dimensions crucial for thinking about the future: the theoretical, teacher, curricular, design, equity and access dimensions, and she stresses how the whole reflection in the study helps to think about what educators can do in order to make digital technologies better serve the cause of mathematics education.

Chapter 21
Design for Transformative Practices

Douglas Butler, Nicholas Jackiw, Jean-Marie Laborde, Jean-Baptiste Lagrange and Michal Yerushalmy

Abstract Software design is a crucial dimension in the educational use of technology and a key for transformative practices. The choice for this chapter, issued of a plenary panel at the study conference, has been to ask creators and designers of well established and widely used environments to contribute from his/her own unique expertise. After introductory remarks by the coordinator of the panel, each contributor exposes what visions drive his/her work and how.

Keywords Software design • Software designers • Exploration • Transformative practices • Dynamic Environments • Computer Algebra

21.1 Introduction

Jean-Baptiste Lagrange *Software design is a crucial dimension in the educational use of technology and a key for transformative practices. This chapter, complementary with Sect. 1, will try to tackle this dimension by considering dynamic educational environments that today attract most attention. Creators and designers of the best established and more widely used of these dynamic environments will expose what visions drive their work and how.*

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Part of the goal is to measure the path since the first ICMI study. At this time dynamic environments did not exist, and Computer Algebra was the technology that attracted attention. It was introduced in the seventies and eighties as means to ease the work of mathematicians, but also with a vision of the whole mathematical practices, including teaching and learning. It seemed that, because it takes in charge up to a certain point algebraic manipulations, it should reorient practices towards more conceptual issues. In a book following the study, Hodgson and Muller (1992) declared "there is no doubt that Symbolic Manipulation Systems must be introduced into the Mathematics curriculum". They saw Computer algebra systems as the "single most powerful tool for compelling change in secondary and university math education in the near future…"

Efforts have been made for 30 years by CAS designers to provide capabilities and user friendly interface. Fifteen years ago, a great improvement was the availability of CAS handheld calculators making possible to use CAS at any moment in the classroom mathematical activity. In spite of the great potential of Computer Algebra and of the designers' efforts, the impact of this technology on most curricula is weak today. Hoyles and Noss (2003) see this phenomenon as a "marginalization of technology" and they think that it "points, in part, to a failure to theorize adequately the complexity of supporting learners to develop a fluent and effective relationship with technology in the classroom".

Designers contributing to this chapter open new paths to get technology out of marginalization, the first one by emphasizing opportunities brought by dynamic technology for teaching challenging but difficult topics and the second by explaining why and how dynamic software should support a visible curricular agenda. The third and fourth authors tackle the necessity of theorizing the learners' relationship with technology especially geometrical environments.

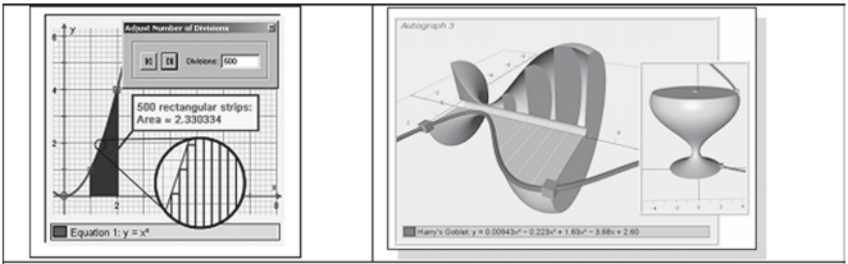
21.2 Potentialities of Dynamic Software for Teaching Challenging But Difficult Topics

Douglas Butler *is the conceptor of Autograph, often regarded as a leading dynamic software for teaching mathematics at secondary level, especially in the UK and making progress overseas. Douglas' concerns as a teacher are deeply reflected in Autograph's design: care is taken to make an efficient tool for classroom use especially by helping students keep the focus on mathematics; topics that Douglas found difficult to teach without technology were specially addressed.*

21.2.1 Making Traditionally Difficult Topics Appear More Straightforward

21.2.1.1 Calculus: Illustrating Integrals, Areas and Volumes

Being able to zoom in on 500 rectangles, under the curve $y = x^2$ from $x = 2$ to 4, illustrates a principle that can be very hard to get across by traditional means. Extend this to 3D and a whole new world opens up.

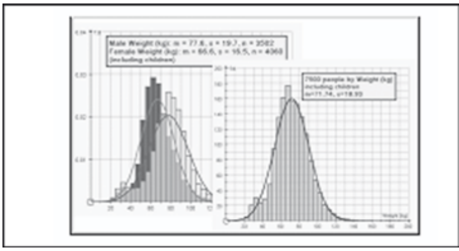


Autograph can explore areas and volumes fully

21.2.1.2 Data Treatment: Understanding the Central Limit Theorem

With so much data readily available, it is important therefore to be able to manage it in the classroom, and to inspire the young to take an interest in how data sets can be interpreted, and how essential they are in everyday decision making.

The basis of a large proportion of data sets is the normal distribution. Why for example, on check-in at the airport, do they weigh our luggage, but not us? The best way to illustrate this is to use the Central Limit Theorem – a nasty theorem to prove, but perfectly understandable if you present it graphically.

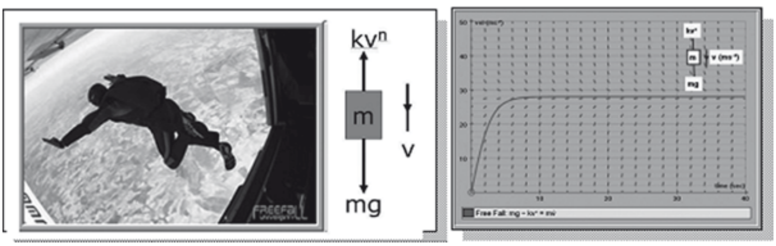


Here is the data from the 7500 patients at a surgery. On the left: male, female and children, and then all together. The total weight of 250 passengers drawn randomly from this population, and turning up for a flight, is very predictable!

21.2.2 Topics That Could be Re-introduced to Mainstream Post-16 Teaching

21.2.2.1 Differential Equations: Seeing What's Going On

Differential equations are usually regarded as the province of only the brightest in high schools. Once the principle of a rate of change of one variable with another has been grasped, together with Newton's First Law, there is no reason why a graphical approach should not be used to bring this topic to life for even modest pupils.



Fascinating applications include falling out of an aeroplane, and predator-prey situations. If they can't solve the equations they can certainly see what's going on – and this is surely the equivalent to the numerical approach that is widely adopted by problem solvers in the 'real' world.

21.2.2.2 Bringing Back the Study of 3D Lines and Planes

The world the students are learning about is 3D, so why not bring the study of 3D lines and planes back? It is very likely that this important topic was lifted up to the advanced levels because it is so difficult to visualise. Not so with the new generation of 3D software such as Autograph and Cabri.

The link with 2D can be enhanced: areas become volumes, arcs become surfaces, reflection in a line becomes reflection in a plane, rotation about a point become rotation about a line, and vectors and matrices simply add a further element.

21.2.3 Making Teaching More Effective and More Fun

We have all watched the phenomenal growth in computer based mathematical technology over the past few years, but the time has surely come for a reassessment of what we teach in secondary mathematics and how we teach it. Up until now the only technology that could be assumed was handheld, and that too is continuing to make strong advances.

With computer hardware becoming more affordable and the opportunities more engaging and exciting, there is a golden chance to save the subject from oblivion: using dynamic software and the Internet can make the teaching far more effective and definitely more fun!

*Related web resources:*http://www.tsm-resources.com

21.3 Software for Mathematical Explorations: Attempting to Make a Curricular Agenda Visible

Michal Yerushalmy *developed with Judah Schwartz the Geometric Supposer in the mid 1980s. It provided solid evidence that it is possible to organize and teach school mathematics in such a way that math students can learn it as they adopt a mathematician's habits of mind. Inspired by this cycle of research and development, her more recent work has been directed towards inventing, designing, teaching and studying the "VisualMath" curriculum*[1] *(1995/2003) which offers new forms of learning algebra and calculus with technology.*

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[1] http://www.cet.ac.il/math-international/first.htm.

A major goal of VisualMath is to help students develop strong modelling and algebraic abilities, learn a variety of standard techniques, develop meaning to signs used and an understanding of the graphical meanings of these techniques, as well as a sense of the purposes for which such techniques are useful. The art and craft of this technology-intensive guided-inquiry curriculum was based on the assumption that any curriculum represents a point of view and that this view could be amplified by especially designed software tools.

An important purpose of the tools was in-line with what Goldenberg (1999) defined as "habits-of-mind orientation" where "a primary purpose of technology will be to help students formulate, express and reason about mathematical ideas;" (p. 212). The orientation of the learning tools was neither to make complex algorithms easy nor to make it serve to reduce the knowledge one needs for manipulating using sophisticated procedures. It does not make solution the central feature and it often provides tools that are explicitly trivial for professionals. The design of the software tools meant to support the design of long-term sequence of learning activities in which the learner jump into what Schwartz (1995) calls the "interesting middle." To do that, the building blocks (the major options offered by the tool) are mathematical objects and processes that are primitive enough to allow construction of new objects by the given processes, but interesting enough to promote uses of higher-order mathematical language, argumentation, and proof.

21.3.1 Clearing the Confusion Regarding the Role of Technology

Many misunderstand curriculum reform with respect to the use of symbol manipulators in teaching algebra and eventually there is still confusion about the role of the four-operation calculator in teaching arithmetic. While CAS just as numerical calculator can support explorations, the design of CAS often delivers the opposite message. Clearing the confusion regarding the role of technology and proficiency in manipulations of equations and inequalities was my major concern.

A central decision of function-based algebra curricula designers is to view any equation and inequality as a comparison of two functions. Numerical (almost-correct) solutions can be read on a graphing screen by reading intersecting points of two graphs or by reading values of zeros of the difference function of the two expressions. However, the major strength of 2D graphing of the two sides of the equation as two functions is its support in viewing the processes involved rather than viewing the solution (Yerushalmy 1999). In the VisualMath curriculum resources and occurrences are designed so that students come to understand what operations on equations are legal ones, and which operations on equations are not mathematically sound and do that while performing manipulations themselves as a way to conjecture and understand "on screen" results. Students are asked to explain how the effect of algebraic operations on the solutions of a comparison depends on the type of comparison (equation, inequality), the type of operation applied to the side/s of the comparison. Software that provides a vertical "rulers" at intersection points, that graphically trace the change of the x-values of the solutions and restricts the free input to well defined algebraic and graphical (e.g. translations and stretch of each or both graphs) operations on the comparison is used to enable explorations of operations on both functions. This design could well be interpreted as awkward and restrictive in comparison to the slick, transparent, and quick operation favoured by "solution tools". But this design is meant to support the construction of a visible map of the point of view of the curriculum.

Several studies of the VisualMath students have been carried out. They suggest that the use of multiple representation technology does not at all omit the structural ideas of expressions from the study of algebra but rather introduces a new style of activities that have a chance at introducing important ideas. A special attention was given to the ways that the tools for explorations may support less successful mathematics students learn inquiry-based curriculum that demand creativity and flexibility. We found differences between the work of these less successful students and the traditional problem-solving patterns of less successful students. The less successful students used the graphing software to obtain a broader view, to confirm conjectures, and to complete difficult operations. However, their process of reaching a solution was found to be relatively long and they delayed using symbolic formalism, and most of their solution attempts focused on numeric and graphic representations. Comparing VisualMath students and equations based algebra students solving algebra problems in context, it was found that the students who were successful students of a traditional algebra sequence which focus on unknowns and stress paper and pencil manipulations procedures were substantially less capable than the function-based students to solve these problems.

