Chapter 8
Series Analysis

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8.1 Objective and General Principles

As we have seen in earlier chapters, the problem of determining the critical behaviour of various generating functions, such as that for SAP and polyominoes is an unsolved problem. One is thus forced to resort to numerical methods, of which the most successful for determining the precise behaviour of a given model on a given lattice, is the method of exact series expansions. In this method, one generates as many terms as possible in the generating function, so that if the generating function is written

$$F(x) = \sum_{n\geq 0} f_n x^n,$$

the coefficient f_n which counts the number of objects with some measure of size indexed by n—typically the perimeter or area—is known for $n \leq N$.

The fundamental problem of series analysis is this: Given a *finite* number of terms in the series expansion of a function $F(x)$ what can one say about the asymptotic and in general singular behaviour of $F(x)$ or f_n? This after all is a property of the *infinite* series. The problem is thus mathematically ill-posed, as given the first N coefficients of a power series expansion, one can add to it the function $x^{N+1}H(x)$, for any function $H(x)$. The behaviour of the modified function is then not reflected in the known series coefficients.

It is thus a (usually) unstated assumption that the coefficients to hand are indeed representative of the underlying function, so that a careful analysis can then reveal something of the true large-n behaviour. For the same reason, quoting error bars in any method of series analysis is fraught with difficulty. It must be understood that quoted error bars are in no sense rigorous. Often unfortunately they reflect the optimism of the investigator in the quality of his/her investigations! Usually the best

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one can do is to calculate some mean and variance of a range of estimates, and try and present evidence that there is no systematic drift of estimates. If there is drift, one can try and estimate that too.

We will show examples of this type of error analysis, which can, in favourable cases, give rise to surprisingly accurate critical parameters. Most of the generating functions pertaining to polygons, polyominoes and polyhedra are believed to have algebraic singularities, though in most cases this has not been proved. That is to say, the generating function above is believed to behave as

$$F(x) \sim A(1-x/x_c)^{\theta} \text{ as } x \to x_c^-. \tag{8.1}$$

Hence it follows that

$$f_n = [x^n]F(x) \sim \frac{An^{-\theta-1}}{\Gamma(\theta)x_c^n}. \tag{8.2}$$

Here A is referred to as the *critical amplitude,* x_c as the *critical point,* and θ as the *critical exponent.*

A more comprehensive review of various methods used to analyse and estimate the asymptotic behaviour of series can be found in [7].

8.2 Ratio Method

The ratio method was perhaps the earliest systematic method of series analysis employed, and is still a useful starting point, prior to the application of more sophisticated methods. It was first used by M F Sykes in his 1951 D Phil studies, under the supervision of C Domb. From equation (8.2), it follows that the *ratio* of successive terms

$$r_n = \frac{f_n}{f_{n-1}} = \frac{1}{x_c}\left(1 - \frac{\theta+1}{n} + \mathrm{O}(\frac{1}{n})\right). \tag{8.3}$$

From this idea, it is then natural to plot the successive ratios $\{r_n\}$ against $1/n$. If the correction term $\mathrm{O}(\frac{1}{n})$ can be ignored, such a plot will be linear, with gradient $-\frac{\theta+1}{x_c}$, and intercept $1/x_c$ at $1/n = 0$.

We show this method in action by considering the application of the ratio method to the polygon generating function for SAP on the triangular lattice. The first few terms in the generating function (in fact from p_3 to p_{26}) are: 2, 3, 6, 15, 42, 123, 380, 1212, 3966, 13265, 45144, 155955, 545690, 1930635, 6897210, 24852576, 90237582, 329896569, 1213528736, 4489041219, 16690581534, 62346895571, 233893503330, 880918093866. Plotting successive ratios against $1/n$ results in the plot shown in Fig. 8.1. The critical point is known [16] to be at $x_c \approx 0.240917574\ldots = 1/4.15079722\ldots$. From Fig. 8.1, one sees that the locus of points, after some initial (low n) curvature becomes linear to the naked eye for $n > 15$ or so, (corresponding to $1/n < 0.067$). Visual extrapolation to $1/x_c$ is quite obvious. A straight line drawn through the last $4-6$ data points intercepts the horizontal axis around $1/n \approx 0.13$. Thus the gradient is approximately $\frac{4.1508-2.8}{-0.13} \approx -10.39$, from which we conclude that the exponent $\theta + 1 = -2.50$. It is known [19] that the exact value is $\theta = -7/2$, which is in complete agreement with this simple graphical analysis.

Various refinements of the method can be readily derived. If the critical point is known exactly, it follows from equation (8.3) that estimators of the exponent θ are given by

$$\theta = n(1 - x_c \cdot r_n) - 1 + \mathrm{O}(1).$$

Similarly, if the exponent θ is known, estimators of the critical point x_c are given by

$$x_c = \frac{1}{r_n}\left(1 - \frac{\theta+1}{n} + \mathrm{O}(\frac{1}{n})\right).$$

One problem with the ratio method is that if the singularity closest to the origin is not the singularity of interest (the so-called *physical singularity*), then the ratio method will not give information about the physical singularity. Worse still, if the closest singularity to the origin is a conjugate pair of singularities, lying in the complex plane and off the real axis, the ratios will vary dramatically in both sign and magnitude. To overcome this difficulty G A Baker Jr [1] proposed the use of Padé approximants applied to the logarithmic derivative of the series expansion.

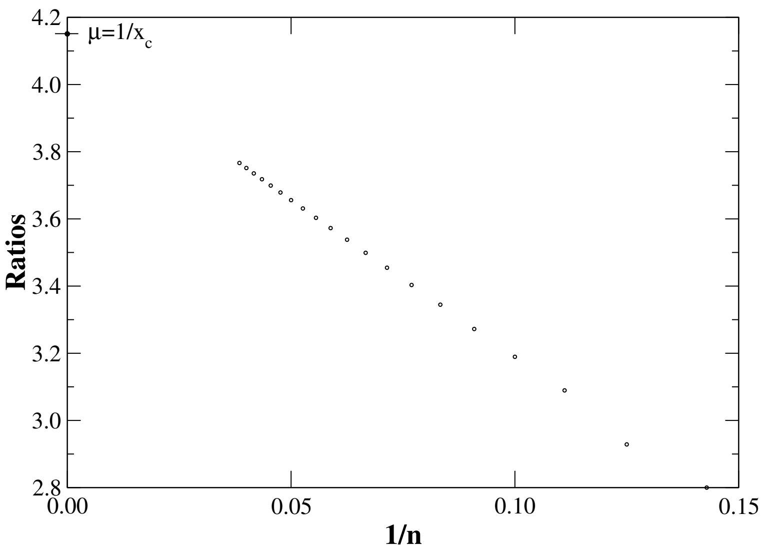

Fig. 8.1 Plot of ratios against $1/n$ for triangular lattice polygons. A straight line through the last few data points intercepts the Ratios axis at $1/x_c$.

8.3 Padé Approximants

The basic idea of Padé approximation is very simple. Given a function $F(x)$ with a simple pole at some point x_c we then use the series expansion of $F(x)$ to form an approximation to $F(x)$ as a ratio of two polynomials,

$$F(x) = \frac{P_i(x)}{Q_j(x)} \tag{8.4}$$

where $P_i(x)$ and $Q_j(x)$ are polynomials of degree i and j, respectively, whose coefficients are chosen such that the first $i+j+1$ terms in the series expansion for $F(x)$ are identical to those of the expansion for $P_i(x)/Q_j(x)$. It is a convention to impose the normalisation condition $Q_j(0) = 1$.

