Chapter 6
Polygons and the Lace Expansion

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6.1 Introduction

The lace expansion was introduced by Brydges and Spencer in 1985 [7] to analyse weakly self-avoiding walks in dimensions $d > 4$. Subsequently it has been generalised and greatly extended, so that it now applies to a variety of problems of interest in probability theory, statistical physics, and combinatorics, including the strictly self-avoiding walk, lattice trees, lattice animals, percolation, oriented percolation, the contact process, random graphs, and the Ising model. A recent survey is [42].

In this chapter, we give an introduction to the lace expansion for self-avoiding walks, with emphasis on self-avoiding polygons. We focus on combinatorial rather than analytical aspects.

The chapter is organised as follows. In Sec. 6.2, we briefly introduce the random walk model underlying our self-avoiding walk models. In Sec. 6.3, we discuss several examples of taking the reciprocal of a generating function, as this is what the lace expansion succeeds in doing for the self-avoiding walk. The lace expansion for self-avoiding walks is derived in Sec. 6.4. Some of the rigorous results for self-avoiding walks and polygons in dimensions $d > 4$, obtained using the lace expansion, are stated without proof in Sec. 6.5. In Sec. 6.6, we indicate how the lace expansion can be used to enumerate self-avoiding walks in all dimensions, as well as to compute coefficients in the $1/d$ expansion for the connective constant μ and certain critical amplitudes. Some heuristic ideas and numerical results concerning

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the series analysis of the lace expansion and its relevance for the antiferromagnetic singularity of the susceptibility are provided in Sec. 6.7. Finally, in Sec. 6.8, we give a brief indication of an extension to a different model, by discussing some of the results for high-dimensional lattice trees that have been obtained using the lace expansion.

6.2 Preliminaries

The self-avoiding walk models we study are based on underlying random walk models. To define the latter, we fix a finite set $\mathscr{N} \subset \mathbb{Z}^d$ that is invariant under the symmetry group of $\mathbb{Z}^d$, i.e., under permutation of coordinates or replacement of any coordinate x_i by $-x_i$. Our two basic examples are the *nearest-neighbour model*

$$\mathscr{N} = \{x \in \mathbb{Z}^d : \|x\|_1 = 1\} \tag{6.1}$$

and the *spread-out model*

$$\mathscr{N} = \{x \in \mathbb{Z}^d : 0 < \|x\|_\infty \le L\}, \tag{6.2}$$

where L is a fixed (usually large) constant. The norms are defined, for $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, by $\|x\|_1 = \sum_{j=1}^d |x_j|$ and $\|x\|_\infty = \max_{1 \le j \le d} |x_j|$.

An n-step walk with steps in $\mathscr{N}$ is a sequence $\omega = (\omega(0), \omega(1), \dots, \omega(n))$ of points in $\mathbb{Z}^d$, with $\omega(j+1) - \omega(j) \in \mathscr{N}$ for $j = 0, 1, \dots, n-1$. The walk ω is a self-avoiding walk if $\omega(i) \ne \omega(j)$ for all $i \ne j$. We will be interested in generating functions for certain classes of walks. These generating functions have the form $G(z) = \sum_{\omega \in \mathscr{C}} z^{|\omega|}$, where $\mathscr{C}$ is some specific class of walks (e.g., all walks with $\omega(0) = 0$), $z \in \mathbb{C}$ is a parameter, and $|\omega|$ denotes the number of steps in the walk ω.

We denote the cardinality of $\mathscr{N}$ by Ω, so that $\Omega = 2d$ for the nearest-neighbour model and $\Omega = (2L+1)^d - 1$ for the spread-out model. We also define

$$D(x) = \begin{cases} 1/\Omega & (x \in \mathscr{N}) \\ 0 & (x \notin \mathscr{N}). \end{cases} \tag{6.3}$$

Thus $D(x)$ is the probability for a random walk on $\mathbb{Z}^d$, which chooses steps uniformly from $\mathscr{N}$, to move from 0 to x in a single step.

The Fourier transform of an absolutely summable function $f : \mathbb{Z}^d \to \mathbb{R}$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) \mathrm{e}^{ik \cdot x}, \quad (k \in [-\pi, \pi]^d). \tag{6.4}$$

For the nearest-neighbour model, direct calculation gives

$$\hat{D}(k) = d^{-1} \sum_{j=1}^{d} \cos k_j \quad \text{(nearest-neighbour model)}.$$

The *convolution* of two absolutely summable functions on $\mathbb{Z}^d$ is defined by

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y)g(x-y). \tag{6.5}$$

The Fourier transform has the convenient property that $\widehat{f*g} = \hat{f}\hat{g}$. The original function $f(x)$ can be recovered from its Fourier transform via the inversion formula

$$f(x) = \int_{[-\pi,\pi]^d} \hat{f}(k) \mathrm{e}^{-ik\cdot x} \frac{\mathrm{d}^d k}{(2\pi)^d}. \tag{6.6}$$

6.3 Generating Functions and Their Reciprocals

In its simplest setting, the lace expansion can be understood as a way to take the reciprocal of the generating function for the number of self-avoiding walks. In this section, we consider four examples of generating functions and their reciprocals.

Suppose that the power series $G(z) = \sum_{n=0}^{\infty} g_n z^n$ has a non-zero radius of convergence, and suppose for simplicity that $g_0 = 1$. Then $G(z)$ is non-zero in a neighbourhood of the origin, so its reciprocal $F(z) = 1/G(z)$ has a power series expansion $F(z) = \sum_{m=0}^{\infty} f_m z^m$ with a non-zero radius of convergence. Knowledge of $f_0, \ldots, f_n$ uniquely determines $g_0, \ldots, g_n$, and vice versa. The identity $F(z)G(z) = 1$ implies that $f_0 = 1$, and, for $n \geq 1$,

$$\sum_{m=0}^{n} f_m g_{n-m} = 0. \tag{6.7}$$

This can be regarded as the recursion relation

$$g_n = -\left(f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n\right). \tag{6.8}$$

Example 1. Let $G(z) = \sum_{\omega:0\to\cdot} z^{|\omega|}$ be the generating function for all random walks on $\mathbb{Z}^d$ that take steps in the set $\mathscr{N}$ and start at 0. There are Ω^n n-step walks, so

$$G(z) = \sum_{n=0}^{\infty} \Omega^n z^n = \frac{1}{1-\Omega z} = \frac{1}{F(z)}. \tag{6.9}$$

In this example, the reciprocal $F(z) = 1 - \Omega z$ takes the very simple form of a linear function.

Example 2. Let $g_0(x) = \delta_{0,x}$, and, for $n \geq 1$, let $g_n(x)$ be the number of n-step walks that take steps in $\mathscr{N}$, start at $\omega(0) = 0$, and end at $\omega(n) = x$. Let $G(x;z) = \sum_{n=0}^{\infty} g_n(x) z^n$ be the generating function for such walks. By conditioning on the first step, we see that for $n \geq 1$,

$$g_n(x) = \sum_{y \in \mathbb{Z}^d} \Omega D(y) g_{n-1}(x-y) = (\Omega D * g_{n-1})(x), \tag{6.10}$$

where we have used the convolution defined in (6.5). The Fourier transform of (6.10) is

$$\hat{g}_n(k) = \Omega \hat{D}(k) \hat{g}_{n-1}(k). \tag{6.11}$$

It follows that

$$\hat{g}_n(k) = (\Omega \hat{D}(k))^n, \tag{6.12}$$

and hence the Fourier transform of $G(x;z)$ is given by

$$\hat{G}(k;z) = \sum_{n=0}^{\infty} (\Omega \hat{D}(k))^n z^n = \frac{1}{1 - \Omega z \hat{D}(k)}. \tag{6.13}$$

Thus the reciprocal of the Fourier transform of the generating function takes the simple form of a linear function. The generating function $G(x;z)$ can then be recovered as an integral using (6.6). Note that the generating function $1/(1-\Omega z)$ of Example 1 is just $\hat{G}(0;z)$ since setting $k = 0$ corresponds to summation over x in (6.4), which counts all n-step walks regardless of their endpoint.

Example 3. Let $G(z)$ be the generating function for nearest-neighbour self-avoiding walks started from the root of an infinite regular tree of degree $\Omega \geq 2$. The self-avoidance constraint merely eliminates immediate reversals, so there are $\Omega(\Omega - 1)^{n-1}$ n-step walks when $n \geq 1$, and therefore

$$G(z) = 1 + \sum_{n=1}^{\infty} \Omega(\Omega - 1)^{n-1} z^n = \frac{1+z}{1-(\Omega-1)z} = \frac{1}{1 - \Omega z - \Pi(z)} \tag{6.14}$$

with

$$\Pi(z) = \frac{-\Omega z^2}{1+z}. \tag{6.15}$$

For $|z| < 1$, $\Pi(z)$ has the power series representation

$$\Pi(z) = -\Omega \sum_{m=2}^{\infty} (-z)^m. \tag{6.16}$$

In particular, the case $\Omega = 2$ is just the nearest-neighbour self-avoiding walk on the 1-dimensional lattice $\mathbb{Z}$, for which

$$G(z) = \frac{1+z}{1-z}. \tag{6.17}$$

In a manner that is not immediately apparent from the formula, the subtracted term $\Pi(z) = \frac{\Omega z^2}{1+z}$ in the reciprocal $F(z) = 1 - \Omega z - \Pi(z)$ serves to eliminate the immediate reversals that are allowed in the generating function $1/(1-\Omega z)$ for simple random walks on the tree. We will see how the lace expansion leads to (6.16) in Sec. 6.4.4.