21.3.2 From Bodily Actions to Symbolizing and Meaning Production

Going beyond the work in multiple representation systems, technology has proved to be a powerful tool for physical interaction. Thus, another important goal of the design was to emphasize relation between bodily actions, artifact mediated activities and the processes of symbolizing and meaning production. The capability of artifacts to be part of semiotic mediations, support experimentation with temporal processes by means of embodied actions, and turning these processes to produce mathematical symbols of space and motion has played a major role in various stages of learning in technology-intensive reform curricula. Using Microcomputer-based laboratory software (MBL), students study the graphs of the process as it changes in time and develop narrative to connect the actions of the situation with the features of the graphs. A planar movement of the hand motion, a motion of an object or an operation of pre-designed simulation provides the input that appears on the screen as a graph of one or two-dimensional path. In order to abstract the data plotted by the software but to still reflect the essential physical actions of the learner, a set of graphical icons is designed in the Function Sketcher environment. The different components of the lexical system (a set of seven icons and a limited verbal list of function properties is designed) are eventually adopted as manipulable objects that support students when solving problems that are too complicated for them to describe symbolically. This intermediate bridging language helped to form a mathematical construction with language that developed from acquaintance with physical scenarios. It supported the abstraction of everyday phenomena into a smaller set of mathematical signs that are manipulated with software tools as "semi-concrete" objects. The tools were designed based on Vygotskian's notion of semiotic mediation, according to which cognitive functioning is intimately linked to the use of signs and tools, and affected by it. Algebra beginners, advanced algebra students and calculus students (Botzer and Yerushalmy 2006) all benefited from the perceptual and visual resources and from the direct and sensual manipulations of graphs.

21.4 Attention to Detail: Broadening Our Design Language

***Nicholas Jackiw** is the original designer and developer of The Geometer's Sketchpad, an educational software environment for the creation, visualization, exploration, and analysis of mathematical models. Sketchpad is an example of effective software, which has made a successful transition from academic research lab to a wide commercial impact. In his contribution he looks at how different perspectives shed specific light on technology design in the field of mathematics education and calls for conceptualization that could help practical decisions in design.*

The perspectives that have dominated the past 30 years' discussion of educational technology design – are largely either of curriculum design and learning science. In curriculum design, technology artifacts – software, devices, and so forth – are seen as relatively thin wrappers around some essentially curricular innovation, and the task of designing them is accounted for by (traditional or novel) practices of curriculum materials development. We celebrate, for example, the return of Euclidean construction to curricular prominence through the vehicles of Dynamic Geometry Environments, and read the latter as a tool for accomplishing the former. From perspectives of the learning sciences, technology concepts, for example programming, debugging, recursive language constructs, are first metaphors for cognitive processes and mathematical practices such as mathematical modeling, problem solving and metacognition. The work of design is to transform these metaphors into actual laboratories for culturing those processes and practices with students. But attention to technology here remains focused less on details and particulars of software packages than on the generalized perspective of an entire technology milieu's potential to enable broad and significant new forms of epistemological activity, such as student programming in LOGO (Papert 1980), or direct manipulation of mathematical constructions through the unbounded parameter space of Dynamic Geometry Environments (Jackiw 1991; Laborde et al. 1990).

21.4.1 Design Detail Counts

While in no way desiring to sleight the importance of these perspectives to our collective effort's past and future, in 20 years ongoing work designing *The Geometer's Sketchpad*, I confess that I find their insights more useful at macroscopic and generalized levels than I do in their contribution to countless specific and practical design considerations that I – and I assume most other designers – encounter every day. And their common tendency toward generalization often passes over interesting – and to my eye, important – design detail. In the hundreds of articles on Dynamic Geometry Systems that claim to scope both *Sketchpad* and *Cabri*, where is the analysis of how these two programs' differ absolutely in the most basic mathematical language they offer for summoning objects into existence – of how the grammar and syntax of their respective user interfaces are almost conceptual mirror images? In lumping *Maple* and *NuCalc* under the common label of Computer Algebra Systems, do we erase the cognitive difference between, on one hand, the transformation of inputs to outputs by writing computer programs (*Maple*'s interface for factoring), and on the other, the embodied, direct manipulation of mathematical expressions (*NuCalc*'s interface for the same)?

From a mathematical perspective, the constructed triangle or factored quadratic is the same at the end of these separate technological trajectories; but from a learning perspective, certainly *how* we construct, or *how* we factor, matters. In seeking always to generalize, to identify the deep structure and the common DNA, our analyses risk mistaking marmots for mammoths. Even if we hesitate to sign on to the creed that design is *everything*, design certainly acts as the first doorway and first doorkeeper to any deeper curricular or epistemological innovation an educational technology might offer. For it is not at the structural level, but rather on the surface – at the designed *interface* – that users interact with technologies; that meanings are negotiated; that cognitive, psychological, educational, and social transformation may, or may not, occur.

At a deep structural level, for example, a world wide web browser is no different than a Gopher agent or a humble FTP client: all are software programs allowing users to browse resources made available by others within a community united by a network whose basic infrastructure has been unchanged for 25 years. But of course, such a summary perspective fails to account for why the first of these technologies has transformed the world, where the others remain only niche tools for a tiny population of techno-cognoscenti. We can only understand that phenomenon by inspecting how they are different, rather than how they are the same. To the degree our work in mathematics technology aspires to educational influence at significant scale, rather than just to the pleasure of small, pre-qualified technological elites, we have first to admit that design matters – that specific design matters, specifically – and, second, to develop a much richer discourse for design analysis.

21.4.2 Well-Developed Design Discourses from Which to Draw

The good news is that beyond our own field there are many well-developed design discourses from which to draw. The industrial and graphic design communities have perhaps the strongest emphasis on the use value of design, on the means by which design limits and enables interpretation and consumption of objects, and on the procedures and techniques for shaping design contours toward intentional effects. All of course are relevant to any educational design enterprise. There is also the long-standing tradition of viewing design in relation to craft, where design is seen as the domain less of trained engineers than of skilled artisans and inspired artists. Here theories of the nature, purpose, and execution of design point toward (encultured or ahistorical) theories of aesthetic and emotional response. Any of us who has delighted in a new program feature or cursed the obtuse logic of an operating system must concede the irresponsibility of ignoring aesthetic relevance. Finally, the recent field of human computer interaction, and within it, the study of interaction design, focuses precisely on the way in which software signifiers are consumed by users, and on how users' conceptual models of technology artifacts grow and change in response to interaction. These design discourses are of course wildly heterogeneous: they both come from, and lead in, different directions. But they unite, in contrast to our own field's traditional analyses, in their willingness to engage with the *specifics* of a particular object's design, with its appearance and form, its motivations and mechanics. It is from them that we learn design is first and foremost in the details.

21.4.3 Paradigms of Embodied Interaction

Such insights are beginning to find root in our own critical discourse. As one promising recent example, Sedig and Sumner (2006) develop a detailed taxonomy of specific techniques for interacting with visual mathematical representations, drawn from an analysis of several dozen educational and professional software packages. They consider less the higher-level purposes of these programs ("manipulate a construction interactively," "solve an ODE graphically") than the intentionally designed surfaces and affordances of them as physical artifacts (the actual language of menu commands, the specific behavioral response of an object to dragging, the deliberate choice between an iconic and verbal representation of potential action). From this survey, and drawing on literature from both computer-human interaction and design psychology, they argue convincingly that all such techniques fall within three basic paradigms of embodied interaction: conversing (the metaphor of the mouth and speaking); manipulating (the metaphor of the hand and grasping); and navigating (the metaphor of the feet and walking). Such a framework lets us usefully "read" a new mathematics technology in terms of its interaction paradigms, and quickly see, where our traditional epistemological and curriculum perspectives might view two technologies as generically similar if not the same, how from the semiotic perspectives of learners and users they may function entirely differently. I look forward to the growth of this type of detailed design critique and argument in our field, both for its analytic value and for its ultimate impact on the tools we find ourselves creating.

21.5 Designing a 3D Dynamic and Interactive Environment

Jean-Marie Laborde *started in 1985 the creation of a rough book for geometry: "Cabri-géomètre". Since then, he brought together researchers in computer science, didactics of mathematics, mathematics, and psychology, and several very successful versions of Cabri were released. In September 2004, at Cabriworld in Roma, he presented, a completely new product Cabri 3D. Drawing on this recent experience as a designer, he focuses on the specifics of direct manipulation and direct engagement in a 3D environment supporting mathematics learning.*

The problem of direct manipulation and direct engagement in a 3D environment arose probably soon in the head of people having started developing dynamic interactive environments in the early 1990s. I applied at my university for a grant to support Research and Development for a 3D type of Cabri. The proposal was rejected because being perceived as too "trivial". For the mathematicians who evaluated the proposal, there was essentially no issue, because designing a 3D Cabri would simply be adding a coordinate to an existing system and going back to an already solved problem… In contrast for people interested in developing new tools, not already existing, in the spirit of direct manipulation and direct engagement, it was an exciting domain with so many new issues to explore. I will look at some of these issues from different points of view.

From a mathematical point of view: 2D environments raise mathematical questions partially solved, although yet no entirely satisfactory, for instance the intrinsic numbering of multiple intersections or the actual implementation of the reversibility principle.

Intrinsic numbering, e.g. of two conics in the same plane, consists in finding a way to attach labels to the various intersections in such a way that the labels do not "jump" among the various intersections, i.e. such a way that a given label is all the time attached to the "same" intersection point, when the configuration of the curves change following the movement of parent objects (actually many "free" or low-cost dynamic Geometry Software do not handle this problem properly even for simple cases as line/circle intersections).

Among the very universal design principles for Machine/Person interface there is one stating that any action should be reversible, allowing the user to take back her/his last mouse action in executing it in the reverse direction and then getting back to the preceding system status. At the same time it is desired that the system changes "continuously" in respect to the mouse moves, in other words that the system does not "jump" from one state to another substantially different. It can be shown that reversibility and continuity cannot coexist globally on the input domain, say the plane of the mouse. So any actual system has to "compromise", in favoring, according to some more or less external cognitive (or ideological…) principles what will be its behavior (Laborde 1999, 2001). Theses questions are harder and thus more challenging in 3D. Along the history of mathematics, mathematicians (e.g. H. Poincaré) already noticed that, beyond regularities, many mathematical questions become more complex when passing from 2D to higher dimensions.

From a user point of view: The challenge is to find the right metaphors to help people reinvest their existing body of knowledge in order that they feel familiar with the new environment. Obviously choosing the mode of perspective representation to adopt is important. Many modes exist and, depending on the culture, some are mode familiar to the user. In Eastern culture Cavalier perspective is extensively used compared to the so-called Western culture. Cavalier perspective, forcing the user to see at a scene from two points a view at the same type (a frontal and an oblique) suffers a clear handicap especially when non static scene are to be represented: an ordinary cube represented in Cavalier perspective (something easy to draw) does not keep, apparently, its cubical shape when rotating around its vertical axis of symmetry. In contrast the “same” cube rotates comfortably when represented in conic perspective (I would love to say “per definition”). In Cabri 3D we decided to use conic perspective as default perspective. Precisely, objects are represented as they would be seen in the hands of the user at a distance of 40 or 50 cm from his/her eyes. We call this perspective “natural”; it is very different from the perspective often used by 3D software - like graphical spreadsheets - where the perspective is exaggerated for questionable aesthetic reasons.