In order to use this basic Padé approximation scheme for polygon problems we must first transform the series into a suitable form, which brings us to the classic method called Dlog-Padé approximation [1]. If we have a function with the expected critical behaviour typical of regular singular points, as given by equation (8.1), then taking the derivative of the log of $F(x)$ gives

$$\widehat{F}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log F(x) \simeq \frac{\theta}{x - x_c} + C. \tag{8.5}$$

This form is perfectly suited for Padé analysis and we see that an estimate x_c^* of the critical point x_c can be obtained from the roots of the denominator polynomial $Q_j(x)$, while an estimate of the critical exponent θ is given by the residue at the pole found at $x = x_c^*$. Such an estimate of the exponent is known as an *unbiased* estimate. If x_c is exactly known, as is sometimes the case, a *biased* estimate of the critical exponent θ can be obtained from the residue of the Padé approximant to $\widehat{F}(x)$ at x_c, that is

$$\theta = \lim_{x \to x_c} (x - x_c) \frac{P_i(x)}{Q_j(x)}. \tag{8.6}$$

Finally, if $F(x) \sim A(1 - x/x_c)^\theta$ as $x \to x_c^-$, then once estimates x_c^* and θ^* of the critical point and critical exponent, respectively, have been obtained, one can then estimate the critical amplitude A by forming Padé approximants to

$$(x_c^* - x)F(x)^{1/\theta^*}|_{x=x_c^*},$$

which should approximate $x_c^* A^{1/\theta^*}$, from which estimates of A follow.

Noting that the Dlog-Padé function $\widehat{F}(x) = F'(x)/F(x)$, we see that forming a Dlog-Padé approximant is simply equivalent to seeking an approximation to $F(x)$ by solving the first order homogeneous differential equation.

$$F'(x)Q_j(x) - F(x)P_i(x) = 0.$$

This observation leads us straight into the more powerful and more general method of differential approximants by noting that we can approximate $F(x)$ by a solution to a higher order ODE (possibly inhomogeneous). This method was first proposed and developed by Guttmann and Joyce [11] in 1972, and was subsequently extended to the inhomogeneous case by Au-Yang and Fisher [5] and Hunter and Baker [13] in 1979.

8.4 Differential Approximants

As we have seen in earlier chapters the majority of polygon and polyomino models in statistical mechanics and combinatorics have generating functions with regular singular points. From the known exact solutions it is clear that the generating functions are often algebraic, or otherwise are given by the solution of simple linear ordinary differential equations. This observation (originally made in the context of the Ising model) forms the nucleus of the method of *differential approximants*. The basic idea is to approximate the function $F(x)$ by solutions to differential equations with polynomial coefficients. The singular behaviour of such ODEs is a well-known classical mathematics problem (see e.g. [6, 14]) and the singular points and exponents are easily calculated. Even if the function *globally* is not a solution of such a linear ODE (as is the case for SAP, as proved in Chapter 6) one hopes that *locally* in the vicinity of the (physical) critical points the generating function can still be well approximated by a solution to a linear ODE.

An M^{th}-order differential approximant (DA) to a function $F(x)$ is formed by matching the coefficients in the polynomials $Q_k(x)$ and $P(x)$ of degree N_k and L, respectively, so that (one) of the formal solutions to the inhomogeneous differential equation

$$\sum_{k=0}^{M} Q_k(x)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \tilde{F}(x) = P(x) \tag{8.7}$$

agrees with the first $N = L + \sum_k (N_k + 1)$ series coefficients of $F(x)$. The function $\tilde{F}(x)$ thus agrees with the power series expansion of the (generally unknown) function $F(x)$ up to the first N series expansion coefficients. We normalise the DA by setting $Q_M(0) = 1$ thus leaving us with N rather than $N+1$ unknown coefficients to find, in order to specify the ODE. From the theory of ODEs, the singularities of $F(x)$ are approximated by zeros x_i, $i = 1, \ldots, N_M$ of $Q_M(x)$, and the associated critical exponent λ_i is estimated from the indicial equation. If there is only a single root at x_i this is just

$$\lambda_i = M - 1 - \frac{Q_{M-1}(x_i)}{x_i Q'_M(x_i)}. \tag{8.8}$$

The physical critical point is the first singularity on the positive real axis.

In order to locate the singularities of the series in a systematic fashion we often use the following procedure: We calculate all $[L; N_0, N_1, N_2]$ and $[L; N_0, N_1, N_2, N_3]$ second- and third-order inhomogeneous differential approximants with $|N_i - N_j| \leq 2$, that is the degrees of the polynomials Q_k differ by at most 2. In addition we demand that the total number of terms used by the DA is at least $N_{\max} - 10$, where

$N_{\max}$ is the total number of terms available in the series. Each approximant yields N_M possible singularities and associated exponents from the N_M zeroes of $Q_M(x)$ (most of these are not singularities of the series but merely spurious zeros). Next these zeros are sorted into equivalence classes by the requirement that they lie at most a distance $1/2^j$ apart, where we typically start with $j = 35$. An equivalence class is accepted as a singularity if an associated zero appears in more than 75% of the total number of approximants, and an estimate for the singularity and exponent is obtained by averaging over the included approximants (the spread among the approximants is also calculated). The calculation is then repeated for $j-1$, $j-2$, ... until a minimum value of 8 or 10. To avoid outputting well-converged singularities at every level, once an equivalence class has been accepted, the data used in the estimate is discarded, and the subsequent analysis is carried out on the remaining data only.

One advantage of this method is that spurious outliers, some of which will almost always be present when so many approximants are generated, are discarded systematically and automatically. Unfortunately, it is not possible to provide rigorous error bounds for differential approximant estimates. In quoting errors we have adopted the following general procedure: For typical individual estimates with a fixed value of L the error is calculated from the spread (basically one or two standard deviations) among the approximants used in obtaining the estimate. Note that these error bounds should *not* be viewed as a measure of the true error as they cannot include possible systematic sources of error. The final estimates (and error bounds) take into account the individual estimates and their error bounds. Note that DA estimates *are not* statistically independent so the true error may exceed the estimated error-bars. This is frequently accommodated by doubling or tripling the calculated error.

8.4.1 The Honeycomb SAP Generating Function

As a first example we apply the differential approximant analysis to the generating function for SAP on the honeycomb lattice. On this lattice the critical point, critical exponent and some universal amplitude ratios are known exactly, so this model provides us with a perfect test-bed for series analysis. In Table 8.1 we have listed the estimates for the critical point x_c^2 and exponent $2-\alpha$ obtained from second- and third-order DAs. We note that all the estimates are in perfect agreement (surely a best case scenario) in that within ‘error-bars’ they take the same value. From this we arrive at the estimate $x_c^2 = 0.2928932186(5)$ and $2-\alpha = 1.5000004(10)$. The final estimates are in perfect agreement with the conjectured [19, 20] exact values $x_c^2 = 1/\mu^2 = 1/(2+\sqrt{2}) = 0.292893218813\ldots$ and $2-\alpha = 3/2$.