Example 4. Now comes the example of most interest. Fix $d \geq 2$, and let $c_n(x)$ denote the number of n-step self-avoiding walks that take steps in $\mathcal{N}$, start at 0, and end at x. Let $c_n = \sum_{x \in \mathbb{Z}^d} c_n(x)$ denote the number of n-step self-avoiding walks that take steps in $\mathscr{N}$, start at 0, and end anywhere. It is a well-known consequence of the simple inequality $c_{m+n} \le c_m c_n$ that the limit $\mu = \lim_{n\to\infty} c_n^{1/n}$ exists, as discussed at greater length in Chapter 2. The limit μ is known as the *connective constant*.

Let $\chi(z) = \sum_{n=0}^{\infty} c_n z^n$ be the generating function for self-avoiding walks that start at 0, and let $G(x;z) = \sum_{n=0}^{\infty} c_n(x) z^n$ be the generating function for those which end at x. It is clear that $\chi(z)$ has radius of convergence $z_c = 1/\mu$. The radius of convergence of $G(x;z)$ cannot be smaller, and it was shown by Hammersley [17] also to be z_c, for any x.

The first two terms of $\chi(z)$ are $\chi(z) = 1 + \Omega z + \cdots$, and hence its reciprocal is of the form

$$\chi(z) = \frac{1}{1 - \Omega z - \Pi(z)}, \tag{6.18}$$

with $\Pi(z) = \sum_{m=2}^{\infty} \pi_m z^m$ for some coefficients π_m. Similarly,

$$\hat{G}(k;z) = \frac{1}{1 - \Omega z \hat{D}(k) - \hat{\Pi}(k;z)}, \tag{6.19}$$

with $\hat{\Pi}(k;z) = \sum_{m=2}^{\infty} \hat{\pi}_m(k) z^m$ for some coefficients $\hat{\pi}_m(k)$. By definition, $\hat{G}(0;z) = \chi(z)$, so (6.18) is a special case of (6.19).

In this example, the problem of determining the coefficients $\hat{\pi}_m(k)$ or $\pi_m = \hat{\pi}_m(0)$ is more difficult. The purpose of the lace expansion is to find a convenient representation for π_m and $\hat{\pi}_m(k)$, which can then be used to better understand $\chi(z)$ and $\hat{G}(k;z)$. As we explain in the next section, π_m can be expressed in terms of the number of self-avoiding polygons and other so-called *lace graphs*. According to (6.8),

$$\hat{c}_n(k) = \Omega \hat{D}(k) \hat{c}_{n-1}(k) + \sum_{m=2}^{n} \hat{\pi}_m(k) \hat{c}_{n-m}(k), \tag{6.20}$$

or, equivalently,

$$c_n(x) = (\Omega D * c_{n-1})(x) + \sum_{m=2}^{n} (\pi_m * c_{n-m})(x). \tag{6.21}$$

The first term on the right-hand side of (6.20) is familiar from (6.11), and by itself would give all random walks, not just the self-avoiding ones. The second term on the right-hand side serves to eliminate self-intersecting walks from the count.

Setting $k = 0$ in (6.20) gives

$$c_n = \Omega c_{n-1} + \sum_{m=2}^{n} \pi_m c_{n-m}, \tag{6.22}$$

and knowledge of the coefficients π_m for $2 \le m \le n$ would allow for the recursive determination of c_n (and vice-versa). This approach to the enumeration of self-avoiding walks has proved fruitful, and is discussed further in Sec. 6.6.

6.4 The Lace Expansion

In this section, we give a quick sketch of the derivation of the lace expansion. We follow the original approach of Brydges and Spencer [7]; an alternate approach based on inclusion-exclusion is discussed e.g. in [42]. Further details can be found in [7] or, for a more recent account, [42]. Our presentation is closely based on [42].

The derivation is essentially unchanged when the setting is generalised to self-avoiding walks on an arbitrary graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ with vertex set $\mathbb{V}$ and edge set $\mathbb{E}$, and we work in this more general setting. For example, $\mathbb{G}$ might be the hypercubic lattice or the honeycomb lattice, but we make no assumption that $\mathbb{G}$ is regular, and it could be finite or infinite. For simplicity, we assume that $\mathbb{G}$ does not contain loops (edges of the form $\{x,x\}$), but it would be easy to relax this assumption.

We set $c_0(x,y) = \delta_{x,y}$, and, for $n \geq 1$, let $c_n(x,y)$ denote the number of n-step self-avoiding walks in $\mathbb{G}$ that take steps in $\mathbb{E}$, begin at $x \in \mathbb{V}$, and end at $y \in \mathbb{V}$. With a little notational effort, it is also possible to include the case of walks which are weighted according to the specific steps they take. We do not work in such generality here, although in the literature it is common to consider weighted steps (see e.g. [23]).

6.4.1 The Recursion Relation

The lace expansion gives rise to a function $\pi_m(x,y)$, defined below, such that for $n \geq 1$,

$$c_n(x,y) = \sum_{v \in \mathbb{V}} c_1(x,v) c_{n-1}(v,y) + \sum_{m=2}^{n} \sum_{v \in \mathbb{V}} \pi_m(x,v) c_{n-m}(v,y). \tag{6.23}$$

In the translation invariant case, e.g. $\mathbb{V} = \mathbb{Z}^d$, we have $c_n(x,y) = c_n(0,y-x) \equiv c_n(y-x)$ and similarly for π_m, the sums over v on the right-hand side reduce to convolutions, and we recover (6.21). All the formulae of Example 4 then also apply. In particular, the lace expansion gives an expression for the reciprocal of the generating functions $\chi(z)$ and $\hat{G}(k;z)$ via (6.18) and (6.19).

6.4.2 Definition of $\pi_m(x,y)$

In this section, we define $\pi_m(x,y)$ and sketch the derivation of (6.23). Let $\mathscr{W}_m(x,y)$ denote the set of all m-step random walk paths (possibly self-intersecting) that start at $x \in \mathbb{V}$ and end at $y \in \mathbb{V}$. Given $\omega \in \mathscr{W}_m(x,y)$, let

$$U_{st}(\omega) = \begin{cases} -1 & \text{if } \omega(s) = \omega(t) \\ 0 & \text{if } \omega(s) \neq \omega(t). \end{cases} \tag{6.24}$$

Then

$$c_n(x,y) = \sum_{\omega \in \mathscr{W}_n(x,y)} \prod_{0 \le s < t \le n} (1 + U_{st}(\omega)), \tag{6.25}$$

since the product is equal to 1 if ω is a self-avoiding walk and is equal to 0 otherwise. We call any set of pairs st, with $s < t$ chosen from $\{0,1,2,\ldots,n\}$, a *graph*. Let $\mathscr{B}_n$ denote the set of all graphs. Expansion of the product in (6.25) gives

$$c_n(x,y) = \sum_{\omega \in \mathscr{W}_n(x,y)} \sum_{\Gamma \in \mathscr{B}_n} \prod_{st \in \Gamma} U_{st}(\omega). \tag{6.26}$$

We call a graph $\Gamma \in \mathscr{B}_n$ *connected*[1] if both 0 and n are endpoints of edges in Γ, and if in addition, for any integer $c \in (0,n)$, there are $s,t \in [0,n]$ such that $s < c < t$ and $st \in \Gamma$. In other words, Γ is connected if, as intervals of real numbers, $\cup_{st \in \Gamma}(s,t)$ is equal to the connected interval $(0,n)$. The set of all connected graphs on $[0,n]$ is denoted $\mathscr{G}_n$. See Fig. 6.1.

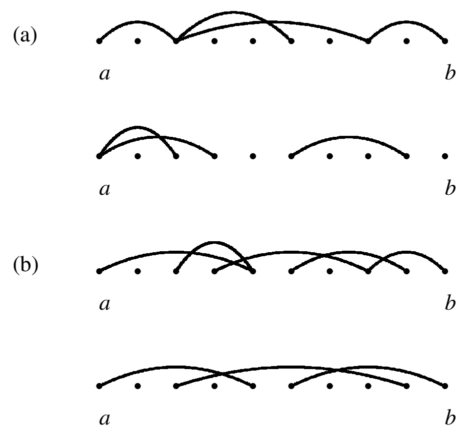

Fig. 6.1 Graphs in which an edge st is represented by an arc joining s and t. The graphs in (a) are not connected, whereas the graphs in (b) are connected.

