UNCERTAINTY MODELING WITH IMPRECISE STATISTICAL REASONING AND THE PRECAUTIONARY PRINCIPLE IN DECISION MAKING

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Abstract: A number of unconventional formal approaches to decision making have been developed to provide mathematical foundations for rational choices under both aleatory and epistemic uncertainty. They challenge a central assumption of the Bayesian theory, that uncertainty should always be gauged by a single (additive) measure, and values should always be gauged by a precise utility function [3].

Decision-making theorists have presented approaches for arriving at rational decisions in spite of imprecision and indeterminacy [4–8, 10]. This paper introduces the theory of upper and lower previsions, provides examples, discusses how to account for unreliable statistical judgements, and reviews the relationships between the Precautionary Principle, indecision, and imprecise statistical reasoning.

1. Introduction

One of the gravest errors in any type of risk management process is the presentation of risk estimates which convey a false impression of accuracy and confidence – disregarding the uncertainties inherent in basic understanding, data acquisition, and statistical analysis. [1]

Decision making concerning human activities with potentially harmful consequences and high uncertainties is based on both scientific findings of the risk assessment and societal norms such as the Precautionary Principle (PP). However, risk assessments along with uncertainty measures complemented by the need to comply with the PP do not compel adoption of a particular course of action. This is usually left to the discretion of decision makers. As the stakes rise, the lack of scientific consistency among all systems analysis constituents preceding the option selection may result in failing to select an acceptable option. Systems analysis constituents include hazard/threat

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identification, risk assessment, uncertainty assessment, account of societal norms, and decision making. Studying each of them as a separate component is necessary, but this is no longer sufficient. An integrated approach, binding them in a formally consistent framework, is a coveted target for risk analysts.

Conceptual and computational structure of analyses of complex systems involves a division of uncertainty into aleatory uncertainty, which arises because the system under study can potentially behave in many different ways; and epistemic uncertainty, which arises from a lack of knowledge about quantities that have fixed but poorly known values. Aleatory uncertainty is also called stochastic and irreducible, while epistemic is called reducible. Such separation plays a particularly important role in risk analyses, where aleatory uncertainty arises from many possible adverse outcomes or consequences and epistemic uncertainty arises from a lack of knowledge with respect to quantities required in the characterization of the frequency, evolution, or consequences of individual potential adverse effects [2].

A number of unconventional formal approaches to decision making have been developed to provide mathematical foundations for rational choices under both aleatory and epistemic uncertainty. They allow for the limited cognitive abilities of human beings and could be regarded as formal variants of the PP. They also give a perspective of how the integrated framework could be built.

Though different in detail, they have a very important point in common. They challenge a central assumption of the Bayesian theory, that uncertainty should always be gauged by a single (additive) probability measure, and values should always be gauged by a precise utility function [3]. This assumption has been referred to as the Bayesian dogma of precision.[1] The opponents of the dogma of precision claim that imprecision, indeterminacy, and indecision are compatible with rational choice [4].

One unconventional theory of rational choice is discussed by Gårdenfors and Sahlin [5]. The point of departure from the conventional theory of rational choice—Bayesian decision theory—is that the amount and quality of information the decision maker has concerning the possible states and outcomes of the decision situation in many cases constitute an important factor when making decisions. To describe this aspect of the decision situation, the authors say that the information available concerning the possible states and outcomes of a decision situation has different degrees of epistemic reliability. The second step is to recognize that the reliability of

[1] Perhaps the most noticeable calls to revise the Bayesian theory for making rational choices were pronounced by Herbert A. Simon (Nobel Prize winner). See, for example, Simon [9].

a probability assignment for states affects the risk of the decision. The less reliable the probability assignment, the more risky the decision, other things being equal.

Another theory of decision making is discussed by Levi [6]. His point of departure is that we often do not know or cannot decide what we most prefer; yet we still have to choose. In such cases, called decision making under unresolved conflict, the requirement that preferences should be logically coherent does not necessarily imply that choices should satisfy properties of consistency such as avoiding sure loss. A rational agent, Levi claims, ought to restrict his choice to the set of admissible options; within this set, any choice is allowed. The theory suggests a formal way of constructing the set of admissible choices.