Before proceeding we will consider possible sources of systematic errors. First and foremost is the possibility that the estimates might display a systematic drift as the number of terms used is increased, and secondly there is the possibility of numerical errors. The latter possibility is quickly dismissed. The calculations were performed using 128-bit real numbers. The estimates from a few approximants were

Table 8.1 Critical point and exponent estimates for self-avoiding polygons.

L	Second order DA		Third order DA	
	x_c^2	$2-\alpha$	x_c^2	$2-\alpha$
0	0.29289321854(19)	1.50000065(41)	0.29289321865(12)	1.50000040(28)
5	0.29289321875(21)	1.50000010(59)	0.29289321852(48)	1.50000041(99)
10	0.29289321855(23)	1.50000060(48)	0.29289321878(32)	1.49999999(97)
15	0.29289321859(19)	1.50000054(43)	0.29289321861(37)	1.50000035(67)
20	0.29289321866(15)	1.50000038(33)	0.29289321860(21)	1.50000049(43)

compared to values obtained using MAPLE with 100 digits accuracy and this clearly showed that the program was numerically stable and rounding errors were negligible. In order to address the possibility of systematic drift and lack of convergence to the true critical values we refer to Fig. 8.2 (this is probably not really necessary in this case but we include the analysis here in order to present the general method).

In the left panel of Fig. 8.2 we have plotted the estimates from third-order DAs for x_c^2 vs. the highest order coefficient index $N < N_{max}$ used by the DA. Each dot in the figure is an estimate obtained from a specific approximant. As can be seen the estimates clearly settle down to the conjectured exact value (solid line) as N is increased and there is little to no evidence of any systematic drift at large N. One curious aspect though is the widening of the spread in the estimates around $N = 140$. We have no explanation for this behaviour but it could quite possibly be caused by just a few 'spurious' approximants. In the right panel we show the variation in the exponent estimates with the critical point estimates. The 'curve' traced out by the estimates passes through the intersection of the lines given by the exact values. We have not been able to determine the reason for the apparent branching into two parts. However, the lower 'branch' contains many more approximants than the upper one, and is therefore the selected branch.

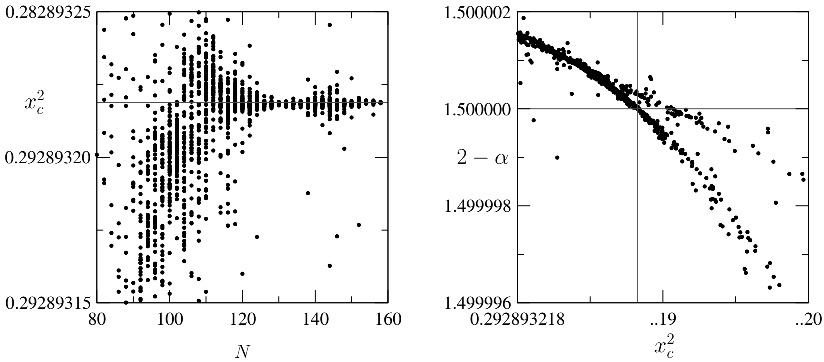

Fig. 8.2 Plot of estimates from third order differential approximants for x_c^2 vs. the highest order term used, and the right panel shows $2-\alpha$ vs. x_c^2. The straight lines are the exact predictions.

The differential approximant analysis can also be used to find possible non-physical singularities of the generating function. Averaging over the estimates from the DAs shows that there is an additional non-physical singularity on the negative x-axis at $x = x_- = -1/\mu_-^2 = -0.41230(2)$, where the estimates of the associated critical exponent α_- are consistent with the exact value $\alpha_- = 3/2$. In the left panel of Fig. 8.3 we have plotted α_- vs. the highest order term used by the DAs and we clearly see the convergence to $\alpha_- = 3/2$. If we take this value as being exact we can get a refined estimate of x_- from the plot in the right panel of Fig. 8.3, where we notice that the estimates for α_- cross the value $3/2$ for $x_- = -0.412305(5)$ which we take as our final estimate. From this we then get $\mu_- = 1.557366(10)$.

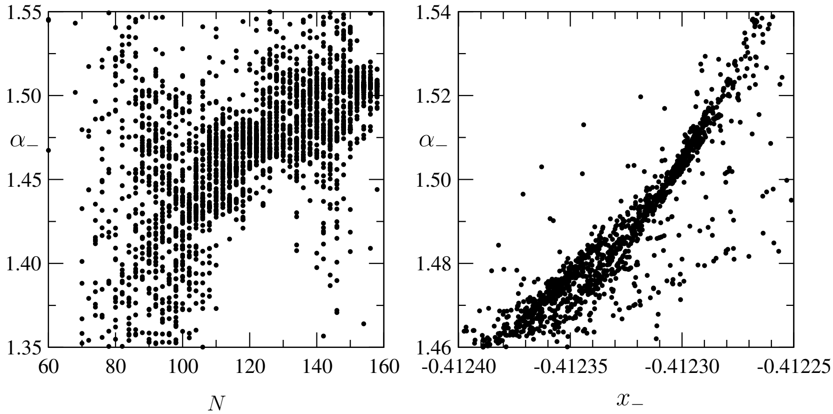

Fig. 8.3 Plot of estimates from third order differential approximants for the location x_- of the non-physical singularity and the associated exponent α_-. The left panel shows α_- vs. the highest order term used, and the right panel shows α_- vs. x_-.

8.5 Amplitude Estimates

Now that the exact values of μ and the exponents have been confirmed we turn our attention to the "fine structure" of the asymptotic form of the coefficients. In particular we are interested in obtaining accurate estimates for the leading critical amplitudes. The method of analysis consists in fitting the coefficients to an assumed asymptotic form. Generally one must include a number of asymptotic terms in order to account for the behaviour of the generating function at both the physical singularity and the non-physical singularities as well as accounting for sub-dominant corrections to the leading order behaviour. As we hope to demonstrate, this method of analysis can not only yield accurate amplitude estimates, but it is often possible to clearly demonstrate which corrections to scaling are present.

Before proceeding with the analysis we briefly consider the kind of terms which occur in the generating functions, and how they influence the asymptotic behaviour of the series coefficients. At the most basic level a function $G(x)$ with a power-law singularity[1]

$$G(x) = \sum_n g_n x^n \sim A(x)(1-\mu x)^{-\xi}, \tag{8.9}$$

where $A(x)$ is analytic in the vicinity of $x = x_c = 1/\mu$, gives rise to the following asymptotic form of the coefficients:

$$g_n \sim \mu^n n^{\xi-1} \left[\tilde{A} + \sum_{i\geq 1} a_i/n^i \right], \tag{8.10}$$

that is, we get the dominant exponential growth given by the term μ^n, modified by a sub-dominant term given by the term $n^{\xi-1}$, involving the critical exponent ξ, followed by analytic corrections. The amplitude $\tilde{A}$ is related to the function $A(x)$ in (8.9) via the relation $\tilde{A} = A(1/\mu)/\Gamma(\xi)$. If $G(x)$ has a non-analytic correction to scaling such as

$$G(x) = \sum_n g_n x^n \sim (1-\mu x)^{-\xi} \left[A(x) + B(x)(1-\mu x)^{\Delta} \right], \tag{8.11}$$

we get the more complicated form,

$$g_n \sim \mu^n n^{\xi-1} \left[\tilde{A} + \sum_{i\geq 1} a_i/n^i + \sum_{i\geq 0} b_i/n^{\Delta+i} \right]. \tag{8.12}$$