If we partition the sum over all graphs according to whether: (a) 0 does not occur in an edge in the graph, or (b) 0 does occur in an edge, then we are led to the identity (6.23) with

$$\pi_m(x,y) = \sum_{\omega \in \mathscr{W}_m(x,y)} \sum_{\Gamma \in \mathscr{G}_m} \prod_{st \in \Gamma} U_{st}(\omega). \tag{6.27}$$

Case (a) gives rise to the first term on the right-hand side of (6.23): the graphs not containing 0 produce a self-avoidance constraint that omits the requirement that

[1] This is not the standard graph-theory definition of a connected graph.

the initial vertex at the origin be avoided subsequently. Case (b) gives rise to the second term on the right-hand side of (6.23), with $[0,m]$ the extent of the connected component containing 0: the lack of an edge that passes over m means that the walk segments before and after time m are independent, and the arbitrary graphs on the interval $[m,n]$ produce a self-avoidance constraint during that interval.

6.4.3 Representation of $\pi_m(x,y)$ via Laces

An important alternate representation for $\pi_m(x,y)$ can be obtained in terms of laces. A *lace* is a minimally connected graph, i.e., a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[0,m]$ is denoted by $\mathscr{L}_m$, and the set of laces in $\mathscr{L}_m$ which consist of exactly N edges is denoted $\mathscr{L}_m^{(N)}$. See Fig. 6.2.

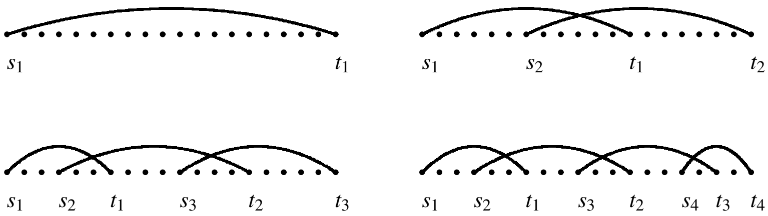

Fig. 6.2 Laces in $\mathscr{L}_m^{(N)}$ for $N = 1,2,3,4$, with $s_1 = 0$ and $t_N = m$.

Given a connected graph $\Gamma \in \mathscr{G}_m$, the following prescription associates to Γ a unique lace $\mathsf{L}_\Gamma \subset \Gamma$: The lace L_Γ consists of edges $s_1t_1, s_2t_2, \ldots$, with $t_1, s_1, t_2, s_2, \ldots$ determined, in that order, by

$$t_1 = \max\{t : 0t \in \Gamma\}, \quad s_1 = 0,$$

$$t_{i+1} = \max\{t : \exists s < t_i \text{ such that } st \in \Gamma\}, \quad s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}.$$

Given a lace L, the set of all edges $st \notin L$ such that $\mathsf{L}_{L\cup\{st\}} = L$ is denoted $\mathscr{C}(L)$. Edges in $\mathscr{C}(L)$ are said to be *compatible* with L. See Fig. 6.3.

We write $L \in \mathscr{L}_m^{(N)}$ as $L = \{s_1t_1, \ldots, s_Nt_N\}$, with $s_l < t_l$ for each l. The fact that L is a lace is equivalent to a certain ordering of the s_l and t_l. For $N = 1$, we simply have $0 = s_1 < t_1 = m$. For $N \geq 2$, $L \in \mathscr{L}_m^{(N)}$ if and only if

$$0 = s_1 < s_2, \quad s_{l+1} < t_l \leq s_{l+2} \quad (l = 1, \ldots, N-2), \quad s_N < t_{N-1} < t_N = m$$

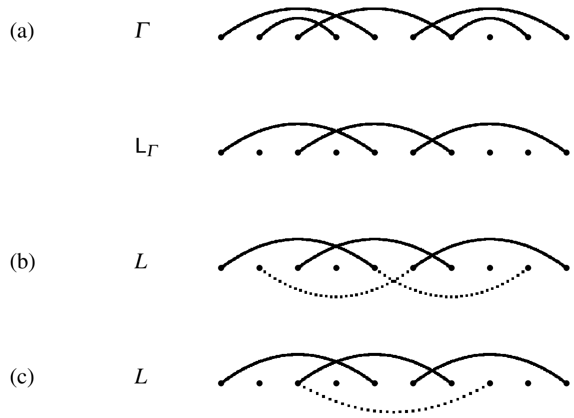

Fig. 6.3 (a) A connected graph Γ and its associated lace $L = \mathsf{L}_\Gamma$. (b) The dotted edges are compatible with the lace L. (c) The dotted edge is not compatible with the lace L.

(for $N = 2$ the vacuous middle inequalities play no role); see Fig. 6.2. Thus L divides $[0,m]$ into $2N-1$ subintervals:

$$[s_1,s_2],\ [s_2,t_1],\ [t_1,s_3],\ [s_3,t_2],\ \dots\ ,[s_N,t_{N-1}],\ [t_{N-1},t_N]. \tag{6.28}$$

Of these, intervals of the form $[t_i,s_{i+2}]$ can have zero length, whereas all others have length at least 1.

The sum over connected graphs in (6.27) can be converted to a double sum, first over all laces L, and then over connected graphs for which the above prescription produces L. This gives

$$\sum_{\Gamma\in\mathscr{G}_m}\prod_{st\in\Gamma}U_{st} = \sum_{L\in\mathscr{L}_m}\prod_{st\in L}U_{st}\sum_{\Gamma\in\mathscr{G}_m:\mathsf{L}_\Gamma=L}\ \prod_{s't'\in\Gamma\setminus L}U_{s't'}. \tag{6.29}$$

The sum over Γ on the right-hand side can then be resummed explicitly (for details, see [7] or [42]) to obtain the formula

$$\pi_m(x,y) = \sum_{\omega\in\mathscr{W}_m(x,y)}\sum_{L\in\mathscr{L}_m}\prod_{st\in L}U_{st}(\omega)\prod_{s't'\in\mathscr{C}(L)}(1+U_{s't'}(\omega)). \tag{6.30}$$

We restrict the sum in (6.30) to laces with N edges, and introduce a minus sign to obtain a non-negative integer, to define

$$\pi_m^{(N)}(x,y) = \sum_{\omega\in\mathscr{W}_m(x,y)}\sum_{L\in\mathscr{L}_m^{(N)}}\prod_{st\in L}(-U_{st}(\omega))\prod_{s't'\in\mathscr{C}(L)}(1+U_{s't'}(\omega)). \tag{6.31}$$

The right hand side of (6.31) is zero unless $N < m$ (since otherwise $\mathscr{L}_m^{(N)}$ is empty), and hence

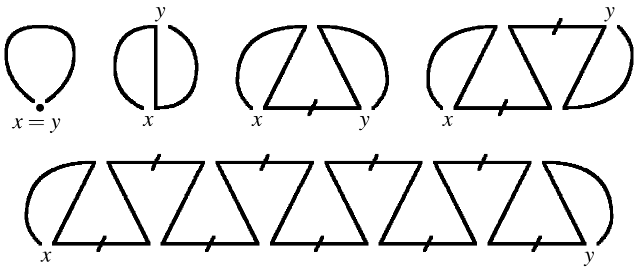

Fig. 6.4 Self-intersections required for a walk ω with $\prod_{st\in L} U_{st}(\omega) \neq 0$, for the laces with $N = 1,2,3,4$ bonds depicted in Fig. 6.2. The picture for $N = 11$ is also shown. A slashed subwalk may have length zero .

$$\pi_m(x,y) = \sum_{N=1}^{m-1} (-1)^N \pi_m^{(N)}(x,y). \qquad (6.32)$$

Note that each term in the sum (6.31) is either 0 or 1. The first product in (6.31) is equal to 1 precisely when $\omega(s) = \omega(t)$ for each edge $st \in L$. The second product is equal to 1 precisely when $\omega(s') \neq \omega(t')$ for each $s't' \in \mathscr{C}(L)$. Thus the edges in the lace require ω to have certain self-intersections, while the compatible edges enforce certain self-avoidance conditions. The self-intersections required are illustrated in Fig. 6.4. We refer to the walk configurations of Fig. 6.4 as *lace graphs*.

The simplest term is $\pi_m^{(1)}(x,y)$, which is zero if $y \neq x$, and which is the number of m-step self-avoiding returns to x when $y = x$. In the translation invariant case, $\pi_m^{(1)}(x,y)$ can be expressed in terms of the number p_m of m-step unrooted unoriented self-avoiding polygons, by $\pi_m^{(1)}(x,y) = 2mp_m\delta_{x,y}$ when $m > 2$ (by convention, $p_2 = 0$).

For $N \geq 2$, $\pi_m^{(N)}(x,y)$ counts m-step walk configurations as indicated in Fig. 6.4. The number of loops in a diagram is equal to the number of edges in the corresponding lace. In these diagrams, each line represents a self-avoiding walk, and the overall walk begins at x and ends at y. The lines which are slashed correspond to subwalks which may consist of zero steps, but the others correspond to subwalks consisting of at least one step. The combined number of steps taken by all the subwalks is m. If the $2N-1$ subwalks in the N-loop diagram are sequentially labeled $1,2,\dots,2N-1$, then the subwalks are mutually avoiding (apart from the required intersections) due to the effect of the compatible edges, with the following patterns: [123] for $N = 2$; [1234], [345] for $N = 3$; [1234], [3456], [567] for $N = 4$; [1234], [3456], [5678], [789] for $N = 5$; and so on for larger N. In the above, e.g., for $N = 4$, the meaning is that subwalks $1,2,3,4$ are mutually avoiding apart from the enforced intersections explicitly depicted, as are subwalks $3,4,5,6$ and subwalks $5,6,7$. However, subwalks not grouped together are permitted to freely intersect, e.g., for $N = 4$, subwalks $1,2$ are permitted to intersect subwalks $5,6,7$, and subwalks 3 and 4 can intersect subwalk 7.