There are some other developed rules of rational choice that accept as a starting point a lack of knowledge for exactly defining utilities and probability assignments for the set of outcomes. They presuppose that numerical input for decision making is interval-valued and suggest different approaches for choosing one option among those permissible. The width of the interval manifests the lack of knowledge concerning utilities and probability assignments [4, 7, 8].

Decision making is the final phase in systems analysis and to all appearances there are mathematically furnished rules to make rational choices under lack of knowledge. The question to ask now is: Are there formal frameworks for uncertainty modeling that are built on the clear distinction between aleatory and epistemic uncertainty?

Reasoning that can accommodate the both types of uncertainties is called imprecise statistical reasoning and is motivated by the idea that the dogma of precision is mistaken and imprecise probabilities are needed in statistical reasoning and decision. The pivotal concept of this reasoning is imprecise probability, which is a generic term for a range of mathematical models that measure chance or uncertainty without sharp numerical probabilities. These models include belief functions, Choquet capacities, comparative probability orderings, convex sets of probability measures, fuzzy measures, interval-valued probabilities, possibility measures, plausibility measures, and upper and lower expectations or previsions [4].

In pursuit of uncertainty representation, aggregation, and propagation through models of reliability and risk, we employ the theory of upper and lower expectations (previsions) as described by Walley [4] and Kuznetsov [10] and build interval statistical models based on it. Generally speaking, to measure aleatory uncertainty, we need some kind of probability; to measure epistemic uncertainty, we need intervals.

This paper introduces the theory of upper and lower previsions in a ‘soft’ way, avoiding heavy formalism. A variety of statistical evidence admitted in the framework is exemplified. A way to account for unreliable statistical judgements is also briefly described. A short passage on the relationships between the PP, indecision, and imprecise statistical reasoning concludes the paper.

2. Discrete Case

Let us look first at what kind of discrete problem can be solved in the framework of the theory of upper and lower expectations.

Assume there are three possible outcomes s_1, s_2, and $s_3$in a subject matter of interest. This is an exhaustive set of events meaning that $P(s_1) + P(s_2) + P(s_3) = 1$, where $P(\cdot)$ stands for a probability. Information on the probabilities of the occurrences of these events is given as three pieces of evidence: (1) $P(s_1) \in [0.1, 0.3]$, (2) s_2 is at least two times as probable as s_3, and (3) s_2 and s_3 is at least as probable as s_1. What probabilities $P(s_2)$ and $P(s_3)$ can one derive based on the provided information?[2]

One can hardly expect that the source imprecise information can result in precise answers in the form of precise probabilities $P(s_2)$ and $P(s_3)$. What is the mechanism for arriving at an answer?

As we have three possible outcomes, the simplex representation can demonstrate well the basic ideas of the approach. In Figure 1, the vertexes 1, 2, and 3 correspond to the three states s_1, s_2, and s_3. The probability simplex is an equilateral triangle with height one unit, in which the probabilities assigned to the three elements are identified with perpendicular distances

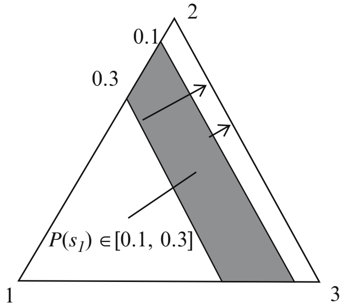

Figure 1. Presentation of the Statistical Evidence $P(s_1) \in [0.1, 0.3]$ on the Simplex.

[2] This Example has been Demonstrated in Greater Detail [11.]

from the three sides of the triangle. Adding up these three distances gives 1. Thus, each point inside of the simplex can be thought of as a precise probability distribution. The simplex representation is especially useful for depicting pieces of statistical evidence and studying their effects on the probabilities of outcomes.

The first piece of evidence, $P(s_1) \infty [0.1, 0.3]$, is depicted in Figure 1; Figure 2 depicts all the source information with the simplex representation.