A singularity on the negative x-axis $\propto (1+\mu_{-}x)^{-\eta}$ leads to additional corrections of the form

$$\sim (-1)^n \mu_{-}^n n^{\eta-1} \sum_{i\geq 0} c_i/n^i. \tag{8.13}$$

Singularities in the complex plane are still more complicated. However, a pair of singularities on the imaginary axis at $\pm\mathrm{i}/\tau$, that is a term of the form $D(x)(1+\tau^2x^2)^{-\eta}$, generally results in coefficients that change sign according to a $++--$ pattern. This can be accommodated by terms of the form

$$\sim (-1)^{\lfloor n/2 \rfloor} \tau^n n^{\eta-1} \sum_{i\geq 0} d_i/n^i. \tag{8.14}$$

All of these possible contributions must then be put together in an assumed asymptotic expansion for the coefficients g_n and we obtain estimates for the unknown amplitudes by directly fitting g_n to the assumed form. That is, we take a sub-sequence of terms $\{g_n, g_{n-1}, \cdots, g_{n-k}\}$, plug into the assumed form and solve

[1] We have rewritten equation (8.1) in a more convenient form for this analysis.

the $k+1$ linear equations to obtain estimates for the first few amplitudes. As we shall demonstrate below this allows us to probe the asymptotic form.

8.5.1 Estimating the Polygon Amplitude $\tilde{A}$

Here we illustrate the method by analysing the coefficients of the generating function for honeycomb lattice polygons,

$$P(x) = \sum_{n=0} p_{2n} x^n.$$

As well as the physical singularity of interest at $x = x_c^2$, there is a non-physical singularity at $x = x_-$, where $|x_-| > x_c^2$. The asymptotic form of the coefficients p_n of the generating function of square and triangular lattice SAP has been previously studied in detail [3, 17, 15]. There is now clear numerical evidence that the leading correction-to-scaling exponent for SAPs is $\Delta_1 = 3/2$, as predicted by Nienhuis [19, 20]. As argued in [3] this leading correction term combined with the $2-\alpha = 3/2$ term of the SAP generating function produces an *analytic* background term as can be seen from equation (8.11). Indeed, in the previous analysis of SAPs there was no sign of non-analytic corrections-to-scaling to the generating function (a strong indirect argument that the leading correction-to-scaling exponent must be half-integer valued). At first we ignore the singularity at x_- (since $|x_-| > x_c^2$ it is exponentially suppressed) and obtain estimates for $\tilde{A}$ by fitting p_n to the form

$$p_n = \mu^n n^{-5/2} \left[\tilde{A} + \sum_{i=1}^{k} a_i / n^i \right]. \tag{8.15}$$

That is, we take a sub-sequence of terms $\{p_n, p_{n-2}, \ldots, p_{n-2k}\}$ (n even), plug into the formula above and solve the $k+1$ linear equations to obtain estimates for the amplitudes. It is then advantageous to plot estimates for the leading amplitude $\tilde{A}$ against $1/n$ for several values of k. The results are plotted in the left panel of Fig. 8.4. Obviously the amplitude estimates are not well behaved and display clear parity effects. So clearly we can't just ignore the singularity at x_- (which gives rise to such effects) and we thus try fitting to the more general form

$$p_n = \mu^n n^{-5/2} \left[\tilde{A} + \sum_{i=1}^{k} a_i / n^i \right] + (-1)^{n/2} \mu_-^n n^{-5/2} \sum_{i=0}^{k} b_i / n^i. \tag{8.16}$$

The results from these fits are shown in the middle panel of Fig. 8.4. Now we clearly have very well-behaved estimates (note the significant change of scale along the y-axis from the left to the middle panel). In the right panel we take a more detailed look at the data and from the plot we estimate that $\tilde{A} = 1.2719299(1)$. We notice that as more and more correction terms are added (k is increased) the plots of the amplitude estimates exhibit less curvature and the slope become less steep. This is very strong evidence that (8.16) indeed is the correct asymptotic form of p_n.

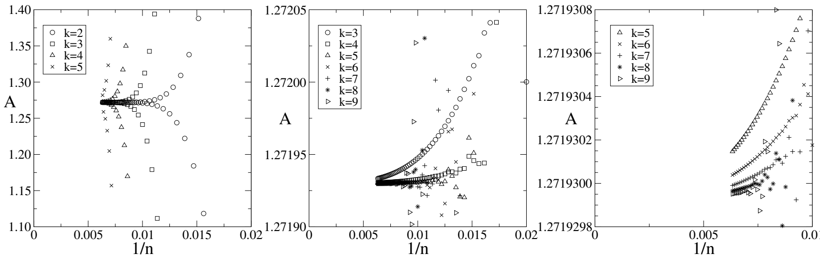

Fig. 8.4 Plots of fits for the self-avoiding polygon amplitude $\tilde{A}$ using in the left panel the asymptotic form (8.15) which ignores the singularity at $x = x_-$, and in the middle panel the asymptotic form (8.16) which includes the singularity at $x = x_-$. The right panel gives a closer look at the data from the middle panel.

8.5.2 The Correction-to-Scaling Exponent

In this section we shall briefly show how the method of direct fitting can be used to differentiate between various possible values for the leading correction-to-scaling exponent Δ_1. There are two competing theoretical predictions, $\Delta_1 = 3/2$ by Nienhuis [19] and $\Delta_1 = 11/16$ by Saleur [21]. As already stated there is now firm evidence from previous work that the Nienhuis result is correct. Here we shall present further evidence. Different values for Δ_1 lead to different assumed asymptotic forms for the coefficients. For the SAP series we argued that a value $\Delta_1 = 3/2$ (or indeed any half-integer value) would result only in *analytic* corrections to the generating function and thus that p_n asymptotically would be given by (8.16). If we have a generic value for Δ_1 we would get

$$p_n = \mu^n n^{-5/2} \left[\tilde{A} + \sum_{i=1}^{k} a_i/n^i + \sum_{i=0}^{k} b_i/n^{\Delta_1 + i} \right] + (-1)^{n/2} \mu_-^n n^{-5/2} \sum_{i=0}^{k} c_i/n^i. \quad (8.17)$$

Fitting to this form we can then estimate the amplitude b_0 of the term $1/n^{\Delta_1}$. We would expect that if we used a manifestly incorrect value for Δ_1 then b_0 should vanish asymptotically thus demonstrating that this term is really absent from (8.17). So let us fit to this form using the value $\Delta_1 = 11/16$. More precisely we fit to the generic form

$$p_n = \mu^n n^{-5/2} \sum_{i=0}^{k} a_i/n^{\alpha_i} + (-1)^{n/2} \mu_-^n n^{-5/2} \sum_{i=0}^{k} b_i/n^i. \quad (8.18)$$

First we include only the leading term arising from Δ_1 using the sequence of exponents $\alpha_i = \{0, 11/16, 1, 2, 3, \ldots\}$. Next we fit to a form including additional analytical corrections arising from Δ_1 leading to the sequence of exponents $\alpha_i = \{0, 11/16, 1, 27/16, 2, 33/16, 3, 49/16, \ldots\}$. More generally one also expects terms of the form $1/n^{m\Delta_1+i}$ with m a non-negative integer. This leads to fits to the form above but with $\alpha_i = \{0, 11/16, 1, 11/8, 27/16, 2, 33/16, 19/8, 43/16, 11/4, 3\ldots\}$. The estimates of the amplitude of the term $1/n^{\Delta_1}$ obtained from fits to these forms are shown in Fig. 8.5. As can be seen from the left panel, where we fit to the first scenario, the amplitude clearly seems to converge to 0, which would indicate the absence of this term in the asymptotic expansion for p_n. In the middle and right panels we show the results from fits to the more general forms. The estimates are consistent with the amplitude being identically zero, though the evidence is not quite as convincing. This is however not really surprising given that the incorrect value $\Delta_1 = 11/16$ gives rise to a plethora of absent terms which will tend to greatly obscure the true asymptotic behaviour.

