Chapter 3
Exactly Solved Models

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3.1 Introduction

3.1.1 Subclasses of Polygons and Polyominoes

This chapter deals with the *exact enumeration* of certain classes of (self-avoiding) polygons and polyominoes. We restrict our attention to the square lattice. As the interior of a polygon is a polyomino, we often consider polygons as special polyominoes. The usual enumeration parameters are the *area* (the number of cells) and the *perimeter* (the length of the border). The perimeter is always even, and often refined into the horizontal and vertical perimeters (number of horizontal/vertical steps in the border). Given a class $\mathscr{C}$ of polyominoes, the objective is to determine the following *complete generating function* of $\mathscr{C}$:

$$C(x,y,q) = \sum_{P \in \mathscr{C}} x^{hp(P)/2} y^{vp(P)/2} q^{a(P)},$$

where $hp(P)$, $vp(P)$ and $a(P)$ respectively denote the horizontal perimeter, the vertical perimeter and the area of P. This means that the coefficient $c(m,n,k)$ of $x^m y^n q^k$ in the series $C(x,y,q)$ is the number of polyominoes in the class $\mathscr{C}$ having horizontal perimeter $2m$, vertical perimeter $2n$ and area k. Several specializations of $C(x,y,q)$ may be of interest, such as the *perimeter generating function* $C(t,t,1)$, its *anisotropic* version $C(x,y,1)$, or the *area generating function* $C(1,1,q)$. From such exact results, one can usually derive many of the asymptotic properties of the polyominoes of $\mathscr{C}$: for instance the asymptotic number of polyominoes of perimeter n,

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or the (asymptotic) average area of these polyominoes, or even the limiting distribution of this area, as n tends to infinity (see Chapter 11). The techniques that are used to derive asymptotic results from exact ones are often based on complex analysis. A remarkable survey of these techniques is provided by Flajolet and Sedgewick's book [33].

The study of sub-classes of polyominoes is natural, given the immense difficulty of the full problem (enumerate all polygons or all polyominoes). The objective is to develop new techniques, and to push the border between solved and unsolved models further and further. However, several classes have an independent interest, other than being an approximation of the full problem. For instance, the enumeration of *partitions* (Fig. 3.2(e)) is relevant in number theory and in the study of the representations of the symmetric group. The first enumerative results on partitions date back, at least, to Euler. A full book is devoted to them, and is completely independent of the enumeration of general polyominoes [2]. Another example is provided by *directed* polyominoes, which are relevant for directed percolation, but also occur in theoretical computer science as binary search networks [54].

All these classes will be systematically defined in Section 3.1.3. For the moment, let us just say that most of them are obtained by combining conditions of *convexity* and *directedness.*

From the perspective of subclasses as an approximation to the full problem, it is natural to ask how good this approximation is expected to be. The answer is quite crude: these approximations are terrible. For a start, all the classes that have been counted so far are *exponentially small* in the class of all polygons (or polyominoes). Hence we cannot expect their properties to reflect faithfully those of general polygons/polyominoes. Why would the properties of a staircase polygon (Fig. 3.2(b)) be similar to those of a general self-avoiding polygon? Indeed, the number of staircase polygons of perimeter $2n$ grows like $2^{2n}n^{-3/2}$ (up to a multiplicative constant), while the number of general polygons is believed to be asymptotically $\mu^{2n}n^{-5/2}$, with $\mu = 2.638\ldots$ [36]. The average width of a staircase polygon is clearly linear in n, while the width of general polygons is conjectured to grow like $n^{3/4}$ (see [42]). And so on! In this context, it may be a pure coincidence that the average area of polygons of perimeter $2n$ is conjectured to scale as $n^{3/2}$ (see [27]), *just as it does for staircase polygons*. But it is also conjectured that the limit distribution of the area of $2n$-step polygons (normalized by its average value) coincides with the corresponding distribution for staircase polygons, and for other exactly solved classes. The universality of this distribution may not be a coincidence (see Chapter 11 for more references and details).

3.1.2 Three General Approaches

In this chapter, we present three robust approaches that can be applied to count many classes $\mathscr{C}$ of polyominoes. The common principle of all of them is to translate a recursive description of the polyominoes of $\mathscr{C}$ into a functional equation satisfied by the generating function $C(x,y,q)$. Some readers may prefer seeing a translation in terms of the *coefficients* of $C(x,y,q)$, namely the numbers $c(m,n,k)$. This translation is possible, but it is usually easier to work with a functional equation than with a recurrence relation. The applicability of each of these three approaches depends on whether the polyominoes of $\mathscr{C}$ have, or don't have, a certain type of recursive structure.

The most versatile approach is probably the third one, as it virtually applies to any class of polyominoes having a convexity property. It was already used by Temperley in 1956 [51] and is often called, in the physics literature, *the Temperley approach*. However, it often produces functional equations that are non-trivial to solve, even when the solution finally turns out to be a simple rational or algebraic series (these terms will be defined in Section 3.1.3 below). From a combinatorics point of view, it is important to get a better understanding of the simplicity of these series, and this is what the first two approaches provide: the first one applies to classes $\mathscr{C}$ having a *linear* structure, and gives rise to rational generating functions. The second applies to classes having an *algebraic* structure, and gives rise to algebraic generating functions.

We have chosen to present these three approaches because, in our opinion, they are the most robust ones, and we want to provide effective tools to the reader. To our knowledge, almost all the classes that have been solved exactly can be solved using one (or several) of these approaches. Still, certain results have been given a beautiful combinatorial explanation via more specific techniques. Let us mention two tools that are often involved in those alternative approaches. The first tool is specific to the enumeration of polygons, and consists in studying classes of possibly self-intersecting polygons, and then using an inclusion-exclusion principle to eliminate the ones with self-intersections. This idea appears in an old paper of Pólya [43] dealing with staircase polygons, and was further exploited to count more general polygons [30, 31], including those in dimensions larger than two [35, 13]. The second tool is the use of *bijections* and is of course not specific to polyomino enumeration. The idea is to describe a one-to-one correspondence between the objects of $\mathscr{C}$ and those of another class $\mathscr{D}$, having a simpler recursive structure. In this chapter, even though we often use encodings of polyominoes by words, these encodings are usually very simple and do not use the full force of bijective methods, which is clearly at work in papers like [16] or [20].

The structure of the chapter is simple: the three approaches we discuss are presented, and illustrated by examples, in Sections 3.2, 3.3 and 3.4 respectively. A few open problems which we consider worth investigating are discussed in Section 3.5.

We conclude this introduction with definitions of various families of polyominoes and formal power series.

3.1.3 A Visit to the Zoo

All the classes studied in this chapter are obtained by combining several conditions of *convexity* and *directedness*. Let us first recall that a polyomino P is a finite set of square cells of the square lattice whose interior is connected. The set of centres of the cells form an *animal* A (Fig. 3.1). The connectivity condition means that any two points of A can be joined by a path made up of unit vertical and horizontal steps, in such a way that every vertex of the path lies in A. The animal A is *North-East directed* (or *directed*, for short) if it contains a point v_0, called the source, such that every other point of A can be reached from v_0 by a path made of North and East unit steps, having all its vertices in A. In this case, the polyomino corresponding to A is also said to be NE-directed. One defines NW, SW and SE directed animals and polyominoes similarly.

A polyomino P is *column-convex* if its intersection with every vertical line is connected. This means that the intersection of every vertical line with the corresponding animal A is formed by consecutive points. The border of P is then a polygon. Row-convexity is defined similarly. Finally, P is d_+-convex if the intersection of A with every line of slope 1 is formed by consecutive points. One defines d_--convex polyominoes similarly.

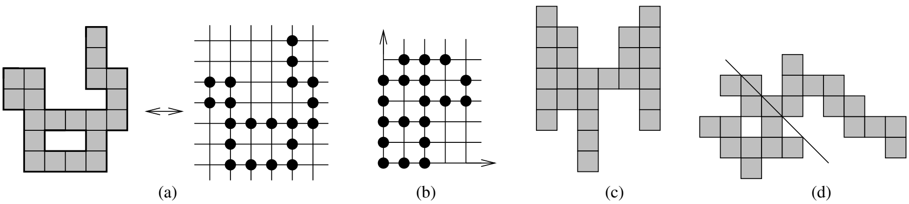


Fig. 3.1 From left to right: (a) a polyomino and the corresponding animal, (b) a NE-directed animal, (c) a column-convex polygon, (d) a d_--convex polyomino.

As discussed in [8], the combination of the four direction conditions and the four connectivity conditions gives rise to 31 distinct (non-symmetric) classes of polyominoes having at least one convexity property. To these 31 classes we must add the 4 different classes satisfying at least one directional property. Some prominent members of this zoo, which will occur in the forthcoming sections, are shown in Fig. 3.2:

- *convex* polyominoes (or polygons): polyominoes that are both column- and row-convex,
- *staircase* polyominoes (or polygons): convex polygons that are NE- and SW-directed,
- *bargraphs*: column-convex polygons that are NE- and NW-directed,
- *stacks*: row-convex bargraphs,

- *partitions,* a.k.a. *Ferrers diagrams,*: convex polygons that are NE-, NW- and SE-directed.

Finally, a formal power series $C(x) \equiv C(x_1, \ldots, x_k)$ with real coefficients is *rational* if it can be written as a ratio of polynomials in the x_i's. It is *algebraic* if it satisfies a non-trivial polynomial equation

$$P(C(x), x_1, \ldots, x_k) = 0.$$

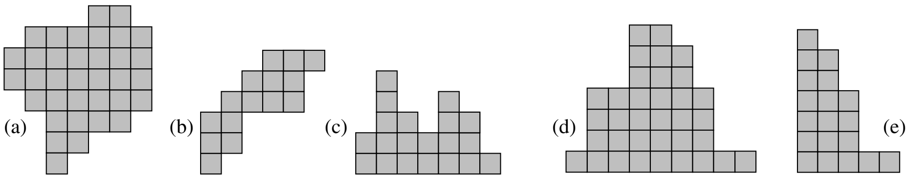


Fig. 3.2 A photo taken at the zoo: (a) a convex polygon, (b) a staircase polygon, (c) a bargraph, (d) a stack, (e) a Ferrers diagram.

3.2 Linear Models and Rational Series

3.2.1 A Basic Example: Bargraphs Counted by Area

Let b_n denote the number of bargraphs of area n. As there is a unique bargraph of area 1, $b_1 = 1$. For $n \geq 2$, there are two types of bargraphs:

1. those in which the last (i.e., rightmost) column has height 1,
2. those in which the last column has height 2 or more.