The source evidence can be rewritten in the form of inequalities (1) $0.1 \leq P(s_1) \leq 0.3$, (2) $P(s_2) \geq 2P(s_3)$, and (3) $P(s_2) + P(s_3) \geq P(s_1)$. These inequalities and condition $P(s_1) + P(s_2) + P(s_3) = 1$ define a constrained area which is shown in black in Figure 2. The calculation of upper and lower bounds for the probabilities of interest becomes a geometric task. The calculated values of the probabilities are $\overline{P}(s_2) = 0.466$, $\overline{P}(s_2) = 0.9$, $\overline{P}(s_3) = 0$, $\overline{P}(s_3) = 0.3$, while $\overline{P}(s_1) = 0.1$ and $\overline{P}(s_1) = 0.3$ remain unchanged.

It can be noticed from Figure 2 that the evidence $P(s_2) + P(s_3) \geq P(s_1)$ does not contribute to the precision and can be discarded without influencing the result. That is, the black area, defining the lower and upper probabilities, does not change if this evidence is removed from the set of evidence. This simply supports the common-sense fact that not all information has a positive contribution to the precision of the result.[3]

The coherent imprecise probabilities are considered a particular case of the theory of imprecise coherent previsions and are based on three fundamental principles: avoiding sure loss, coherence, and natural extension. A probability model *avoids sure loss* if it cannot lead to behavior that

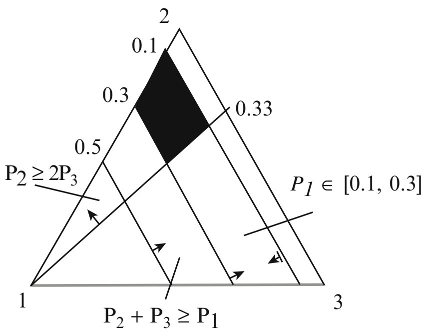

Figure 2. Presentation of All Available Statistical Evidence on the Simplex.

[3] Precision is considered the value of difference between the upper and lower bound of the probability of interest.

is certain to be harmful. This is a basic principle of rationality. *Coherence* is a stronger principle, which characterizes a type of self-consistency. Coherent models can be constructed from any set of probability assessments that avoid sure loss through a mathematical procedure of *natural extension* which effectively calculates the behavioral implications of the assessments [4].

The principle of avoiding sure loss for the lower and upper probabilities is equivalent to holding the following inequalities:

$$0 \le \underline{P}(A_i) \le \overline{P}(A_i) \le 1 \forall i = 1,\ldots,n, \tag{1}$$

$$\underline{P}(\Omega) = \sum_{i=1}^{n} \underline{P}(A_i) \le 1 \text{ and}$$

$$\overline{P}(\Omega) = \sum_{i=1}^{n} \overline{P}(A_i) \ge 1$$

where A_i are pairwise-disjoint subsets for any $i,j = 1,\ldots,n$ whose union is Ω, the possibility space.

The construction of coherent imprecise statistics and probabilities of events different from A_i is performed through the natural extension. The natural extension for this particular case is the solution of two linear programming problems

$$\underline{M}g = \sum_{i=1}^{n} g(A_i)P(A_i) \to \min \tag{2}$$

$$\overline{M}g = \sum_{i=1}^{n} g(A_i)P(A_i) \to \max \tag{3}$$

subject to

$$\left.\begin{array}{l} \sum_{i=1}^{n} P(A_i) = 1 \\ \underline{P}(A_i) \le P(A_i) \le \overline{P}(A_i), \;\; \mathrm{i = 1,...,n} \end{array}\right\} \tag{4}$$

Function g can be, for example, $g = x$ and then $\underline{M}g$ and $\overline{M}$ are lower and upper mean values of x.

If g is a characteristic function of an event B, i.e., $g = I_B(A_i) = 1$ if $A_i \in B$ and $I_B(A_i) = 0$ if $A_i \notin B$, then the natural extension is

$$\underline{P}(B) = \sum_{i=1}^{n} I_B(A_i)P(A_i) \to \min \tag{5}$$

$$\overline{P}(B) = \sum_{i=1}^{n} I_B(A_i)P(A_i) \to \max \tag{6}$$

subject to (4).

The lower and upper mean values $\underline{M}g$ and $\overline{M}g$ or $\overline{p}(B)$ and $\underline{P}(B)$ obtained as the solutions of linear programming problems (2) and (3) subject to (4) and (5) and (6) subject to (4) are referred to as coherent. In [4] and [10] other definitions of the natural extension can be found.