Since 3D movements are to be performed by way of a 2D pointing device, non-trivial decisions have also to be taken relatively to how a user can drive points in space. He or she must feel “at home” while moving objects within the scene. Most of the pointing devices are 2D devices… Pointing device is evidently a key to direct manipulation. For direct manipulation, as introduced by the engineers at the Rank Xerox Research Park in the late 1970s essentially for their desktop metaphor, an ordinary mouse is OK. For 3D one could think of 3D pointing device; they exist and are still quite expensive; one could also think of user full immersion in a 3D virtual reality environment. To keep technology affordable and widely available, we decided for Cabri 3D to stick with ordinary 2D pointing devices and make use of the old typewriter metaphor: pressing the Shift key actually causes a vertical motion of the carriage. In Cabri 3D moving the mouse normally produces a movement of the dragged object in a horizontal plane while pressing the Shift key changes this into a movement along the vertical axis.

From a computer science point of view: Designing a 3D environment brings issues also for a work in Computer Science. The first one is the rendering of objects, an issue already raised by Hilbert and Cohn-Vossen in their book “Geometry and the Imagination” around 1922. In Hilbert and Cohn-Vossen’s book, points or lines are far from being “ideal” infinitely thin objects: they are depicted as serious spheres and solids rods, actually shown as slightly converging lines, due to conic perspective. Adding fog in the scene in order to increase the depth perception has also been for Cabri 3D an innovative idea. Finally I will only mention here the particular issue about decisions to be taken with regard to infinite objects: how to dynamically represent a plane. Because a plane is infinite and in most cases covers the domain view (the screen) some 3D environments decided no to limit the representation and then, practically, do not display any representation for planes. It does not seem to me, from an educational point of view, very reasonable.

21.5.1 A Vision: Technology to Operate an Epistemological Shift

Until the seventeenth century Geometry has been the queen of sciences and then decayed. Why this decay? It might be because of the too poor quality of drawing at that time, making "formal" approaches more efficient. For instance, although he used methods for reporting on a front plane the intersection of a circle based cone and a plane, Dürer saw this intersection as some kind of egg rather than as an ellipse. With new computer based tools geometrical thinking can return to be a central source of insights when exploring new domains of knowledge and modelling. This movement is already visible in the way engineers not only use computer aided design to conceive their systems but also heavily rely on software environments where geometry modeling is central.

I would then favor the idea of technology tools as directly impacting knowledge contents and shaping them backwards. For me, Maths is a multiformed body of knowledge and culture. Current Technology Environments like Cabri and others are making easier to widely share this math culture.

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Chapter 22
Connectivity and Virtual Networks for Learning

Celia Hoyles, Ivan Kalas, Luc Trouche, Laurent Hivon, Richard Noss and Uri Wilensky

Abstract We present papers that indicate the potential and challenge of connectivity within or between mathematics classroms.

Keywords Collaboration • Virtual networks

22.1 Introduction

Celia Hoyles Digital technologies are already changing the ways we think about interacting with mathematical objects, especially in terms of dynamic visualizations and the multiple connections that can be made between different kinds of symbolic representation. At the same time, we are seeing rapid developments in the ways that it is possible for students to share resources and ideas and to collaborate through technological devices both in the same physical space and at a distance. Given that these developments are becoming more and more available to all students as the Web becomes increasingly accessible across the world, ICMI Study 17 was keen to explore the potential and challenges for mathematics education of these new levels of connectivity, both within and between classrooms. It was envisaged that there would be considerable impact on teaching and learning in the short,

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medium, and long term. A theme of connectivity and virtual networks was therefore set out in the original plans for the Study and the following questions presented as guides to the submission of proposals within this theme:

- How can theoretical frameworks and methodologies developed for interpreting activity, learning, and teaching in technology-integrated classrooms be extended to assist in understanding the distance-learning context? What kinds of changes and refinements are needed?
- What is the potential contribution to mathematics learning of different levels of interactivity and different modalities of interaction, and how might this potential be realized?
- What is special about the potential of physically separated collaborative study of mathematics, and how might this potential be harnessed so as to support mathematics learning?
- What is the potential for creating virtual communities for mathematics learning and permitting communication between individuals from different educational settings?

In the event, there were rather few papers submitted to this theme, no doubt due to the fact that rather little research had been completed at that time around the impact of connectivity on mathematics teaching and learning. But rather than drop the theme, a group of researchers in this field was invited to participate in a plenary panel at the Study Conference – either at the meeting itself or through a video link: as mentioned in the introduction to this volume this latter mode was considered to be particularly appropriate for this panel as an illustration of the potential of this form of communication.

This chapter comprises the papers written by the four panelists following from their contributions to the plenary panel. There are common threads running through the papers. All point to the importance of design: of the technical aspects that shape what students can do with the technology, what they can share and how they can interact; and of the activities themselves, how they exploit connectivity and stimulate student participation. Some contributions describe experiments that take advantage of connectivity within one classroom while others focus on between-classroom interactivity. In both scenarios, teachers' actions in supporting new communities of practice are recognized as crucial, and new roles for the teacher are noted while acknowledging that these roles had as yet been under theorized.

To complete this summary, we note that other ongoing research in this area (see for example Hegedus and Penuel 2008) supports some of the ideas presented in this chapter, most notably in suggesting how "networks can link private cognitive efforts to public social displays thus – potentially at least – enhancing students' metacognitive ability to reflect upon their own work in reference to others" (Moreno-Armella et al. 2008). More radically, these authors argue in a similar way to panel members that this type of connectivity means that the introduction of technology will lead – at last – to a real transformation of practice in classrooms. This remains to be seen. There is no doubt that connectivity will transform how students interact with each other – simply consider the widespread ownership and use of the mobile phone – a technology that is truly personal for a rapidly increasing number of students. Yet if and how connectivity, in whatever form, transforms mathematical practices in school is a matter of future investigation. It is clear from the papers in this chapter that design will continue to be a crucial research theme in the future, as it will be design decisions that will shape what can be shared in terms of resources, information, student solutions, or part-solutions. But an even more fundamental theme emerges that concerns how the technology, activities, and teacher strategies together can motivate students to engage in and take responsibility for mathematical discussion of the process by which they construct their own knowledge and the justifications they propose for solutions to mathematical conjectures.

22.2 Developing Microworlds for On-Line Collaborative Learning

Ivan Kalas

22.2.1 Background Issues

In our department we have considerable experience in developing flexible software platforms for learning, such as Super Logo, Thomas the Clown, Imagine Logo and others, and in the process of their development we have tried to create effective opportunities for communication including being able to work in a common learning space. Such spaces have different forms: a physically common learning space in one place like an interactive smart board; a virtual common learning space within one classroom, such as several computers within one classroom with groups of learners collaborating with and between groups, and a virtual common learning space shared over the network. In our research projects we are trying to address questions like:

- What are the properties of a flexible software platform that support the development of microworlds for effective collaboration?
- What are the important criteria for developing collaborative microworlds?[1]

The aim of our CoLabs project (see http://matchsz.inf.elte.hu/Colabs/) was to examine obstacles that obstruct collaborative learning, see for example Turcsanyi-Szabo and Kalas (2005). In CoLabs, we used Imagine Logo as a platform for developing and exploring collaborative microworlds – called *collaboratories* – which would allow children to communicate and cooperate – either locally in one place or through the network among different schools, towns, or even among different countries in spite of the many technical, linguistic, and cultural obstacles.

As one such *collaboratory* we created Visual Fractions – a complex dynamic interactive computer environment, which allowed groups of children to explore and discover fractions and fractional relations, see Fig. 22.1. Visual Fractions provides dynamic jigsaw puzzle pieces for children to build their own understanding of the topic, see Lehotska and Kalas, 2005. The evaluation of the Visual Fractions environment by a group of future teachers suggested that building and exploring these dynamic playgrounds of dependent

[1] In Kalas and Winczer 2006, we presented our attempt to summarize all known aspects in a framework for the development of collaborative microworlds. We do however accept the argument presented in the panel by Hivon and Trouche that a complete list is probably impossible to generate.

Fig. 22.1 Visual fractions: dynamic environment for discovering fractions and fractional relations

visual and interactive representations of fraction objects and relations required (and further developed) the same competencies as programming, see Lehotska (2006).[2]

22.2.2 A Further Example

Since the CoLabs project, we have concentrated mainly on how to provide support for collaboration, again within an Imagine Logo environment. Our ambition is to build an environment that could be offered to teachers, researchers, and enthusiastic amateur developers who want small, immersive, open, interactive, flexible, and collaborative microworlds developed for everyday learning situations.

During this process, we distinguished four dimensions that each needed to be addressed: the technicalities, the connection interface, the aspects to be shared and the features of the activities. We were particularly interested in building proper metaphors that would mediate computational support for collaboration among teachers and learners in the most intuitive and inspiring ways.

Figure 22.2 illustrates an experimental microworld in which several connected users (here represented by letters A, B, C…) own their personal technical panels in which they are given several visually represented parts of a whole, that is a selection of fractions. All users additionally share a common workspace in which they are expected to piece together a given quantity, expressed in shaded circles as an improper fraction. In the example shown in Fig. 22.2, the goal is produce 2 and ¼. The users (in this case four users A, B, C, and D) can bring their own pieces into the common workspace by dragging them into their local representation of that space, or by manipulating the pieces of other children (comparing them, rearranging, or rotating them etc.). However, only a user's own pieces can be dragged back to his/her personal technical panel or individual workspace, to be "weighted" there[3] or divided into smaller pieces and then reused in the common space.

[2]Other researchers (Pratt and his colleagues) used Visual Fractions for other kind of observations more closely related to mathematics education (see, for example, Jones and Pratt 2006).

[3] When a child drags a piece or several pieces one by one (i.e. the visual representations of fractions), into the dark area of his/her technical panel, the environment "weighs" or "measures" them all together and shows the total sum (value) for example 5/12 or 1/6 or 1/6.

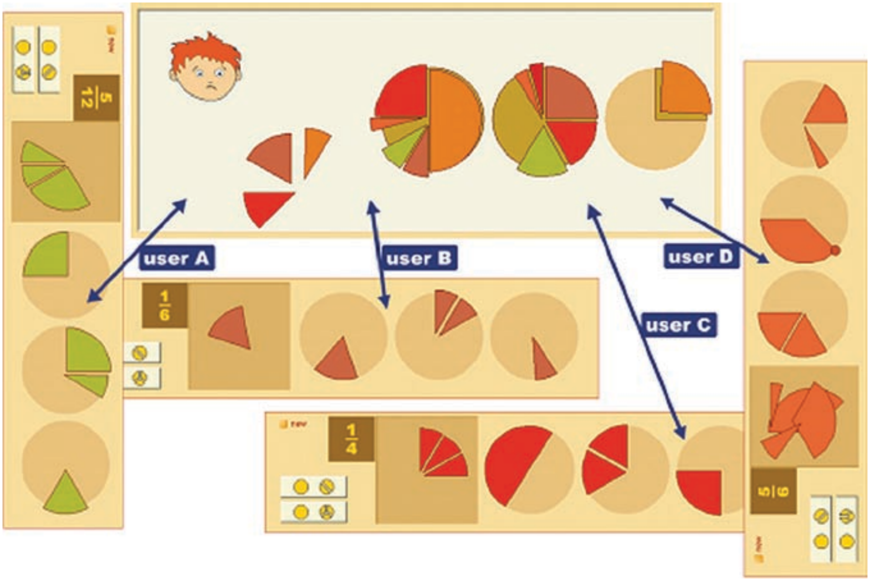

Fig. 22.2 A common virtual learning space (the rectangle in the top middle) shared by four collaborating distant learners, A, B, C, and D

Based on this work, we conjecture that an approach to collaboration that involves constructing common spaces in which children can compose together, explore, construct, communicate, pose, or solve problems can be employed successfully with children from preschool to upper secondary stages; and not only in mathematics learning but also in the development for example of language skills, in art and design, science, and citizenship.