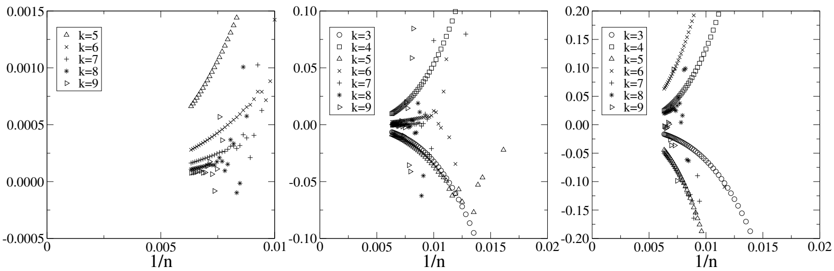

Fig. 8.5 Plots of estimates for the amplitude of the term $1/n^{\Delta_1}$. The left panel shows results from fits to the form (8.18) where only the leading order term $1/n^{\Delta_1}$ is included (as well as analytical corrections). In the middle panel additional terms of the form $1/n^{\Delta_1+i}$ are included and in the right panel terms like $1/n^{m\Delta_1+i}$ are included.

8.6 Exact Fuchsian ODEs for Polygon Models

In recent work Zenine et al. [22, 23, 24] obtained, by experimental computer search, the linear differential equations whose solutions give some quantities of interest in the study of the Ising model of ferromagnetism. Adopting their methods, Guttmann and Jensen [8, 9] used the same ideas to find linear differential equations which have as a solution the generating function $\mathscr{T}(x)$ for three-choice polygons and $\mathscr{P}(x)$ for punctured staircase polygons.

Punctured staircase polygons [10] are staircase polygons with internal holes which are also staircase polygons (the polygons are mutually- as well as self-avoiding). Here we will study only the case with a *single* hole (see Fig. 8.6), and

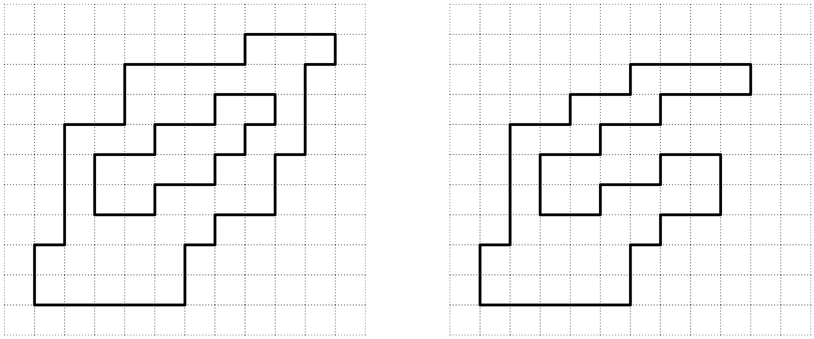

Fig. 8.6 A punctured staircase polygon and an imperfect staircase polygon.

we will refer to these objects as punctured staircase polygons. The perimeter length of staircase polygons is even and thus the total perimeter (the outer perimeter plus the perimeter of the hole) is also even. We denote by p_n the number of punctured staircase polygons of perimeter $2n$.

Three-choice self-avoiding walks on the square lattice were introduced by Manna [18] and can be defined as follows: Starting from the origin one can step in any direction; after a step upward or downward one can head in any direction (except backward); after a step to the left one can only step forward or head downward, and similarly after a step to the right one can continue forward or turn upward. As usual one can define a polygon version of the walk model by requiring the walk to return to the origin. So a three-choice polygon [12] is simply a three-choice self-avoiding walk which returns to the origin, but has no other self-intersections. There are two distinct classes of three-choice polygons. The three-choice rule either leads to staircase polygons or *imperfect staircase polygons* [4] (see Fig. 8.6). The three-choice rules produce imperfect staircase polygons in two ways and staircase polygons of perimeter n in n ways. We denote by t_n the number of three-choice polygons of perimeter $2n$.

Here we briefly outline the method used to find the exact ODE, which we will illustrate by looking at the perimeter generating function of three-choice polygons. Assume we have a function $F(x)$ with a singularity at $x = x_c = 1/\mu$. Starting from a (long) series expansion for the function $F(x)$ we look for a linear differential equation of order M of the form

$$\sum_{k=0}^{M} P_k(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) = 0, \tag{8.19}$$

such that $F(x)$ is a solution of this homogeneous linear differential equation, where the $P_k(x)$ are polynomials. In order to make it as simple as possible we start by searching for a Fuchsian [14] equation. Such equations have only regular singular points. There are several reasons for searching for a Fuchsian equation, rather than a more general D-finite equation. Computationally the Fuchsian assumption simplifies the search for a solution. One may also argue, less precisely, that for "sensible" combinatorial models one would expect Fuchsian equations, as non-Fuchsian equations are characterized by explosive, super-exponential behaviour. Such behaviour is not normally characteristic of combinatorial problems. (The point at infinity may be an exception to this somewhat imprecise observation.) One may also ask the question whether most of the problems in combinatorics with D-finite solutions have Fuchsian solutions? While we have not made an exhaustive study, we know of no counter-example to this suggestion.

From the general theory of Fuchsian [14] equations it follows that the degree of $P_k(x)$ is at most $N_M - M + k$ where N_M is the degree of $P_M(x)$. To simplify matters (reduce the order of the unknown polynomials) it is often advantageous to explicitly assume that the origin and $x = x_c$ are regular singular points and to set $P_k(x) = Q_k(x)S(x)^k$, where $S(x) = xR(x)$ and $R(x)$ is a polynomial of minimal degree having x_c as a root (in our case we have $R(x) = 1-4x$). $S(x)$ could be generalised to include more regular singular points if some were known from other methods of analysis, but we have not found this to be particularly advantageous. Thus when searching for a solution of Fuchsian type there are only two parameters: namely the order M of the differential equation and the degree q_M of the polynomial $Q_M(x)$. Let ρ be the degree of $S(x)$ (2 in our case), then for given M and q_M there are $L = (M+1)(q_M+1)+\rho M(M+1)/2-1$ unknown coefficients, where we have assumed without loss of generality that the leading order coefficient in $P_M(x) = Q_M(x)S(x)^M$ is 1. We can then search systematically for solutions by varying M and q_M.