6.4.4 Walks Without Immediate Reversals

The algebra used in deriving the lace expansion does not depend on the precise form of the interaction $U_{st}(\omega)$, and other choices are possible. For example, we could instead take

$$U_{st}(\omega) = \begin{cases} -1 \text{ if } \omega(s) = \omega(t) \text{ and } t = s+2 \\ \ \ 0 \text{ otherwise.} \end{cases} \tag{6.33}$$

With this choice, $c_n(x,y)$ of (6.25) simply counts the number of walks from x to y that do not make any immediate reversals. For a non-zero contribution to $\pi_m(x,y)$ in (6.30), laces must have all edges of length 2, and thus there is a unique lace on the interval $[0,m]$ and this lace contains $m-1$ edges. In this case, the lace graphs consist of successive immediate reversals. For a vertex transitive (translation invariant) graph with vertices of degree Ω, an examination of (6.31) shows that

$$\sum_{v \in \mathbb{V}} \pi_m^{(N)}(v) = \begin{cases} \Omega \text{ if } N = m-1 \\ 0 \ \ \text{otherwise.} \end{cases} \tag{6.34}$$

This reproduces the formula (6.16) of Example 3, with $\Pi(z) = \sum_{m=2}^{\infty} \pi_m z^m$ and $\pi_m = \sum_{N=1}^{m-1} (-1)^N \sum_{u \in \mathbb{V}} \pi_m^{(N)}(v) = (-1)^{m-1}\Omega$, as it leads to

$$\Pi(z) = \sum_{m=2}^{\infty} \pi_m z^m = -\Omega \sum_{m=2}^{\infty} (-z)^m. \tag{6.35}$$

6.5 Self-Avoiding Walks and Polygons in Dimensions $d > 4$

The major mathematical problem for self-avoiding walks on $\mathbb{Z}^d$ is to prove the existence and compute the values of the universal critical exponents γ, ν, α which appear in the predicted asymptotic formulas

$$c_n \sim A\mu^n n^{\gamma-1}, \quad \langle |\omega(n)|^2 \rangle_n \sim D n^{2\nu}, \quad c_n(e) \sim B\mu^n n^{\alpha-2}. \tag{6.36}$$

Here $\langle \cdot \rangle_n$ denotes expectation with respect to the uniform measure on the set of n-step self-avoiding walks started from the origin. In the last formula, n is required to be odd for the nearest-neighbour model, and e represents a neighbour of the origin.

In this section, we describe some of the results that have been obtained in this direction using the lace expansion, for self-avoiding walks in dimensions $d > 4$. The hypothesis $d > 4$ is used to ensure convergence of the lace expansion, e.g., in the sense that $\sum_{m=2}^{\infty} m|\pi_m| z_c^m < \infty$, where $z_c = 1/\mu$. There are now several different approaches to proving convergence of the lace expansion, and we make no attempt here to explain them. Perhaps the simplest approach, and many references to other approaches, can be found in [42].

All known convergence proofs require a small parameter to ensure convergence. To prove that the critical exponent γ is equal to 1 amounts to proving that $\frac{\mathrm{d}}{\mathrm{d}z}[1/\chi(z)]|_{z=z_c}$ is finite and non-zero, since $c_n \sim A\mu^n$ corresponds to $\chi(z) \sim A(1-\mu z)^{-1}$ as $z \nearrow z_c$. This, in turn, amounts to analysing $z_c \frac{\mathrm{d}}{\mathrm{d}z}\Pi(z_c)$ or, equivalently, $\sum_{m=2}^{\infty} m\pi_m z_c^m$. The $m=2$ term in this sum is equal to $2\Omega z_c^2$, and assuming that z_c is close to $(\Omega-1)^{-1}$ and that Ω is large, this is close to $2\Omega^{-1}$. For the nearest-neighbour model with d large, or for the spread-out model with $d>4$ and L large, this approximation turns out to be reasonably accurate and $\sum_{m=2}^{\infty} m|\pi_m|z_c^m$ is $O(d^{-1})$ or $O(L^{-d})$. In [19, 21], Hara and Slade used a computer assisted proof to show that even for the nearest-neighbour model when $d=5$ the small parameter is small enough to allow for a proof of convergence of the lace expansion.

In particular, the following three theorems were proved in [19, 21].

Theorem 1. *For the nearest-neighbour model in dimensions $d \geq 5$, there are positive constants A, D such that the following hold:*

(a) $c_n = A\mu^n[1+O(n^{-\varepsilon})]$ *as $n \to \infty$, for any $\varepsilon < 1/2$.*
(b) $\langle|\omega(n)|^2\rangle_n = Dn[1+O(n^{-\varepsilon})]$ *as $n \to \infty$, for any $\varepsilon < 1/4$.*

For $d=5$, $A \in [1, 1.493]$ and $D \in [1.098, 1.803]$.

Theorem 1 is alluded to in the general discussion of the asymptotic behavior of c_n in Chapter 1, and additionally provides explicit bounds on the error terms. A corollary of (a) is that $\lim_{n\to\infty} c_{n+1}/c_n = \mu$. This is believed to be true in all dimensions, but remains unproved for $d = 2,3,4$. It was proved by Kesten [32] that $\lim_{n\to\infty} c_{n+2}/c_n = \mu^2$ in all dimensions.

Let $C_d[0,1]$ denote the set of continuous functions $f : [0,1] \to \mathbb{R}^d$, equipped with the supremum norm. Given an n-step self-avoiding walk ω, we define a rescaled version $X_n \in C_d[0,1]$ of the self-avoiding walk by setting $X_n(k/n) = (Dn)^{-1/2}\omega(k)$ for $k = 0,1,2,\ldots,n$, and taking $X_n(t)$ to be the linear interpolation of this. We denote by $\mathrm{d}W$ the Wiener measure on $C_d[0,1]$. The following theorem shows that for $d \geq 5$ the scaling limit of the self-avoiding walk is Brownian motion.

Theorem 2. *For the nearest-neighbour model in dimensions $d \geq 5$, X_n converges in distribution to Brownian motion, i.e., for any bounded continuous function f : $C_d[0,1] \to \mathbb{C}$,*

$$\lim_{n\to\infty} \langle f(X_n)\rangle_n = \int f \mathrm{d}W.$$

Perhaps the most basic application of the above theorem is to the case $f(X) = \mathrm{e}^{ik\cdot X(1)}$. In this case, Theorem 2 gives

$$\lim_{n\to\infty} \langle \mathrm{e}^{ik\cdot\omega(n)/\sqrt{Dn}}\rangle_n = \mathrm{e}^{-|k|^2/2d}, \tag{6.37}$$

i.e., the scaling limit of the endpoint of the self-avoiding walk has a Gaussian distribution. Note that the expression under the limit in the above equation can also be written as $\hat{c}_n(k/\sqrt{Dn})/c_n$; this shows the relevance of the Fourier transform of $c_n(x)$ in understanding the scaling limit.

The results for self-avoiding polygons in dimensions $d \geq 5$ are less complete than those above. Since $p_n = \frac{1}{2n}\sum_{e\in\mathcal{N}} c_{n-1}(e)$, the study of $c_n(x)$ is more general than the study of self-avoiding polygons. Ideally one would like a result which states that $c_{n-1}(e) \sim B\mu^n n^{-d/2}$ for $d > 4$, but this has not been proved. The following theorem from [21] proves an upper bound on the generating function for $c_n(x)$ which is consistent with this asymptotic behaviour. Madras [36] has proved bounds valid for general d, believed not to be sharp.

Theorem 3. *For the nearest-neighbour model in dimensions $d \geq 5$, for any $a < (d-2)/2$,*

$$\sup_{x\in\mathbb{Z}^d} \sum_{n=0}^{\infty} n^a c_n(x)\mu^{-n} < \infty.$$

Stronger results for $c_n(x)$ have been obtained for the spread-out model in dimensions $d > 4$. The best results can be found in [24], and include the following as a very special case. In the statement of the theorem, μ denotes the connective constant for the spread-out model.

Theorem 4. *Consider the spread-out model in dimensions $d > 4$. Let $0 < \delta < \min\{1, \frac{d-4}{2}\}$. There is an L_0 such that for $L \geq L_0$ the following statements hold:*
(a) There exist positive constants a and b (depending on d and L), such that for all $k \in \mathbb{R}^d$ with $|k|^2$ bounded by a constant, as $n \to \infty$,

$$\hat{c}_n(k/\sqrt{bn}) = a\mu^n \mathrm{e}^{-|k|^2/2d}\left[1 + O(n^{-(d-4)/2}) + O(|k|^2 n^{-\delta})\right]. \tag{6.38}$$

(b) There are constants C_1, C_2 (depending on d but not L) such that

$$C_1\mu^n L^{-d} n^{-d/2} \leq \sup_{x\in\mathbb{Z}^d} c_n(x) \leq C_2\mu^n L^{-d} n^{-d/2}. \tag{6.39}$$

Note that for $k = 0$, Theorem 4(a) gives

$$c_n = a\mu^n\left[1 + O(n^{-(d-4)/2})\right], \tag{6.40}$$

which is a better error bound than that proved for the nearest-neighbour model in Theorem 1. It was predicted in [15] that the $O(n^{-(d-4)/2})$ is sharp, and by universality we expect it to hold also for the nearest-neighbour model (this is confirmed numerically in [8]). In [24], a version of Theorem 4 is obtained in the much more general setting of cycle-free networks of mutually-avoiding self-avoiding walks. For arbitrary networks, possibly with cycles, see [26].