22.2.3 Some Reflections and Observations

In our research on developing microworlds for on-line collaborative learning we have also examined whether digital technologies can motivate children to collaborate and communicate and how specifically designed microworlds support these phenomena. Most of all, we want to identify the critical factors for efficient collaboration, motivation, and engagement in the learning process. Below is a summary of our observations distinguished by some widely held claims (for more detail see Kalas and Winczer 2006):

Claim 1 "Our interfaces for collaborative learning always have an amateur look and are therefore far less attractive than the professional activities and games that many children know and use elsewhere". Although this claim is probably more true for boys than for girls, I wanted to say that we conducted a survey and we found following that children were asked to rate the importance of several aspects of collaborative environments that a *clear and intuitive interface scored more highly* than the professional look of the environment. We thus concluded that an intuitive interface was the key factor in the children's motivation and engagement in the cooperation.

Claim 2 "*The intensity of communication increases during an activity as does its efficiency*". We found on the contrary that the number of brief interchanges of information communicated between the children was high at the beginning of an activity but decreases considerably during the activity, and finally reaches nearly minimal, yet optimal, flow. This can be explained since initially children always explored all the possible communication channels of the environment and exploited them heavily – even without any obvious reason. For example when they discovered that it was possible "to chat" in the environment, they immediately paused their main activity to exchange messages with nearly no content with each participant. Only after these phases did they resume the primary tasks.

Claim 3 "*Competition is important for motivation*". Perhaps surprisingly, we found that competitiveness was not in conflict with collaboration. Rather both phenomena could be stimulated in parallel in activities with two or more competing teams.

22.2.4 Some Concluding Remarks

Although we are rather successful in overcoming a range of technical, linguistic, and cultural obstacles in our experimental collaborative microworlds, we have to admit we still need to find ways to face the hardest obstacle of all, namely the educational obstacles to implementation. It seems to us that our formal educational systems are not yet quite prepared to assimilate computational support for effective on-line collaboration.

22.3 Connectivity: New Challenges for the Ideas of Webbing and Orchestrations

Luc Trouche and Laurent Hivon

22.3.1 Introduction

It is not easy to speak about the implications of connectivity since the word itself calls up a set of connected questions for research:

- What is possible to do for mathematics learning with ICT either face-to-face or through distance learning that transcends just the ability to communicate?
- What are the implications for each learner of the potential of "cognitive connectivity," that is being able to establish links between a situation and an idea and being able to move more or less easily from one mathematical frame to another?

And more generally:

- What are the relationships between what we call *orchestrations* (Trouche 2004; Drijvers and Trouche 2008) – the intentional organization by the teacher of the

Fig. 22.3 TI Navigator system

various tools available in a learning environment, and creativity of the learners who form part in this situation?

In this short contribution, we focus on an environment dedicated to a particular type of connectivity, namely the TI Navigator, providing wireless communication between students' TI graphing calculators and the teacher's personal computer (Fig. 22.3), with activities designed following the collaborative work of a team of teachers and tried out in ordinary classrooms.

We introduce the following questions that guided the investigation:

- How should orchestrations be conceived in order to optimize the chances that the tools serve as efficient instruments for mathematics learning?
- What new difficulties and opportunities become evident for students, using the technology to interact each other, and with the teacher?
- What new difficulties and opportunities become evident for teachers, and what new professional practices are necessary?

Finally, from a theoretical point of view:

- What are the challenges that need to be addressed in new formulations of the two theoretical concepts: of *webbing* – "a structure that learners can draw upon and reconstruct for support – in ways that they choose as appropriate for their struggle to construct meaning for some mathematics" (Noss and Hoyles 1996; p. 108) – and of orchestrations?

22.3.2 Some Elements on the Experiment

Working with INRP and IREM,[4] a team of six high-school teachers near Orleans, France, studied how to introduce and work with the TI Navigator System in their classrooms. The team had two main hypotheses; namely that the integration of this new device into classrooms:

- Would lead to new orientation to mathematics teaching, particularly from the point of view of orchestrations
- Would foster interactions between students, and motivate peer debate

The research began with studying the device and its integration into the French school system (10th grade), sorting out issues of installation and familiarization with the device, and then the design of some specific activities. The research focused also on the development of collaborative work that integrated the new technological tools. The device incorporates many new technical developments, allowing three main configurations:

- Displaying all (or some) of the pupils' calculator screens in quasi-real time (*screen mosaic* configuration)
- Displaying all of the pupils' data, for example, points or curves, in a single coordinate system (*common coordinate system* configuration)
- Displaying immediately the results of a class vote between two (or more) contradictory proposals (*consultation* configuration)

These three configurations have the common property of establishing a *common workspace* on the class screen. The teacher can choose between several ways of using these configurations such as:

- For the screen mosaic configuration, he/she can choose whether or not to display the name of the corresponding pupil on each screen (Fig. 22.4).
- For the common coordinate system configuration the teacher can decide whether or not to give the pupils the option to change their answer, make one or more uploads and whether or not to perform these uploads simultaneously (Fig. 22.5).

To test the two hypotheses of the research, some specific activities (mathematical problems and orchestrations) were designed and tested in five classrooms (Hivon et al. 2008).

[4]INRP: National Institute for Pedagogical Research; IREM: Research Institutes on Mathematics Teaching.

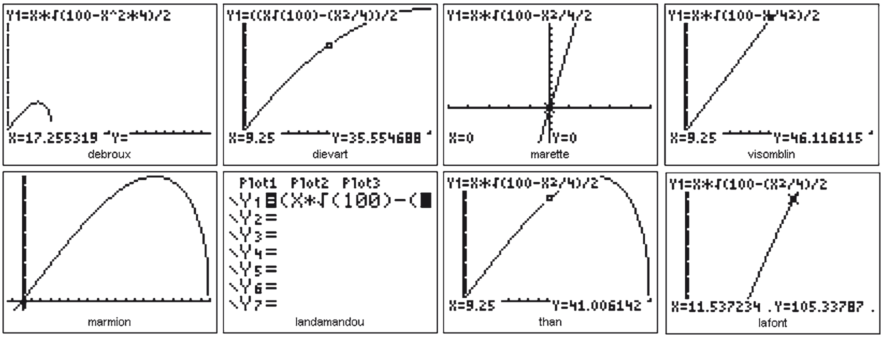

Fig. 22.4 Screen mosaic sent to the common workspace by different students

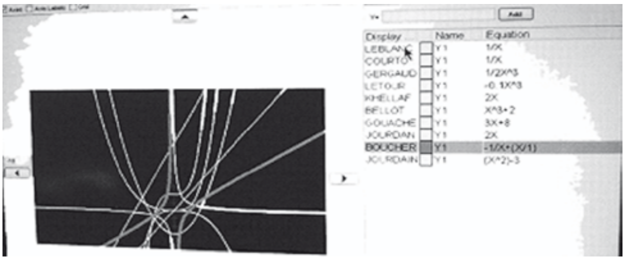

Fig. 22.5 Example of common screen configuration

22.3.3 Some Results

The work with the TI Navigator was found to foster an emergent real community of practice (Wenger 1998) in the classroom in which we could distinguish three fundamental aspects, *participation*, *reification*, and the existence of shared resources, whose major elements are summarized below:

- Participation with the engagement of students in the mathematical activity and debate
- Reification with the collaborative creation of mathematical objects (a good example being the collective creation of the graph of a function that gradually becomes an easily identifiable object, cf. Figure 22.6, see also Wilensky in this chapter)
- Shared resources most notably the public shared board, which is a place where every student can show her/his mathematical creation. Each student is confronted with her/his production and those of other pupils

In traditional classrooms, speech or writing on the board are the ways students can express themselves and share with others, at the request of the teacher. With TI Navigator, the situation is very different, for two main reasons:

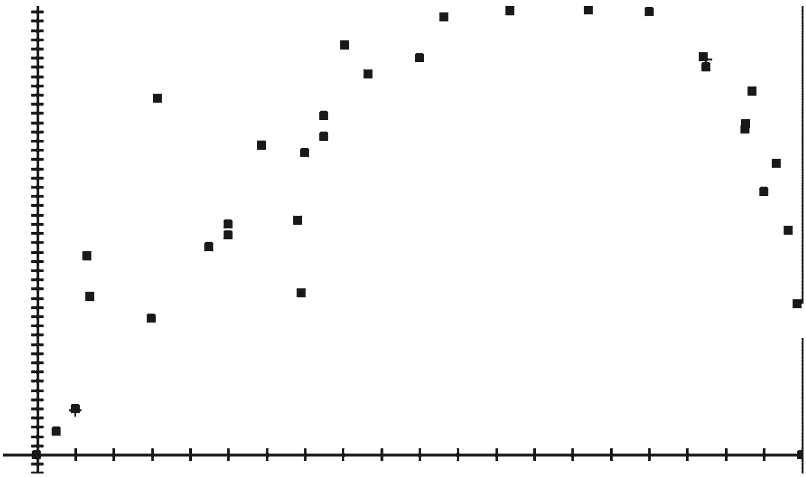

Fig. 22.6 Points sent by every student as a result of a modeling problem, collected in one space

- A new interactivity is fostered between the artefact and the student, and between students themselves: students convey their messages through the artefact, the artefact acts on the students enabling them to distance themselves from their productions thus freeing them to become more easily involved in peer exchanges. Thus the common space becomes a space of debate and exchange that aims to elaborate a "social mathematical truth".
- Each student becomes detached from his/her production as a distance is created between student and the expression of her/his creation and this distance seemed to improve collective reflection on practice. The student becomes involved in the class activity in a different way as the tool maintains this distance between a student and the results proposed to the class and to the teacher.

Thus our first conclusions point to the renewal of relationships and exchanges inside the class. However, other elements must be borne in mind:

(a) Daily use of the device is difficult due to the complex equipment. Thus the device was not often used which has two contradictory consequences. On the one hand each new usage of TI Navigator needs time for re-appropriation, and on the other, these rare moments of use tend to be remembered by all students.
(b) As the responses proposed by the students were often very different and there were many solution processes opened up for discussion, not just the one used in a traditional course, the students often tried to produce the most sophisticated solution they could possibly find.
(c) The use of the devices deeply changed the way of the class had to be managed, that is the way the class mathematical activity was orchestrated. This added a

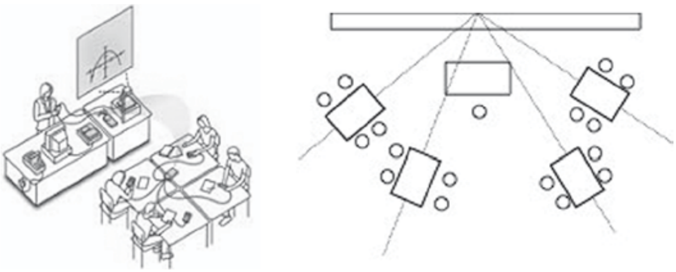

Fig. 22.7 From the intended configuration by the manufacturer to the configuration chosen by the teachers

new complexity to the teacher's work. The complexity of facing the integration of ICT in mathematics education is well known (see Guin et al. 2004), but the necessity to manage both students' tools and the collective tool (the calculator network) makes this integration much more complex. For example, students' activity is deeply sensitive to the organization of the classroom space, for example where teachers change the orientation of the students' desks (Fig. 22.7).

(d) The collaborative work of the teachers involved in this experiment seemed absolutely necessary to face all this complexity and to produce the design of appropriate mathematical situations.

(e) The cross-observations of teachers in their own classes helped them to create a distance from their own practice and to develop a reflexive attitude to the orchestration of students' activity.

