Chapter 5
The Anisotropic Generating Function of Self-Avoiding Polygons is not D-Finite

Andrew Rechnitzer

5.1 Introduction

The enumeration of self-avoiding polygons, and other families of lattice animals, is one of the most famous problems in enumerative combinatorics, and despite many years of intensive study these problems remain completely open.

Let p_n be the number of self-avoiding polygons on the square lattice of perimeter $2n$ and let $G(z) = \sum p_n z^n$ be the corresponding generating function. Neither an explicit nor a useful implicit expression is known for either the $G(z)$ or p_n. The most efficient means of computing p_n is the finite-lattice method (see Chapter 7). This method is exponentially faster than brute-force enumeration (and considerably so!), but still requires exponential time and space.

One of the most fruitful approaches has been the study of simpler combinatorial models in which extra conditions such as directedness or convexity are imposed (see Chapter 3). Almost all these models when enumerated by their number of bonds, however, share the property that their generating functions are the solutions of ordinary linear differential equations with polynomial coefficients—*differentiably finite* or *D-finite* functions.

Definition 1. Let $F(z)$ be a formal power series in z with coefficients in $\mathbb{C}$. It is said to be *differentiably finite* or *D-finite* if there exists a non-trivial differential equation:

$$P_d(z)\frac{\mathrm{d}^d}{\mathrm{d}z^d}F(z) + \cdots + P_1(z)\frac{\mathrm{d}}{\mathrm{d}z}F(z) + P_0(z)F(z) = 0, \tag{5.1}$$

with P_j a polynomial in z with complex coefficients [12].

D-finite functions have many nice properties including having a finite number of singularities [22]. Additionally a knowledge of the differential equation is sufficient

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to compute the coefficients of the generating function in linear time and also their asymptotic behaviour.

In this work we seek to show that the generating function of self-avoiding polygons is distinctly different from those of models that have been solved to date, in that it is not D-finite, and so give some explanation why the problem remains unsolved. One way of doing this would be show that it has an infinite number of singularities; while there is strong numerical data for the location of the dominant singularity (see [2, 10] for example), very little is known about subdominant singularities.

Guttmann and Enting [8] devised a numerical method for examining the singularity structure of solved and unsolved lattice models based on their *anisotropic* generating function. Their survey of these generating functions demonstrated a distinct difference between solved and unsolved bond animal problems. They observed a similar difference for thermodynamic functions of lattice models of magnets such as the Ising model free-energy and susceptibility. They proposed that this could be used as a test of “solvability”; it provides compelling evidence that the anisotropic generating functions of many unsolved problems are not D-finite. In this article we prove that this is indeed the case for self-avoiding polygons.

To form this generating function we distinguish between vertical and horizontal bonds, and so count according to the vertical and horizontal half-perimeters

$$G(x,y) = \sum_{P\in\mathscr{G}} x^{|P|_{\leftrightarrow}} y^{|P|_{\updownarrow}}, \tag{5.2}$$

where $\mathscr{G}$ is the set of all self-avoiding polygons, $|P|_{\leftrightarrow}$ and $|P|_{\updownarrow}$ respectively denote the horizontal and vertical half-perimeters of a polygon P. By partitioning $\mathscr{G}$ according to the vertical half-perimeter we may resum the above generating function as

$$G(x,y) = \sum_{n\geq 1} y^n \sum_{P\in\mathscr{G}_n} x^{|P|_{\leftrightarrow}} = \sum_{n\geq 1} H_n(x) y^n, \tag{5.3}$$

where $\mathscr{G}_n$ is the set of SAPs with $2n$ vertical bonds, and $H_n(x)$ is its horizontal half-perimeter generating function.

In some sense, the anisotropic generating function is a more manageable object than the isotropic. Splitting the set of animals, $\mathscr{G}$, into separate simpler subsets, $\mathscr{G}_n$, breaks the problem into smaller pieces, each of which is easier to study than the whole. If one seeks to compute the *isotropic* generating function then one must examine *all* possible configurations that can occur in $\mathscr{G}$. Arguably, this is the reason that the only families of bond animals that have been solved are those with severe topological restrictions (such as directedness or convexity). On the other hand, if we examine the generating function of $\mathscr{G}_n$, then the number of different shapes that can occur is always finite (this idea will be made more precise below). Similarly, instead of trying to study the properties of the whole (possibly unknown) generating function, the anisotropy separates the generating function into separate simpler pieces, $H_n(x)$, that can be calculated exactly (for small n). By studying the properties of these coefficients, particularly their singularities, we can obtain some idea of the properties of the generating function as a whole.

While no solution is known for $G(x,y)$, the first few coefficients of y may be computed exactly [1]. This was done for self-avoiding polygons (and a range of other problems) by Guttmann and Enting [8] and they observed the following:

- $H_n(x)$ is a rational function of x,
- the degree of the numerator of $H_n(x)$ is equal to the degree of its denominator,
- the denominators of $H_n(x)$ (we denote them $D_n(x)$) are products of cyclotomic polynomials [2] and the first few are:

$$\begin{aligned}
D_1(x) &= (1-x)\\
D_2(x) &= (1-x)^3\\
D_3(x) &= (1-x)^5\\
D_4(x) &= (1-x)^7\\
D_5(x) &= (1-x)^9(1+x)^2\\
D_6(x) &= (1-x)^{11}(1+x)^4\\
D_7(x) &= (1-x)^{13}(1+x)^6(1+x+x^2)\\
D_8(x) &= (1-x)^{15}(1+x)^8(1+x+x^2)^3\\
D_9(x) &= (1-x)^{17}(1+x)^{10}(1+x+x^2)^5\\
D_{10}(x) &= (1-x)^{19}(1+x)^{12}(1+x+x^2)^7(1+x^2).
\end{aligned} \tag{5.4}$$

The singularities of the $H_n(x)$, if this pattern persists will become dense on $|x| = 1$. A similar pattern was observed for many unsolved models and is absent in solved models such as staircase or convex polygons. Guttmann and Enting suggested that this pattern of singularities becoming dense on $|x| = 1$ was the hallmark of an unsolvable problem, and that it could be used as a test of *solvability*; D-finite functions of two variables do not display this behaviour.

Then definition of D-finite can be extended to encompass multivariate functions [12] and so cover the anisotropic generating functions considered here.

Definition 2. Let $G(x,y)$ be a formal power series in y with coefficients that are rational functions of x. Such a series is said to be D-finite if there exists a non-trivial differential equation:

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[1] More precisely, the first hundred (or so) terms of the expansion of $H_n(x)$ can be computed using either brute force or the finite-lattice method. The first few tens of these can then be fitted using Padé approximants. and the remaining terms can be used to "verify" the conjectured form. We also note that one can show that $H_n(x)$ is rational using transfer matrix arguments and bounds are given for the numerator and denominator degrees in [15, 16], and so the conjectured forms are exact.

[2] The cyclotomic polynomials $\Psi_k(x)$ are the factors of the polynomials $(1-x^n)$. More precisely

$$(1-x^n) = \prod_{k|n} \Psi_k(x).$$

If k is a prime number then $\Psi_k(x) = 1+x+x^2+\cdots x^{k-1}$.

$$Q_d(x,y)\frac{\partial^d}{\partial y^d}G(x,y)+\cdots+Q_1(x,y)\frac{\partial}{\partial y}G(x,y)+Q_0(x,y)G(x,y)=0, \qquad (5.5)$$

with Q_j a polynomial in x and y with complex coefficients

One can show that such functions cannot have a dense set of singularities.