Bargraphs of the first type are obtained by adding a column of height 1 to the right of any bargraph of area $n-1$. Bargraphs of the second type are obtained by adding one square cell to the top of the last column of a bargraph of area $n-1$. Since a bargraph cannot be simultaneously of type 1 and 2, this gives

$$b_1 = 1 \quad \text{and for } n \geq 2, \quad b_n = 2b_{n-1},$$

which implies $b_n = 2^{n-1}$. The area generating function of bargraphs is thus a rational series:

$$B(q) := \sum_{n \geq 1} b_n q^n = \frac{q}{1-2q}.$$

3.2.2 Linear Objects

The above enumeration of bargraphs is based on a very simple recursive description of bargraphs. This description only involves the following two constructions:

1. taking disjoint unions of sets,
2. concatenating a new cell with an already constructed object.

In terms of generating functions (g.f.s), taking the disjoint union of sets means summing their g.f.s, while concatenating a new cell (of size 1) to all elements of a set means multiplying its g.f. by q. Hence the above description of bargraphs translates directly into a linear equation for the g.f. $B(q)$:

$$B(q) = q + qB(q) + qB(q).$$

This equation reflects the fact that the set of bargraphs is the union of three disjoint subsets (the unique bargraph of area 1, bargraphs of type 1, bargraphs of type 2), and that the second and third subsets are both obtained by adding a cell to any bargraph.

More generally, we will say that a class of objects, equipped with a size, is *linear* if these objects can be obtained from a finite set of initial objects using disjoint union and concatenation of one cell, or *atom*. It is assumed that the concatenation of an atom increases the size by 1. The construction must be *non-ambiguous*, meaning that each object of the class is obtained only once. The construction may involve several classes of objects simultaneously. For instance, the class $\tilde{\mathscr{B}}$ of bargraphs whose last column has height 1 is linear: the objects of $\tilde{\mathscr{B}}$, other than the one-cell bargraph, are obtained by adding one cell to the right of any bargraph. The associated series $\tilde{B}(q)$ is defined by the linear system:

$$\begin{aligned} \tilde{B}(q) &= q + qB(q), \\ B(q) &= q + qB(q) + qB(q). \end{aligned}$$

In general, the generating function of a linear class of objects is the first component of the solution of a system of k linear equations of the form

$$B_i(q) = P_i(q) + q\sum_{j=1}^{k} a_{i,j} B_j(q) \quad 1 \le i \le k, \tag{3.1}$$

where $a_{i,j} \in \mathbb{N}$ and each $P_i(q)$ is a polynomial in q with coefficients in $\mathbb{N}$. The polynomial $P_i(q)$ counts the initial objects of type i, and there are $a_{i,j}$ ways to aggregate an atom to an object of type j to form an object of type i. The system (3.1) uniquely defines each series $B_i(q)$, which is rational. The series obtained in this way are called $\mathbb{N}$-*rational*. Their study is closely related to the theory of *regular languages* [50].

3.2.3 More Linear Models

In this section we present three typical problems that can be solved via a linear recursive description. The first one is the perimeter enumeration of Ferrers diagrams (and stacks). The second one generalizes the study of bargraphs performed in Section 3.2.1 to all column-convex polygons (and to the subclass of directed column-convex polygons) counted by area. The third one illustrates the role of linear models in the approximation of hard problems, and deals with the enumeration of self-avoiding polygons confined to a narrow strip. In passing, we illustrate the two following facts:

1. it may be useful to begin by describing a size-preserving bijection between polyominoes and other objects (having a linear structure),
2. linear constructions are conveniently described by a directed graph when they become a bit involved.

3.2.3.1 Ferrers Diagrams by Perimeter

The set of Ferrers diagrams can be partitioned into three disjoint subsets: first, the unique diagram of (half-)perimeter 2; then, diagrams of width at least 2 whose rightmost column has height 1; finally, diagrams with no column of height 1. The latter diagrams can be obtained by duplicating the bottom row of another diagram (Fig. 3.3).

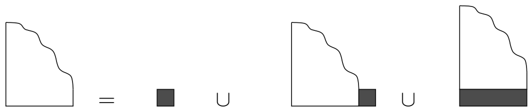


Fig. 3.3 Recursive description of Ferrers diagrams.

From this description, it follows that the set of words that describe the North-East boundary of Ferrers diagrams, from the NW corner to the SE one, admits a linear construction. This boundary is formed by East and South steps, and will be encoded by a word over the alphabet $\{e,s\}$. Any word over this alphabet that starts with an e and ends with an s corresponds to a unique Ferrers diagram. Let $\mathscr{F}$ be this class of words, and let $\mathscr{L}$ be the set of all non-empty prefixes of words of $\mathscr{F}$. Then $\mathscr{F}$ and $\mathscr{L}$ admit the following linear description:

$$\mathscr{F} = \mathscr{L}s \quad \text{and} \quad \mathscr{L} = \{e\} \cup \mathscr{L}e \cup \mathscr{L}s.$$

In these equations, the notation $\mathscr{L}s$ means $\{us,\ u \in \mathscr{L}\}$, and the unions are disjoint. The series that count the words of these sets by their length (number of letters) are thus given by the linear system

$$F(t) = tL(t) \quad \text{and} \quad L(t) = t + 2tL(t).$$

Since the length of a coding word is the half-perimeter of the associated diagram, this provides the length g.f.:

$$F(t) = \frac{t^2}{1-2t} = \sum_{n \geq 1} 2^{n-2} t^n.$$

By separately counting East and South steps, we obtain the equations

$$F(x,y) = yL(x,y) \quad \text{and} \quad L(x,y) = x + xL(x,y) + yL(x,y), \tag{3.2}$$

and hence the anisotropic perimeter g.f. of these diagrams:

$$F(x,y) = \frac{xy}{1-x-y} = \sum_{m,n \geq 1} \binom{m+n-2}{m-1} x^m y^n.$$

A similar treatment can be used to determine the perimeter g.f. of stack polygons: the construction schematized in Fig. 3.4 gives:

$$S(x,y) = xy + xS(x,y) + S_+(x,y), \quad S_+(x,y) = yS(x,y) + xS_+(x,y)$$

which yields

$$S(x,y) = \frac{xy(1-x)}{(1-x)^2 - y}.$$

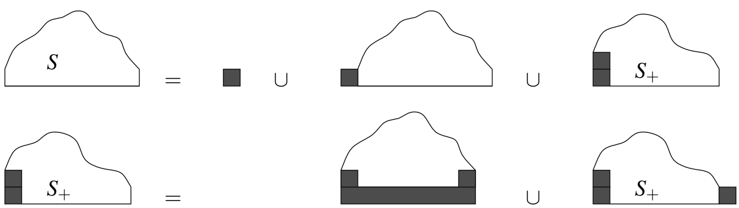


Fig. 3.4 Recursive description of stack polygons.

3.2.3.2 Column-Convex Polygons by Area

Consider a column-convex polygon P having n cells. Let us number these cells from 1 to n as illustrated in Fig. 3.5. The columns are visited from left to right. In the first column, cells are numbered from bottom to top. In each of the other columns, the lowest cell that has a left neighbour gets the smallest number; then the cells lying below it are numbered from top to bottom, and finally the cells lying above it are

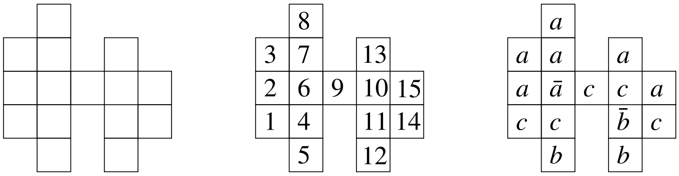


Fig. 3.5 A column-convex polygon, with the numbering and encoding of the cells.

numbered from bottom to top. Note that for all i, the cells labelled $1,2,\ldots,i$ form a column-convex polygon. This labelling describes the order in which we are going to aggregate the cells.

Associate with P the word $u = u_1 \cdots u_n$ over the alphabet $\{a,b,c\}$ defined by
– $u_i = c$ (like Column) if the i^{th} cell is the first to be visited in its column,
– $u_i = b$ (like Below) if the i^{th} cell lies below the first visited cell of its column,
– $u_i = a$ (like Above) if the i^{th} cell lies above the first visited cell of its column.
Then, add a bar on the letter u_i if the i^{th} cell of P has a South neighbour, an East neighbour, but no South-East neighbour. (In other words, the barred letters indicate where to start a new column, when the bottommost cell in this new column lies above the bottommost cell of the previous column.) This gives a word v over the alphabet $\{a,b,c,\bar{a},\bar{b},\bar{c}\}$, and P can be uniquely reconstructed from v.

We now focus on the enumeration of these coding words. Let $\mathscr{L}$ be the set of all prefixes of these words, including the empty prefix ε. By considering which letter can be added to the right of which prefix, we are led to partition $\mathscr{L}$ into five disjoint subsets $\mathscr{L}_1,\ldots,\mathscr{L}_5$, subject to the following linear recursive description:

$$\begin{aligned}
&\mathscr{L}_1 = \{\varepsilon\}, \\
&\mathscr{L}_2 = \mathscr{L}_1 c \cup \mathscr{L}_2 a \cup \mathscr{L}_3 a \cup \mathscr{L}_4 c, \quad \mathscr{L}_4 = \mathscr{L}_2 \bar{a} \cup \mathscr{L}_3 \bar{a} \cup \mathscr{L}_4 a \cup \mathscr{L}_5 b, \\
&\mathscr{L}_3 = \mathscr{L}_2 c \cup \mathscr{L}_3 b \cup \mathscr{L}_3 c, \qquad\qquad \mathscr{L}_5 = \mathscr{L}_2 \bar{c} \cup \mathscr{L}_3 \bar{b} \cup \mathscr{L}_3 \bar{c} \cup \mathscr{L}_5 b.
\end{aligned} \tag{3.3}$$

The words of $\mathscr{L}_4$ and $\mathscr{L}_5$ are those in which a barred letter (the rightmost one) still waits to be "matched" by a letter c creating a new column. The words of $\mathscr{L}_2 \cup \mathscr{L}_3$ are those that encode column-convex polygons. This construction is illustrated by a directed graph in Fig. 3.6: every path starting from 1 and ending at i corresponds to a word of $\mathscr{L}_i$, obtained by reading edge labels. The series counting the words of $\mathscr{L}_i$ by their length satisfy:

$$\begin{aligned}
&L_1 = 1, \\
&L_2 = q\left(L_1 + L_2 + L_3 + L_4\right), \quad L_4 = q\left(L_2 + L_3 + L_4 + L_5\right), \\
&L_3 = q\left(L_2 + L_3 + L_3\right), \qquad\qquad L_5 = q\left(L_2 + L_3 + L_3 + L_5\right).
\end{aligned}$$

The area g.f. of column-convex polygons is $C(q) = L_2(q) + L_3(q)$. Solving the above system gives:

$$C(q) = \frac{q\,(1-q)^3}{1 - 5q + 7q^2 - 4q^3}.$$

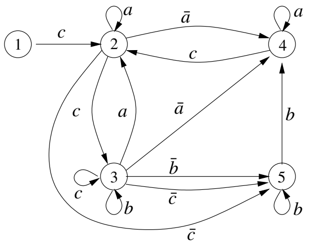


Fig. 3.6 Linear construction of the words of $\mathscr{L}$. The words of $\mathscr{L}_i$ encode the paths starting from 1 and ending at i.