22.3.4 Questions to Be Considered in More Depth

In the future, when students will be used to working with the system, we intend to undertake a deeper analysis of students' learning processes in this environment although we recognize the difficulties in doing this for three major reasons:

- *The complexity of the orchestrations:* as a lesson is made up of many stages (personal work, interactions within each group, interactions in the whole classroom and debates), it is not easy to observe the way a single student changes her/his mind.
- *The multiple instruments used in the students' work*: the students use paper, screen calculator, public screens, so it is difficult to know what they do and in which order they do it.
- *The interaction between the phases of classroom work*: we could classify the students' work in the classroom into three stages: first, they expressed their

personal point of view (especially at the beginning of the lesson); second, they expressed themselves as members of their group (they worked in groups of four); and third, they expressed themselves as members of the class. Of course, the three ways are mixed, which adds to the complexity of analysis.

Other questions arose for future investigation are:

(a) *The teacher's behavior and professional development*. S/he created the conditions for students to build a mathematical object, but this object would, partly, be built by the community of students. Thus a student does no longer only plays the music written by the conductor, rather s/he is writing part of the music. The question then has to be faced as to how the teacher can create conditions to make the music not too different from what s/he wanted it to be, or to enrich his/her own partition with the – sometimes – unexpected students' improvizations.
(b) *The teachers' use of the computer*. A recent report of the European Commission[5] showed that French teachers "do not use computers and the internet very frequently and intensively in schools." Could connectivity tools like the Navigator change this situation?
(c) *The sharing of students' conceptions*. In a situation of connectivity, a student constructs her/his knowledge in collaboration with other students. As everyone takes part in this construction, will the others' conceptions help her/him to build her/his own knowledge? How will the students learn to manage this new situation? What influence will it have on the way they build their conceptions of an object?
(d) *The influence of the private practice of connectivity* (blogs, chat, MSN) on how connectivity is used inside the classroom? For example, will this private practice of connectivity make the instrumentalization processes (the way a user appropriates, modifies, a given artefact) more important than in a nonconnectivity activity in the school?

All these questions need to be addressed in new experiments.

22.3.5 Some More General Considerations

This first experiment was derived from a particular context (a classroom in a high school, in a given technological environment), but our conviction, based on other experimentation (Guin and Trouche 2005) is that several elements of this context are more generally relevant, such as to distance learning. These elements are:

(a) The idea of a *common workspace*, for the pupils as well as for the teacher, in which each learner has to orchestrate the part of the game over which s/he is in charge (see also Kalas this chapter). This part is much more important than in an ordinary classroom given there are many results, many mathematical objects and semiotic registers all appearing at the same time on her/his own machine and in the common work place. As we know from students, such an approach appears to motivate,

[5] http://ec.europa.eu/information_society/newsroom/cf/itemlongdetail.cfm?item_id =2888

with mathematics appearing like a game. But of course it must be recognized that this is not the case for all the pupils, as some will remain passive. Also the teacher's role is complex as s/he has to manage the different instruments used in the classroom, as well as the collective instruments (in our experiment the network of calculators). The question of time is crucial: many results appear very quickly and the teacher has to make didactical choices swiftly and all the time. We have thus observed, as also noted by Noss and Hoyles, this chapter, in most of our experiments, that *successful knowledge construction is critically dependent on teacher intervention directly to facilitate, encourage, and foster interactions*.

(b) The idea of *collaboratories*. Connectivity and collaborative work are strongly connected. Connectivity enriches and is enriched by collaborative work, both among learners and among teachers: that is to say, the emergence of a community of practice is a condition for connectivity to work, while connectivity in turn facilitates such communities to emerge.

(c) The possibility of *building a detailed map of all aspects important for developing collaborative microworlds* (as suggested by Kalas in this chapter). It is certainly possible to suggest some features for developing collaborative microworlds, but agreeing on all aspects is certainly impossible. Rather it must be recognized that some aspects are necessarily dependent on the community using them and many are simply not predictable. Therefore, there is a necessity for flexible adaptive environments. Behind this, there are ideas of distributed design, between designers themselves and users, what Rabardel (1995), French ergonomist, calls *conceptions in use*; the need to rethink the notion of orchestration and the notion of webbing (see Noss and Hoyles 1996). As Hoyles et al. (2004) point out, these two metaphors are not referring to the same thing and are not exactly at the same level. On the one hand, it is important to have in mind a necessary assistance (the notion of orchestration) of students' mathematical activity, and on the other, it is crucial to let the students free to think and establish connections (the idea of webbing): Hadamard (1954), a French mathematician put in evidence some extraordinary moments of *illumination*, based on very quick internal and external connections. For example, the image of the concept of function, as a teacher said, appears, for each student, at once, as the result of the sum of the contributions of the whole class. It is a sort of depersonalization: the object is no more on my screen; it is in the common work place, enriched by all the community. But it is certainly the result of a given mathematical situation and of a particular orchestration by the teacher, which makes necessary new processes of *documentation* (Gueudet and Trouche, online) for teachers.

In this sense and as a summary of our work, connectivity raises many new didactical challenges for the teaching of mathematics.

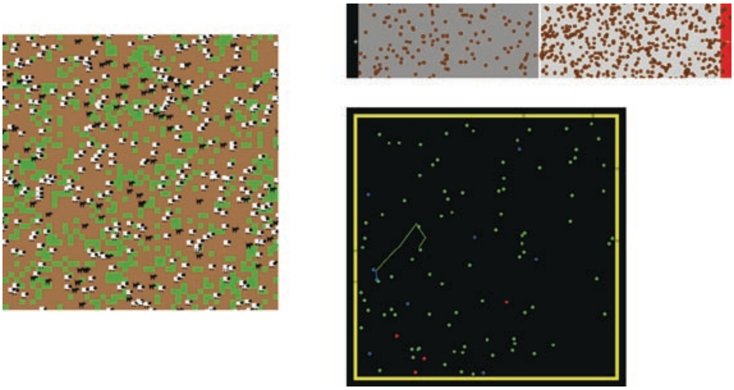

Fig. 22.8 NetLogo models of predator and prey, electricity, and ideal gases

22.4 Concurrent Connectivity: Using Netlogo's Hubnet Module to Enact Classroom Participatory Simulations

Uri Wilensky

In the panel, I presented an outline of our 20 years of work with agent-based modeling and NetLogo (Fig. 22.8) (Wilensky 1999) and described how this work can be enhanced though classroom connectivity.

Much of the discussion of connectivity in education has focused on the potential of asynchronous collaboration and distance learning. Moreover, the vision for connectivity is usually about connecting people who are geographically separated and need such connectivity in order to work together. But, there is another, more neglected affordance of connectivity: the ability to give people a shared interactive experience in a classroom context. This use builds on gamelike scenarios wherein players interact with each other in a simulated world. Such games have great holding power for children and that same holding power can be leveraged for educational benefits in the classroom.

In our many years of working with NetLogo in middle and secondary classrooms, we have endeavored to bring to students descriptions of complex systems at a micro-level and connect those micro-level descriptions to macro-level and observable phenomena. Typically when we have taught students about systems that can be construed as complex, we have concentrated on aggregate equations that summarize system behavior. For example, to describe the behavior of ideal gases, we rely on equations such as $PV = nRT$. But agent-based modeling enables students to more directly control and examine the behavior of elements of the system and connect this behavior to the system emergent behavior. Thus in NetLogo's GasLab model suite, students come to understand the ideal gas as composed of myriad interacting gas molecules and see that $PV = nRT$ is an emergent result of these interactions.

There are hundreds of NetLogo models we have used in classrooms. Students examine a wide range of phenomena such as the spread of a disease through a population, or the interactions of predator and prey in an ecosystem or the flow of electricity through a circuit or traffic on a highway, etc. There is considerable research that shows that it is hard for students (and people in general) to reason about such systems (Centola et al. 2000; Penner 2000; Wilensky and Reisman 2006; Wilensky and Resnick 1999). We have argued that this is largely because the aggregate descriptions do not shed light on the mechanisms of action and, conversely, it has been impractical to have students do the extensive computations required for the micro-level approach.

The use of agent-based modeling (ABM) has changed the terms of use – both in scientific practice and for classrooms. ABM languages and environments enable students to focus on the systems parts and their interactions and to rapidly compute the emergent results and experiment with a host of alternative scenarios. In recent years, a number of ABM-based curricula have been developed that have been quite successful in classrooms, especially at the secondary school and university levels (Abrahamson and Wilensky 2002; Blikstein and Wilensky 2005; Levy et al. in press; Sengupta and Wilensky 2005; Stieff and Wilensky 2003; Wilensky et al. 2006).

However, despite considerable efforts to “lower the threshold” of entry into agent-based modeling, it remains difficult for elementary students to master both the programming and modeling skills needed. A remedy for this that we and others have tried is to have the teacher present and explore a model with the entire class. This approach has considerable merits, but it leaves the student somewhat passive as only a few can be engaged at any one time and they are limited to discussion of model behaviour.

One possible solution to this dilemma is to enable students to collectively participate within the simulation, controlling elements of it and collectively observing and discussing the results of their actions. This approach enables all classroom students to be simultaneously active while giving them an experience of a complex system that they all share. It also empowers them to try to change the system by their actions and to see how much they can affect the system and how much they are constrained by it.

To accomplish this aim, we added a networked architecture to the NetLogo software. This added module, HubNet (Fig. 22.9) (Wilensky and Stroup 1999a), enables a host of devices to connect to a NetLogo simulation and control agents within that simulation. We designed HubNet to be able to accept a range of client devices, including computers, graphing calculators, handheld devices and phones. All of these devices have bee implemented with HubNet, but the two most robust client devices are full computers, which use the computer-HubNet interface and TI graphing calculators, which use the Calc-HubNet interface. We worked with Texas Instruments for many years on networked calculator products which has led to the current TI-Navigator interface which includes HubNet activities.

By adding synchronous connectivity to NetLogo, the modeling activity is transformed into a participatory simulation (Wilensky and Stroup 1999b). This

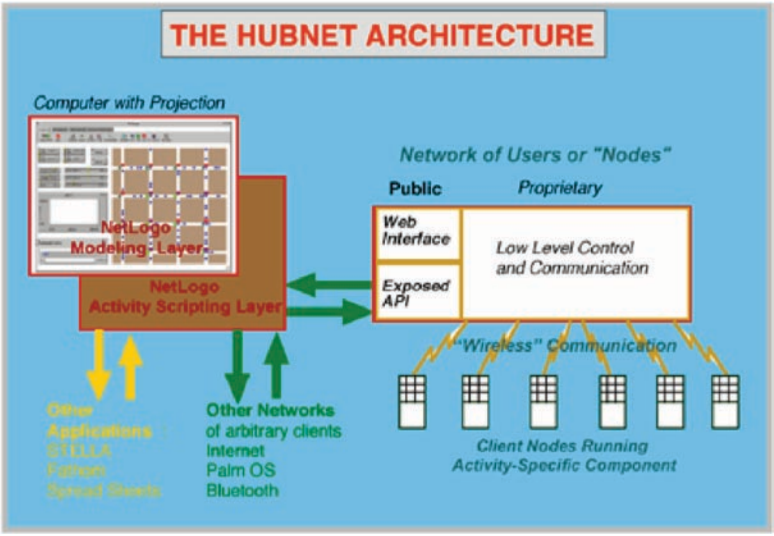

Fig. 22.9 The HubNet architecture

transformation has several important benefits for learning. For example the modeling activity:

- Becomes more engaging – especially for younger learners. It becomes a social activity and captures much of the same draw as online games.
- Promotes greater student participation. Every student can be actively involved at the same time. Because they often require continuous action on the part of the students, they are "in-the moment" motivated to participate. Such universal participation is very hard to achieve in a traditional classroom.
- Enables a shared experience of a complex system. There are very few opportunities, in the classroom or in life, for students to collectively witness the same complex system unfolding. Focal attention to such a system is hard to achieve outside of the virtual and, even when achieved, if the viewing does not connect the micro-level behavior to the macro-level outcomes, then only the appearance is shared, not the mechanisms of action.
- Facilitates classroom discussion of the system and examination of "what-ifs." Student can suggest experiments with varying critical system parameters and/or agent-rules, hypothesize the observed behavioral change, run the simulation and refine the experiment.
- Scaffolds individual modeling and analysis. Once students have experienced several opportunities to collectively model and analyze complex systems, they become much better prepared (and motivated) to conduct such inquiry on their own. Often students have suggestions for model experiments that are not

explored in class. These questions are potent seeds of further student inquiry, experimentation, and model revision.