Theorem 1 (from [5]). *Let $f(x,y) = \sum_{n\geq 0} H_n(x)y^n$ be a D-finite series in y with coefficients $H_n(x)$ that are rational functions of x. Let S_n be the set of poles of $H_n(x)$ and $S = \cup_n S_n$. Then S has only a finite number of singularities.*

Ideally we would like to determine all of the singularities of the coefficients, $H_n(x)$, but unfortunately we have not been able to do so. Instead, we are able to prove that the first occurrence of each cyclotomic factor in the denominators of $H_n(x)$ does not cancel with the corresponding numerator.

Theorem 2. *Write $G(x,y) = \sum H_n(x)y^n$. The function $H_{3k-2}(x)$ has simple poles at the zeros of $\Psi_k(x)$ except when $k = 2$.*

This then implies that the set of singularities of the $H_n(x)$ form a dense set on $|x| = 1$ and so we have the corollary and our main result:

Corollary 1. *Let S_n be the set of singularities of the coefficient $H_n(x)$. The set $S = \bigcup_{n\geq 1} S_n$ is dense on the unit circle $|x| = 1$. Consequently the self-avoiding polygon anisotropic half-perimeter generating function is not a D-finite function of y.*

In Section 5.2 we describe the "haruspicy" techniques that are used to define equivalence classes on the set of polygons. Each equivalence class has a simple rational generating function. Adding these generating functions together proves that $H_n(x)$ is rational and that its denominator is a product of cyclotomic polynomials (see Theorem 3). In Section 5.3 we determine which of these equivalence classes cause the first appearance of $\Psi_k(x)$ as a denominator factor. It turns out that these equivalence classes have a simple description and we can use this to find a functional equation satisfied by their generating function. Analysing this functional equation then completes the proof of Theorem 2. Finally in Section 5.4 we describe extensions of this work to other problems.

5.2 Haruspicy

In [15], the author developed techniques which allow us to determine properties of the coefficients, $H_n(x)$, whether or not they are known in some nice form. The central idea is to reduce the set of polygons to some sort of minimal set; various properties of the $H_n(x)$ may be inferred by examining the bond configurations of those minimal polygons. Since polygons are a type of bond animal, we refer to this approach as *haruspicy*; the word refers to techniques of divination based on the examination of the forms and shapes of the organs of animals.

5.2.1 Sections, Squashing and Posets

We start by describing how polygons may be cut up into simpler pieces that can be reduced in a consistent way. In particular we cannot violate self-avoidance, and we also need to conserve enough information to recover the original polygon. We cut the polygon into *pages* each of which may be expanded or reduced independently from the rest of the polygon.

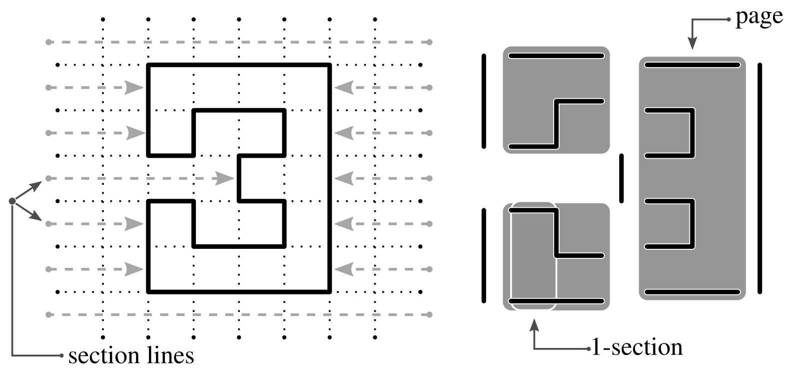

Fig. 5.1 *Section lines*, indicated by the grey dashed lines in the left-hand figure, split the polygon into *pages*. The pages are shown in right-hand figure. Each column in a page is a *section*. This polygon is split into three pages, each containing two sections; a 1-section is highlighted. Ten vertical bonds lie between pages and four vertical bonds lie within the pages.

Definition 3. We construct the *section lines* of a polygon in the following way. Draw horizontal lines from the extreme left and the extreme right of the lattice towards the polygon so that the lines run through the middle of each lattice cell. The lines are terminated when they first touch a vertical bond (see Fig. 5.1).

Cut the lattice along each section line from infinity until it terminates at a vertical bond. Then from this vertical bond cut vertically in both directions until another section line is reached. In this way the polygon is split into *pages* (see Fig. 5.1); we consider the vertical bonds along these vertical cuts to lie *between* pages, while the other vertical bonds lie *within* the pages.

We cannot stretch or expand the horizontal bonds within a page independently of each other without violating self-avoidance. Instead we can expand or delete the horizontal bonds in a given column of a page together.

Definition 4. We call a *section* the set of horizontal bonds within a single column of a given page. Equivalently, it is the set of horizontal bonds of a column of a polygon between two neighbouring section lines. A section with $2k$ horizontal bonds is a k-section. The number of k-sections in a polygon, A, is denoted by $\sigma_k(A)$.

A polygon can now be encoded as a list of pages and sections within those pages. Many of these sections, however, are not really needed to encode the shape (in some loose sense of the word) of the polygon. If two neighbouring sections are the same we can remove one of them and still leave the shape of the polygon unchanged.

Definition 5. We say that a section is a *duplicate section* if the section immediately on its left (without loss of generality) is identical and there are no vertical bonds between them (see Fig. 5.2).

One can squash or reduce polygons by *deletion* of duplicate sections by slicing the polygon on either side of the duplicate section, removing the section and re-combining the polygon, as illustrated in Fig. 5.2. By reversing the section deletion process we define *duplication* of a section.

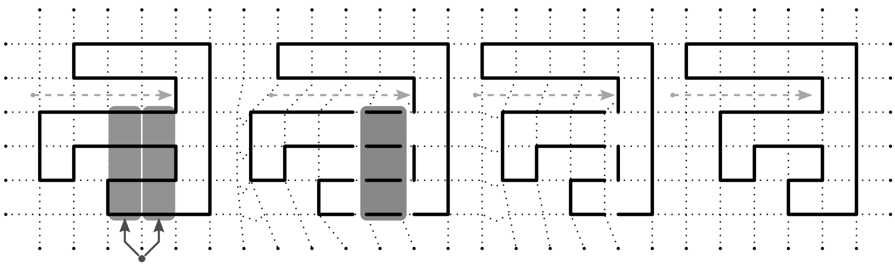

Fig. 5.2 The two indicated sections are duplicates. We can delete the duplicate by slicing on either side separating the polygon into three pieces. The middle piece, being the duplicate, is removed and the remainder of the polygon is recombined. Reversing the steps duplicates the section duplication. Also indicated is a section line which separates the duplicate sections from the rest of the columns in which they lie.

Section-deletion defines a partial order, $\preceq$, on the set of polygons (see Fig. 5.2). Hence polygons together with this partial order form a partially ordered set, or poset.

Lemma 1. *Let P and Q be two polygons in $\mathscr{G}_n$. Write $P \preceq Q$ if P can be obtained from Q by a sequence of section-deletions. This relation is a partial order on the set of polygons.*

Proof. Let A, B and C be polygons. A partial order must be reflexive, anti-symmetric and transitive.

- reflexive—By definition $A \preceq A$.
- anti-symmetric—If $A \preceq B$, then either $A = B$ or A can be obtained from B by a sequence of deletions. This implies that either $A = B$ or $|A|_\leftrightarrow < |B|_\leftrightarrow$. Similarly if $B \preceq A$ then either $B = A$ or $|B|_\leftrightarrow < |A|_\leftrightarrow$. Hence if $A \preceq B$ and $B \preceq A$ then $A = B$.
- transitive—If $A \preceq B$ then there exists a sequence of section-deletions that takes B to A. Similarly if $B \preceq C$, then there exists another sequence of section-deletions that takes C to B. Concatenating these gives a sequence of deletions that takes C to A, and hence $A \preceq C$.