The sense of the natural extension in precise mathematical terms is to estimate the interval $[\underline{M}g, \overline{M}g]$ of possible values of Mg for all probability distributions for which $\underline{P}(A_i) \leq P(A_i) \leq \overline{P}(A_i)$, $i = 1,\ldots,n$. That is, we assume that any probability distribution consistent with the initial judgements $\underline{P}(A_i) \leq P(A_i) \leq \overline{P}(A_i)$ for $i = 1,\ldots,n$ is possible and base our inferences on this assumption without preferring a particular distribution.

An example is provided below.

3. Interpretation of Upper and Lower Probabilities

For many people, the first time they heard of the Pentagon's plan to accept bets on terrorist activities was when the bizarre-sounding idea was abandoned. …The Defence Advanced Research Projects Agency (DARPA) would have traded futures contracts that paid out if particular events, including terrorist attacks, took place. It was widely attacked as both ghoulish and nonsensical. [26]

Expressions (5) and (6) give us a formal definition (mathematical representation) of the upper and lower bounds for probabilities as maxima and minima of the objective functions subject to a set of constraints. In turn, the set of constraints also includes upper and lower probabilities. Where do they come from? How can one acquire them?

To answer these questions we need to distinguish first the issue of interpretation from that of mathematical representation. There are many kinds of mathematical models for uncertainty, such as additive probability measures, upper and lower probabilities, and comparative probability ordering. Any of these models can be given various interpretations. Similarly, any single interpretation of probability can be given various mathematical representations. De Finetti's work is a valuable example of how interpretation can profoundly affect the mathematical theory. His emphasis on finite (rather than countable) additivity and on exchangeability is a consequence of his operational interpretation [4].

Let the possibility space Ω be the set of possible states of the world that are of interest. The elements of Ω are assumed to be mutually exclusive and exhaustive. A gamble is a bounded real-valued function defined on the domain Ω. A gamble X should be interpreted as a random or uncertain reward; if the true state of the world turns out to be ω, then the reward is X(ω) units of an appropriate asset. The reward may be negative, in which case it represents a loss of X(Ω) units. The value of the reward X is uncertain, because it is uncertain which element of Ω is the true state.

Essentially, gambles are risky investments in which the utility values of the possible outcomes are known precisely [12]. The subject's uncertainty about a domain can be measured through his attitudes to gambles X defined on that domain, and particularly by determining whether he will buy or sell a gamble X for a specified price x. In principle, we could measure the subject's uncertainty concerning Ω to any desired accuracy by offering him sufficiently many gambles and observing which of them are accepted. Equivalently, we could measure the subject's lower and upper previsions for a particular Gamble X, which are defined to be the supremum acceptable buying price and infimum acceptable selling price for X. The transaction in which a Gamble X is bought at price x has reward function X – x, a new gamble. A subject's supremum acceptable buying price for X is the largest real number c such that he is committed to accept the gamble X – x for all x < c. Similarly, the transaction in which a gamble X is sold for price x has reward function x – X, and a subject's infimum acceptable selling price for X is the smallest real number d such that he is committed to accept the gamble x – X for all x > d. This leads to the theory of upper and lower previsions in [10]. The marginal buying and selling prices (lower and upper previsions) for a gamble may differ because the subject is indecisive or because he has little information about the gamble. As the amount of relevant information increases, the difference between the marginal buying and selling prices typically decreases. In the special case where every gamble X has a 'fair price,' meaning that the supremum acceptable buying price agrees with the infimum acceptable selling price, one obtains the theory of linear previsions [13].

Subsets of Ω, which are called events, can be identified with their indicator functions, which are gambles as well. When A is a subset of Ω, buying and selling prices (lower and upper previsions) for the indicator function A can be regarded as betting rates on and against A (lower and upper probabilities).

4. Judgements Admitted in Imprecise Statistical Reasoning: Continuous Case

The thesis that, "all available statistical evidence in risk and reliability analyses is to be utilized" is repeated in numerous guidelines in risk and reliability analysis. Everybody agrees but nobody knows how to make this true. As the remedy, Bayesian updating is usually brought up. Unfortunately, many people seem to believe that this is the only way of producing coherent statistical inferences. That is not so, for two reasons [14].