Theorem 4(a) provides a central limit theorem for the endpoint of the spread-out self-avoiding walk in dimensions $d > 4$. It is natural to wonder if Theorem 4(b) extends to a local central limit theorem, i.e., a statement that $c_n^{-1}c_n(x\sqrt{bn})$ is asymptotically Gaussian (when x has the same parity as n). Such an extension follows from the results of [23, 24]. Some care is needed with such a statement, since $c_n(0) = 0$ for all $n \geq 1$, and to eliminate this local effect we average over a region that grows with n. For the averaging, we denote the cube of radius R centred at $x \in \mathbb{Z}^d$ by

$$C_R(x) = \{y \in \mathbb{Z}^d : \|x-y\|_\infty \le R\}, \tag{6.41}$$

with cardinality $|C_R(x)|$. Let $\lfloor x \rfloor$ denote the closest lattice point in $\mathbb{Z}^d$ to $x \in \mathbb{R}^d$ (with some arbitrary rule to break ties).

Theorem 5. *Consider the spread-out model in dimensions $d > 4$. Let R_n be any sequence with $\lim_{n\to\infty} R_n = \infty$ and $\lim_{n\to\infty} n^{-1/2} R_n = 0$. There is an L_0 such that for $L \ge L_0$, and for any $x \in \mathbb{R}^d$ with $|x|^2 [\log R_n]^{-1}$ sufficiently small, as $n \to \infty$,*

$$\frac{1}{|C_{R_n}(\lfloor x\sqrt{bn}\rfloor)|} \sum_{y \in C_{R_n}(\lfloor x\sqrt{bn}\rfloor)} \frac{c_n(y)}{c_n} \sim \left(\frac{d}{2\pi bn}\right)^{d/2} \mathrm{e}^{-d|x|^2/2}, \tag{6.42}$$

in the sense that the limit of the ratio of the two sides is 1.

The Gaussian limit (6.42) does not follow directly from the convergence of the Fourier transform in Theorem 4. The latter implies that sums over cubes of side $\sqrt{n}$ converge to integrals of the Gaussian density, whereas (6.42) permits arbitrarily slow growth of R_n.

Finally, we mention that rigorous results have been proved for the scaling of the weakly self-avoiding walk on a 4-dimensional *hierarchical* lattice, using renormalisation group methods [5, 6]. The 3-dimensional cubic lattice appears to be well beyond the reach of any currently known methods. For $d = 2$, there is very strong evidence [34] that the scaling limit is given by $\mathrm{SLE}_{8/3}$, but the problem of proving existence of the scaling limit remains open.

6.6 Self-Avoiding Walk Enumeration and $1/d$ Expansions

6.6.1 Self-Avoiding Walk Enumeration via the Lace Expansion

For the nearest-neighbour model on $\mathbb{Z}^d$, (6.22) states that

$$c_n = 2dc_{n-1} + \sum_{m=2}^{n} \pi_m c_{n-m}. \tag{6.43}$$

Let

$$\pi_m^{(N)} = \sum_{x \in \mathbb{Z}^d} \pi_m^{(N)}(x) \tag{6.44}$$

denote the number of N-loop lace graphs of length m, so that $\pi_m = \sum_{N=1}^{m-1} (-1)^N \pi_m^{(N)}$. Equation (6.43) recursively expresses the number of self-avoiding walks of length n in terms of π_m, and thus allows for the determination of c_n from the number of lace graphs with $m \le n$ and $N \le n-1$. The lace graph trajectories, shown in Fig. 6.4, are less spatially extended than SAWs of the same length, and are hence less numerous.

In [8], nearest-neighbour self-avoiding walks on $\mathbb{Z}^d$ were enumerated in dimensions $d \geq 3$ by enumeration of lace graphs together with (6.43). The value of c_n was determined for $n \leq 30$ for $d = 3$ and for $n \leq 24$ for *all* $d \geq 4$ (knowledge of π_m for $m \leq 24$ and $d \leq 12$ determines π_m also for $d > 12$, since lace graphs with at most 24 steps can occupy at most 12 dimensions). In practice, for the cubic lattice it was found that there are approximately 525 times as many 30-step self-avoiding walks as compared to 30-step lace graphs. This factor was found to get much larger as the dimension is increased: the factor for $d = 4$, $n = 24$ is approximately 1700, for $d = 5$, $n = 24$, it is approximately 6200, while for $d = 6$, $n = 24$, it is approximately 20000.

The simplest lace graphs are the self-avoiding returns counted by $\pi_m^{(1)}$, and their enumeration is equivalent to the enumeration of self-avoiding polygons. Polygons were counted in [8] using the so-called *two-step method*. The two-step method is an innovation for the direct enumeration of self-avoiding walks and reduces the exponential complexity of the enumeration problem. For details, we refer to [8]. In [8], the two-step method was used to count polygons and, more generally, to enumerate lace graphs needed to determine π_m. For polygons, the results of [8] are slightly better than those for general lace graphs: p_{32} is determined for $d = 3$, and p_{26} is determined for $d = 4$.

6.6.2 $1/d$ *Expansions via the Lace Expansion*

Next, we indicate how knowledge of the values of π_m for the d-dimensional nearest-neighbour model can be combined with estimates on the lace expansion to derive $1/d$ expansions for the connective constant μ and for the amplitudes A and D of Theorem 1.

Let $z_c = 1/\mu$ denote the radius of convergence of the susceptibility $\chi(z) = \sum_{n=0}^{\infty} c_n z^n$. It is a consequence of the simple inequality $c_{n+m} \leq c_n c_m$ that $c_n \geq \mu^n$ for all n, and from this it follows that $\chi(z) \nearrow \infty$ as $z \nearrow z_c$. Therefore $1/\chi(z) = 1 - 2dz - \Pi(z) \searrow 0$ as $z \nearrow z_c$. In particular, $\lim_{z \nearrow z_c} \Pi(z) = 1 - 2dz_c$. It is proved in [21] that, for $d \geq 5$, $\lim_{z \nearrow z_c} \Pi(z) = \Pi(z_c)$, and therefore z_c obeys the equation

$$z_c = \frac{1}{2d}\left[1 - \sum_{m=2}^{\infty} \pi_m z_c^m\right]. \tag{6.45}$$

This equation was used recursively in [22] to prove that z_c has an asymptotic expansion $z_c \sim \sum_{i=1}^{\infty} a_i (2d)^{-i}$, to all orders, with integer coefficients a_i.

Well-developed lace expansion methods (see [8]) show that for each $N \geq 1$ and $j \geq 2$ there are constants $C_N, C_{N,j}$, independent of sufficiently large d, such that

$$\sum_{m=2}^{\infty} \sum_{M=N}^{m-1} \pi_m^{(M)} z_c^m \leq C_N d^{-N}, \qquad \sum_{m=j}^{\infty} \pi_m^{(N)} z_c^m \leq C_{N,j} d^{-j/2}. \tag{6.46}$$

It then follows from (6.45) and (6.46) that

$$z_c = \frac{1}{2d}\left[1 - \sum_{m=2}^{2N}\sum_{M=1}^{N}(-1)^M \pi_m^{(M)} z_c^m\right] + O(d^{-N-2}), \tag{6.47}$$

where we have used the fact that z_c has an asymptotic expansion in powers of d^{-1}, to replace an error term of order $d^{-N-3/2}$ by one of order d^{-N-2}. Knowledge of the coefficients $\pi_m^{(M)}$ (as polynomials in d) for $m \le 2N$ and $M \le N$ permits the recursive calculation of the terms in the $1/d$ expansion for z_c up to and including order d^{-N-1}. The enumerations of [8] with $m \le 24$, $M \le 12$ give

$$z_c = \frac{1}{2d} + \frac{1}{(2d)^2} + \frac{2}{(2d)^3} + \frac{6}{(2d)^4} + \frac{27}{(2d)^5} + \frac{157}{(2d)^6} + \frac{1065}{(2d)^7} + \frac{7865}{(2d)^8} + \frac{59665}{(2d)^9} + \frac{422421}{(2d)^{10}} + \frac{1991163}{(2d)^{11}} - \frac{16122550}{(2d)^{12}} - \frac{805887918}{(2d)^{13}} + O\left(\frac{1}{(2d)^{14}}\right). \tag{6.48}$$

Taking the reciprocal gives

$$\mu = 2d - 1 - \frac{1}{2d} - \frac{3}{(2d)^2} - \frac{16}{(2d)^3} - \frac{102}{(2d)^4} - \frac{729}{(2d)^5} - \frac{5533}{(2d)^6} - \frac{42229}{(2d)^7} - \frac{288761}{(2d)^8} - \frac{1026328}{(2d)^9} + \frac{21070667}{(2d)^{10}} + \frac{780280468}{(2d)^{11}} + O\left(\frac{1}{(2d)^{12}}\right). \tag{6.49}$$

Equations (6.48) and (6.49) more than double the length of the previously known series [10, 22, 39], which were known up to and including the term $-102(2d)^{-4}$ in (6.49). The error estimates are rigorous. The above expansions would appear to have radius of convergence zero, but there is no proof of this; it would be of interest to study their Borel summability. The critical temperature of the spherical model is known to have an asymptotic $1/d$ expansion with radius of convergence zero [13], and the suggestion that this is true rather generally for $1/d$ expansions of critical points was made in [11]. Note the change in sign at the term $(2d)^{-10}$; a similar sign change is observed in [13] for the critical temperature of the spherical model.