We believe that this result was first published by Temperley [51].

A column-convex polygon is directed if and only if its coding word does not use the letter b. We obtain a linear description of the prefixes of these words by deleting all terms of the form $\mathscr{L}_i b$ in the description (3.3). The class $\mathscr{L}_5$ becomes irrelevant. Solving the associated system of linear equations gives the area g.f. of directed column-convex polygons:

$$DC(q) = \frac{q(1-q)}{1-3q+q^2}.$$

As far as we know, this result was first published by Klarner [38].

3.2.3.3 Polygons Confined to a Strip

Constraining polyominoes or polygons to lie in a strip of fixed height endows them with a linear structure. This observation gives a handle to attack difficult problems, like the enumeration of general self-avoiding polygons (SAP), self-avoiding walks, or polyominoes [1, 5, 48, 55, 56]. As the size of the strip increases, the approximation of the confined problem to the general one becomes better and better. This widely applied principle gives, for instance, lower bounds on growth constants that are difficult to determine. We illustrate it here with the perimeter enumeration of SAP confined to a strip.

Before we describe this calculation, let us mention a closely related idea, which consists of considering anisotropic models (for instance, SAP counted by vertical and horizontal perimeters), and fixing the number of atoms lying in one direction, for instance the number of horizontal edges. Again, this endows the constrained objects with a linear structure. The denominators of the rational generating functions that count them often factor in terms $(1-y^i)$. The number of exponents i that occur can be seen as a measure of the complexity of the class. This is often observed only at an experimental level, and is further discussed in Chapter 4. However, this observation has been pushed in some cases to a proof that the corresponding generating function is not *D-finite* and in particular not algebraic (see for instance [49], and Chapter 5).

But let us return to SAP in a strip of height k (a *k-strip*). A first observation is that a polygon is completely determined by the position of its horizontal edges. Consider the intersection of the polygon with a vertical line lying at a half-integer abscissa (a *cut*): the strip constraint implies that only finitely many configurations (or *states*) can occur. The number of such states is the number of even subsets of $\{0,1,\ldots,k\}$. This implies that SAP in a strip can be encoded by a word over a finite alphabet. For instance, the polygon of Fig. 3.7 is encoded by the word $\tilde{b}\tilde{b}\tilde{b}aabaa\tilde{b}a$.

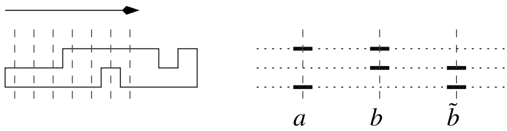


Fig. 3.7 A self-avoiding polygon in a strip of height 2, encoded over a 3-letter alphabet.

It is not hard to see that for all k, the set of words encoding SAP confined to a strip of height k has a linear structure. To make this structure clearer, we refine our encoding: for every vertical cut, we not only keep track of its intersection with the polygon, but also of the way the horizontal edges that meet the cut are connected to the left of the cut. This does not change the size of the alphabet for $k=2$, as there is a unique way of coupling two edges. However, if $k=3$, the configuration where 4 edges are met by the cut gives rise to 2 states, depending on how these 4 edges are connected (Fig. 3.8). The number of states is now the number of non-crossing couplings on $\{0,1,\ldots,k\}$. This is also the size of our encoding alphabet A.

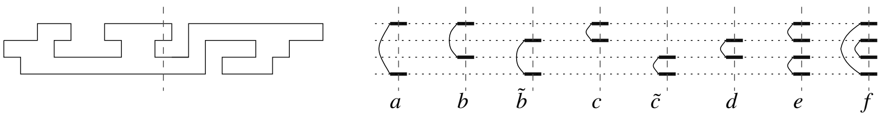


Fig. 3.8 A self-avoiding polygon in a strip of height 3, encoded by the word $d\tilde{b}af\tilde{c}\tilde{c}eaaf\tilde{c}aceeabcc$.

Fix k, and let $\mathscr{S}$ be the set of words encoding SAP confined to a k-strip. The set $\mathscr{L}$ of *prefixes* of words of $\mathscr{S}$ describes incomplete SAP, and has a simple linear structure: for every such prefix w, the set of letters a such that wa lies in $\mathscr{L}$ only depends on the last letter of w. In other words, these prefixes are Markovian with memory 1. For every letter a in the encoding alphabet, we denote by $\mathscr{L}_a$ the set of prefixes ending with the letter a. The linear structure can be encoded by a graph, from which the equations defining the sets $\mathscr{L}_a$ can automatically be written. This graph is shown in Fig. 3.9 (left) for $k=2$. Every path in this graph starting from the initial vertex 0 corresponds to a word of $\mathscr{L}$, obtained by reading vertex labels. The linear structure of prefixes reads:

$$\mathscr{L}_a = (\varepsilon + \mathscr{L}_a + \mathscr{L}_b + \mathscr{L}_{\tilde{b}})a, \quad \mathscr{L}_b = (\varepsilon + \mathscr{L}_a + \mathscr{L}_b)b, \quad \mathscr{L}_{\tilde{b}} = (\varepsilon + \mathscr{L}_a + \mathscr{L}_{\tilde{b}})\tilde{b}.$$

From this we derive linear equations for incomplete SAP, where every horizontal edge is counted by $\sqrt{x}$, and every vertical edge by $\sqrt{y} = z$:

$$L_a = (z^2 + L_a + zL_b + zL_{\tilde{b}})x, \quad L_b = (z + zL_a + L_b)x, \quad L_{\tilde{b}} = (z + zL_a + L_{\tilde{b}})x.$$

These equations keep track of how many edges are added when a new letter is appended to a word of $\mathscr{L}$. They can be schematized by a weighted graph (Fig. 3.9, middle). Now the (multiplicative) weight of a path starting at 0 is the weight of the corresponding incomplete polygon. Finally, the completed polygons are obtained by adding vertical edges to the right of incomplete polygons. This gives the generating function of SAP in a strip of height 2 as:

$$S_2(x,y) = z^2 L_a + zL_b + zL_{\tilde{b}}.$$

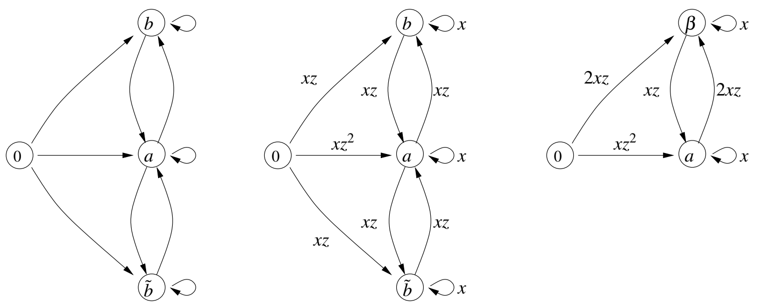


Fig. 3.9 The linear structure of SAP in a 2-strip.

Clearly, we should exploit the horizontal symmetry of the model to obtain a smaller set of equations. The letters b and $\tilde{b}$ playing symmetric roles, we replace them in the graph of Fig. 3.9 by a unique vertex β, such that the generating function of paths ending at β is the sum of the g.f.s of paths ending at b and $\tilde{b}$ in the first version of the graph (Fig. 3.9, right). Introducing the series $L_\beta = L_b + L_{\tilde{b}}$, we have thus replaced the previous system of four equations by

$$L_a = x(z^2 + L_a + zL_\beta), \quad L_\beta = x(2z + 2zL_a + L_\beta), \quad S_2(x,y) = z^2 L_a + zL_\beta,$$

from which we obtain

$$S_2(x,y) = \frac{xy(2-2x+y+3xy)}{(1-x)^2-2x^2y}.$$

Note that this series counts polygons of height 1 twice, so that we should subtract $S_1(x,y) = xy/(1-x)$ to obtain the g.f. of SAP of height at most 2, defined up to translation.

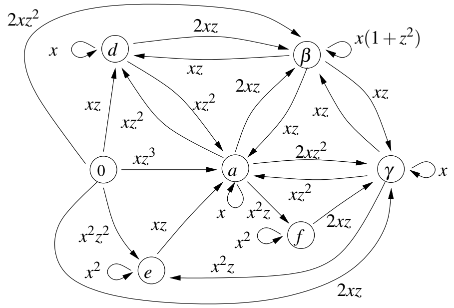


Fig. 3.10 The linear structure of SAP in a 3-strip.