□

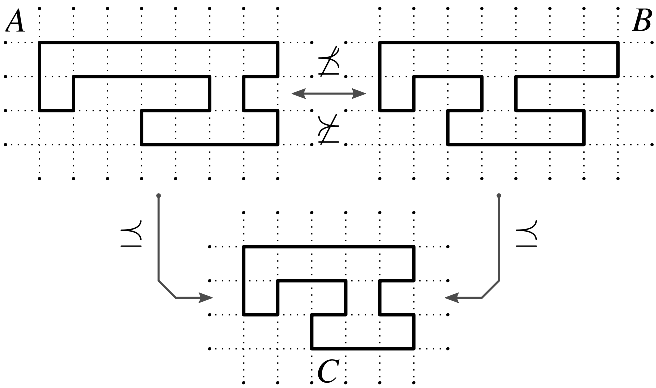

Fig. 5.3 Polygons A and B can be reduced by section-deletions to C, so $C \preceq A, B$. However A cannot be reduced to B or vice-versa. Also C does not contain any duplicate sections and so is section-minimal. Since all three polygons reduce to the same minimal polygon, they are all section-equivalent.

5.2.2 Minimal Polygons, Equivalence Classes and Generating Functions

It is clear that a polygon cannot be reduced to nothing. We quickly reach a polygon without duplicate sections—an example is given in Fig. 5.3. These minimal elements of the self-avoiding polygon poset can be used to reconstruct any polygon by duplicating sections; a knowledge of the minimal polygons is sufficient to reconstruct the entire set and its generating function.

Definition 6. A *section-minimal polygon*, A, is a polygon such that for all polygons, B, satisfying $B \preceq A$, then $B = A$. *i.e.* A cannot be reduced any further.

Lemma 2. *Every polygon C reduces by section-deletions to a unique section-minimal polygon.*

Proof. Number the pages of a given polygon B from $1, 2, \ldots$ (from left to right, top to bottom). Consider, without loss of generality, the first page. We can encode the sections that lie within this page as a sequence $(s_1^{\alpha_1}, s_2^{\alpha_2}, \ldots s_j^{\alpha_j})$, where $s_i^{\alpha_i}$ denotes α_i repetitions of the section s_i. If we enforce the condition that $s_i \neq s_{i+1}$ then the α_i are unique. Deleting all the duplicate sections within this page reduces it to the unique sequence $(s_1^1, s_2^2, \ldots, s_j^1)$.

Note that section-deletion does not delete pages, nor does it move sections between pages, and so repeating this process for each page will reduce B to a unique minimal polygon. □

If two polygons reduce to the same minimal polygon then they have (roughly speaking) similar shapes (see Fig. 5.3). We use this idea to define an equivalence relation.

Lemma 3. *If two polygons, A and B reduce to the same minimal polygon then we say that they are section-equivalent and write $A \approx B$. Section-equivalence is an equivalence relation.*

Proof. It follows almost directly from the definition that section-equivalence is reflexive, symmetric and transitive. □

This equivalence relation induces equivalence classes each of which has a simple rational generating function whose denominator is a product of cyclotomic polynomials. This shows the link between the minimal polygons and the structure of $H_n(x)$.

Definition 7. Section-equivalence partitions the set of polygons into equivalence classes each of which can be characterised by the minimal polygon within the class. We refer to the equivalence class of a section-minimal polygon, A, as the section-expansion of A. We write:

$$\mathscr{X}(A) = \{B \in \mathscr{G} \mid A \preceq B\}. \tag{5.6}$$

Note that all the elements in such an expansion must have the same number of vertical bonds. We write the horizontal bond generating function of the expansion of a minimal element, A, as

$$G(A) = \sum_{B \in \mathscr{X}(A)} x^{|B|_\leftrightarrow} \qquad \text{if } A \text{ is section-minimal.} \tag{5.7}$$

Lemma 4. *Let P be a section-minimal polygon; its expansion has the following generating function:*

$$G(P) = \prod_k \left(\frac{x^k}{1-x^k} \right)^{\sigma_k(P)} \tag{5.8}$$

Proof. Let P be a section-minimal polygon. Each page of the polygon can be encoded as a sequence of sections $(s_1, \dots, s_j)$, with $s_i \neq s_{i+1}$. Since we can duplicate any section in P any number of times, given any $(\alpha_1, \dots, \alpha_j) \in \mathbb{Z}^{+j}$ there exists a polygon Q whose corresponding page is encoded by a sequence of sections $(c_1^{\alpha_1}, \dots c_j^{\alpha_j})$. So

$$\begin{aligned} G(P) &= \prod_{\text{pages}} \prod_i \sum_{\alpha_i} |x_k|_\leftrightarrow (x^{|s_i|_\leftrightarrow})^{\alpha_i} \\ &= \prod_{\text{pages}} \prod_i \frac{x^{|s_i|_\leftrightarrow}}{1-x^{|s_i|_\leftrightarrow}} \end{aligned} \tag{5.9}$$

where $|s_i|_\leftrightarrow$ is the number of horizontal bonds in s_i. The result follows. □

Lemma 5. *If $\mathscr{G}_n$ is a set of polygons with n vertical bonds, then the set of section-minimal elements in $\mathscr{G}_n$ is finite.*

Proof. Let A be a section-minimal polygon in $\mathscr{G}_n$. It is clear that A cannot contain more than n rows. Between any two columns of A there must be at least a single vertical bond. If there is no vertical bond between two columns, then the horizontal bond configuration in each column must be the same and so they will be duplicates of each other and so A is not minimal. Hence A contains at most $2n+1$ columns. Since there are a finite number of bond configurations containing at most n rows and $2n+1$ columns there are only a finite number of section-minimal polygons. □

We can now prove two theorems about the coefficients y^n in the polygon generating function.

Theorem 3. *If $G(x,y) = \sum_{n\geq 0} H_n(x) y^n$ is the anisotropic generating function of self-avoiding polygons, $\mathscr{G}$, then*

- *$H_n(x)$ is a rational function,*
- *the degree of the numerator of $H_n(x)$ cannot be greater than the degree of its denominator, and*
- *the denominator of $H_n(x)$ is a product of cyclotomic polynomials.*

Proof. Let $\mathscr{M}$ be the set of section-minimal polygons of $\mathscr{G}_n$. Since each polygon in $\mathscr{G}_n$ is an element in the expansion of exactly one element in $\mathscr{M}$ we can write

$$H_n(x) = \sum_{B \in \mathscr{G}_n} x^{|B|_\leftrightarrow} = \sum_{A \in \mathscr{M}} G(A) \tag{5.10}$$

Lemmas 5 and 4 imply that this sum is a finite sum of rational functions with the desired properties. The result follows. □

Looking a little more carefully at the number of k-sections present in minimal polygons gives the following theorem.