It is proved in [21] that for $d \ge 5$ the amplitudes A and D of Theorem 1 are given by the formulas

$$\frac{1}{A} = 2dz_c + \sum_{m=2}^{\infty} m\pi_m z_c^m, \qquad D = A\left[2dz_c + \sum_{m=2}^{\infty} r_m z_c^m\right], \tag{6.50}$$

where $r_m = \sum_{x\in\mathbb{Z}^d} |x|^2 \pi_m(x)$. The formula for A can be understood from the fact that $\gamma = 1$ for $d \ge 5$ and hence the susceptibility $\chi(z) = 1/F(z)$ should be given approximately by $[F'(z_c)(z-z_c)]^{-1}$, according to Taylor's theorem. The coefficient c_n of z^n is given in this approximation to be $[-z_cF'(z_c)]^{-1}z_c^{-n}$. This gives $A^{-1} = -z_cF'(z_c)$, and using the formula $F(z) = 1 - 2dz - \Pi(z)$ from (6.18) gives the above formula for A. The formula for D can be found similarly, using the fact that $\sum_x |x|^2 c_n(x)$ is the coefficient of z^n in $-\nabla^2\hat{G}(k;z)|_{k=0}$, where ∇ represents the gradient with respect to the vector k. To leading order, if we write $\hat{G}(k;z) = 1/\hat{F}(k;z)$, this is given by

$$-\nabla^2 \hat{G}(0;z) = \frac{\nabla^2 \hat{F}(0;z)}{\hat{F}(0;z)^2} \approx \frac{\nabla^2 \hat{F}(0;z_c)}{[F'(z_c)(z-z_c)]^2}, \tag{6.51}$$

where we have used $\hat{F}(0;z) = F(z)$. Expansion of the right-hand side in powers of z then gives the desired formula for D.

It can be argued from (6.50) using an extension of (6.46) (see [8]) that

$$\frac{1}{A} = 2dz_c + \sum_{m=2}^{2N} \sum_{M=1}^{N} (-1)^M m \pi_m^{(M)} z_c^m + O(d^{-N-1}) \tag{6.52}$$

and

$$D = A \left[2dz_c + \sum_{m=2}^{2N} \sum_{M=1}^{N} (-1)^M r_m^{(M)} z_c^m \right] + O(d^{-N-1}), \tag{6.53}$$

with $r_m^{(M)} = \sum_{x \in \mathbb{Z}^d} |x|^2 \pi_m^{(M)}(x)$. Insertion of (6.48) and the enumerations of [8] for $m \le 24$, $M \le 12$ into (6.52) and (6.53) then give

$$\begin{aligned} A = 1 &+ \tfrac{1}{2d} + \tfrac{4}{(2d)^2} + \tfrac{23}{(2d)^3} + \tfrac{178}{(2d)^4} + \tfrac{1591}{(2d)^5} + \tfrac{15647}{(2d)^6} + \tfrac{164766}{(2d)^7} + \tfrac{1825071}{(2d)^8} \\ &+ \tfrac{20875838}{(2d)^9} + \tfrac{240634600}{(2d)^{10}} + \tfrac{2684759873}{(2d)^{11}} + \tfrac{26450261391}{(2d)^{12}} + O\left(\tfrac{1}{(2d)^{13}}\right), \end{aligned} \tag{6.54}$$

$$\begin{aligned} D = 1 &+ \tfrac{2}{2d} + \tfrac{8}{(2d)^2} + \tfrac{42}{(2d)^3} + \tfrac{284}{(2d)^4} + \tfrac{2296}{(2d)^5} + \tfrac{21024}{(2d)^6} + \tfrac{210306}{(2d)^7} + \tfrac{2242084}{(2d)^8} \\ &+ \tfrac{24909542}{(2d)^9} + \tfrac{280764914}{(2d)^{10}} + \tfrac{3079111998}{(2d)^{11}} + \tfrac{29964810674}{(2d)^{12}} + O\left(\tfrac{1}{(2d)^{13}}\right). \end{aligned} \tag{6.55}$$

This extends the series up to and including order $(2d)^{-5}$ that were reported in [12, 39] and [38] for A and D, respectively, and also provides rigorous error estimates.

6.7 The Antiferromagnetic Singularity

There is strong numerical evidence that the number of self-avoiding walks on the lattice $\mathbb{Z}^d$ for $d = 2$ is given asymptotically by

$$c_n \sim \mu^n n^{\gamma-1} \left(A + \frac{a_1}{n} + \frac{a_2}{n^{3/2}} \cdots \right) + (-\mu)^n n^{\alpha-2} \left(b_0 + \frac{b_1}{n} + \cdots \right) \tag{6.56}$$

and for $d \ge 3$ by

$$c_n \sim \mu^n n^{\gamma-1} \left(A + \frac{a_1}{n^\theta} + \cdots \right) + (-\mu)^n n^{\alpha-2} \left(b_0 + \frac{b_1}{n^\theta} + \cdots \right) \tag{6.57}$$

(with a log correction when $d = 4$); see [28] for $d = 2$ and [8] for $d \geq 3$. Similar asymptotic behaviour applies also for the honeycomb lattice [30]. The $\mu^n n^{\gamma-1}$ term is the familiar leading asymptotic form for c_n, but the $(-\mu)^n n^{\alpha-2}$ term also appears, with the polygon exponent α. These two terms are reflections of singularities of the generating function $\chi(z)$: one of the form $(1 - z/z_c)^{-\gamma}$ at $z = z_c$, and another of the form $(1 + z/z_c)^{1-\alpha}$ at $z = -z_c$. The latter is referred to as the *antiferromagnetic singularity*.

The direct theoretical evidence for existence of the antiferromagnetic singularity seems to be rather thin, despite the fact that its existence has been recognised for decades [14]. The bipartite nature of $\mathbb{Z}^d$ and the honeycomb lattice plays an important role in the existence of the antiferromagnetic singularity: for example, numerical evidence shows that the antiferromagnetic singularity does not occur for the triangular lattice [29].

The hyperscaling relation $2 - \alpha = d\nu$ allows the leading behaviour in the second term of (6.57) to be rewritten as $(-\mu)^n n^{-d\nu}$, and the predicted value of ν implies that $d\nu > 1$ for all $d \geq 2$. This corresponds to a finite value for $\chi(-z_c)$, but indicates that derivatives of χ of order $d\nu - 1 = 1 - \alpha$ or higher will diverge at $-z_c$.

In this section, we first make a connection between the numerically observed sign alternation in the sequence π_m and the existence of the antiferromagnetic singularity. We then draw parallels between the role of the polygon exponent α in the asymptotic behaviour of π_m and in the asymptotic behaviour of the susceptibility near the antiferromagnetic singularity. Finally, we report the results of series analysis of $1/\chi(z)$ (equivalent to an analysis of $\Pi(z)$) via differential approximants, which provides the locations of the zeroes of the susceptibility.

This section is not devoted to rigorous results, but is a combination of heuristic arguments and numerical observations.

6.7.1 Sign Alternation of π_m

There is now significant numerical evidence that π_m alternates in sign for nearest-neighbour self-avoiding walks on $\mathbb{Z}^d$ ($d \geq 1$) and on the honeycomb lattice, which are all bipartite lattices. For $d = 1$, the sign alternation is immediate from Example 3 of Section 6.3, where $\Pi(z) = -2\sum_{m=2}^{\infty}(-z)^m$. The values of π_m can be computed for $m \leq 71$ on the square lattice and for $m \leq 105$ on the honeycomb lattice using (6.22) and the enumeration of c_n given in [28, 30]. In both cases, the signs are strictly alternating. For $d \geq 3$, the values of π_m are known for $m \leq 30$ when $d = 3$ and for $m \leq 24$ for *all* $d \geq 4$, due to the lace graph enumerations of [8]. In all these cases, the signs are strictly alternating.

On the other hand, for the triangular lattice direct enumeration shows that $\pi_2 = \pi_3 = -6$, and the strict sign alternation fails. A similar result is observed for the fcc lattice. The triangular and fcc lattices are not bipartite.