For $k=3$, the original alphabet, shown in Fig. 3.8, has 8 letters, but two pairs of them play symmetric roles. After merging the vertices b and $\tilde{b}$ on the one hand, c and $\tilde{c}$ on the other, the condensed graph, with its x,z weights, is shown in Fig. 3.10. The corresponding equations read

$$\begin{aligned}
L_a &= x\left(z^3 + L_a + zL_\beta + z^2L_\gamma + z^2L_d + zL_e\right),\\
L_\beta &= x\left(2z^2 + 2zL_a + (1+z^2)L_\beta + zL_\gamma + 2zL_d\right),\\
L_\gamma &= x\left(2z + 2z^2L_a + zL_\beta + L_\gamma + 2zL_f\right),\\
L_d &= x\left(z + z^2L_a + zL_\beta + L_d\right),\\
L_e &= x^2\left(z^2 + zL_\gamma + L_e\right),\\
L_f &= x^2\left(zL_a + L_f\right),
\end{aligned}$$

and the generating function of completed polygons is

$$S_3(x,y) = z^3L_a + z^2L_\beta + zL_\gamma + zL_d + z^2L_f = \frac{xyN(x,y)}{D(x,y)}$$

where

$$\begin{aligned}
N(x,y) = {} & 3\,(x+1)^2\,(1-x)^5 + (5x+2)\,(2x-1)\,(x+1)^2\,(x-1)^3\,y\\
& - (x-1)\left(6x^6 + 4x^5 - 18x^4 - 6x^3 + 11x^2 + 8x + 1\right)y^2\\
& - x\,(x+1)\left(2x^4 + 6x^3 - 8x^2 + 4x + 1\right)y^3
\end{aligned}$$

and

$$D(x,y) = (x+1)^2(x-1)^6 - x(1+4x)(x+1)^2(x-1)^4 y \\ + x^2\left(3x^4+4x^3-6x^2-8x-3\right)(x-1)^2 y^2 \\ + x^3(x+1)\left(x^3+3x^2-5x+3\right)y^3.$$

By setting $x=y=t$, we obtain the half-perimeter generating function of SAP in a 3-strip,

$$S_3(t) = \frac{t^2\left(-8t^9+4t^8+10t^7-20t^6-t^5-t^4+7t^3+3t^2-7t+3\right)}{4t^{10}-2t^9-5t^8+8t^7-t^6+2t^5-4t^4+2t^3+3t^2-4t+1}$$

and, by looking at the smallest pole of this series, we also obtain the (very weak) lower bound $1.68\ldots$ on the growth constant of square lattice self-avoiding polygons.

The above method has been automated by Zeilberger [55]. It is not hard to see that the number of states required to count polygons in a k-strip grows like 3^k, up to a power of k. This prevents one from applying this method for large values of k. Better bounds for growth constants may be obtained via the *finite lattice method* described in Chapter 7, and implemented in Chapter 10. A further improvement is obtained by looking at a cylinder rather than a strip [5].

3.2.4 q-Analogues

By looking at the height of the rightmost column of Ferrers diagrams, we have described a linear construction of these polygons that proves the rationality of their perimeter g.f. (Fig. 3.3). Let us examine what happens when we try to keep track of the area in this construction.

They key point is that the area *increases by the width of the polygon* when we duplicate the bottom row. (In contrast, the half-perimeter simply increases by 1 during this operation.) This observation gives the following functional equation for the complete g.f. of Ferrers diagrams:

$$F(x,y,q) = xyq + xqF(x,y,q) + yF(xq,y,q).$$

This is a *q-analogue* of the equation defining $F(x,y,1)$, derived from (3.2). This equation is no longer linear, but it can be solved easily by iteration:

$$\begin{aligned} F(x,y,q) &= \frac{xyq}{1-xq} + \frac{y}{1-xq}F(xq,y,q) \\ &= \frac{xyq}{1-xq} + \frac{y}{1-xq}\frac{xyq^2}{1-xq^2} + \frac{y}{1-xq}\frac{y}{1-xq^2}F(xq^2,y,q) \qquad (3.4) \\ &= \sum_{n\geq 1} \frac{xy^n q^n}{(xq)_n} \end{aligned}$$

with

$$(xq)_0 = 1 \quad \text{and} \quad (xq)_n = (1-xq)(1-xq^2)\cdots(1-xq^n).$$

Similarly, for the stack polygons of Fig. 3.4, one obtains:

$$\begin{aligned} S(x,y,q) &= xyq + xqS(x,y,q) + S_+(x,y,q), \\ S_+(x,y,q) &= yS(xq,y,q) + xqS_+(x,y,q). \end{aligned}$$

Eliminating the series S_+ gives

$$\begin{aligned} S(x,y,q) &= \frac{xyq}{1-xq} + \frac{y}{(1-xq)^2} S(xq,y,q) \\ &= \sum_{n\geq 1} \frac{xy^n q^n}{(xq)_{n-1}(xq)_n}. \end{aligned}$$

In Section 3.4 we present a systematic approach for counting classes of column-convex polygons by perimeter and area.

3.3 Algebraic Models and Algebraic Series

3.3.1 A Basic Example: Bargraphs Counted by Perimeter

Let us return to bargraphs. The linear description used in Section 3.2.1 to count them by area cannot be directly recycled to count them by perimeter: indeed, when we add a cell at the top of the last column, how do we know if we increase the perimeter, or not? Instead, we are going to scan the polygon from left to right, and *factor* it into two smaller bargraphs as soon as we meet a column of height 1 (if any). If there is no such column, deleting the bottom row of the polygon leaves another bargraph. This description is schematized in Fig 3.11.

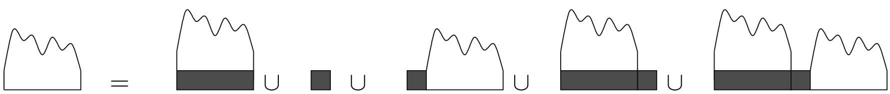


Fig. 3.11 A second recursive construction of bargraphs.

Let $\mathscr{B}$ be the set of words over the alphabet $\{n,s,e\}$ that naturally encode the top boundary of bargraphs, from the SW to the SE corner. Fig. 3.11 translates into the following recursive description, where the unions are disjoint:

$$\mathscr{B} = n\mathscr{L}s \quad \text{with} \quad \mathscr{L} = n\mathscr{L}s \cup \{e\} \cup e\mathscr{L} \cup n\mathscr{L}se \cup n\mathscr{L}se\mathscr{L}. \tag{3.5}$$

This implies that the anisotropic perimeter g.f. of bargraphs satisfies

$$\begin{cases} B(x,y) = yL(x,y), \\ L(x,y) = yL(x,y) + x + xL(x,y) + xyL(x,y) + xyL(x,y)^2. \end{cases}$$

These equations are readily solved and yield:

$$B(x,y) = \frac{1-x-y-xy-\sqrt{(1-y)((1-x)^2-y(1+x)^2)}}{2x}. \tag{3.6}$$

Thus the perimeter g.f. of bargraphs is algebraic, and its algebraicity is explained combinatorially by the recursive description of Fig. 3.11.

Note that one can directly translate this description into an algebraic equation satisfied by $B(x,y)$, without using the language $\mathscr{B}$. This language is largely a convenient tool to highlight the *algebraic structure* of bargraphs. The translation of Fig. 3.11 into an equation proceeds as follows: there are two types of bargraphs, those that have at least one column of height 1, and the others, which we call *thick* bargraphs. Thick bargraphs are obtained by duplicating the bottom row of a general bargraph, and are thus counted by $yB(x,y)$. Among bargraphs having a column of height 1, we find the single cell bargraph (g.f. xy), and then those of width at least 2. The latter class can be split into 3 disjoint classes:

– the first column has height 1. These bargraphs are obtained by adding a cell to the left of any general bargraph, and are thus counted by $xB(x,y)$,
– the last column is the only column of height 1. These bargraphs are obtained by adding a cell to the right of a thick bargraph, and are thus counted by $xyB(x,y)$,
– the first column of height 1 is neither the first column, nor the last column. Such bargraphs are obtained by concatenating a thick bargraph, a cell, and a general bargraph; they are counted by $xB(x,y)^2$.

This discussion directly results in the equation

$$B(x,y) = yB(x,y) + xy + xB(x,y) + xyB(x,y) + xB(x,y)^2. \tag{3.7}$$

3.3.2 Algebraic Objects

The above description of bargraphs involved two constructions:

1. taking disjoint unions of sets,
2. taking Cartesian products of sets.

For two classes $\mathscr{A}_1$ and $\mathscr{A}_2$, the element (a_1, a_2) of the product $\mathscr{A}_1 \times \mathscr{A}_2$ is seen as the concatenation of the objects a_1 and a_2. We will say that a class of objects is *algebraic* if it admits a non-ambiguous recursive description based on disjoint unions and Cartesian products. It is assumed that the size of the objects is *additive* for the concatenation. For instance, (3.5) gives an algebraic description of the words of $\mathscr{L}$ and $\mathscr{B}$.

In the case of linear constructions, the only concatenations that were allowed were between one object and a single atom. As we can now concatenate two objects, algebraic constructions generalize linear constructions. In terms of g.f.s, concatenating objects of two classes means taking the product of the corresponding g.f.s. Hence the g.f. of an algebraic class will always be the first component of the solution of a polynomial system of the form:

$$A_i = P_i(t, A_1, \ldots, A_k) \quad \text{for} \quad 1 \leq i \leq k,$$

where P_i is a polynomial with coefficients in $\mathbb{N}$. Such series are called $\mathbb{N}$-*algebraic*, and are closely related to the theory of *context-free languages.* We refer to [50] for details on these languages, and to [12] for a discussion of $\mathbb{N}$-algebraic series in enumeration.

3.3.3 More Algebraic Models

In this section we present three problems that can be solved via an algebraic decomposition: staircase polygons, then column-convex polygons counted by perimeter (and the subclass of directed column-convex polygons), and finally directed polyominoes counted by area.

3.3.3.1 Staircase Polygons by Perimeter

In Section 3.1.3 we defined staircase polygons through their directed and convexity properties. See Fig. 3.2(b) for an example. We describe here a recursive construction of these polygons, illustrated in Fig. 3.12. It is analogous to the construction of bargraphs described at the end of Section 3.3.1 and illustrated in Fig. 3.11. Denote by $S(x,y)$ the anisotropic perimeter generating function of staircase polygons.

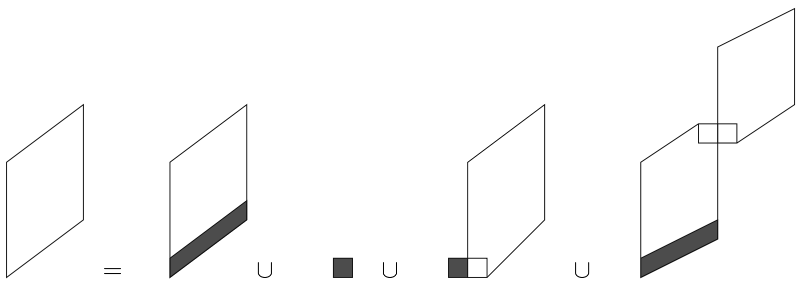


Fig. 3.12 A recursive construction of staircase polygons.