In this way we first found a solution with $M = 10$ and $q_M = 12$, which required the determination of $L = 206$ unknown coefficients. We have 260 terms in the half-perimeter series and thus have more than 50 additional terms with which to check the correctness of our solution. Having found this conjectured solution we then turned the ODE into a recurrence relation and used this to generate more series terms in order to search for a lower order Fuchsian equation. The lowest order equation we found was eighth order ($M = 8$) and with $q_M = 30$, which requires the determination of $L = 321$ unknown coefficients. Thus from our original 260 term series we could not have found this 8^{th} order solution since we did not have enough terms to determine all the unknown coefficients in the ODE. This raises the question as to whether perhaps there is an ODE of lower order than 8 that generates the coefficients? The short answer to this is no. Further study of our differential operator revealed that it can be factorised. In fact we found a factorization into three first-order linear operators, a second order and a third order. The generating function is a solution of the 8^{th} order operator, not of any of the smaller factors.

So the (half)-perimeter generating function $\mathscr{T}(x)$ for three-choice polygons is conjectured to be a solution of the linear differential equation of order 8,

$$\sum_{k=0}^{8} P_k(x)\frac{\mathrm{d}^k}{\mathrm{d}x^k}F(x) = 0 \tag{8.20}$$

with

$$
\begin{aligned}
&P_8(x) = x^3(1-4x)^4(1+4x)(1+4x^2)(1+x+7x^2)Q_8(x),\\
&P_7(x) = x^2(1-4x)^3Q_7(x), \quad P_6(x) = 2x(1-4x)^2Q_6(x),\\
&P_5(x) = 6(1-4x)Q_5(x), \quad P_4(x) = 24Q_4(x),\\
&P_3(x) = 24Q_3(x), \;\; P_2(x) = 144x(1-2x)Q_2(x),\\
&P_1(x) = 144(1-4x)Q_1(x), \; P_0(x) = 576Q_0(x),
\end{aligned}
\tag{8.21}
$$

where $Q_8(x)$, $Q_7(x)$, ..., $Q_0(x)$, are polynomials of degree 25, 31, 32, 33, 33, 32, 29, 29, and 29, respectively. See [8] for further details.

The singular points of the differential equation are given by the roots of $P_8(x)$. One can easily check that all the singularities (including $x = \infty$) are *regular singular points* so equation (8.20) is indeed of the Fuchsian type. It is thus possible, using the method of Frobenius, to obtain from the indicial equation the critical exponents at the singular points. These are listed in Table 8.2.

Table 8.2 Critical exponents for the regular singular points of the Fuchsian differential equation satisfied by $\mathscr{T}(x)$.

Singularity	Exponents
$x=0$	$-1, 0, 0, 0, 1, 2, 3, 4$
$x=1/4$	$-1/2, -1/2, 0, 1/2, 1, 3/2, 2, 3$
$x=-1/4$	$0, 1, 2, 3, 4, 5, 6, 13/2$
$x=\pm i/2$	$0, 1, 2, 3, 4, 5, 6, 13/2$
$1+x+7x^2=0$	$0, 1, 2, 2, 3, 4, 5, 6$
$x=\infty$	$-2, -3/2, -1, -1, -1/2, 1/2, 3/2, 5/2$
$Q_8(x)=0$	$0, 1, 2, 3, 4, 5, 6, 8$

We shall now consider the local solutions of the differential equation around each singularity. Recall that in general it is known [6, 14] that if the indicial equation yields k critical exponents which differ by an integer, then the local solutions *may* contain logarithmic terms up to $\log^{k-1}$. However, for the Fuchsian equation (8.20) *only* multiple roots of the indicial equation give rise to logarithmic terms in the local solution around a given singularity, so that a root of multiplicity k gives rise to logarithmic terms up to $\log^{k-1}$.

In particular this means that near any of the 25 roots of $Q_8(x)$ the local solutions have no logarithmic terms and the solutions are thus *analytic* since all the exponents are positive integers. The roots of $Q_8(x)$ are thus *apparent singularities* [6, 14] of the Fuchsian equation (8.20). There are methods for distinguishing real and apparent singularities (see, e.g, [6] §45) and in principle one should check that the roots of $Q_8(x)$ satisfy the conditions for being apparent singularities. However, this theoretical method is quite cumbersome. An easier numerical way to see that the roots of $Q_8(x)$ must be apparent singularities is as follows: We already found a 10th order

Fuchsian equation for which the polynomial $P_{10}(x)$ was of a form similar to $P_8(x)$ as listed in equation (8.21), but with the degree of $Q_{10}(x)$ being only 7. That is all the singularities as tabulated in Table 8.2 also appear in this higher order equation with the exception of the 25 roots of $Q_8(x)$ (at most 7 of these could appear in the order 10 Fuchsian equation). In fact we can find a solution of order 14 of the same form as above but with $Q_{14}(x)$ being just a constant. So at this order none of the roots of $Q_8(x)$ appear. Clearly any real singularity of the system cannot be made to vanish and we conclude that the 25 roots of $Q_8(x)$ must indeed be apparent singularities.

Assuming that only repeated roots give rise to logarithmic terms, and thus that a sequence of positive integers give rise to *analytic* terms, then near the physical critical point $x = x_c = 1/4$ we expect the singular behaviour

$$\mathscr{T}(x) \sim A(x)(1-4x)^{-1/2} + B(x)(1-4x)^{-1/2}\log(1-4x), \qquad (8.22)$$

where $A(x)$ and $B(x)$ are analytic in the neighbourhood of x_c. Note that the terms associated with the exponents $1/2$ and $3/2$ become part of the analytic correction to the $(1-4x)^{-1/2}$ term. Near the singularity on the negative x-axis, $x = x_- = -1/4$ we expect the singular behaviour

$$\mathscr{T}(x) \sim C(x)(1+4x)^{13/2}, \qquad (8.23)$$

where again $C(x)$ is analytic near x_-. We expect similar behaviour near the pair of singularities $x = \pm\mathrm{i}/2$, and finally at the roots of $1+x+7x^2$ we expect the behaviour $\mathscr{T}(x) \sim D(x)(1+x+7x^2)^2\log(1+x+7x^2)$.

Next we turn our attention to the asymptotic behaviour of the coefficients of $\mathscr{T}(x)$. To standardise our analysis, we assume that the critical point is at 1. The growth constant of three-choice polygons is 4, so we normalise the series by considering a new series with coefficients r_n, defined by $r_n = t_{n+2}/4^n$. Thus the generating function we study is $\mathscr{R}(y) = \sum_{n\geq 0} r_n y^n = 4 + 3y + 2.625y^2 + \cdots$. From equations (8.22) and (8.23) it follows that the asymptotic form of the coefficients is

$$[y^n]\mathscr{R}(y) = r_n = \frac{1}{\sqrt{n}} \sum_{i\geq 0} \left(\frac{a_i \log n + b_i}{n^i} + (-1)^n \left(\frac{c_i}{n^{7+i}} \right) \right) + \mathrm{O}(\lambda^{-n}). \qquad (8.24)$$

The last term includes the effect of other singularities, further from the origin than the dominant singularities. These will decay exponentially since $\lambda > 1$ in the scaled variable $y = x/4$.