A bipartite graph is characterised by the absence of odd cycles, which in the translation invariant case is equivalent to the vanishing of $\pi_m^{(1)}$ for all odd m. This means that we can write (recall (6.32) and (6.44)) for integer m

$$\pi_{2m} = -\sum_{M=1}^{m} \left(\pi_{2m}^{(2M-1)} - \pi_{2m}^{(2M)}\right) \tag{6.58}$$

$$\pi_{2m-1} = \sum_{M=1}^{m-1} \left(\pi_{2m-1}^{(2M)} - \pi_{2m-1}^{(2M+1)}\right). \tag{6.59}$$

For $\mathbb{Z}^d$, the values of $\pi_m^{(N)}$ are enumerated in [8] for $m \le 30$, $d = 2,3$, and $m \le 24$, $d \ge 4$. For this range of parameters, we have explicitly verified that $\pi_m^{(2M-1)} - \pi_m^{(2M)} > 0$ for even m, and that $\pi_m^{(2M)} - \pi_m^{(2M+1)} > 0$ for odd m. With (6.58) and (6.59), this gives rise to sign alternation for π_m. (It also raises the question of whether $\pi_m^{(N)} - \pi_m^{(N+1)}$ has a combinatorial interpretation.) The enumerations of [8] actually give a more refined version of the above two inequalities: they are obeyed also if $\pi_m^{(N)}$ is replaced by $\pi_{m,\delta}^{(N)}$, which counts the number of lace graphs which occupy exactly δ dimensions.

If the sequence π_m does indeed strictly alternate in sign for sufficiently large m, then its generating function $\Pi(z) = \sum_{m=2}^{\infty} \pi_m z^m$ will have its dominant singularity on the *negative* real axis. We expect that on lattices such as $\mathbb{Z}^d$, the honeycomb lattice and the triangular lattice, the singularity of $\chi(z)$ at z_c is not merely a pole, so that $\Pi(z)$ will also be singular at z_c. (Note, however, that for self-avoiding walks on a tree of degree $\Omega \ge 3$, Example 3 shows that $\chi(z)$ has a simple pole at $z_c = 1/(\Omega - 1) < 1$ while $\Pi(z)$ has its closest singularity at -1.) A singularity of $\Pi(z)$ at $-x_c$ with $0 < x_c < z_c$ is possible a priori, but since $\chi(z)$ is analytic in the disk of radius z_c such a singularity of $\Pi(z)$ could only be a pole, corresponding to a zero of $\chi(z)$. We find no numerical evidence for this possibility on $\mathbb{Z}^d$ or the honeycomb lattice, as we discuss below. Assuming the absence of such a pole, we conclude that sign alternation of π_m corresponds to a singularity of $\Pi(z)$ at $-z_c$, and that this is the origin of the antiferromagnetic singularity.

There is currently no proof that π_m remains strictly alternating on the honeycomb lattice or on $\mathbb{Z}^d$ for any $d \ge 2$. It is an appealing though perhaps difficult problem to prove this.

6.7.2 π_m and the Polygon Exponent α

The asymptotic form (6.57) implies that $1/\chi(z)$ should vanish like $(1 - z/z_c)^{\gamma}$ near $+z_c$, while near $-z_c$ its singular part should behave like $(1 + z/z_c)^{1-\alpha} = (1 + z/z_c)^{d\nu - 1}$. Consider first the dimensions $d = 2,3$. For $d = 2,3$, the numerical values of γ and ν are such that the antiferromagnetic singularity is the dominant one, i.e., $d\nu - 1 < \gamma$. Since π_m is the m^{th} coefficient in the series for $1/\chi(z)$, this suggests that

$$\pi_m \sim c(-\mu)^m m^{\alpha-2} \quad (d = 2,3). \tag{6.60}$$

Recall that $\pi_m = \sum_{N=1}^{\infty}(-1)^N \pi_m^{(N)}$ is the alternating sum of lace graph counts. The term $\pi_m^{(1)}$ just counts self-avoiding returns, whose asymptotic behaviour is predicted to be given by $\mu^m m^{\alpha-2}$, again with the polygon exponent. It is tempting to guess that $\pi_m^{(N)}$ has this same asymptotic form for all N, but we have insufficient data to verify this numerically. If it does, then the form (6.60) would arise from an alternating sum of counts of objects which are each governed by the polygon exponent α.

The case $d = 4$ is delicate due to logarithmic corrections, but dimensions $d > 4$ are also more subtle. For $d > 4$, the error term in (6.40) can be expected to be sharp, and to be the source of a singularity $(1 - z/z_c)^{(d-2)/2}$ in the susceptibility. Since the power $1 - \alpha$ is also equal to $\frac{d-2}{2}$, this indicates that the ferromagnetic and antiferromagnetic singularities are of equal strength, so that (6.60) should be replaced by

$$\pi_m \sim [c_+ + c_-(-1)^m]\,\mu^m m^{-d/2} \quad (d > 4). \tag{6.61}$$

The observed fact that the sequence π_m does alternate in sign for $m \leq 24$ suggests that the antiferromagnetic singularity still dominates, in the sense that $|c_-| > |c_+|$.

For dimensions $d > 4$, it is proved in [23, 24] that for spread-out models with $d > 4$ and L large (see (6.2)), $\pi_m^{(N)} \leq (cL^{-d})^N \mu^m m^{-d/2}$ and therefore $|\pi_m| \leq \sum_{N=1}^{\infty} \pi_m^{(N)} \leq CL^{-d}\mu^m m^{-d/2}$. This verifies (6.61) as a one-sided bound, and supports the belief that $\pi_m^{(N)}$ may have the same asymptotic form for all N. In particular, it proves that $\Pi(z)$ cannot have a singularity inside the disk of radius z_c. Universality suggests that the same behaviour should apply also for the nearest-neighbour model.

For dimensions $d = 2,3$, there are no rigorous results to help verify (6.60), so it is useful to turn to series analysis. The extensive methodology that has been developed for series analysis is discussed in Chapter 8 by Guttmann and Jensen (a classic earlier reference is [16]). An important method of series analysis is that of differential approximants, a powerful technique which is quite generally applicable and often (although not always) more effective than any other known method. The method of differential approximants can provide information about the singularities of a generating function for which we know a finite number of series coefficients, including poles, power law singularities and their confluent corrections.

We have applied the method of differential approximants to analyse the reciprocal series $1/\chi(z)$ for the square, honeycomb and triangular lattices, and for $\mathbb{Z}^d$ with $3 \leq d \leq 8$ (the susceptibility series itself was analysed in the original papers [8, 28, 29, 30]). Long series which are approaching the asymptotic regime are the most amenable to series analysis, and we obtain our most accurate results for the 2-dimensional lattices. Information for $\mathbb{Z}^d$ with $d \geq 3$ is far more difficult to extract owing to the availability of only relatively short series [8], and the existence of strong corrections to scaling for $d = 3$.

In all cases, we find that $\Pi(z)$ clearly has radius of convergence z_c with leading singularity at the ferromagnetic singularity $-z_c$. This confirms the leading exponential growth $(-\mu)^m$ of (6.60). However, we find that information about critical exponents is degraded with reciprocal series, and more accurate estimates for the power law correction to the leading exponential behaviour $(-\mu)^n$ for π_m are obtained from the prior analyses of $\chi(z)$ than we find via analysis of $1/\chi(z)$. (Dlog Padé approximants give the same results for $\chi(z)$ and $1/\chi(z)$, but the dlog Padé method is not as accurate as more general differential approximants, which give different results.)

6.7.3 Zeroes of the Susceptibility

Although the analysis of $1/\chi(z)$ via differential approximants did not prove to be fruitful for an accurate determination of the exponent α, such analysis does yield the location of susceptibility zeroes, which correspond to poles of $1/\chi(z)$. In this section, we report the location of zeroes of $\chi(z)$ found in this manner.

Square lattice. For the square lattice, the analysis of $\chi(z)$ in [28] clearly confirms the existence of the antiferromagnetic singularity at $-z_c = -0.379052\ldots$ with the polygon exponent $\alpha = \frac{1}{2}$. We find that the differential approximants for $1/\chi(z)$, most clearly the first-order inhomogeneous approximants, detect a pole at $z^* \approx -0.3758$. This would seem to imply the unexpected result that the radius of convergence of $\Pi(z)$ is strictly less than z_c. However, direct integration of the differential equations of the differential approximant method to determine the amplitude of $\chi(z)$ at this point shows that $\chi(z^*) > 0$. The apparent contradiction can be resolved as follows:

Let us assume that

$$\chi(z) \sim A(z)(1+z/z_c)^{1/2} + B(z) \tag{6.62}$$

near $z = -z_c$ (with the critical amplitude $A(z)$ and background amplitude $B(z)$ analytic at $-z_c$), and also assume that $\chi(z) > 0$ for $z \in (-z_c, 0]$. If we integrate the differential equation from the differential approximant for $\chi(z)$ along the negative axis to a point $z = -z_c + \varepsilon$, we obtain a function of the form (6.62), with approximate values for the exponent and z_c. If we then integrate the differential equation around the circle $|z + z_c| = \varepsilon$, starting and ending at $z = -z_c + \varepsilon$, the non-analytic part will pick up a factor of $\exp(\pi i) = -1$. We are now on a different Riemann sheet, and it is possible for $\chi(z)$ to have a zero provided that there is a solution of $-A(z)(1+z/z_c)^{1/2} + B(z) = 0$. We confirm this numerically by using the procedure of Velgakis et al. [45] to find $A(z)$ and $B(z)$ in the vicinity of $z = -z_c$, and observe that there is a zero at a value close to $z^* = -0.3758$. Thus the pole observed in the analysis of $1/\chi(z)$ in fact corresponds to a zero of $\chi(z)$ in the Riemann sheet visited by circling around the singularity at $-z_c$. We do not detect any other poles in $1/\chi(z)$.