We say that a staircase polygon is *thick* if deleting the bottom cell of each column gives a staircase polygon of the same width. These thick polygons are obtained by duplicating the bottom cell in each column of a staircase polygon, so that their generating function is $yS(x,y)$.

Among non-thick staircase polygons, we find the single cell polygon (g.f. xy), and then those of width at least 2. Let P be in the latter class, and denote its columns $C_1,\dots,C_k$, from left to right. The fact that P is not thick means that there exist two consecutive columns, C_i and C_{i+1}, that overlap by one edge only. Let i be minimal for this property. Two cases occur:

– the first column has height 1. In particular, $i = 1$. These polygons are obtained by adding a cell to the bottom left of any general staircase polygon, and are thus counted by $xS(x,y)$.
– otherwise, the columns $C_1,\dots,C_i$ form a thick staircase polygon, and $C_{i+1},\dots,C_k$ form a general staircase polygon. Concatenating these two polygons in such a way that they share only one edge gives the original polygon P. Hence the g.f. for this case is $S(x,y)^2$.

This discussion gives the equation

$$S(x,y) = yS(x,y) + xy + xS(x,y) + S(x,y)^2$$

so that

$$\begin{aligned} S(x,y) &= \frac{1}{2}\left(1-x-y-\sqrt{1-2x-2y-2xy+x^2+y^2}\right) \\ &= \sum_{p,q\geq 1} \frac{1}{p+q-1}\binom{p+q-1}{p}\binom{p+q-1}{q} x^p y^q. \end{aligned}$$

This expansion can be obtained using the Lagrange inversion formula [9]. The isotropic semi-perimeter g.f. is obtained by setting $t = x = y$:

$$S(t,t) = \frac{1}{2}\left(1-2t-\sqrt{1-4t}\right) = \sum_{n\geq 1} C_n t^{n+1}$$

where $C_n = \binom{2n}{n}/(n+1)$ is the n^{th} Catalan number. The same approach can be applied to more general classes of convex polygons, like directed convex polygons and general convex polygons. See for instance [9, 23].

3.3.3.2 Column-Convex Polygons by Perimeter

We now apply a similar treatment to the perimeter enumeration of column-convex polygons (cc-polygons for short). Their area g.f. was found in Section 3.2.3. Let $\mathscr{C}$ denote the set of these polygons, and $C(x,y)$ their anisotropic perimeter generating function. Our recursive construction requires us to introduce two additional classes of polygons. The first one, $\mathscr{C}_1$, is the set of cc-polygons in which one cell of the last column is marked. The corresponding g.f. is denoted $C_1(x,y)$. Note that, by symmetry, this series also counts cc-polygons where one cell of the *first* column is marked. Then, $\mathscr{C}_2$ denotes the set of cc-polygons in which one cell of the first column is marked (say, with a dot), and one cell of the last column is marked as well (say, with a cross). The corresponding g.f. is denoted $C_2(x,y)$. Our recursive construction of the polygons of $\mathscr{C}$ is illustrated in Fig. 3.13.

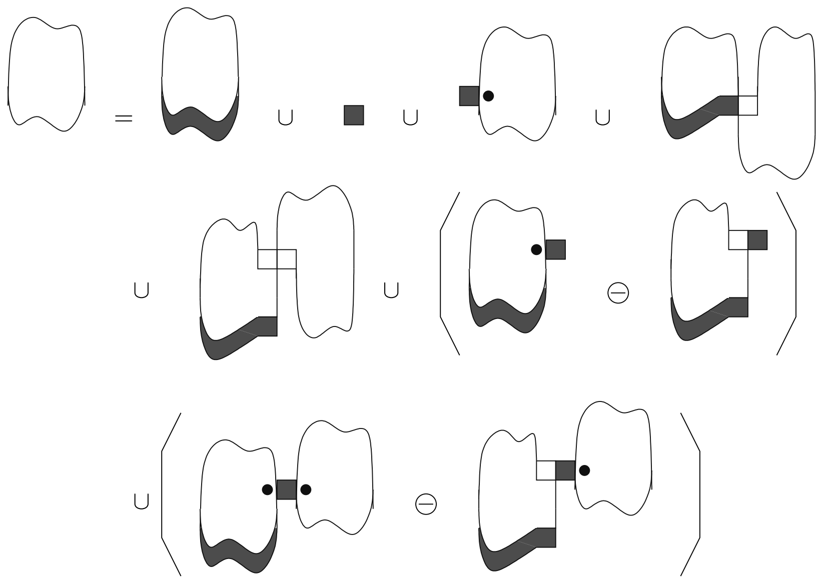


Fig. 3.13 A recursive construction of column-convex polygons.

We say that a cc-polygon is *thick* if deleting the bottom cell of each column gives a cc-polygon of the same width. These thick polygons are obtained by duplicating the bottom cell in each column of a cc-polygon, so that their generating function is $yC(x,y)$.

Among non-thick cc-polygons, we find the single cell polygon (g.f. xy), and then those of width at least 2. Let P be in the latter class, and denote its columns $C_1,\ldots,C_k$, from left to right. The fact that P is not thick means that there exist two consecutive columns C_i and C_{i+1} that overlap by one edge only. Let i be minimal for this property. Two cases occur:

- the first column has height 1. In particular, $i = 1$. These polygons are obtained by adding a cell to the left of any cc-polygon having a marked cell in its first column, next to the marked cell. They are thus counted by $xC_1(x,y)$.
- otherwise, the columns $C_1,\ldots,C_i$ form a thick cc-polygon P_1, and the columns $C_{i+1},\ldots,C_k$ form a general cc-polygon P_2. There are several ways of concatenating these two polygons in such a way they share only one edge:

– either the shared edge is at the bottom of C_i and at the top of C_{i+1}. Such polygons are counted by $C(x,y)^2$.
– or the shared edge is at the top of C_i and at the bottom of C_{i+1}. Such polygons are also counted by $C(x,y)^2$.
– if C_{i+1} has height at least 2, there are no other possibilities. However, if C_{i+1} consists of one cell only, this cell may be adjacent to *any* cell of C_i, not only to the bottom or top ones. The case where C_{i+1} is the last column of P is counted by $xy(C_1(x,y) - C(x,y))$. The case where $i+1 < k$ is counted by $x(C_1(x,y)^2 - C(x,y)C_1(x,y))$.

Let us drop the variables x and y in the series C, C_1 and C_2. The above discussion gives the equation:

$$C = yC + xy + xC_1 + 2C^2 + xy(C_1 - C) + x(C_1^2 - CC_1).$$

The construction of Fig. 3.13 can now be recycled to obtain an equation for the series C_1, counting cc-polygons with a marked cell in the last column. Note that the first case of the figure (thick polygons) gives rise to two terms, depending on whether the marked cell is one of the duplicated cells, or not:

$$C_1 = y(C + C_1) + xy + xC_2 + 2CC_1 + xy(C_1 - C) + x(C_1C_2 - CC_2).$$

We need a third equation, as three series (namely C, C_1 and C_2) are now involved. There are two ways to obtain a third equation:

– either we interpret C_1 as the g.f. of cc-polygons where one cell is marked in the *first* column. The construction of Fig. 3.13 gives:

$$C_1 = y(C + C_1) + xy + xC_1 + 2(C + C_1)C + xy\big((C_1 - C) + (C_2 - C_1)\big) \\ + x\big((C_1^2 - CC_1) + (C_2C_1 - C_1^2)\big).$$

Note that now many cases give rise to two terms in the equation.
– or we work out an equation for C_2 using the decomposition of Fig. 3.13. Again, many of the cases schematized in this figure give rise to several terms. In particular, the first case (thick polygons) gives rise to 4 terms:

$$C_2 = y(C + 2C_1 + C_2) + xy + xC_2 + 2(C + C_1)C_1 + xy\big((C_1 - C) + (C_2 - C_1)\big) \\ + x\big((C_1C_2 - CC_2) + (C_2^2 - C_1C_2)\big).$$

Both strategies of course give the same equation for $C \equiv C(x,y)$, after the elimination of C_1 and C_2:

$$\begin{aligned}&\left(-5xy-18+2xy^2-18y^2+36y+2x\right)C^4\\&\qquad+(y-1)\left(5xy^2-21y^2+42y-14xy+5x-21\right)C^3\\&\qquad+2\,(y-1)^2\left(-4y^2+2xy^2+8y-7xy-4+2x\right)C^2\\&\qquad\quad+(y-1)^3\left(xy^2-y^2+2y-6xy+x-1\right)C-xy\,(y-1)^4=0.\end{aligned}$$

This quartic has 4 roots, among which the g.f. of cc-polygons can be identified by checking the first few coefficients. This series turns out to be unexpectedly simple:

$$C(x,y)=(1-y)\left(1-\frac{2\sqrt{2}}{3\sqrt{2}-\sqrt{1+x+\sqrt{(1-x)^2-16\frac{xy}{(1-y)^2}}}}\right).$$

Feretić has provided direct combinatorial explanations for this formula [28, 29]. The algebraic equation satisfied by $C(t,t)$ was first[1] obtained (via a context-free language) in [22]. The method we have used is detailed in [26].

3.3.3.3 Directed Column-Convex Polygons by Perimeter

It is not hard to restrict the construction of Fig. 3.13 to *directed* cc-polygons. This is illustrated in Fig. 3.14. Note that the case where the columns C_i and C_{i+1} share the bottom edge of C_i (the fourth case in Fig. 3.13) is only possible if C_{i+1} has height 1. Moreover, only one additional series is needed, namely that of directed cc-polygons marked in the last column (D_1).

One obtains the following equations:

$$\begin{aligned}D&=yD+xy+xD+xD^2+xyD+D^2+xy(D_1-D)+x(D_1-D)D,\\D_1&=y(D+D_1)+xy+xD_1+xDD_1+xyD+DD_1+xy(D_1-D)+x(D_1-D)D_1.\end{aligned}$$

Eliminating D_1 gives a cubic equation for the series $D\equiv D(x,y)$:

$$D^3+2\,(y-1)\,D^2+(y-1)\,(x+y-1)\,D+xy\,(y-1)=0.$$

This equation was first obtained in [21]. The first few terms of the semi-perimeter generating function are

$$D(t,t)=t^2+2t^3+6t^4+20t^5+71t^6+263t^7+1005t^8+3933t^9+\cdots$$

[1] Eq. (32) in [22] has an error: the coefficient of t^5c^3 in p_2 should be -40 instead of $+40$.