NetLogo comes with a bundle of HubNet activities. At Northwestern's Center for Connected Learning and Computer-Based Modeling (CCL), we have authored many of these and tested them in classrooms. We have explored a wide range of content domains and simulation forms including simulations of ecologies, economies, disease transmission, traffic patterns, and many more. Some of these activities can be freely downloaded with NetLogo from ccl.northwestern.edu/netlogo. Many more are in classroom tests and in development.

22.5 Designing for Exploiting Connectivity Across Classrooms

Richard Noss and Celia Hoyles

This section is based around two large-scale projects that have occupied us for much of the last decade: the Playground project and the WebLabs project, both funded by the European Union, and co-directed by ourselves. Both projects set out to investigate ways that students could be motivated to collaborate while physically separated.

22.5.1 The Playground Project

In the Playground project (Noss et al. 2002), we attempted to tap in to children's games culture by adding a new dimension whereby they built their *own* games. The central idea was to design and try out computational worlds – playgrounds – in which the objects in a game *and* the means for expressing them are engaging; where the programming of a game is itself a game (we used *ToonTalk*[6] as the major programming environment, and we also created an icon-based language of our own, called *Pathways*[7]).

Fig. 22.10 The stones combined into rules for a monster

[6]See http://www.toontalk.com

[7]This prototype system was subsequently published as *Magic Forest*. See http"//www.logo.com/cat/view/magicforest.html

We set ourselves the task of working with young children (aged as young as 4 and at most 8) where it was obvious that we could not rely on the written word as a means of communication. This challenged us considerably and forced us to take seriously other modalities of interaction, such as speech as well as direct manipulation.

Children populated their games with objects which had "behaviors" – sets of rules that determined their actions. Behaviors were defined using collections of iconic rules, which could be viewed by opening a "scroll of paper" attached to the object. Each rule was expressed as a visible "sentence" or string of graphic icons which combined a condition and a series of actions to be executed whenever the condition was true. The icons representing the conditions and actions were represented as "stones," small concrete manifestations of the concepts that could be strung together to constitute a rule (see Fig. 22.10). Action stones had a convex left side so that conditions with their concave right sides could naturally fit to their left. Any object could accept any number of these iconic rules, all of which would be executed in parallel whenever the conditions for their execution were satisfied. Figure 22.1, for example, illustrates three rules for a "monster." Pathways provided 13 conditions and 25 actions, together with a wide range of object parameters (such as speed and heading) that could be set by using sliders and other manipulable tools. Pathways also included predrawn objects, backgrounds and – in the final version – a mobile phone icon that allowed players to send messages to each other. Objects could be edited (e.g. size and color changed), copied, deleted, and pasted. For examples of children's activities with these rules, see Hoyles et al. (2002).

We gave the children the opportunity to construct creative and fun games (see Fig. 22.11), and at the same time, offered them an appreciation of – and a language for – the rules which underpin them, and the mathematical structures that they had to engage with in order to make their games function. The motor for this latter stage was that one group of children would share their games either face-to-face in their own classroom, or with another group in a remote classroom, either synchronously or asynchronously using the Web. In this latter scenario, the remote group could comment on the game, and amend and extend it as they saw fit by changing the

Fig. 22.11 A space game built by children, involving the monsters from Fig. 22.10, together with new elements (spaceship, scoring – see *top left* and *right* – and a space background)

rules, introducing new ones, and typically, merging existing objects (and their corresponding behaviors) into the games to add complexity and interest.

Our findings confirmed that while working both face-to-face and remotely on their games, children could collaboratively explain phenomena arising from rules we characterized as either *player rules* (an agreed regulation), or *system rules* (a formal condition and action for the behavior of the game). We found that in face-to-face collaboration, the children centered their attention on narrative, and addressed the problem of translating the narrative into system rules which can be programmed into the computer. This allowed the children to debug any conflicts between system rules in order to maintain the flow of the game narrative.

When we added remote communication to the system by enabling the sending and receiving of games from within the *Playground* system – we found that children were encouraged to add complexity and innovative elements to their games, not by the addition of socially-constructed or "player" rules but rather through additional system rules which elaborate the formalism (games were created using two different kinds of programming system, neither of which employed textual modality. This shift of attention to system rules occurs at the same time, and perhaps as a result of, a loosening of the game narrative that is a consequence of the remoteness of the interaction.

This phenomenon was particularly evident in the case of asynchronous interaction where, stripped of even the semantics of gestures, our extremely young students found it increasingly natural to try to communicate meaning via the various formalisms we provided. Thus a key historical claim for programming, that it offers a key motivation and model for immersion in a formal system, came to life as children struggled to modify and add rules of their programs that achieved the effects they desired. And it is worth stressing that asynchronous communication, while somewhat less attractive to the students at the time (we should not underestimate the impact of online synchronous video communication, in 2000, with children in other countries), allowed students to reflect on, and therefore use more effectively, the formal rules of their games.

The Playground project left us with a modest set of corroborative data that leads to the general conclusion that online collaboration catalyzed some interesting outcomes. The shift from narrative to system/formal rules does, in fact, seem to be a direct result of the necessity to formalize in the absence of all the normal richness of interaction that characterizes face-to-face collaboration. Moreover, the contrast was all the more vivid when we compared the children's later work with their initial constructions, in which the narrative was clearly foregrounded, and the focus of attention was necessarily the translation of the narrative into a form that the computer could accept. This initial form of engagement made it possible to debug the system rule conflict that occurred. There were, inevitably, some difficulties. First, we noted that harder games did not necessarily mean harder mathematics – sometimes the games simply became more complicated rather than more complex. Second, peer-to-peer connectivity was severely limited in scope for knowledge building and sharing (the project began in the previous century!).

Fig. 22.12 A WebReport. Rita (in Portugal) has challenged other students to find the mathematical function that generates her sequence. Nasko (in Bulgaria) answers in an interesting – and surprising – way

22.5.2 *The Weblabs Project*

In a second project, we decided to address these issues directly. The *WebLabs* Project (http://www.weblabs.eu.com, European Union, Grant # IST-2001–32200), aimed to explore new ways of constructing, expressing, and sharing mathematical and scientific knowledge in communities of young learners. Some of the ideas we asked students to engage with were indeed sophisticated (for example, the convergence of infinite sequences and the properties of infinite decimals). Once again, we used ToonTalk as our primary platform for construction, building open toolsets for students to construct models, and supplementing these with other appropriate tools as necessary (for example, *Excel*). From the point of view of the panel and this chapter, a key focus was our ambition to design and build a web-based collaboration system for sharing and discussing student constructions. This was considerably more sophisticated than the primitive system of sending files back and forth as in *Playground*, and consisted of a set of functionalities – named *Web Reports* (http://www.weblabs.org.uk/wlplone) – that allowed models to be shared and included prompts for students to add comments, conjecture about the best approaches, and most efficient models (see Fig. 22.12 for an example). It is worth noting that this project was ahead of its time: had we engaged in this a few years later, we would have been able to tap into web 2.0 applications that are now taken for granted: essentially we had to invent a genre of mathematical social software! (While we were constructing WebReports, we became aware of Knowledge Forum, and drew inspiration from this work: for recent work, see, for example, Scardamalia and Bereiter 2006).

WebReports allowed students to embed seamlessly their models in free-form text documents and publish them on the web. Thus the central tenet of the approach was that students simultaneously *build* and *share* models of their emerging mathematical knowledge.

Our pedagogical approach was based on encouraging students to propose conjectures or derive concrete questions to explore (real, and complex ones: e.g. are there more integers than even integers?), which were then formulated by us into modeling/programming tasks.

Students completed these tasks individually or in pairs and published their individual models (*ToonTalk* programs) along with their observations about them, in their personal webreports, commenting on each other's models, which were then used as input to an instructor-led group discussion. The product of this discussion was a *group* webreport which represented the shared understandings of the group, a process that we intended would encourage students to reflect on their work, to acknowledge the need to construct rigorous arguments for their claims, and to negotiate socio-mathematical and socio-technical norms within the (international) community (in the sense of Cobb et al. 2002). As an (ideal) final step, *Web report* would be reviewed by another group, perhaps in another country, and an inter-group online discussion would ensue: (we would now probably call this a math-blog and the students would need little, if any, tutoring on how to use it!).

Once again, we found that collaboration and discussion played a central role in the construction of individual and group knowledge. The need to publish their thoughts in writing, and in a public medium, provoked students to reflect on their experiences and intuitions. The process of writing a joint report required that they find a shared mathematical language, and revisit their arguments. Reading others' reports critically, encouraged attention to detail. Yet all these results were contingent on two major facets: that the students had *something engaging to talk about*, and that they had a *reason to talk about it*. In our case, the former consisted of their models and conjectures, and the latter was built into the activity structure.

In fact, we rather seldom succeeded in orchestrating lively discussion, largely due to pragmatic limitations but also because of the difficulty in establishing a distributed community of practice. The modal thread length of interaction when building a webreport was 1, and the average only slightly greater than one. However, we had some outstanding successes – for example, "Guess my Robot" (in which the challenge is to write the program/robot/function that generates a given sequence of numbers) had a modal thread length of more than 20. In considering why we achieved this kind of success, we identified the teacher as critical: as a facilitator who maintained and supported the interaction, and as a mechanism for validating what did and did not make sense in terms of knowledge building.

22.5.3 Concluding Remarks

Alongside overcoming not inconsiderable technical challenges, establishing an appropriate set of socio-technical/mathematical norms that prioritized collaboration was crucial in exploiting connectivity. We found that the school culture – with its relative lack of expectation to reuse knowledge, and the difficulty, in assessment-heavy systems, of simply "being wrong" – was a formidable challenge. Here again, the teacher could play a crucial role. And finally, the role of the teacher was essential in finding ways to exploit connectivity to encourage students to permeate the layers of our system: to move, for example, from running models to writing or modifying the programs that generated them.

We knew already, of course, that the teacher is crucial. But here we are delineating new, even more demanding roles for the teacher, to be aware – across not only her own classroom but those in remote locations – of the evolution of discussion, the mathematical substance of what is and what is not discussed, and the need all the while to find ways to keep students on task without removing the exploratory and fun elements of the work. This is, surely, a demanding set of roles for the teacher! And it is with this in mind that we have begun new research, in which we are exploring the extent to which the technical system may be able to assist in helping teachers in these roles, by working on building intelligence into the system to achieve this. See www.migen.org and for some early results, see Pearce et al. (2008a, b), Noss (2008).

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Chapter 23
The Future of Teaching and Learning Mathematics with Digital Technologies

Michèle Artigue

Abstract In this text, directly inspired by my closing lecture at the ICMI Study Conference in Hanoi, I use first my personal experience for analyzing the evolution of relationship with digital technologies in mathematics education along the last two decades, and for situating the reflection about the future into an historical dynamics. Then, I focus on some dimensions that I consider crucial for thinking the future of teaching and learning with digital technologies: the theoretical, teacher, curricular, design, equity and access dimensions. These have been extensively addressed during the ICMI Study Conference and I use the perception I have of its outcomes for thinking about the challenges we have to face, and about what we can do in order to make digital technologies better serve the cause of mathematics education.