Honeycomb lattice. For the honeycomb lattice, the antiferromagnetic singularity is clearly seen in [30]. Without performing a careful error analysis we find, via differential approximant analysis, a complex conjugate pair of simple poles of $1/\chi(z)$ at $z = -0.262426 \pm 0.676916i$, with the error likely to be confined to the final digit. These poles lie outside the circle $|z| = z_c$. We confirm that these are genuine zeroes of $\chi(z)$ by integrating the differential equations obtained via the differential approximant method to obtain a representation of $\chi(z)$, and confirming that the amplitude of $\chi(z)$ is very small in the vicinity of these points.

Triangular lattice. For the triangular lattice there is, as expected, no sign of the antiferromagnetic singularity [28]. Analysis of $1/\chi(z)$ reveals two complex conjugate pairs of poles at $z = -0.464 \pm 0.331i$, and $z = -0.204 \pm 0.611i$, well outside the radius of convergence determined by $z_c = 0.24091\ldots$. We have again confirmed that these poles correspond to zeroes of the susceptibility.

Dimensions $3 \le d \le 8$. Using differential approximants and the enumerations of [8], we clearly observe the antiferromagnetic singularity at $-z_c$, but we find no evidence of any poles for $1/\chi(z)$ anywhere in the complex plane, and thus no evidence of zeroes of the susceptibility.

As a technical point, we find that the first-order inhomogeneous approximants seem to be much more effective than higher order approximants at pinpointing the location of the poles of $1/\chi(z)$. This is probably due to the fact that the more restrictive functional form, which does not allow for confluent corrections, is more appropriate for fitting the function in the immediate vicinity of the pole.

6.8 Lattice Trees

The lace expansion has been applied to a wide range of models [42]. The simplest extension beyond self-avoiding walks is to lattice trees. In this section we discuss some results for lattice trees in dimensions $d > 8$, without entering into details about the methods of proof.

A *lattice tree* on $\mathbb{Z}^d$ is defined to be a finite connected[2] set of bonds which contains no cycles (closed loops). Bonds are pairs $\{x,y\}$ of vertices of $\mathbb{Z}^d$, with $y-x \in \mathscr{N}$, where $\mathscr{N}$ is given either by the nearest-neighbour set (6.1) or the spread-out set (6.2). Although a lattice tree T is defined as a set of bonds, we will write $x \in T$ if x is an element of a bond in T. The number of bonds in T is denoted $|T|$, and the number of vertices in T is thus $|T|+1$.

A basic combinatorial problem is to count the number of lattice trees of fixed size. Let $t_n^{(1)}$ denote the number of n-bond lattice trees that contain the origin. It is customary to count lattice trees modulo translation, namely to consider t_n defined by

$$t_n = \frac{1}{n+1} t_n^{(1)}. \tag{6.63}$$

A sub-additivity argument [33] shows that there is a positive constant λ such that $\lim_{n\to\infty} t_n^{1/n} = \lambda$. The precise asymptotic behaviour of t_n as $n \to \infty$ is believed to be given by

$$t_n \sim A\lambda^n n^{-\theta}, \tag{6.64}$$

where θ is a universal critical exponent. The bounds

[2] This is the standard graph-theory definition of a connected graph.

$$c_1 \lambda^n n^{-c_2 \log n} \le t_n \le c_3 \lambda^n n^{-(d-1)/d}, \tag{6.65}$$

were proved respectively in [27] and [37] for general dimensions $d \ge 2$. The upper bound does provide a power law correction, but it is predicted that $\theta > (d-1)/d$ for all $d \ge 2$.

Let $\bar{x}_T = (|T|+1)^{-1} \sum_{x \in T} x$ denote the centre of mass of T (considered as a set of equal masses at the *vertices* of T), and let

$$R(T)^2 = \frac{1}{|T|+1} \sum_{x \in T} |x - \bar{x}_T|^2 \tag{6.66}$$

denote the squared radius of gyration of T. The typical length scale of a lattice tree is characterized by the average radius of gyration R_n, defined by

$$R_n^2 = \frac{1}{t_n^{(1)}} \sum_{T:|T|=n, T \ni 0} R(T)^2. \tag{6.67}$$

It is predicted that there is a universal critical exponent ν such that

$$R_n \sim D n^{\nu}. \tag{6.68}$$

Based on a field theoretic representation, it was argued in [35] that the upper critical dimension for lattice trees is 8. Further evidence for this was given in [2, 43, 44]. The mean-field values of the exponents are $\theta = \frac{5}{2}$ and $\nu = \frac{1}{4}$. The value $\nu = \frac{1}{4}$ corresponds in (6.68) to n being asymptotic to a multiple of R_n^4, which is a statement of 4-dimensionality. The fact that two 4-dimensional objects generically do not intersect above eight dimensions gives a quick prediction that $d = 8$ is the upper critical dimension for lattice trees.

The lace expansion has been used to prove a number of results for lattice trees in dimensions $d > 8$. The following theorem from [20] proves that $\theta = \frac{5}{2}$ and $\nu = \frac{1}{4}$ in high dimensions.

Theorem 6. *For nearest-neighbour lattice trees with d sufficiently large, or for spread-out lattice trees with $d > 8$ and L sufficiently large, there are positive constants A and D (depending on d, L) such that for every $\varepsilon < \min\{\frac{1}{2}, \frac{d-8}{4}\}$,*

$$t_n = A \lambda^n n^{-5/2} [1 + O(n^{-\varepsilon})], \tag{6.69}$$

$$R_n = D n^{1/4} [1 + O(n^{-\varepsilon})]. \tag{6.70}$$

A *lattice animal* is a finite connected set of bonds which may contain closed loops. It is believed that lattice animals belong to the same universality class as lattice trees, so that both models have the same critical exponents and scaling limits. Results related to Theorem 6 have been obtained for lattice animals, in terms of generating functions [18].

Information about the spatial distribution of lattice trees is contained in the number $t_n(x)$ of n-bond lattice trees containing the vertices $0, x \in \mathbb{Z}^d$. The scaling behaviour of the Fourier transform of $t_n(x)$ in high dimensions is given in the following theorem from [9].

Theorem 7. *For nearest-neighbour lattice trees with d sufficiently large, or for spread-out lattice trees with $d > 8$ and L sufficiently large, as $n \to \infty$,*

$$\hat{t}_n^{(2)}(kD_1^{-1}n^{-1/4}) \sim A\lambda^n n^{-1/2} \int_0^\infty \mathrm{d}t\, t\, \mathrm{e}^{-t^2/2}\mathrm{e}^{-|k|^2 t/2d}, \tag{6.71}$$

where $D_1 = 2^{3/4}\pi^{-1/4}D$, and where A and D are the constants of Theorem 6.

The scaling of the Fourier variable k by $kn^{-1/4} = kn^{-\nu}$ in (6.71) corresponds to rescaling the lattice $\mathbb{Z}^d$ to $n^{-1/4}\mathbb{Z}^d$, and (6.71) is a statement about the scaling limit of $t_n^{(2)}(x)$ in Fourier language. Theorem 7 provides a first step in understanding the scaling limit of lattice trees in dimensions $d > 8$—the full scaling limit has been obtained by proving corresponding statements for the number of n-bond lattice trees containing vertices $x_1, \ldots, x_l$ for all $l \geq 1$. Under the hypotheses of Theorem 7, the scaling limit for $d > 8$ has been shown to be given by the random measure on $\mathbb{R}^d$ known as integrated super-Brownian excursion (ISE) [9, 42], as was first conjectured by Aldous [1]. In a somewhat different formulation, the scaling limit can also be interpreted as the measure-valued stochastic process known as the canonical measure of super-Brownian motion [25]. This is part of a larger story in which the scaling limit of various high-dimensional critical branching models can be understood in terms of super-Brownian motion [41, 42].

A mathematically rigorous analysis of critical exponents for lattice trees in low dimensions appears to be beyond the reach of current methods. However, there has been recent progress by Brydges and Imbrie [3, 4] on a very natural model of continuous branched polymers in $\mathbb{R}^d$, which is expected to be in the same universality class as lattice trees. Inspired by ideas of Parisi and Sourlas [40], in their remarkable paper [3] Brydges and Imbrie proved existence of critical exponents for their continuum model in dimensions $d = 2$ and 3 (with partial results for $d = 4$), with values $\theta = 1$ for $d = 2$, and $\theta = \frac{3}{2}$, $\nu = \frac{1}{2}$ for $d = 3$. An alternate approach to the results of Brydges and Imbrie has recently been obtained by Kenyon and Winkler [31].

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