Theorem 4. *If $H_n(x)$ has a denominator factor $\Psi_k(x)$, then $\mathscr{G}_n$ must contain a section-minimal polygon containing a K-section for some $K \in \mathbb{Z}^+$ divisible by k. Further if $H_n(x)$ has a denominator factor $\Psi_k(x)^\alpha$, then $\mathscr{G}_n$ must contain a section-minimal polygon that contains α sections that are K-sections for some (possibly different) $K \in \mathbb{Z}^+$ divisible by k.*

Proof. Let $\mathscr{M} = \{M_i\}$ be the set of section-minimal polygons $\in \mathscr{G}_n$.

$$\begin{aligned} H_n(x) &= \sum_i GM_i \\ &= \sum_i \prod_K \left(\frac{x^K}{1-x^K} \right)^{\sigma_K(M_i)} \\ &= \sum_i x^{|M_i|_\leftrightarrow} \prod_k \Psi_k(x)^{-\sum_d \sigma_{kd}(M_i)} \\ &= \frac{< \text{some polynomial in } x >}{\prod_k \Psi_k(x)^{\mu_k}} \end{aligned} \tag{5.11}$$

where $\mu_k \leq \max_i\{\sum_d \sigma_{kd}(M_i)\}$—this is an inequality since the numerator and denominator could share common cyclotomic factors. Consequently if there is no minimal element M_i containing a K-section (for some K divisible by k) then $\mu_k = 0$, and the denominator cannot contain $\Psi_k(x)$. □

The above theorems can be generalised to most interesting sets of bond animals—such as self-avoiding walks, bond animals, bond trees and directed bond animals (see [15, 16]).

By determining how many vertical bonds are required to construct a section-minimal polygon with α k-sections, one can use the above theorems to show that

$$\text{The denominator of } H_n(x) \text{ divides } \prod_{k=1}^{\lceil n/3 \rceil} \Psi_k(x)^{2n-6k+5}. \tag{5.12}$$

In fact the denominator of $H_n(x)$ appears (as far as available data permits us to observe) to be exactly the right hand side of the above expression divided by a single power of $\Psi_2(x)$.

There is a similar result [15] for the corresponding generating function of general bond animals (in which x is conjugate to the total number of horizontal bonds), namely

$$\text{The denominator of } H_n(x) \text{ divides } \Psi_1(x)^{3n+1} \prod_{k=2}^{\lfloor n/2 \rfloor} \Psi_k(x)^{2n-3k+4}. \tag{5.13}$$

The denominator of $H_n(x)$ appears to be exactly equal to the right-hand side of the above expression.

These results can be considered upper bounds on the exponents of $\Psi_k(x)$ in the denominator of $H_n(x)$. These are bounds, rather than equalities, since denominator factors might cancel with terms in the numerator. Demonstrating that a given factor does or does not cancel is considerably more difficult and we have only been able to do so in the case of the first occurrence of $\Psi_k(x)$. This is what we do below.

5.3 Analysing *2-4-2* Polygons

In this section we study the first occurrence of a given cyclotomic factor in the denominators of the $H_n(x)$. We start by characterising the section-minimal polygons that give rise to them. These polygons turn out to be significantly easier to construct than general self-avoiding polygons and we can find a functional equation for their generating function. The singularities of the solution of this equation give the singularities of the $H_n(x)$ and so lead us to Theorem 2.

5.3.1 The First **k**-*Section*

Examining the denominators (see (5.4)) of the first few $H_n(x)$ we see that $\Psi_k(x)$ first appears in the denominator of $H_{3k-2}(x)$ (with the exception of Ψ_2 which first appears in $H_5(x)$). We start by showing that it takes $6k-4$ vertical bonds to build a polygon that contains a k-section, and so $\Psi_k(x)$ cannot occur in the denominators of $H_n(x)$ for $n < 3k-2$.

Lemma 6. *To the left (without loss of generality) of a k-section there are at least* $3k-2$ *vertical bonds, of which at least* $2k-1$ *obstruct section lines. Hence no polygon with fewer than* $6k-4$ *vertical bonds may contain a k-section. Further, it is always possible to construct a polygon with* $6k-4$ *vertical bonds and a single k-section.*

Proof. Consider a vertical line drawn through a k-section (as depicted in the left-hand side of Fig. 5.4). The line starts outside the polygon and then as it crosses horizontal bonds it alternates between the inside and outside of the polygon. More precisely, there are $k+1$ segments of the line that lie outside the polygon and k segments that lie inside the polygon. Call the segments that lie within the polygon "*inside gaps*" and those that lie outside "*outside gaps*".

Draw a horizontal line through an inside gap (as depicted in the top-right of Fig. 5.4). This line must cross at least one vertical bond to the left of the gap (since it is inside the polygon) and then another to the right of the gap. Hence to the left of any inside gap there must be at least one vertical bond. Similarly there must be at least one vertical bond to the right of any inside gap.

Draw a horizontal line through the topmost of the $k+1$ outside gaps. Since the line need not intersect the polygon it need not cross any vertical bonds at all. Similarly for the bottommost outside gap.

Now consider a horizontal line through one of the other outside gaps (as depicted in the bottom-right of Fig. 5.4). Traverse this line from the left towards the outside gap. If no vertical bonds are crossed then a section line may be drawn from the left into the outside gap. This splits the k-section into two smaller sections and so contradicts our assumptions. Hence the line must cross at least one vertical bond to block section lines. If only a single vertical bond is crossed before reaching the gap then the gap would lie inside the polygon. Hence the line must cross at least two (or any even number) vertical bonds before reaching the gap. Similar reasoning shows that it must also cross an even number of vertical bonds to the right of the gap.

Since any k-section contains k inside gaps, a topmost outside gap, a bottommost outside gap and $k-1$ other outside gaps, there must be at least $k \times 1 + 2 \times 0 + 2 \times (k-1) = 3k-2$ vertical bonds to its left and $3k-2$ vertical bonds to its right. The polygons depicted in Fig. 5.5 are constructed by adding "*hooks*". In this way it is possible to construct a section-minimal polygon with $(6k-4)$ vertical bonds and exactly one k-section. □

Now that we have established that Ψ_k cannot occur before H_{3k-2}, we bound the exponent with which it occurs in H_{3k-2} by bounding the number of k-sections that a section-minimal polygon can have.

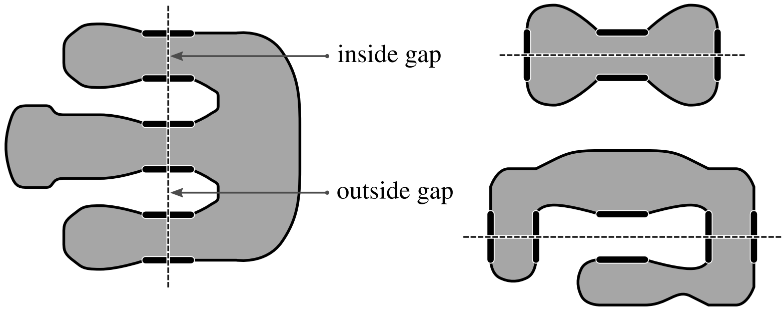

Fig. 5.4 Vertical and horizontal lines drawn through a k-section show the minimum number of vertical bonds required in their construction.

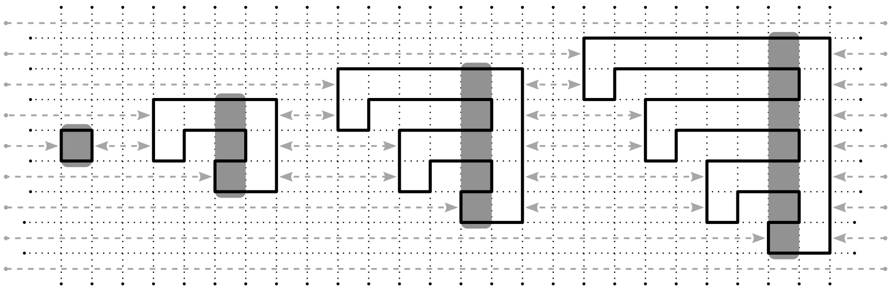

Fig. 5.5 Section-minimal polygons with $6k-4$ vertical bonds and a single k-section may be constructed by concatenating such "*hook*" configurations.