First, coherent statistical inferences need not be based on any assessment of prior probabilities. Second, even when inference proceeds by updating prior probabilities, imprecise prior probabilities can be presented by several mathematical models other than a set of prior probability distributions. In many problems it is difficult to identify a suitable prior distribution or set of prior distributions to perform Bayesian sensitivity analysis. Coherent imprecise previsions constitute an alternative method that in some problems is more convenient and traceable.

In this section I will give some examples of the judgments that can be easily utilized by the method and that are relevant for a continuous set of possible outcomes. (More on admitted judgments can be found elsewhere [15, 16].) Examples will usually involve the notion of time to failure (a continuous variable), this being a favorite target for reliability analysts. I will try to avoid giving too much mathematical formalism, but some of it cannot be avoided. To utilize a judgment it has to be represented in a mathematical form that is then used as a constraint for a properly constructed objective function.

Direct judgements on the lower and upper probabilities of events or—in general—lower and upper previsions are a straightforward way to elicit the imprecise probability characteristics of interest. Constraint $\underline{a} \leq \int_{R_+} f(x)\rho(x)dx \leq \overline{a}$ is the model of a direct judgement. If, for instance, $f_i(X) = X$, then $\underline{a}_i$, $\bar{a}_i$ are the lower and upper expected values of $\underline{X}$, correspondingly. If X is time to failure, then $\underline{a}_i$, $\bar{a}_i$ are the lower and upper bounds for the mean time to failure. If $f_i(X) = I_{[t,\infty]}(X)$, where $I_{[t,\infty]}(X)$ is an indicator function such that $I_{[t,\infty]}(X) = 1$ if $X \in [t,\infty]$ and $I_{[t,\infty]}(X) = 0$ otherwise, then $\underline{a}_i$, $\bar{a}_i$ are the lower and upper bounds for the probability of failure occurrence within $[t, \infty]$.

On a general note, direct judgements can be elicited and utilized for any probability characteristic that can be represented as an expectation to a properly chosen gamble.

Being able to utilize *comparative judgements* is a good feature of the theory of imprecise previsions. They could be, for example, "the failure of component A within the time interval [0,10] is at least as probable as the failure of component B within [0,20]," or "the mean time to failure of component B is less than the mean time to failure of component A." The first judgement is modeled as follows:

$$\int_0^\infty \int_0^\infty (I_{[0,10]}(x_A) - I_{[0,20]}(x_B))\rho(x_A, x_B)dx_A dx_B \geq 0,$$

and the second:

$$\int_0^\infty \int_0^\infty (x_A - x_B)\rho(x_A, x_B)dx_A dx_B \geq 0.$$

Another kind of judgement is a *structural judgement*. Informally, a structural judgement is a hypothetical judgement that if you were willing to accept Gamble X, then you would be willing also to accept Gamble Y [4]. Structural judgements may involve the notions (properties) of independence and permutability, and both types can be modeled.

If the objective function for computing the lower bound of the expected value of a random function g appears in a form like this

$$\overline{M}(g) = \sup_P \int_{R_+^n} g(\boldsymbol{x})\rho(\boldsymbol{x})d\boldsymbol{x},\ \underline{M}(g) = \inf_P \int_{R_+^n} g(\boldsymbol{x})\rho(\boldsymbol{x})d\boldsymbol{x},$$

where $x = (x_1, \ldots, x_n)$, then this models the complete ignorance with regard to independence. The infimum is sought over the set P of all possible joint probability density functions $\rho(x)$. No structural judgement is introduced here. If there is a ground on which to judge independence among x_i, then $\rho(x) = \rho(\mathbf{x}_1) \ldots \rho(x_n)$. It is clear that in this case set P is reduced and consists only of densities which can be represented as a product. As set P becomes smaller, then the imprecision, $\Delta = \overline{M}(g) - \underline{M}(g)$, is reduced.

In fact, the scope of judgements that can be utilized by the method is very wide (for more examples see [4], page 169). This, therefore, makes the thesis "all available statistical evidence in risk and reliability analyses is to be utilized" persuasive. This is because a tool really exists that can utilize a wide spectrum of evidence.