Keywords Theories • Teacher • Curriculum • Design • Equity

23.1 Introduction

In 1985, ICMI launched a first Study on technology entitled "The influence of computers and informatics on mathematics and its teaching". As explained by Jean Pierre Kahane who was the President of ICMI in 1985 in a recent interview in a series that was done for the ICMI 100th anniversary,[1] the choice of this title was motivated by the following reason: at that time it seemed evident that informatics was likely to have an important influence on mathematics education but many professional mathematicians were not already convinced that informatics would

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[1] This interview can be downloaded on the historical website of ICMI: http://www.icmihistory.unito.it.

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have a substantial influence on their mathematical practices. As was also pointed out by Jean Pierre Kahane in the same interview, 20 years after this first Study, the situation is quite different: no one would deny the influence of informatics and digital technologies on the professional practices and life of mathematicians and on the mathematical sciences themselves, but regarding the influence on mathematics education, the situation is not so brilliant and no one would claim that the expectations expressed at the time of the first study have been fulfilled.

Having this in mind, there is no doubt that Seymour Papert was perfectly right when, in his opening lecture at the ICMI Study 17 Conference, he said: “We need a vision!” The day after, we were sat at the same table for dinner and he asked me: “Do you think that a vision will emerge from this ICMI Study?” I answered him that I was rather optimistic about our collective ability at expressing such a vision about the future of teaching and learning mathematics with digital technologies, and in my closing lecture I tried to show how the week of collaborative work in Hanoi had reinforced my optimism, through a diversity of resonances and insights.

This contribution is directly inspired by this lecture and reproduces its structure. In the first part, I briefly evoke the main episodes of my research life with technology, using this personal example for reflecting on the evolution of our relationship with digital technologies in mathematics education along the last decades. I focus then on some evolutions and ideas I see especially promising for the future, evolutions and ideas that, in my opinion, can help us develop the vision that Papert was asking for in his lecture. My personal history will make understandable I hope why I privilege some perspectives, some issues among the diversity of those evoked and worked in this Study, and which certainly are not so less important.

23.2 A Personal Journey with Digital Technologies

As many of those involved in this ICMI Study, I have had a long story with technology, and I will focus here just on some episodes of this story, some milestones along a personal journey.

23.2.1 From Programming to Visualization and Experimentation: A First University Experience

I began to work in that area as a young university teacher in the early eighties, using technology, mainly through programming activities, in an experimental mathematics-physics course for first year university students (Artigue 1981). At that time, the graphical capabilities offered by technology where still quite limited, and the idea of enhanced technology learning in mathematics was mainly attached to the programming affordances of technology. From an educational point of view, such uses were supported by the increasing interest induced by technological evolution on algorithmic and constructive mathematics on the one hand, and also by emerging theoretical approaches such as APOS and more generally the so-called theories of reification (Dubinsky 1991). Programming activities were expected to favour the building of processes and their encapsulation into objects. The book resulting from the first ICMI Study re-edited by UNESCO in 1992 well reflects this state of the art (Cornu and Ralston 1992).

Very quickly, the improvement of computer graphical capabilities increased the affordances of technology, and I saw in this technological improvement an opportunity for making accessible to students in their first university year a qualitative approach to the study of differential equations. I was using such capabilities in my personal research but they were reserved at that time to master and doctoral courses. A first research project was implemented in an experimental course at the University of Lille and resulted quite successful (Artigue 1989, 1992). We proved that, thanks to technology, a new balance could be found in elementary ODE courses between the qualitative, algebraic, and numerical solving of differential equations. This made possible to design and implement courses more respectful of the epistemology of the field. Design required a non trivial transpositive work as the qualitative approach had to be adapted to the limited familiarity of first year students with Analysis concepts and techniques, but such a transpositive work was proved to be possible. In this project, programming facilities were no longer the main affordance of technology; its potential for visualization, for supporting the articulation of conjectures and their test, for supporting reasoning and interaction between settings and semiotic registers was much more essential. As reported above, the experiment reproduced several years, was successful, but I soon understood that it would not be easily up-scaled. Success required a radical change in the institutional status given to the graphical semiotic register in university courses. This register could no longer be limited to its function of representation; it had to be acknowledged as a legitimate register for mathematical reasoning. In the experimental setting, we had experienced the strength of the cultural resistance to this change, all the more as for coherence reason change could not be limited to the sole topic of differential equations. We had used our quality of expert mathematicians in that area for legitimating the change of status among our colleagues but this was only a strategy of local value. We had also discovered that most of our university colleagues were themselves poorly familiar with the qualitative solving of ODE, and that this new course put them in a situation quite new and destabilizing: facing open problems when teaching beginners. A cultural and systemic change was needed that certainly would need time, and much more than what our isolated piece of research could allow.

23.2.2 Working with Low Achievers in Geometry with Logo Technology

Soon after, as a member of the IREM Paris 7, I was engaged in another research project, that time with low achievers in grade 8. In this project, the software

Euclide, a software derived from Logo with specific geometric macro constructions was used for reconciling students with mathematics, and for supporting the development of mathematical rationality (Artigue et al. 1989). Students for instance were asked to reproduce a given figure with Euclide, infer from the program of their construction the geometrical properties used in the construction, and list properties conjectured as true (using the software) but not used. The comparison of the different constructions was expected help distinguish between what Duval calls the epistemic and the logic status of properties (Duval 1995) and after a collective discussion, students organized in small groups were asked to produce different exercises based on the same figure, and propose them to other groups. Problems and strategies for solving these were then collectively discussed. As one can easily imagine, such mathematical activities are rather ambitious and are not generally proposed to low achievers, but once more the project was a very successful project, and the majority of these students entered high school after the two experimental years. Nevertheless, the software Euclide used had evident limitations when compared with DGS such as Cabri-Géomètre just entering the scene. Euclide in some sense had no educational future, and I experienced with this project the pressure put by technological evolution on didactic research for the first time. Moreover this experiment made me sensitive to the changes in teachers' practices required by technological integration, especially in terms of management of the classroom and role, in the words of today the new orchestration needs (Trouche 2004) and on the inadequacy of the ordinary militant discourse developed in teacher training courses. This discourse obviously underestimated the required changes and could not support the building of the new competences needed from teachers (Artigue 1998).

23.2.3 The CAS Experience

At the beginning of the nineties, I was asked by the Ministry of Education to join, as a didactic expert a group of teachers, experts in digital technology, working at identifying the potential offered by Computer Algebra Systems (CAS) for the teaching and learning of mathematics at secondary level, and at planning the curricular changes that the integration of CAS at senior high school level would require. This was a new kind of technology for secondary schools, much more disturbing than the graphical calculators compulsory at that time for the ordinary norms and values of secondary mathematics education, much more complex too. This was also a new kind of technology for me.

The contrast between the idealistic discourse of the experts of the group, totally coherent with the literature on the educational use of CAS at that time and what was revealed by observations made in their classrooms, turned for us quickly into a research question: how to understand such a gap?

A first research project carried out with DERIVE in computer labs, allowed us to detect some possible sources (Artigue 1997):

- The negative effects of the conceptual/technical opposition permeating the literature and the experts' discourse
- A poor sensitivity to the changes in the economy of mathematical practices induced by the use of CAS
- And the underestimation of instrumental issues

A second research project, involving several teams in France was then carried out with the first symbolic calculator: the TI92 from Texas just coming out. It allowed us to test our conjectures, deepen and structure our reflection.

This work led to the development of an instrumental approach integrating the affordances of the instrumental perspective developed in cognitive ergonomy, and today associated with Rabardel's name (Vérillon and Rabardel 1995), and the anthropological didactic perspective developed by Chevallard (1992).

The connection between these two complementary perspectives and the reflection it supported had two important consequences:

- First, the anthropological approach helped us overcome the problem posed by the conceptual/technical dichotomy by making clear that mathematical techniques have both an epistemic value and a pragmatic value. As a consequence, we were obliged to seriously consider the change in the balance between these two values induced by the use of CAS, which was especially insightful
- Simultaneously, the instrumental approach by the distinction it introduces between an artefact and the instrument it can become for an individual, a group or an institution, made us aware of the complexity of instrumental genesis processes, and of the personal and the institutional dimension of these. It led us to put at the right place in the research agenda the determination of the mathematical and technological needs of such processes; to investigate how these were taken into account in curricular choices and teacher training programs, and understand the impact this had on the technological integration

I cannot enter here in the details of this theoretical elaboration and its outcomes (Artigue 2002; Guin et al. 2004) but I would like to stress that the change in perspective regarding technological issues that resulted from the development of this approach was for me a very strong experience. I could no longer see the question posed by the integration of digital technologies as I did before. The resistance to digital technologies, the incredible recurrence of debates on topics such as the famous long division quoted by Papert in his lecture, could be re-interpreted in terms of balance between epistemic and pragmatic values. Let us me elaborate this point. Digital technologies boil over the traditional balance between the pragmatic and the epistemic value of techniques which was built within a paper and pencil culture. An essential reason for that is the way educational systems tend to adapt to digital technologies, without reconsidering their fundamental values, treating technology as simple pedagogical adjuvant. Such an adaptation leads to play on the pragmatic power of technology at the expense of its epistemic power. But what makes legitimate a technique at school cannot only be its pragmatic power, and this makes an essential difference between school and the outside world. Making technology legitimate and mathematically useful requires modes of integration allowing a reasonable balance between the pragmatic and the epistemic power of instrumented techniques. This, as shown by research (Guin et al. 2004; Laborde 2001), if one correctly reads its results, requires tasks and situations that are not simple adaptation of paper and pencil tasks, often tasks without equivalent in the paper and pencil environment, thus tasks not so easy to design when you enter in the technological world with your paper and pencil culture. Comparing with an example given by Papert in his lecture, you are more or less in the same situation as a Greek or Roman who cannot imagine what could be counting with Arabic numbers and numeration system.

This is just one particular example but I hope to have shown that thinking in such terms changes one's mind, obliges to look at educational resistances differently, and obliges also to question the resources that, as researchers, we provide to teachers and institutions for overcoming these difficulties.

23.2.4 From Microworlds and Open Software to Tutorial and on Line Resources

This is not the end of the story. In the recent years, my life with technology has taken new ways, due to the technological evolution. I have indeed been asked to pilot a regional project involving more than 5,000 students and 100 teachers, and using on-line resources (Artigue and Groupe TICE IREM Paris 7 2008). In France, regions are in charge of senior high schools. They pay for the buildings, for the computers, for the textbooks. Three years ago, the region Ile-de-France, the biggest in the country decided to launch a new project, paying the access to on-line mathematics resources to grade ten students living in poor social areas in order to try to compensate the little access these students have to the existing services for accompanying personal school work and improving academic results. The region also decided that this project would be evaluated by a university team and our IREM was proposed. Both free and commercial resources were used, built with the teacher or the student mainly in mind.

This project was challenging for us for at least two reasons: its size and the kind of technology used. Passing from open software to on line resources as those used in this project, one can have the impression of a dramatic didactic regression, due to the didactic strategies implemented and the poor quality of interaction. We nevertheless accepted this challenge considering that, as researchers, we could not avoid to consider technologies that are more and more pervasive in our societies, and risk in a near future to influence more the learning and teaching of mathematics than micro-world technologies have been able to do in more than 20 years.

This project made us face another important and different change in the economy and ecology of learning processes. It also obliged us to adapt the instrumental approach to this new context: what does it mean for a student to instrumentalize such a technology as a learning tool? What does it mean for a teacher to turn it into a professional instrument? As you can imagine, the answers here are not the same as they could be considering open technologies such as CAS, spreadsheets, DGS or graphic calculators. And once more the contrast between the two categories of answers is for a researcher something really insightful, raising a lot of interrogations, showing that there is there a wide technological space that is nearly terra incognita.