Using the recurrence relations for t_n (derived from the ODE) it is easy and fast to generate many more terms r_n. We generated the first 100000 terms and saved them as floats with 500 digit accuracy (this calculation took less than 15 minutes). With such a long series it is possible to obtain accurate numerical estimates of the first 20 amplitudes a_i, b_i, c_i for $i \leq 19$ with precision of more than 100 digits for the dominant amplitudes, shrinking to 10–20 digits for the the case when $i = 18$ or 19. In making these estimates we have ignored the exponentially decaying term, which is the last term in equation (8.23). In this way we confirmed an earlier conjecture [4] that $a_0 = \frac{3\sqrt{3}}{\pi^{3/2}}$, (where we have taken into account the different normalisation used in that paper). We also find that $b_0 = 3.17327538458989848 1765\ldots$ and $c_0 = \frac{-24}{\pi^{3/2}}$, though we have not been able to identify b_0. However, we have successfully identified further sub-dominant amplitudes, and find $a_1 = \frac{-89}{8\sqrt{3}\pi^{3/2}}$, $a_2 = \frac{1019}{384\sqrt{3}\pi^{3/2}}$, and $a_3 = \frac{-10484935}{248832\sqrt{3}\pi^{3/2}}$, and $c_1 = \frac{225}{\pi^{3/2}}$, $c_2 = \frac{-16575}{16\pi^{3/2}}$, and $c_3 = \frac{389295}{128\pi^{3/2}}$. It seems possible that the amplitudes $\pi^{3/2}\sqrt{3}a_i$ and $\pi^{3/2}c_i$ are rational.

Estimates for the amplitudes were obtained by fitting r_n to the form given above using an increasing number of amplitudes. 'Experimentally' we find we need about the same total number of terms at x_c and $-x_c = x_-$.

So in the fits we used the terms with amplitudes a_i, and b_i, $i = 0,\ldots,K$ and c_i, $i = 0,\ldots,2K$. Going only to $i = K$ with the c_i amplitudes results in much poorer convergence and going beyond $2K$ leads to no improvement. For a given K we thus have to estimate $4K+3$ unknown amplitudes. So we use the last $4K+3$ terms r_n with n ranging from 100000 to $100000-4K-2$ and solve the resulting system of $4K+3$ linear equations. We find that the amplitudes are fairly stable up to around $2K/3$. We observed this by doing the calculation with $K = 30$ and $K = 40$ and then looking at the difference in the amplitude estimates. For a_0 and b_0 the difference is less than 10^{-131}, while for c_0 the difference is less than 10^{-123}. Each time we increase the amplitude index by 1 we lose around 10^6 in accuracy. With $i = 20$ the differences are respectively around 10^{-16} and 10^{-8}.

The excellent convergence is solid evidence (though naturally not a proof) that the assumptions leading to equation (8.24) are correct. Further evidence was obtained as follows: We can add extra terms to the asymptotic form and check what happens to the amplitudes of the new terms. If the amplitudes are very small it is highly likely that the terms are not truly present (if the calculation could be done exactly these amplitudes would be zero). One possibility is that our assumption about integer exponents leading only to analytic terms is incorrect. To test this we fitted to the form

$$\frac{1}{\sqrt{n}}\sum_{i\geq 0}\left(\frac{\tilde{a}_i \log n + \tilde{b}_i}{n^{i/2}} + (-1)^n\left(\frac{\tilde{c}_i}{n^{7+i}}\right)\right) + \mathrm{O}(\lambda^{-n})$$

(as above, in making these estimates we have ignored the exponentially decaying term, which is the last term in the above equation). With $K = 30$ we found that the amplitudes $\tilde{a_1}$ and $\tilde{b_1}$ of the terms $\log n/n$ and $1/n$, respectively, were less than 10^{-60}, while the amplitudes $\tilde{a_3}$ and $\tilde{b_3}$ were less than 10^{-50}. We think we can safely say that all the additional terms we just added are *not* present. We found similar results if we added terms like $\log^2 n$ or additional $\log n$ terms at $y = -1$. That is, we found that those terms were not present. So this fitting procedure provides convincing evidence that the asymptotic form (8.24), and thus the assumption leading to this formula, is correct.

8.6.1 Exact ODEs *Modulo* a Prime

In the above calculations we searched for the ODE using the full series. However, if the size of the ODE is large then this is very time consuming both in terms of generating the actual series and then searching through values of M and D looking for the ODE. In very recent work [2] we have adopted a different and much more efficient strategy. Rather than perform the search on the full series we search only for a solution *modulo* a specific prime (in practice we used the prime $p_0 = 32749 = 2^{15} - 19$). The advantages of this approach are obvious. Firstly we only need generate a long series for a single prime (at least initially) and secondly solving the system determined by (8.25) amounts to finding whether or not the system of linear equations has a zero determinant. This is easily done using Gaussian elimination, and if a zero-determinant is found one can then proceed to solve the system, which yields the ODE *modulo* the prime p_0. In theory one has to worry about possible false positive results, but in widespread use we have never encountered this situation in practice (and in most cases one can check the result using a different prime). Below we give a brief outline of the procedure developed in [2]. Further details can be found there.

Here we illustrate the method by looking at a different generating function for punctured and imperfect staircase polygons. We study the case where the counting variable is the 'length' (extent along the main diagonal) of the polygons. The 'length' is equal to the sum of the coordinates of the point of the polygon furthest from the origin. For punctured staircase polygons this is also equivalent to counting according to the half-perimeter of the outer staircase polygon (rather than the total perimeter of the outer and inner polygons combined). We denote the associated generating functions as $\mathscr{L}(x)$ in the punctured case and $\mathscr{I}(x)$ in the imperfect case.

Starting from a (long) series expansion for some function $F(x)$ we look for a linear differential equation of order M satisfied by $F(x)$. An essential constraint on the ODE of the type we shall consider here is that it be Fuchsian. In particular this means that $x = 0$ and $x = \infty$ are regular singular points. A form for the ODE that automatically satisfies this constraint is

$$\sum_{k=0}^{M} Q_k(x) \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^k F(x) = 0, \tag{8.25}$$

where the $Q_k(x) = \sum_{j=0}^{D} q_{k,j} x^j$ are polynomials of degree D. The condition $a_{M,0} \neq 0$ makes $x = 0$ a regular singular point and the use of the operator $(x\mathrm{d}/\mathrm{d}x)$ rather than just $\mathrm{d}/\mathrm{d}x$ makes analysis around $x = \infty$ simple. Finding the ODE (if it exists) then essentially amounts to solving a system of $(M+1) \times (D+1)$ linear equations.

To determine the coefficients $q_{k,j}$ of the polynomials in (8.25) we arrange the set of linear equations in a well-defined order. There exists a non-trivial solution if the determinant of the matrix of the system of $(M+1) \times (D+1)$ linear equations vanishes. We test this by standard Gaussian elimination, creating an upper triangular matrix U in the process. If we find that a diagonal element $U(N,N) = 0$ for some N, then a non-trivial solution exists. If $N < N_{MD} = (M+1) \times (D+1)$ we set to zero all $q_{k,j}$ in the ordered list beyond N. Of the remaining $q_{k,j}$ we set $q_{M,0} = 1$,

Table 8.3 M is the order of the ODE, D is the degree of each polynomial multiplying each derivative, $N_{MD} = (M+1)(D+1)$, N is the actual number of terms predicted by (8.26) as necessary to find an ODE of the given order M, and Δ is the difference $N_{MD} - N$. The first five columns gives this data for $\mathscr{L}(x)$ while the next five columns gives this data for $\mathscr{I}(x)$.