Lemma 7. *A section-minimal polygon with* $6k-4$ *vertical bonds cannot contain more than one k-section. Hence the factor* $\Psi_k(x)$ *in the denominator of* $H_{3k-2}(x)$ *cannot occur with exponent greater than* 1.

Proof. Let A be a section-minimal polygon with $6k-4$ vertical bonds and more than one k-section. We show that A cannot exist. The second statement of the lemma then follows from the first by Theorem 4.

Assume that A has only a single k-section in each column. By Lemma 6, there are $3k-2$ vertical bonds to the left of the leftmost k-section and $3k-2$ vertical bonds to the right of the rightmost k-section. Between any two k-sections there must be at least one vertical bond (or they would be duplicates). Hence A contains more than $6k-4$ vertical bonds. If, on the other hand, A contains a column with two or more k-sections, then to the left of this column there must be at least $6k-4$ vertical bonds and similarly to its right. Hence A contains at least $12k-8$ vertical bonds. Hence A cannot exist. □

In order to proceed we need to split the set of polygons with $6k-4$ vertical bonds into those that contain a k-section and those which do not. While we can, in principle define these sets of polygons, it is much easier to define a superset of those that contain a k-section, and this does not significantly alter the subsequent analysis.

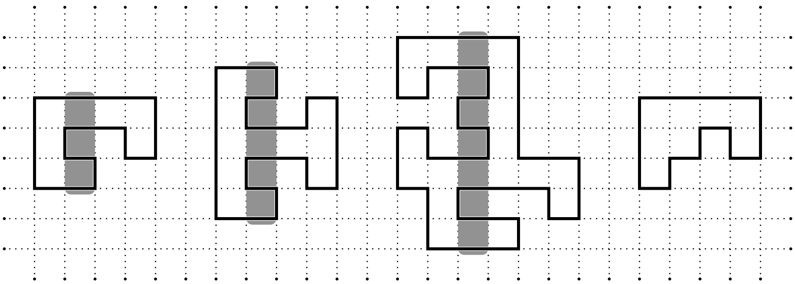

Fig. 5.6 Four section-minimal *2-4-2* polygons. The first three contain a 2-, 3- and 4-section respectively, while the rightmost only contains 1-sections.

Definition 8. Number the rows of a polygon P starting from the topmost row (row 1) to the bottommost (row r). Let $v_i(P)$ be the number of vertical bonds in the i^{th} row of P. If $(v_1(P),\dots,v_r(P)) = (2,4,2,\dots,4,2)$ then we call P a *2-4-2* polygon. We denote the set of such *2-4-2* polygons with $2n$ vertical bonds by $\mathscr{P}_n^{242}$. Note that this set is empty unless $2n = 6k-4$.

Lemma 8. *A section-minimal polygon with* $(6k-4)$ *vertical bonds that contains one k-section must be a* 2-4-2 *polygon. On the other hand, a section-minimal* 2-4-2 *polygon need not contain a k-section.*

Proof. The first statement follows by arguments given in the proof of Lemma 6. The rightmost polygon in Fig. 5.6 shows that a *2-4-2* polygon need not contain a k-section. □

Now that we have isolated the polygons that contain a k-section, the following lemma shows that we can ignore the effect of the remaining polygons.

Lemma 9. *The factor* $\Psi_k(x)$ *appears in the denominator of the generating function* $\sum_{P\in\mathscr{P}_{3k-2}^{242}} x^{|P|_\leftrightarrow}$ *with exponent exactly equal to* 1 *if and only if it appears in the denominator of* $H_{3k-2}(x)$ *with exponent exactly equal to one.*

Proof. The set of *2-4-2* polygons is closed under section-deletion (since it does not move vertical bonds between rows). Similarly the complement of this set is closed under section-deletion. One can then prove that similar results to Theorems 3 and 4 hold for these sets.

Hence the horizontal half-perimeter generating functions of these sets are rational and their denominators are products of cyclotomic factors. Since $\mathscr{P}_{3k-2} \setminus \mathscr{P}_{3k-2}^{242}$ does not contain a polygon with k-section (or indeed, by Lemma 6, any section with more than $2k$ horizontal bonds), it follows (by similar results to Theorem 4) that the denominator of the horizontal half-perimeter generating function of this set is a product of cyclotomic polynomials $\Psi_j(x)$ for $j < k$. Consequently this generating function is not singular at the zeros of $\Psi_k(x)$.

By Lemma 7 every section-minimal polygon in $\mathscr{P}_{3k-2}^{242}$ contains at most one k-section, and so the exponent of $\Psi_k(x)$ in the denominator of the horizontal half-perimeter generating function of $\mathscr{P}_{3k-2}^{242}$ is either one or zero (due to cancellations with the numerator). The result follows since this denominator factor may not be cancelled by adding the other generating function. □

5.3.2 Hadamard Products and Functional Equations

Lemma 9 tells us that in order to prove that the denominator of $H_{3k-2}(x)$ has a factor of $\Psi_k(x)$ it suffices to examine the generating function of P^{242}_{3k-2}. Since *2-4-2* polygons have simpler structure than general self-avoiding polygons, this task is much easier. The technique we use is a variation of the Temperley method [21] and leads to functional equations very similar to those in [3]. It also appears in [4].

We construct *2-4-2* polygons by cutting them into smaller *2-4-2* polygons (see Fig. 5.7). In particular we decompose them into a rectangle of unit height and a sequence of *2-4-2* polygons each of height 3. Call these *2-4-2* polygons of height three "building blocks". We then glue these pieces back together. A functional equation for the generating function of all *2-4-2* polygons can then be obtained from the generating function of the building blocks.

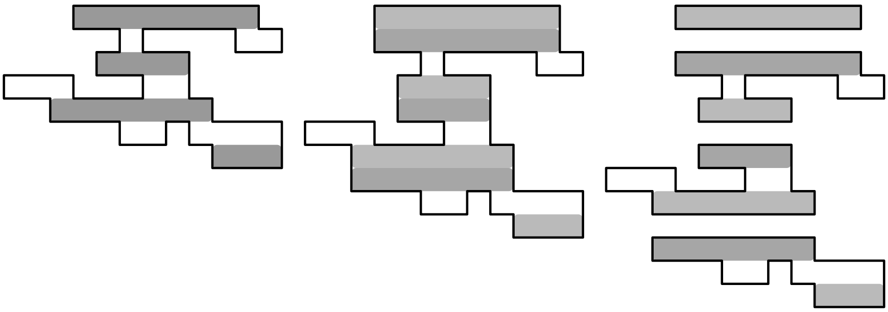

Fig. 5.7 Decomposing *2-4-2* polygons into a sequence of building blocks (*2-4-2* polygons of height three). Highlight each row with 2 vertical bonds. Then duplicate each of these rows excepting the bottommost. By cutting along each of the duplicated rows, the polygon is then uniquely decomposed into a rectangle of unit height and a sequence of building blocks.