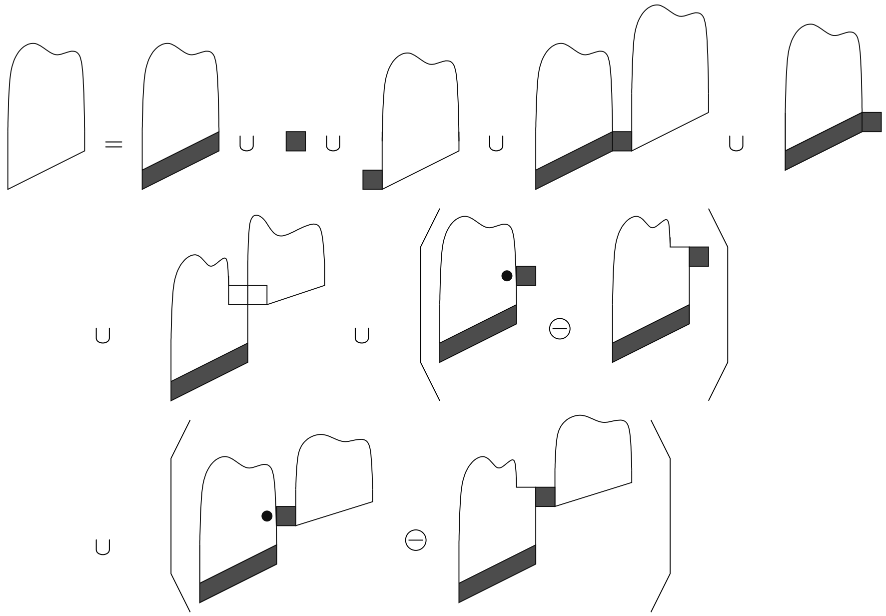


Fig. 3.14 A recursive construction of directed column-convex polygons.

3.3.3.4 Directed Polyominoes by Area

Let us move to a class that admits a neat, but non-obvious, algebraic structure: directed polyominoes counted by area. This structure was discovered when Viennot developed the theory of *heaps* [52]. Intuitively, a heap is obtained by dropping vertically some solid pieces, one after the other. Thus, a piece lies either on the "floor" (when it is said to be *minimal*), or at least partially covers another piece.

Directed polyominoes *are*, in essence, heaps. To see this, replace every cell of the polyomino by a *dimer*, after a 45 degree rotation (Fig. 3.15). This gives a heap with a unique minimal piece. Such heaps are called *pyramids*. If the columns to the left of the minimal piece contain no dimer, we say we have a *half-pyramid* (Fig. 3.15, right).

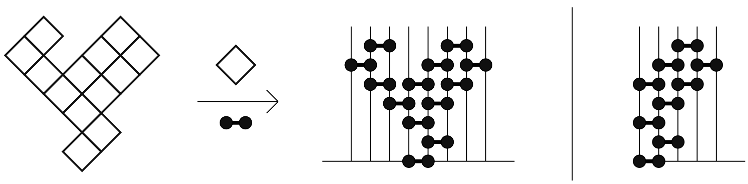


Fig. 3.15 *Left*: A directed polyomino and the associated pyramid. *Right*: a half-pyramid.

The interest in heaps lies in the existence of a *product* of heaps: The product of two heaps is obtained by putting one heap above the other and dropping its pieces. Conversely, one can factor a heap by pushing upwards one or several pieces. See an example in Fig. 3.16. This product is the key in our algebraic description of directed polyominoes, or, equivalently, of pyramids of dimers, as we now explain.

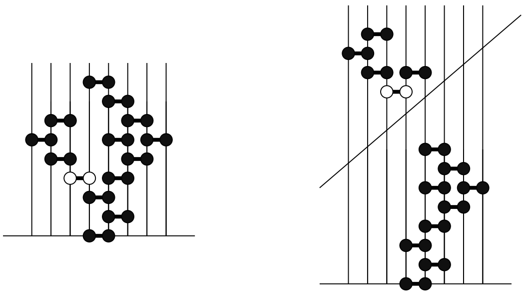


Fig. 3.16 A factorization of a pyramid into a pyramid and a half-pyramid. Observe that the highest dimer of the pyramid moves up as we lift the white dimer.

A pyramid is either a half-pyramid, or the product of a half-pyramid and a pyramid (Fig. 3.17, top). Let $D(q)$ denote the g.f. of pyramids counted by the number of dimers, and $H(q)$ denote the g.f. of half-pyramids. Then $D(q) = H(q)(1 + D(q))$.

Now, a half-pyramid can be a single dimer. If it has several dimers, it is the product of a single dimer and of one or two half-pyramids (Fig. 3.17, bottom), which implies $H(q) = q + qH(q) + qH^2(q)$. Note that $D(q)$ is also the area g.f. of directed polyominoes. A straightforward computation gives:

$$D(q) = \frac{1}{2}\left(\sqrt{\frac{1+q}{1-3q}} - 1\right) \tag{3.8}$$

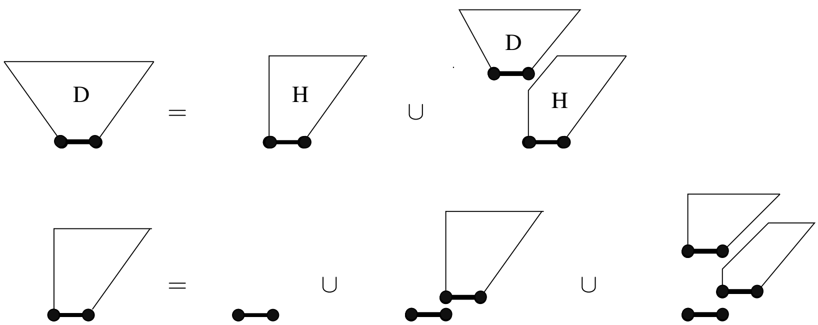


Fig. 3.17 Decomposition of pyramids (top) and half-pyramids (bottom).

This was first proved by Dhar [24]. The above proof is adapted from [7].

3.3.4 *q*-Analogues

By looking for the first column of height 1 in a bargraph, we have described an algebraic construction of these polygons (Fig. 3.11) that proves that their perimeter g.f. is algebraic (Section 3.3.1). Let us now examine what happens when we try to keep track of the area of these polygons.

As in Section 3.2.4, the key observation is that the area behaves additively when one concatenates two bargraphs, but *increases by the width of the polygon* when we duplicate the bottom row. (In contrast, the half-perimeter simply increases by 1 during this operation.) This observation gives rise to the following functional equation for the complete g.f. of bargraphs:

$$B(x,y,q) = yB(xq,y,q) + xyq + xqB(x,y,q) + xyqB(xq,y,q) \\ + xqB(xq,y,q)B(x,y,q). \quad (3.9)$$

This is a q-analogue of Equation (3.7) defining $B(x,y,1)$. This equation is no longer algebraic, and it is not clear how to solve it. It has been shown in [44] that it can be linearized and solved using a certain Ansatz. We will show in Section 3.4.1 a more systematic way to obtain $B(x,y,q)$, which does not require any Ansatz.

3.4 Adding a New Layer: a Versatile Approach

In this section we describe a systematic construction that can be used to find the complete g.f. of many classes of polygons having a convexity property [10]. The cost of this higher generalization is twofold:

- it is not always clear how to solve the functional equations obtained in this way,
- in contrast with the constructions developed in Sections 3.2 and 3.3, this approach does not provide combinatorial explanations for the rationality/algebraicity of the corresponding g.f.s.

This type of construction is sometimes called *Temperley's approach* since Temperley used it to write functional equations for the generating function of column-convex polygons counted by perimeter [51]. But it also occurs, in a more complicated form, in other "old" papers [6, 40]. We would prefer to see a more precise terminology, like *layered approach*.

3.4.1 A Basic Example: Bargraphs by Perimeter and Area

We return to our favourite example of bargraphs, and we now aim to find the complete g.f. $B(x,y,q)$ of this class of polygons. We have just seen that the algebraic description of Fig. 3.11 leads to the q-algebraic equation (3.9), which is not obvious to solve. The linear description of Section 3.2.1 cannot be directly exploited either: in order to decide whether the addition of a cell at the top of the last column increases the perimeter or not, we need to know which of the last two columns is higher.

We present here a variation of this linear construction that allows us to count bargraphs by area and perimeter, *provided we also take into account the right height* by a new variable s. By *right height*, we mean the height of the rightmost column. The g.f. we are interested in is now

$$B(x,y,q,s) = \sum_{h\geq 1} B_h(x,y,q)s^h,$$

where $B_h(x,y,q)$ is the complete g.f. of bargraphs of right height h.

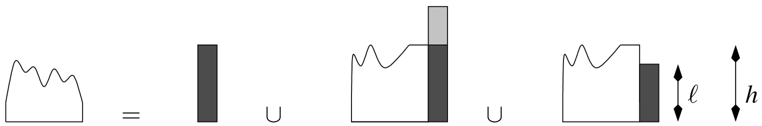


Fig. 3.18 A third recursive construction of bargraphs.