23.2.5 European Cooperation and Theoretical Connections

The last experience I would like to mention is still on-going. It began in 2003 in the frame of the European network of excellence Kaleidoscope, and more specifically of a European research team of this network called TELMA (Technology Enhanced Learning in Mathematics) which led more recently to a STREP called ReMath.[2] Kaleidoscope has among its main aims to provide tools for improving the exchange and mutualization of knowledge between teams working in technology enhanced learning. The research team TELMA, involving six different teams from four different countries is the sole structure of this network focused on mathematics. One hypothesis we made in TELMA was that the multiplicity and fragmented character of theoretical approaches used in that area is an obstacle to communication and productive collaboration and, beyond that, to the necessary capitalization of knowledge. We thus decide to look for possible networking between theories and approaches, and to develop specific methodologies and constructs for doing so:

- The notion of didactical functionality enabling us to connect theoretical approaches and practice
- The meta-language of concerns enabling us to organize the networking efforts around shared sensitivities
- The cross-experiment methodology enabling us to confront and compare our respective ways of identifying didactical functionalities, designing scenarios of use, analyzing and interpreting experimental data, through the experimentation by each team of an alien technology, that is to say a technology developed by another team in another educational context and under a different theoretical approach (Artigue 2006, 2007)

This networking work was really a fascinating experience, and it deeply influenced my vision of theoretical issues, and of the ways these can be fruitfully approached today.

This personal story with technology is the background with which I attended the Study Conference, with which I perceived its possible outcomes, and the vision that could emerge from it.

[2] All documents related to these projects are accessible on the associated websites: http://telma.noe-kaleidoscope.org and www.remath.cti.gr.

In the next part of this contribution, I would like to point out now some directions in the work carried out that, in my opinion, are crucial for such a vision. I have selected five different dimensions: the theoretical perspective, the teacher perspective, the institutional and curricular perspective, the design perspective, equity and access issues.

23.3 Towards a Vision: Some Crucial Directions

23.3.1 The Theoretical Perspective

I have the feeling, certainly reinforced by my recent experience within TELMA and ReMath, that as a community we are more mature today for facing the challenge of theoretical diversity. The educational field we work with, even restricted to mathematics, has so many different facets, is so dependent on contexts and cultures that theoretical diversity, even if it has to be controlled, imposes to us as an evidence. Building or choosing some theoretical approach is choosing some coherent lens for looking at this field, and theories are powerful because they renounce to be holistic. The different groups and individuals that constitute our community have different sensitivities shaped by the social, cultural and educational contexts they live in, as well as by their particular history. At the same time nevertheless, they face also common problems, educational dynamics and phenomena that are not so different, they are subjected to similar global influences. They share thus some common concerns, even if they approach these differently, building on the approaches and constructs they are familiar with.

I will illustrate this point with two examples. In the last decades, instrumental issues have become a more and more common concern in this area of research. This does not mean that we all approach these issues in the same way. As evidenced by TELMA work already mentioned, those whose main theoretical reference is Activity Theory do not integrate this concern in the same way as those as myself which have grown up under the influence of Brousseau's and Chevallard's theories. Nevertheless, we all share this common concern, we also often share some common external references as for instance that to Rabardel's (1995) work, and with some effort, we can communicate more than at a superficial level. Several of us, coming from very different countries have already this experience here.

A second example, not completely independent from the first one is that of semiotics. Globally the field of mathematics education is more and more influenced by semiotic perspectives (Saenz-Ludlow and Presmeg 2006). This increasing attention paid to the semiotic dimension of mathematics activity also expresses in different ways according to our respective didactic cultures. In many cases, it is integrated in more general perspectives but, in some others, it constitutes a full theoretical approach by itself. But whatever be the conceptual tools we use for approaching this common concern, and the importance we give to it with respect to others, we are all enriched by the research work of those who focus on this perspective. They oblige us to consider the diversity and richness of the semiotic systems and mediations involved in mathematical activity and learning processes, beyond the standard semiotic registers, and offer us new ways for connecting the perceptive-sensorial world and the symbolic world.

These are just two examples but they show that we have now the means and maturity for better controlling theoretical diversity, benefiting from the richness it can offer and overcoming the fragmentation it tends to generate. This is of course not a personal task; it is a collective task that has to be taken in charge at an international level, and I hope that this ICMI Study will offer a decisive contribution.

23.3.2 The Teacher Perspective

As stressed in the discussion document associated with this ICMI Study, at the time of the first ICMI Study, research and reflection focused on the technology itself and its mathematical potential, and on the student seen as a cognitive entity. This is no longer the case, and following a general trend in mathematics education (Sfard 2005), teachers' practices in technological environments have become an object of systematic enquiry (Monaghan 2004). This evolution has been very well represented at the Conference Study, which has clearly shown up to what point knowledge had progressed in the last two decades, allowing us to understand better how digital technologies modify teacher professional work, requiring new competences, up to what point too the usual discourse accompanying the promotion of technology has been misleading and counterproductive, the educational resources and training strategies poorly appropriate.

But we find also in the different contributions some evidence that we are now ready to enter a new phase, and that the Study can efficiently contribute to this new phase through the analysis it provides of current practices and of their resulting effects, through the methodological and conceptual tools it proposes, through the positive and substantial examples it presents of teacher preparation and professional development programs. These examples moreover show that the technology itself offers now new and powerful tools for supporting and accompanying the professional development of teachers in that area, seen as a collective and collaborative enterprise within communities of practice (Wenger 1998). I am convinced that from this point of view this Study will be especially insightful, at a time when everyone acknowledges that the quality of teacher preparation and teacher professional development are the key of any possible evolution of our educational systems.

23.3.3 The Institutional and Curricular Perspectives

The introduction to the second edition by UNESCO of the first ICMI Study book on computers in mathematics and mathematics education stressed the point that, at that time, in spite of the existence of a number of interesting and successful experiments, the proof of the potential of technology for improving at large mathematics teaching and learning remained to be done. Up-scaling the results obtained in experimental environments was pointed out as a major challenge to be faced.

More than 20 years later, we cannot say that we have successfully faced this challenge. Contributions to this Study include some very interesting large scale studies which all confirm what we intimately know: successful technological integration at large scale level is still a major problem, and this seems to be a general phenomenon. But we neither can say that we are regarding this challenge in the same state where we were two decades ago. Two decades years ago we were very naïve. We understand better now the reasons for such a difficulty, and the theoretical frames we have developed for approaching technological issues allow us today to take into account the socio-cultural and institutional dimensions of integration that have shown to be so important.

In his plenary lecture, Seymour Papert stressed one important reason of the observed failure: the fact that technology has been put at the service of mathematics curricula thought in a paper and pencil culture and he suggested to turning down them. I agree with the diagnostic. I do not fully agree with the suggestion. Educational systems are complex systems, with the scientific acceptance given to the notion of complexity today. Educational research shows that radical curricular changes produce generally results that are far from those expected. There is no doubt that institutional decisions that encourage or even require the introduction of digital technologies in the curriculum, without paying attention to the needs of an effective technological integration beyond the material needs, that remain blind to the fact that technology both affects what is learnt and the form in which it is learnt, have a heavy responsibility in the failure generally observed. From this point of view, the term itself of integration can be considered a misleading term inducing that there is some permanent entity to which technology has to be integrated. There is no doubt also that minimal curricular changes favor assimilation processes where accommodation is at least required. But we all know today that radical changes imposed in a top-bottom process (I do not say that Seymour had this in mind when using the expression "turn down") are far from being a solution. We need to build adequate synergies between top-down and bottom-up processes, and imagine dynamics that preserve all along the way an acceptable distance between the new and the old in order to be acceptable, to be viable, not to collapse or deviate (Assude and Gelis 2002; Haspekian and Artigue 2007).

We need also to take seriously into account the complexity metaphor, and the capacity that complex systems have to auto-organize and structure in bottom-up processes under favorable conditions. Some contributions to this conference especially those related to design (Sect. 1 and Chap. 21) beautifully illustrate this point, showing us that patient and coherent evolutions in the long run can be achieved. Also with regard to complexity, the recent technological evolution brings new challenges related to collaboration and connectivity that I will outline in the next section.

23.3.4 Collaboration and Connectivity

From the time of the first ICMI Study, digital technologies have tremendously evolved. Beyond the reification of mathematical objects and relationships between these, beyond the potential for direct manipulation on these, what they offer us today are reifications of the social dimension of learning processes (Wilensky 2003), and this is not by chance that, going from one discussion group to another one all during the Conference, I have regularly heard the word “collaborative”. The way digital technologies can support and foster today collaborative work, at the distance or not, between students or between teachers, and also between teachers and researchers, and the consequences that this can have on students’ learning processes, on the evolution of teachers’ practices is certainly one essential technological evolution that educational research has to systematically explore in the future. As mentioned above, most of this space is still for us nearly terra incognita. We observe an intense creativity which very often develops independently of research and this is a very stimulating situation. But we also have to be careful. As stressed by Richard Noss in the panel on connectivity (Chap. 22), connectivity does not necessarily imply collaborative work and collaborative work does not necessarily imply better mathematics learning, or I would add, better mathematics teaching. We are submerged by an avalanche of information, data and possibilities of connections and the way this avalanche can be organized, treated and transformed into knowledge or means for productive action is an open problem.

23.3.5 Equity and Accessibility

The last point I would like to evoke, not the least for me, far from it, it that of equity and accessibility, and that of technology in developing countries. The Study Conference has taken place in Vietnam, and at the opening ceremony I pointed out that this choice had for ICMI a high symbolic value. At the end of the Conference, we all had understood up to what point such a choice was important. The ICMI Study book certainly reflects this experience. Those who as myself come from developed countries (I use this word even if I don’t like it), often tend to think that the normal way of development, including technological development, is more or less to copy our development, and are prompt to export our educational technologies. In terms of digital technology and mathematics education, I am personally convinced that other ways have to be explored and are currently explored. The emphasis that many emerging countries put today in the development of distance education for instance through a mixture of digital technologies and more classical ones, or in the development of internet resources for supporting teachers' professional development, make me think that in a not too far future we will learn from their advances in that area, as we are now learning from colleagues from Latino-America and Africa how we can approach the socio-cultural and linguistic heterogeneity which is quickly increasing in the suburb classrooms of our rich countries. From this point of view, the regional panel at the conference was especially insightful (Chap. 17).

23.4 Some Concluding Comments

In this conclusion I would like to come back to the Seymour Papert lecture. Seymour Papert ended his talk asking us to spend reasonable part of our time and energy thinking about possible futures, freeing our minds of the current constraints. I fully support his demand. This is an important role of research, whatever be the area it deals with, to explore avenues beyond those already possible, but in the precise case of technology and mathematics education, this is an imperative necessity. Several examples in the contributions to this Study show how fascinating can be the results of taking such a position both for the design of digital media and the design of their use. They are part of the vision that this ICMI Study has to built, as are part of it all the other facets of our research and design work contributing to our understanding of learning and teaching processes with digital technologies, and designing realistic strategies for the evolution of mathematics education at the light of this understanding.

I would like to add that for this ICMI Study it is not enough to propose a critical and insightful reflection analyzing what has been achieved in the last two decades and what has failed, to develop a vision and show possible ways for making it reality. The Study has also to make this reflection, this vision widely accessible beyond the sole community of researchers to all those who are professionally interested and whose contribution and support is needed. An ICMI Study book does not aim to be only a handbook of research more in a given area. Its aims also at making possible productive exchanges and collaborations between all those who are part of the community of mathematics education, with the richness and diversity of expertise this community reflects. It aims at being a source of insights for this wide community. This is all the more important regarding the theme of this study for which competences and creativity are so much distributed. I am confident that the Study Book will face this challenge successfully.

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