Lemma 10. *Let $T(t,s;x,y)$ be the generating function of* 2-4-2 *polygon building blocks, where t and s are conjugate to the length of top and bottom rows (respectively). Then T may be expressed as*

$$T(t,s;x,y) = 2\left(\hat{T}(t,s;x,y) + \hat{T}(s,t;x,y)\right), \tag{5.14}$$

where the generating function $\hat{T}(t,s;x,y)$ is given by

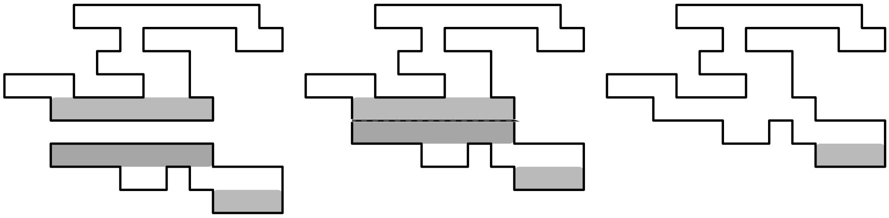

Fig. 5.8 Constructing a *2-4-2* polygon from a (shorter) *2-4-2* polygon and a building block. When the building block and the polygon are squashed together, the total vertical perimeter is reduced by 2, and the total horizontal perimeter is reduced by twice the width of the joining row.

$$\begin{aligned}\hat{T}(t,s;x,y) = y^4 \big(& A(s,t;x)\cdot [\![stx]\!][\![tx]\!]^2 \cdot B(s,t;x) \\ & + A(s,t;x)\cdot [\![stx]\!][\![stx^2]\!][\![tx]\!]^2 \cdot B(s,t;x) \\ & + A(s,t;x)\cdot [\![stx]\!][\![tx]\!]^3 \cdot B(s,t;x) \\ & + C(s,t;x)\cdot [\![sx]\!][\![tx]\!]^3 \cdot B(s,t;x) \\ & + C(s,t;x)\cdot [\![sx]\!][\![x]\!][\![tx]\!]^3 \cdot B(s,t;x)\big)\,. \end{aligned} \tag{5.15}$$

We have used $[\![f]\!]$ *as shorthand for* $\frac{f}{1-f}$, *and the generating functions A, B and C are:*

$$\begin{aligned} A(s,t;x) &= 1 + [\![x]\!] + 2[\![sx]\!] + 2[\![tx]\!] + [\![sx]\!][\![tx]\!] + \\ & \quad [\![sx]\!]^2 + [\![sx]\!][\![x]\!] + [\![tx]\!]^2 + [\![tx]\!][\![x]\!] \end{aligned} \tag{5.16a}$$

$$B(s,t;x) = 1 + [\![tx]\!] + [\![x]\!] \tag{5.16b}$$

$$C(s,t;x) = 1 + [\![sx]\!] + [\![x]\!]. \tag{5.16c}$$

Proof. Figures 5.9 and 5.10 show how to construct the generating function $\hat{T}$ of building blocks in one orientation; each building block can be placed in one of four orientations.To obtain all building blocks we must reflect the blocks counted by $\hat{T}$ about both horizontal and vertical lines. Reflecting about a vertical line multiplies $\hat{T}$ by 2. Reflecting about a horizontal line interchanges the roles of s and t. This proves the first equation.

We compute $\hat{T}$ by considering all the section-minimal polygons that contribute to it. This is done in detail in [16]. One can decompose each section-minimal polygon into one of the five polygons given in Fig. 5.9. The left and right ends of these polygons are made up of the *frills* given in Fig. 5.10. This calculation can be (and has been) verified using the P-partition techniques in [20].

The above equation for $\hat{T}(t,s;x,y)$ follows by expanding each of the sections in the minimal polygons. □

Larger *2-4-2* polygons can be constructed by gluing a building block onto a smaller *2-4-2* polygon as illustrated in Fig. 5.8. Gluing combinatorial objects together usually corresponds to multiplying their generating functions. However, when we glue together these objects we require that the top row of the building

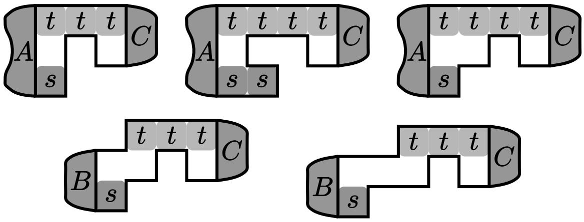

Fig. 5.9 The section-minimal building blocks of *2-4-2* polygons. The "*frills*", denoted A, B and C are given in Fig. 5.10.

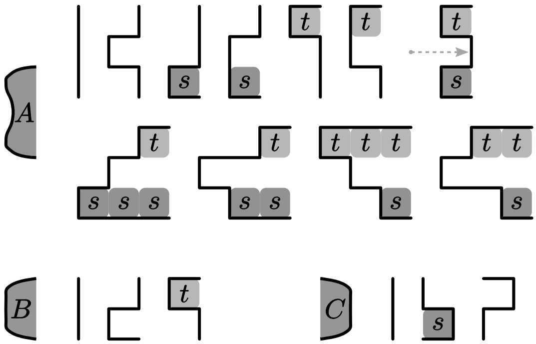

Fig. 5.10 The "*frills*" of the building blocks in Fig. 5.9.

block has the same length as the bottom row of the *2-4-2* polygon. The corresponding operation on their generating functions is a type of Hadamard product.

Definition 9. Let $f(t) = \sum_{t\geq 0} f_n t^n$ and $g(t) = \sum_{t\geq 0} g_n t^n$ be two power series in t. We define the (restricted) Hadamard product $f(t) \odot_t g(t)$ to be

$$f(t) \odot_t g(t) = \sum_{n\geq 0} f_n g_n. \tag{5.17}$$

It is generally quite difficult to (explicitly) calculate the Hadamard product of two functions. However when one of the functions is rational the problem is much simpler.

Lemma 11. *Let $f(t) = \sum_{t\geq 0} f_n t^n$ be a power series, then*

$$f(t) \odot_t \frac{1}{1-\alpha t} = f(\alpha) \tag{5.18a}$$

$$f(t) \odot_t \frac{k! t^k}{(1-\alpha t)^{k+1}} = \left.\frac{\partial^k f}{\partial t^k}\right|_{t=\alpha}. \tag{5.18b}$$

Proof. The second equation follows from the first by differentiating with respect to α. The first equation follows because

$$f(t) \odot_t \frac{1}{1-\alpha t} = f(t) \odot_t \sum_{n \geq 0} \alpha^n t^n = \sum_{n \geq 0} f_n \alpha^n = f(\alpha). \tag{5.19}$$

□

We can now use the building block generating function and the above Hadamard product to find an equation (though not yet in a usable form) for the generating function of *2-4-2* polygons.

Lemma 12. *Let $f(s;x,y)$ be the generating function of* 2-4-2 *polygons, where s is conjugate to the length of the bottom row of the polygon. This generating function satisfies the following equation*

$$f(s;x,y) = \frac{ysx}{1-sx} + f(t;x,y) \odot_t \left(\frac{1}{y} T(t/x,s;x,y) \right), \tag{5.20}$$

where $T(t,s;x,y)$ is the generating function of the 2-4-2 *building blocks.*

Proof. Write $f(s;x,y) = \sum_n f_n(x,y)s^n$ and $T(t,s;x,y) = \sum_n T_n(s;x,y)$, where $f_n(x,y)$ is the generating function of *2-4-2* polygons whose *bottom* row has length n, and $T_n(s;x,y)$ is the generating function of *2-4-2* building blocks, whose *top* row has length n. The above recurrence becomes:

$$f(s;x,y) = \frac{ysx}{1-sx} + \sum_{n \geq 1} f_n(x,y) T_n(s;x,y)/(yx^n). \tag{5.21}$$

This follows because a *2-4-2* polygon is either a rectangle of unit height (counted by $\frac{ysx}{1-sx}$) or may be constructed by gluing a *2-4-2* polygon, whose last row is of length n (counted by $f_n(x,y)$) to a *2-4-2* polygon whose top row is of length n (counted by $T_n(s;x,y)$).