Our new construction is illustrated in Fig. 3.18. The class $\mathscr{B}$ of bargraphs is split into three disjoint subsets:

1. bargraphs of width 1 (columns). The g.f. of this class is $xysq/(1-ysq)$,
2. bargraphs in which the last column is at least as high as the next-to-last column. These bargraphs are obtained by duplicating the last column of a bargraph (which boils down to replacing s by sq in the series $B(x,y,q,s)$), and adding a (possibly empty) column at the top of the newly created column. The corresponding g.f. is thus
$$\frac{x}{1-ysq}B(x,y,q,sq).$$
3. bargraphs in which the last column is lower than the next-to-last column. To obtain these, we start from a bargraph, say of right height h, and add a new column of height $\ell < h$ to the right. The g.f. of this third class is:

$$x\sum_{h\geq 1}\left(B_h(x,y,q)\sum_{\ell=1}^{h-1}(sq)^{\ell}\right) = x\sum_{h\geq 1}\left(B_h(x,y,q)\,\frac{sq-(sq)^h}{1-sq}\right)$$
$$= x\,\frac{sqB(x,y,q,1)-B(x,y,q,sq)}{1-sq}. \tag{3.10}$$

Writing $B(s) \equiv B(x,y,q,s)$, and putting together the three cases, we obtain:

$$B(s) = \frac{xysq}{1-ysq} + \frac{xsq}{1-sq}B(1) + \frac{xsq(y-1)}{(1-sq)(1-ysq)}B(sq). \tag{3.11}$$

This equation is solved in two steps: first, an iteration, similar to what we did for Ferrers diagrams in (3.4) (Section 3.2.4), provides an expression for $B(s)$ in terms of $B(1)$:

$$B(s) = \sum_{n\geq 1} \frac{(xs(y-1))^{n-1} q^{\binom{n}{2}}}{(sq)_{n-1}(ysq)_{n-1}} \left(\frac{xysq^n}{1-ysq^n} + \frac{xsq^n}{1-sq^n}B(1) \right).$$

Then, one sets $s = 1$ to obtain the complete g.f. $B(1) \equiv B(x,y,q,1)$ of bargraphs:

$$B(x,y,q,1) = \frac{I_+}{1-I_-} \tag{3.12}$$

with

$$I_+ = \sum_{n\geq 1} \frac{x^n (y-1)^{n-1} q^{\binom{n+1}{2}}}{(q)_{n-1}(yq)_n} \quad \text{and} \quad I_- = \sum_{n\geq 1} \frac{x^n (y-1)^{n-1} q^{\binom{n+1}{2}}}{(q)_n (yq)_{n-1}}.$$

3.4.2 More Examples

In this section, we describe how to apply the layered approach to two other classes of polygons: staircase and column-convex polygons, counted by perimeters and area simultaneously. In passing we show how the *difference* of g.f.s resulting from a geometric summation like (3.10) can be explained combinatorially by an inclusion-exclusion argument.

3.4.2.1 Staircase polygons

As with the bargraph example above, we define an extended generating function which tracks the height of the rightmost column of the staircase polygon,

$$S(x,y,q,s) = \sum_{h\geq 1} S_h(x,y,q)s^h,$$

where $S_h(x,y,q)$ is the generating function of staircase polygons with right height h. The set of all staircase polygons can be partitioned into two parts (Fig. 3.19):

1. those which have only one column. The g.f. for this class is $xyqs/(1-yqs)$.
2. those which have more than one column. Their g.f. is obtained as the difference of the g.f. of two sets as follows.

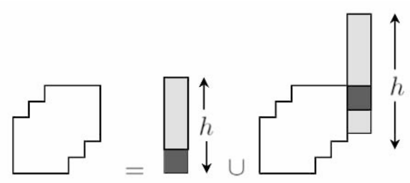


Fig. 3.19 The two types of staircase polygons.

Staircase polygons of width $\ell \geq 2$ can be split into two objects: a staircase polygon formed of the $\ell - 1$ first columns, and the rightmost column. The left part has generating function $S(x,y,q,1)$ (ignoring the rightmost height), to which we then attach a column of cells. The attached column is constrained in that it must not extend below the bottom of the rightmost column of the left part. It is generated (see Fig. 3.19) by gluing a descending column (with g.f. $1/(1-qs)$) and an ascending column (with g.f. $1/(1-yqs)$) to a single square (with g.f. $xyqs$). The single square is required to ensure that the column is not empty and is glued to the immediate right of the topmost square of the left part. An important observation is that only the ascending column contributes in the increase of the vertical perimeter. This gives the generating function

$$S(x,y,q,1) \cdot xyqs \cdot \frac{1}{1-qs} \cdot \frac{1}{1-yqs}.$$

This construction however results in configurations which might have the rightmost column extending below the rightmost column of the left part. We must thus subtract the contribution of these "bad" configurations from the above g.f. We claim that they are generated by

$$S(x,y,q,sq) \cdot xyqs \cdot \frac{1}{1-qs} \cdot \frac{1}{1-yqs}.$$

The replacement of s with sq in $S(x,y,q,sq)$ is interpreted as adding a copy of the last column of the left part, as illustrated in Fig. 3.20. The $xyqs$ factor is interpreted as attaching a new cell to the bottom of the duplicated column (thus ensuring the rightmost column is strictly below the rightmost column of the left part). Finally, we add a descending and an ascending column. Again, the height of the latter must not be taken into account in the vertical perimeter.

Thus the final equation for the generating function is

$$S(x,y,q,s) = \frac{xyqs}{1-yqs} + \big(S(x,y,q,1) - S(x,y,q,sq)\big)\frac{xyqs}{(1-qs)(1-yqs)}.$$

It can also be obtained via geometric sums, as was done for (3.10). The equation is solved with the same two step process as for bargraphs. First we iterate it to obtain $S(x,y,q,s)$ in terms of $S(x,y,q,1)$, and then we set s to 1, obtaining

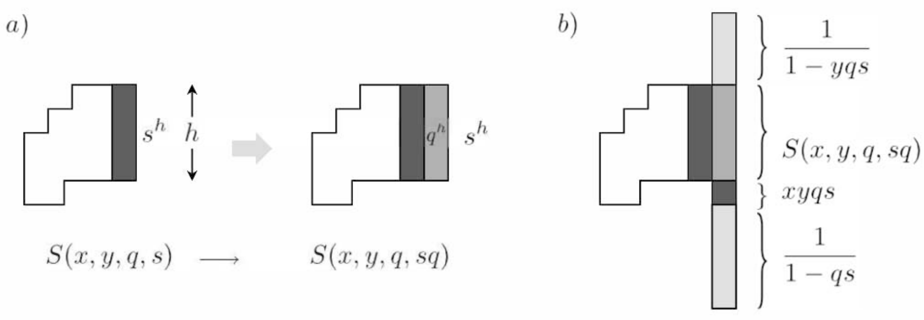


Fig. 3.20 a) Replacing s by sq in $S(x,y,q,s)$ duplicates the last column of the polygon. b) Generating function of "bad" configurations.

$$S(x,y,q,1) = y\frac{J_1}{J_0},$$

where J_0 and J_1 are two q-Bessel functions [34]:

$$J_1(x,y,q) = \sum_{k\geq 1}(-1)^{k+1}\frac{x^k q^{\binom{k+1}{2}}}{(q)_{k-1}(yq)_k}$$

and

$$J_0(x,y,q) = \sum_{k\geq 0}(-1)^{k}\frac{x^k q^{\binom{k+1}{2}}}{(q)_{k}(yq)_k}.$$

Note, the appropriate limit as $q \to 1$ leads to standard Bessel functions which are related to the generating function for semi-continuous staircase polygons—see [18] for details.

3.4.2.2 Column-Convex Polygons

The case of column-convex polygons is more complex and we will not give all the details but discuss only the primary additional complication. We refer to [10] for a complete solution. As in the case of staircase polygons, a functional equation for column-convex polygons can be obtained by considering the rightmost (last) column. The position of the last column compared with the second last column must be carefully considered. Again there are several cases depending on whether the top (resp. bottom) of the last column is strictly above, at the same level or below the top (resp. bottom) of the second-last column. The case that leads to a type of term that does not appear in the equation for staircase polygons is the case where the top (resp. bottom) of the last column cannot be above (resp. below) the top (resp. bottom) of the second-last column. Thus we will only explain this case which we will refer to as the *contained* case, as the last column is somehow contained in the previous one.

If the generating function for the column-convex polygons is $C(s) = C(x,y,q,s)$ then we claim that the polygons falling into the contained case are counted by the generating function

$$\frac{xsq}{1-sq}\frac{\partial C}{\partial s}(1) - \frac{xs^2q^2}{(1-sq)^2}\big(C(1) - C(sq)\big). \tag{3.13}$$

Thus we see we now need a derivative of the generating function. As a polygon of right height h contributes h times to the series $\partial C/\partial s(1)$, this series counts polygons with a marked cell in the rightmost column.

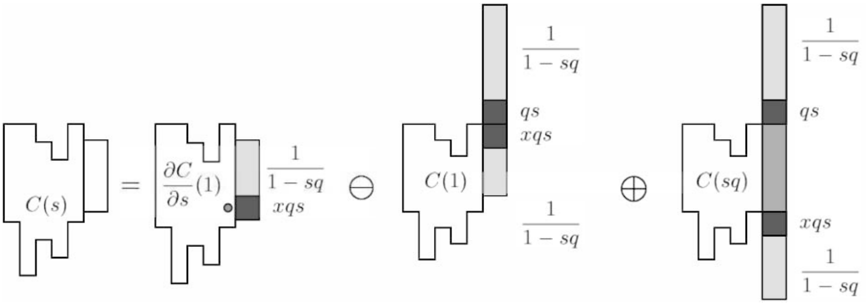


Fig. 3.21 A schematic representation of the equation for the case where the rightmost column does not extend above or below the second-last column.

Let us now explain this expression, which is illustrated in Fig. 3.21. We consider a polygon as the concatenation of a left part with a new (rightmost) column C. In the left part, we mark the cell of the rightmost column that is at the same level as the bottom cell of C. So, starting from a marked polygon, we first add a single square to the right of the marked cell—this gives a factor xsq. Above this square we then add an ascending column which is generated by $1/(1-sq)$. However, as with the staircase polygons, the resulting series counts "bad" configurations, where the last column ends strictly higher than the second last column. We subtract the contribution of these bad configurations by generating them as shown on the second picture of Fig. 3.21. This results in subtracting the term $xq^2s^2C(1)/(1-sq)^2$. However, we have now subtracted too much! Indeed, some configurations counted by the latter series have a rightmost column that ends below the second last column. We correct this by adding the contribution of these configurations, which is $xq^2s^2C(sq)/(1-sq)^2$ (Fig. 3.21, right). This establishes (3.13) for the g.f. of the contained case.