Terms needed to find $\mathscr{L}(x)$					Terms needed to find $\mathscr{I}(x)$				
M	D	N_{MD}	N	Δ	M	D	N_{MD}	N	Δ
11	53	648	648	0	14	92	1395	1395	0
12	31	416	415	1	15	52	848	848	0
13	23	336	336	0	16	39	680	679	1
14	20	315	312	3	17	32	594	594	0
15	17	288	288	0	18	28	551	551	0
16	16	289	286	3	19	26	540	536	4
17	15	288	284	4	20	24	525	521	4
18	14	285	280	5	21	22	506	506	0
19	13	280	280	0	22	21	506	505	1
20	13	294	289	5	23	20	504	504	0
21	13	308	298	10	24	20	525	517	8
22	12	299	296	3	25	19	520	516	4

thus guaranteeing that $x = 0$ is a regular singular point and determine the remaining coefficients by back substitution. The N for which $U(N,N) = 0$ is the minimum number of series coefficients needed to find the ODE within the constraint of a given M and D. Obviously, $N \le N_{MD} = (M+1)\cdot(D+1)$. Henceforth, D will always refer to the minimum D for which a solution can be found for a given M. Then, for example, we can define a unique non-negative deviation Δ by $N = N_{MD} - \Delta = (M+1)\cdot(D+1) - \Delta$. Examples of such constants are given in Table 8.3 based on our analysis of $\mathscr{L}(x)$ and $\mathscr{I}(x)$. A very striking empirical observation was made in [2], namely that the numbers N in Table 8.3 are given by a simple linear relation

$$N = A\cdot M + B\cdot D - C = (M+1)\cdot(D+1) - \Delta \tag{8.26}$$

where A, B and C are constants depending on the particular series. For $\mathscr{L}(x)$ they are $A = 9$, $B = 11$, $C = -34$ and for $\mathscr{I}(x)$ they are $A = 13$, $B = 14$, $C = -75$ as can be verified from Table 8.3. Note that (8.26) has no (positive) solution for D if $M < B$. Thus $B = M_0$ is the minimum order possible for the ODE. Similarly, $A = D_0$ is the minimum possible degree and thus we can rewrite (8.26) in the more definitive form

$$N = D_0\cdot M + M_0\cdot D - C = (M+1)\cdot(D+1) - \Delta. \tag{8.27}$$

The minimum order M_0 and degree D_0 can be inferred directly from the ODE independently of (8.27). The head polynomial $Q_M(x)$ in (8.25) can be factored *modulo* a prime and the greatest common divisor of these from several different orders M is the polynomial $Q(x)$ whose zeros are the true singularities of the linear ODE. In all cases we have tested, the degree of this polynomial factor is the D_0 in (8.27).

8.6.2 Reconstructing the Exact ODE from Modular Results

Finding the minimal exact ODE for $\mathscr{I}(x)$ using the exact series coefficients would be a difficult task since the size of the coefficients grow as 2^{4n}, so we would have to handle integers of some 1700 digits using an array of size 1396^2 in order to solve the set of linear equations arising out of equation (8.25). This would be stretching the capacity of our current algorithms. So instead we decided to use a different and as we shall see much more efficient approach. It is possible to reconstruct the exact ODE using the results from several *modulo* prime calculations (actually as we shall show only 10 primes are needed). Here we schematically outline the procedure for finding the exact minimum order ODE.

Procedure for ODE reconstruction:

1. Generate a long series modulo a single prime.
2. Find ODEs at different orders and identify the constants A, B, and C of (8.26).
3. Then use this formula to identify both the minimal order ODE and the ODE requiring the least number of terms.
4. Generate series for more primes p_i long enough to find the minimal term ODEs.
5. Turn these ODEs into recurrences and generate longer series.
6. Use these series to find the minimal order ODE mod p_i.
7. Combine to find the exact minimal order ODE:
   (a) Use the Chinese Remainder Theorem to get coefficients a_{ij}.
   This gives us $b_{ij} = a_{ij}$ *modulo* P, where $P = \prod p_i$.
   (b) Find the exact rational coefficients say by using the Maple call $a_{ij} =$ iratrecon(b_{ij}, P).

We managed to reconstruct the exact ODE for $\mathscr{I}(x)$ using 18 primes of the form $p_j = 2^{30} - r_j$. Reconstructing the exact series coefficients up to the length needed to find the exact ODE by the more traditional approach would require at least 10 times as many primes. We note that it takes only a few minutes to find the ODEs mod the primes and then reconstruct the exact ODE coefficient. Even fewer primes are actually needed. In general the numerators are much smaller than the denominators so we can modify the call to read $r/s = a_{ij} =$ iratrecon(b_{ij}, P, R, S), where R and S are positive integers such that $|r| \le R$ and $0 < s \le S$ with $2RS \le P$. If we assume that $s < \sqrt{r}$ then we may choose $S = \sqrt[4]{P}$ and $R = P/(2S)$ and we can then find the a_{ij} using only 12 primes.

A further refinement is possible by generating the a_{ij} starting from a_{MD}. We then multiply all the residues by the denominator of a_{MD} *modulo* the respective primes. We then run through the remaining coefficients by decreasing first j so as to generate all a_{Mj}. Whenever a fraction is encountered we multiply all residues by its denominator (after this we found that the only remaining denominator was 9). We then repeat for $i = M - 1$ and so on until all a_{ij} have been exhausted. After this the modified residues for the a_{ij} are representations of integer coefficients which we then reconstruct. This procedure can generate the exact integer coefficients of the ODE using only 10 primes. We note that for this problem the new procedure is at least 1000 times faster than the original one described above in Section 8.6.1.

For completeness and comparison to the results for three-choice polygons we list in Table 8.4 the critical points and exponents of $\mathscr{I}(x)$.

Table 8.4 Critical exponents at the regular singular points of the Fuchsian differential equation satisfied by $\mathscr{I}(x)$ as obtained from the exact ODE.

Singularity	Exponents
$x = 0$	0, 1, 2, 2, 7/3, 8/3, 3, 3, 3, 4, 5, 6, 7, 8
$x = 1/16$	0, 1, 2, 3, 4, 4, 5, 6, 13/2, 7, 8, 9, 10, 11
$x = 1/5$	0, 1/2, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
$x = 1/4$	−1, −1/2, 0, 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 5, 6, 7
$x = 1$	−2, −3/2, −1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
$x = -1/4$	0, 1, 2, 3, 4, 5, 6, 13/2, 7, 8, 9, 10, 11, 12
$x = \infty$	−2, −3/2, −7/6, −1, −1, −5/6, −1/4, 0, 0, 1/4, 1, 2, 3, 4
$P_{14}(x) = 0$	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14

8.7 Conclusion

In this chapter we have given an outline of the principal methods used for the analysis of series. The method of series analysis was originally developed as a numerical tool, designed to estimate the various constants and exponents that appear in asymptotic estimates. More recently, with the enhancement of both algorithms and computational hardware, it has been possible in some cases to obtain very long series expansions. Then by use of the techniques outlined in Section 8.6, it is sometimes possible to actually obtain the exact ODE whose solution gives the generating function. In such cases the method becomes not just an approximate tool, but an exact one. This development is quite recent, and is likely to enable us to solve hitherto unsolvable problems.

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