To explain the factor of $1/(yx^n)$ see Fig. 5.8; when the building block is joined to the polygon (centre) and the duplicated row is "*squashed*" (right), the total vertical half-perimeter is reduced by 1 (two vertical bonds are removed) and the total horizontal half-perimeter is reduced by the length of the join (two horizontal bonds are removed for each cell in the join). Hence if the join is of length n, the perimeter weight needs to be reduced by a factor of (yx^n). □

In order to turn the Hadamard equation in the above lemma into a more standard functional equation we use Lemma 11, and rewrite $T(t/x,s;x,y)/y$ in (a non-standard) partial fraction form:

$$y^3 \left[c_0 \cdot t^0 + \sum_{k=0}^{5} c_{k+1} \frac{k! t^k}{(1-t)^{k+1}} + c_7 \frac{1}{1-st} + c_8 \frac{1}{1-stx} \right], \tag{5.22}$$

where the c_i are (large and ugly) rational functions of s and x. We will need c_8:

$$c_8 = -\frac{2sx^2(s^2x^2+sx-s+1)}{(1-sx)^4(1-x)^2}. \tag{5.23}$$

We do not require the other coefficients in the analysis that follows. We note that their denominators are products of $(1-x),(1-s)$ and $(1-sx)$. When $s=1$ some singularities of T coalesce and we have

$$T(t/x,1;x,y)/y = y^3\left[\hat{c}_0\cdot t^0+\sum_{k=0}^{6}\hat{c}_{k+1}\frac{k!t^k}{(1-t)^{k+1}}+\hat{c}_8\frac{1}{1-tx}\right], \tag{5.24}$$

where the $\hat{c}_i$ are (slightly simpler) rational functions of x. Again, we will need $\hat{c}_8$:

$$\hat{c}_8 = -2\frac{x^3(1+x)}{(1-x)^6} = c_8|_{s=1}. \tag{5.25}$$

We note that the denominators of the $\hat{c}_i$ are products of $(1-x)$. Applying Lemma 11, we find:

$$f(t;x,y)\odot_t T(t/x,s;x,y)/y = y^3\left[\sum_{k=0}^{5}c_{k+1}\frac{\partial^k f}{\partial t^k}(1;x,y)+c_7 f(s;x,y)+c_8 f(sx;x,y)\right], \tag{5.26}$$

where we have made use of the fact that $[t^0]f(t;x,y)=0$ (there are no rows of zero length). When $s=1$ the coalescing poles change equation (5.26) to:

$$f(t;x,y)\odot_t T(t/x,1;x,y)/y = y^3\left[\sum_{k=0}^{6}\hat{c}_{k+1}\frac{\partial^k f}{\partial t^k}(1;x,y)+\hat{c}_8 f(x;x,y)\right] \tag{5.27}$$

These equations give the following lemma:

Lemma 13. *Let $f(s;x,y)$ be the generating function for* 2-4-2 *polygons enumerated by bottom row-width, half-horizontal perimeter and half-vertical perimeter (s,x and y respectively). Write $f(s;x,y)=\sum_{n\geq 1}f_n(s;x)y^{3n-2}$, where the coefficient $f_n(s;x)$ is the generating function for $\mathscr{P}^{242}_{3n-2}$. These coefficients satisfy the following equations:*

$$f_1(s;x) = \frac{sx}{1-sx} \tag{5.28a}$$

$$f_{n+1}(s;x) = \sum_{k=0}^{5}c_{k+1}\frac{\partial^k f_n}{\partial s^k}(1;x)+c_7 f_n(s;x)+c_8 f_n(sx;x) \tag{5.28b}$$

$$f_{n+1}(1;x) = \sum_{k=0}^{6}\hat{c}_{k+1}\frac{\partial^k f_n}{\partial s^k}(1;x)+\hat{c}_8 f_n(x;x). \tag{5.28c}$$

The second of these is only valid when $s\neq 1$; when $s=1$ it reduces to the last.

Proof. Apply Lemma 11 to the partial fraction form of $T(t,s;x,y)$ for general s, and when $s=1$. Extracting the coefficients of y^{3n+1} from these equations gives the above recurrences. □

5.3.3 *Proof of Theorem 2*

We complete the proof of Theorem 2 by showing that $f_n(1;x)$ is singular at the zeros of $\Psi_n(x)$. We are able to do this by induction on the recurrences in the previous lemma. It turns out that we are able to disregard most of these recurrences except for the terms involving $f_n(sx;x)$ and $f_n(x;x)$; these are the only terms that introduce new denominator factors. We will require the following lemma to show that the coefficients c_8 and $\hat{c}_8$ cannot cancel these factors since they do not contain cyclotomic factors (except $\Psi_2(x)$).

Lemma 14. *Consider the coefficient $c_8(s;x)$ defined above. When $s=x^k$, $c_8(x^k,x)$ has a single zero on the unit circle at $x=-1$ when k is even. When k is odd $c_8(x^k,x)$ has no zeros on the unit circle.*

Proof. When $s=x^k$, the coefficient c_8 is

$$c_8(x^k,x) = \frac{2x^{k+2}(k^{2k+2}+x^{k+1}-x^k+1)}{(1-x^{k+1})^4(1-x)^2}. \tag{5.29}$$

Let ξ be a zero of $c_8(x^k,x)$ that lies on the unit circle; ξ must be a solution of $x^{2k+2}+x^{k+1}-x^k+1=0$. Hence:

$$\begin{aligned} \xi^k - \xi^{k+1} &= \xi^{2k+2}+1 \\ 1/\xi - 1 &= \xi^{k+1} + \xi^{-k-1}. \end{aligned} \tag{5.30}$$

Since ξ lies on the unit circle we may write $\xi = e^{i\theta}$:

$$\begin{aligned} e^{-i\theta} - 1 &= e^{i(k+1)\theta} + e^{-i(k+1)\theta} \\ &= 2\cos((k+1)\theta). \end{aligned} \tag{5.31}$$

Since the right hand-side of the above expression is real the left-hand side must also be real. Therefore $\theta = 0, \pi$ and $\xi = \pm 1$. If $\xi = 1$ then $p_k(\xi) = 2$. On the other hand, if $\xi = -1$ then $p_k(\xi) = 4$ if k is odd and is zero if k is even.

Since the denominator of $c_8(x^k,x)$ is not zero when k is even and $x=-1$ the result follows. One can verify that there are no multiple zeros at $x=-1$ by examining the derivative of the numerator.

□

Proof of Theorem 2 :
Consider the recurrence given in Lemma 13. This implies that $f_n(s;x)$ is a rational

function of s and x. Further, since $f_n(1;x)$ is a well-defined (and rational) function, the denominator of $f_n(s;x)$ does not contain any factors of $(1-s)$.

Let $\mathbb{C}_n(s;x)$ be the set of polynomials of the form

$$\prod_{k=1}^{n} \Psi_k(x)^{a_k}(1-sx^k)^{b_k}, \tag{5.32}$$

where a_k and b_k are non-negative integers. We define $\mathbb{C}_n(x) = \mathbb{C}_n(0;x)$ (i.e. polynomials which are products of cyclotomic polynomials). We first prove by induction on n that f_n may be written as

$$f_n(s;x) = \frac{N_n(s;x)}{(1-sx^n)D_n(s;x)}, \tag{5.33}$$

where $N_n(s;x)$ and $D_n(s;x)$ are polynomials in s and x with the restriction that $D_n(s;x) \in \mathbb{C}_{n-1}(s;x)$. Then we consider what happens when $s=1$ and x is a zero of Ψ_k.