The other cases are simpler, and in the same vein as what was needed for staircase polygons. Considering all cases gives

$$C(s) = \frac{xsyq}{1-syq} + \frac{xsq}{1-sq}\frac{\partial C}{\partial s}(1) + \frac{xs^2q^2(2y-syq-1)}{(1-sq)^2(1-syq)}C(1) + \frac{xs^2q^2(1-y)^2}{(1-sq)^2(1-syq)^2}C(sq). \quad (3.14)$$

In order to solve this equation, we first iterate it to obtain $C(s)$ in terms of $C(1)$ and $C'(1) = \partial C/\partial s(1)$. Setting $s = 1$ gives a linear equation between $C(1)$ and $C'(1)$. Setting $s = 1$ *after having differentiated with respect to s* gives a second linear equation between $C(1)$ and $C'(1)$. We end up solving a linear system of size 2, and obtain $C(1)$ as a ratio of two 2×2 determinants. The products of series that appear in these determinants can be simplified, and the final expression reads

$$C(x,y,q,1) = y\frac{(1-y)X}{1+W+yX}$$

where

$$X = \frac{xq}{(1-y)(1-yq)} + \sum_{n\geq 2} \frac{(-1)^{n+1}x^n(1-y)^{2n-4}q^{\binom{n+1}{2}}(y^2q)_{2n-2}}{(q)_{n-1}\,(yq)_{n-2}\,(yq)^2_{n-1}\,(yq)_n\,(y^2q)_{n-1}}$$

and

$$W = \sum_{n\geq 1} \frac{(-1)^n x^n(1-y)^{2n-3}q^{\binom{n+1}{2}}(y^2q)_{2n-1}}{(q)_n\,(yq)^3_{n-1}\,(yq)_n\,(y^2q)_{n-1}}.$$

The first solution, involving a more complicated expression, was given in [17]. The one above appears as Theorem 4.8 in [10].

3.4.3 The Kernel Method

In Sections 3.2 and 3.3, we have explained combinatorially why the area g.f. of bargraphs, $B(1,1,q)$, and the perimeter g.f. of bargraphs, $B(x,y,1)$, are respectively rational and algebraic. It is natural to examine whether these properties can be recovered from the construction of Fig. 3.18 and the functional equation (3.11).

As soon as we set $y = 1$ in this equation, the main difficulty, that is, the term $B(sq)$, disappears. We can then substitute 1 for s and solve for $B(x,1,q,1)$, the width and area g.f. of bargraphs. This series is found to be

$$B(x,1,q,1) = \frac{xq}{1-q-xq}.$$

From this, one also obtains a rational expression for the series $B(x,1,q,s)$. The rationality of $B(x,1,q,1)$ also follows directly from the expression (3.12): setting $y = 1$ shrinks the series I_+ and I_- to simple rational functions.

How the *perimeter* g.f. of bargraphs can be derived from the functional equation (3.11) is a more challenging question. Setting $q = 1$ gives

$$B(s) = \frac{xys}{1-ys} + \frac{xs}{1-s}B(1) + \frac{xs(y-1)}{(1-s)(1-ys)}B(s). \tag{3.15}$$

This equation cannot be simply solved by setting $s = 1$. Instead, the solution uses the so-called *kernel method*, which has proved useful in a rather large variety of enumerative problems in the past 10 years [3, 4, 14, 19, 32, 47]. This method solves, in a systematic way, equations of the form:

$$K(s,x)A(s,x) = P(x,s,A_1(x),\ldots,A_k(x))$$

where $K(s,x)$ is a polynomial in s and the other indeterminates $x = (x_1,\ldots,x_n)$, P is a polynomial, $A(s,x)$ is an unknown series in s and the x_i's, while the series $A_i(x)$ only depend on the x_i's. (It is assumed that the equation uniquely defines all these unknown series.) We refer to [14] for a general presentation, and simply illustrate the method on (3.15). We group the terms involving $B(s)$, and multiply the equation by $(1-s)$ to obtain:

$$\left(1 - s - \frac{xs(y-1)}{1-ys}\right)B(s) = \frac{xys(1-s)}{1-ys} + xsB(1). \tag{3.16}$$

Let $S \equiv S(x,y)$ be the only formal power series in x and y that satisfies

$$S = 1 - \frac{xS(y-1)}{1-yS}.$$

That is,

$$S = \frac{1-x+y+xy-\sqrt{(1-y)((1-x)^2-y(1+x)^2)}}{2y}.$$

Replacing s by S in (3.16) gives an identity between series in x and y. By construction, the left-hand side of this identity vanishes. This gives

$$\begin{aligned} B(1) \equiv B(x,y,1,1) &= \frac{y(S-1)}{1-yS} \\ &= \frac{1-x-y-xy-\sqrt{(1-y)((1-x)^2-y(1+x)^2)}}{2x}, \end{aligned}$$

and we have recovered the algebraic expression (3.6) of the perimeter g.f. of bargraphs.

3.5 Some Open Questions

We conclude this chapter with a list of open questions. As mentioned in the introduction, the combination of convexity and direction conditions gives rise to 35 classes of polyominoes, not all solved. But all these classes are certainly not equally interesting. The few problems we present below have two important qualities: they do not seem completely out of reach (we do not ask about the enumeration of all polyominoes) and they have some special interest: they deal either with large classes of polyominoes, or with mysterious classes (that have been solved in a non-combinatorial fashion), or they seem to lie just at the border of what the available techniques can achieve at the moment.

3.5.1 The Quasi-Largest Class of Quasi-Solved Polyominoes

Let us recall that the growth constant of n-cell polyominoes is conjectured to be a bit more than 4. More precisely, it is believed that p_n, the number of such polyominoes, is equivalent to $\mu^n n^{-1}$, up to a multiplicative constant, with $\mu = 4.06\ldots$ [37]. The techniques that provide lower bounds on μ involve looking at *bounded* polyominoes (for instance polyominoes lying in a strip of fixed height k) and a concatenation argument. See [5] for a recent survey and the best published lower bound, $3.98\ldots$. It is not hard to see that for k fixed, these bounded polyominoes have a linear structure, and a rational generating function. This series is obtained either by adding recursively a whole "layer" to the polyomino (as we did for self-avoiding polygons in Section 3.2.3.3), or by adding one cell at a time. The latter approach is usually more efficient (Chapter 6).

What about solved classes of polyominoes that do not depend on a parameter k, and often have a more subtle structure? We have seen in Section 3.3.3 that the g.f. of directed polyominoes is algebraic, with growth constant 3. This is "beaten" by the growth constant $3.20\ldots$ derived from the rational g.f. of column-convex polyominoes (Section 3.2.3). A generalization of directed polyominoes (called multi-directed polyominoes) was introduced in [15] and proved to have a fairly complicated g.f., with growth constant about 3.58. To our knowledge, this is the largest growth constant reached from exact enumeration (again, apart from the rational classes obtained by bounding column heights).

However, in 1967, Klarner introduced a "large" class of polyominoes that seems interesting and would warrant a better understanding [39]. His definition is a bit unclear, and his solution is only partial, but the estimate he obtains of the growth constant is definitely appealing: about 3.72. Let us mention that the triangular lattice version of this mysterious class is solved in [15]. The growth constant is found to be about 4.58 (the growth constant of triangular lattice animals is estimated to be about 5.18, see [53]).

3.5.2 Partially Directed Polyominoes

This is another generalization of directed polyominoes, with a very natural definition: the corresponding animal A contains a source point v_0 from which every other point can be reached by a path formed of North, East and West steps, only visiting points of A (Fig. 3.22(a)). This model has a slight flavour of *heaps of pieces*, a notion that has already proved useful in the solution of several polyomino models (see Section 3.3.3 and [7, 16, 15]). The growth constant is estimated to be around 3.6, and, if proved, would thus improve that of multi-directed animals [45].

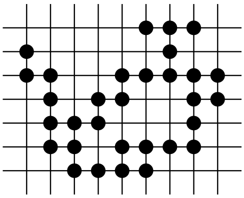
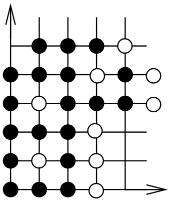
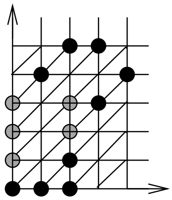
(a)  (b)  (c) 

Fig. 3.22 (a) A partially directed animal. The source can be any point on the bottom row. (b) A directed animal on the square lattice, with the right neighbours indicated in white. (c) A directed animal A on the triangular lattice. The distinguished points are those having (only) their South neighbour in A.

3.5.3 The Right Site-Perimeter of Directed Animals

We wrote in the introduction that *almost all* solved classes of polyominoes can be solved by one of the three main approaches we present in this chapter. Here is one simple-looking result that we do not know how to prove via these approaches (and not via any combinatorial approach, to be honest).

Take a directed animal A, and call a *neighbour* of A any point that does not lie in A, but could be added to A to form a new directed animal. The number of neighbours is the *site-perimeter* of A. The *right site-perimeter* of A is the number of neighbours that lie one step to the right of a point of A. It was proved in [11] that the g.f. of directed animals, counted by area and right site-perimeter, is a very simple extension of (3.8):

$$D(q,x) = \frac{x}{2}\left(\sqrt{\frac{(1+q)(1+q-qx)}{1-q(2+x)+q^2(1-x)}}-1\right).$$

The proof is based on an equivalence with a one-dimensional gas model, inspired by [25]. It is easy to see that the right site-perimeter is also the number of vertices v

of A whose West neighbour is not in A. (By the West neighbour, we mean the point at coordinates $(i-1,j)$ if $v=(i,j)$).

Described in these terms, this result has a remarkable counterpart for triangular lattice animals (Fig. 3.22). Let us say that a point (i,j) of the animal has a West (resp. South, South-West) neighbour in A if the point $(i-1,j)$ (resp. $(i,j-1)$, $(i-1,j-1)$) is also in A. Then the g.f. that counts these animals by the area and the number of points having a SW-neighbour (but no W- or S-neighbour) is easy to obtain using heaps of dimers and the ideas presented in Section 3.3.3:

$$\tilde{D}(q,x) = \frac{1}{2}\left(\sqrt{\frac{1+q-qx}{1-3q-qx}}\right).$$

What is less easy, and is so far only proved via a correspondence with a gas model, is that $\tilde{D}(q,x)$ also counts directed animals (on the triangular lattice) by the area and the number of points having a South neighbour (but no SW- or W-neighbour). Any combinatorial proof of this result would give a better understanding of these objects. One possible starting point may be found in the recent paper [41], which sheds some combinatorial light on the gas models involved in the proof of the above identities.

3.5.4 Diagonally-Convex Polyominoes

Let us conclude with a problem that seems to lie at the border of the applicability of the third approach presented here (the layered approach). In the enumeration of, say, column-convex polyominoes (Section 3.4.2), we have used the fact that deleting the last column of such a polyomino gives another column-convex polyomino. This is no longer true of a d_--convex polyomino from which we would delete the last diagonal (Fig. 3.1(d)). Still, it seems that this class is sufficiently well structured to be exactly enumerable. Note that this difficulty vanishes when studying the restricted class of *directed* diagonally-convex polyominoes [8, 46], which behave approximately like column-convex polyominoes.

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