For $n=1$, equation (5.33) is true, since $f_1(s;x) = \frac{sx}{1-sx}$. Now assume equation (5.33) is true up to n and apply the recurrence. The only term that may introduce a new zero into the denominator is $c_8(s;x)f_n(sx;x)$. By assumption $f_n(s;x) = \frac{N_n(sx;x)}{(1-sx^{n+1})D_n(sx;x)}$, and $D_n(sx;x) \in C_n(s;x)$. Hence equation (5.33) is true for $n+1$, and so is also true for all $n \geq 1$.

Let ξ be a zero of $\Psi_k(x)$. We wish to prove that $f_n(1;x)$ is singular at $x=\xi$ and we do so by proving that for $k=1,\dots,n$, the generating function $f_k(x^{n-k};x)$ is singular at $x=\xi$, and then setting $k=n$. We proceed by induction on k for fixed n.

If we set $k=1$, then we see that

$$f_1(x^{n-1};x) = \frac{x^n}{1-x^n}, \tag{5.34}$$

and so the result is true. Now let $k \geq 2$ and assume that the result is true for $k-1$, i.e. $f_{k-1}(x^{n-k+1};x)$ is singular at $x=\xi$. The recurrence relation and equation (5.33) together imply

$$f_k(s;x) = \frac{N(s;x)}{D(s;x)} + c_8(s;x)f_{k-1}(sx;x), \tag{5.35}$$

where N and D are polynomials in s and x and $D(s;x) \in \mathbb{C}_{k-1}(s;x)$. Setting $s = x^{n-k}$ yields

$$f_k(x^{n-k};x) = \frac{N(x^{n-k};x)}{D(x^{n-k};x)} + c_8(x^{n-k};x)f_{k-1}(x^{n-k+1};x), \tag{5.36}$$

and we note that $D(x^{n-k};x) \in \mathbb{C}_{n-1}(x)$. In the case $k=n$ the above equation is still true, since $\hat{c}_8 = c_8|_{s=1}$.

Equation (5.36) shows that $f_k(x^{n-k})$ is singular at $x = \xi$ only if the contribution from $c_8(x^{n-k};x)f_{k-1}(x^{n-k+1};x)$ is singular at $x = \xi$. This is true (by assumption) unless $c_8(x^{n-k};x) = 0$ at $x = \xi$. By Lemma 14, $c_8(x^{n-k};x)$ is non-zero at $x = \xi$, except when $n = k = 2$.

In the case $n = k = 2$ this proof breaks down, and indeed we see that $H_4(x)$ is not singular at $x = -1$. Excluding this case, $f_k(x^{n-k};x)$ is singular at $x = \xi$ and so $f_n(1;x)$ is also singular at $x = \xi$. By Lemma 9, $H_{3k-2}(x)$ is singular at $x = \xi$. □

This theorem then allows us to prove the main aim of this chapter; the anisotropic generating function of self-avoiding polygons is not a D-finite function.

Corollary 1. *Let S_n be the set of singularities of the coefficient $H_n(x)$. The set $S = \bigcup_{n\geq 1} S_n$ is dense on the unit circle $|x| = 1$. Consequently the self-avoiding polygon anisotropic half-perimeter generating function is not a D-finite function of y.*

Proof. For any $q \in \mathbb{Q}$, there exists k, such that $\Psi_k(e^{2\pi iq}) = 0$. By Theorem 2, $H_{3k-2}(x)$ is singular at $x = e^{2\pi iq}$, excepting $x = -1$. The set S is dense on $|x| = 1$ and so has an infinite number of accumulation points. By Theorem 1 $G(x,y) = \sum H_n(x)y^n$ is not a D-finite power series in y. □

Since the specialisation of a D-finite power series is D-finite, the above result extends to self-avoiding polygons on hypercubic lattices.

Corollary 2. *Let G_d be the generating function of self-avoiding polygons on the d-dimensional hyper-cubic lattice defined by:*

$$G_d(x_1,\dots,x_{d-1},y) = \sum_P y^{|P|_d} \prod_{i=1}^{d-1} x_i^{|P|_i},$$

where $|P|_i$ is half the number of bonds in parallel to the unit vector $\tilde{e}_i$. If $d = 1$, then this generating function is zero, and otherwise is a non-D-finite power series in y.

Proof. When $d = 1$ then there are no self-avoiding polygons and so the generating function is zero. Now consider $d \geq 2$. The square lattice generating function $G(x,y)$ can be recovered from G_d by setting $x_2 = \cdots = x_{d-1} = 0$. Since any well-defined specialisation of a D-finite power series is itself D-finite [12], it follows that if G_d were D-finite, then so would $G(x,y)$. This contradicts Corollary 1 and so G_d is not D-finite. □

5.4 Discussion

We have shown above that the anisotropic generating function of self-avoiding polygons on the square lattice, $G(x,y)$, is not a D-finite function of y. This result then extends to prove that the anisotropic generating function of self-avoiding polygons on any hypercubic lattice is either trivial (in one dimension) or a non-D-finite function (in dimensions 2 and higher). Similar results hold for directed-bond animals

[17], general bond animals and bond trees [14]. Unfortunately, work on a similar result for self-avoiding walks appears to be beyond the scope of these techniques. The self-avoiding walk analogue of *2-4-2* polygons appear to be quite complicated [18] and it is at all not clear that one can find recurrences such as those in Lemma 13.

There are several non-D-finiteness results for generating functions of other combinatorial problems, such as bargraphs enumerated by their site-perimeter [6], a number of lattice animal models related to heaps of dimers [5] and certain types of matchings [11]; these results rely upon a knowledge of the generating function—either in closed form or via some sort of recurrence. The result for self-avoiding polygons is, as far as we are aware, the first result on the D-finiteness of a completely unsolved model.

Unfortunately we are not able to use this result to obtain information about the nature of the isotropic generating function $G(z,z)$; one can easily construct a two-variable function that is not D-finite, that reduces to a single variable D-finite function. Consider, for example, the following function

$$F(x,y) = \sum_{n\geq 1} \frac{y^n}{(1-x^n)(1-x^{n+1})} \tag{5.37}$$

By Theorem 1 this is not a D-finite function of y. However, setting $x = y = z$ reduces F to a simple rational, and hence D-finite, function:

$$\begin{aligned} F(z,z) &= \frac{1}{1-z}\sum_{n\geq 1}\left(\frac{z^n}{1-z^n} - \frac{z^{n+1}}{1-z^{n+1}}\right) \\ &= \frac{z}{(1-z)^2}. \end{aligned} \tag{5.38}$$

On the other hand, the anisotropisation of *solvable* lattice models does not alter the nature of the generating function. Unfortunately we are unable to determine how far this phenomenon extends since we know so little about the nature of the generating functions of unsolved models.

We note that if the isotropic generating function is indeed not D-finite then it will not be found using computer packages such as GFUN [19] or differential approximants [7] which can only find D-finite solutions. At best one might hope that the solution may satisfy some sort of q-linear equation.

As noted above, the techniques developed for self-avoiding polygons have been successfully applied to other families of bond animals. Recent series expansion work by I. Jensen [9] shows that there is some possibility that these techniques can be extended to families of site animals (such as self-avoiding polygons enumerated by their area).

It would also be very interesting to apply these ideas to pattern-avoiding permutations—though it is not entirely clear how to "anisotropise" a permutation. Noonan and Zeilberger conjecture that the generating function of permutations avoiding a given pattern is D-finite [13]. This conjecture has helped drive developments in this field and any progress towards its resolution would constitute a major advance.

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