5. Unreliable Judgements (Hierarchical Models)

Good is prepared to define second order probability distributions..., and third order probability distributions over these, etc., until he gets tired. [17]

The quality of information that a decision maker has concerning the possible states and outcomes of a decision situation is in many cases an important factor when making decisions. Experts providing judgements have different levels of expertise and their sources of information may not be equally reliable. So it is natural to assign different degrees of plausibility or probability to opinions by different experts. To allow for this, a kind of hierarchical model can be used. In general, hierarchical models arise when there is a "correct" or "ideal" (first-order) uncertainty model about a phenomenon of interest, but the modeler is uncertain about what it is. The modeler's uncertainty is then called second-order uncertainty [12]. The hierarchical model is, in many applications, a useful assessment strategy for constructing a first-order prior distribution [14].

The most common hierarchical model is the Bayesian one, where both the first and the second-order model are (precise) probability measures [18–22]. Other models allow imprecision in the second-order model, but still assume that the first-order model is precise. Examples are the robust Bayesian models [18], models involving second-order possibility distributions [14, 23, 24], and the Gardenfors and Sahlin epistemic reliability model [5]. In [12] de Cooman introduced and studied a particular type of imprecise behavioral second-order model in terms of so-called lower desirability functions.

We have studied hierarchical uncertainty models of a general form: imprecise first- and second-order uncertainty models. Both models of uncertainty, first-order and second-order, are coherent interval statistical models.

Suppose that we have a set of unreliable interval-valued expert judgements on a parameter of interest b. To be more specific, we have n intervals $B_i = \left[b_1^i, b_2^i\right]$ provided by n experts, where b_1^i and b_2^i are the lower and upper bound of the interval B_i, respectively. The intervals provided are thought of as covering the true value of b, and are the models of uncertainty of the first order. The levels of confidence in the judgements depend on available information about experts' performance and their competences and may be subject to their own self-assessment. Suppose that each of n experts or each of their judgements is characterized by a subjective probability γ_i or, in general, by an interval-valued probability $[\underline{\gamma}_i, \overline{\gamma}_i]$, i = 1, …, n. Now a hierarchical model can be written as follows:

$$\Pr\left\{b_1^i \le b \le b_2^i\right\} \in [\ \underline{\gamma}_i,\ \ \overline{\gamma}_i\ \ ], i = 1,\ldots,n$$

The hierarchical model is introduced to become a useful assessment strategy for constructing first-order uncertainty intervals. Its implementation is illustrated by the problem of combining expert opinions.

As given above, the information concerning a parameter b is given by a collection of n intervals B_i. Combined lower, $\underline{b}$, and upper, $\overline{b}$, bounds for b are the goals.

The result will definitely depend on the degree of credibility to each of the provided judgements. Say, the analyst is absolutely (100%) and equally confident about all the judgements. In terms of the formalism introduced above this means that $\Pr\left\{b_1^i \le b \le b_2^i\right\} = 1 \forall i = 1,\ldots,n$, that is, $\underline{\gamma}_i = \overline{\gamma}_i = 1 \forall i = 1,\ldots,n$. As proven in [15], this case yields a simple rule of combination called the conjunction rule [4]:

$$\underline{b} = \max_{i=1,\ldots,n} b_1^i \text{ and } \overline{b} = \min_{i=1,\ldots,n} b_2^i$$

This rule is valid only for nonconflicting judgements ("consistent collection of intervals") and if the analyst is prepared to accept the modeling of the linguistic expression "equally credible" as $\underline{\gamma}_i = \overline{\gamma}_i = 1\ \forall\ i = 1,\ldots, n$. Consistency as well as the absence of conflict mean that $\cap_{i},\ {}_{i}B_i \neq \emptyset$.

Another rule of combination is valid if all intervals in the collection are nested ("consonant"), that is, if

$$\left[b_1^1, b_2^1\right] \subseteq \left[b_1^2, b_2^2\right] \subseteq \ldots \subseteq \left[b_1^n, b_2^n\right] \quad \text{and}$$

the credibility to the judgements is expressed in the different form $\underline{\gamma}_i = \gamma_i$, $\overline{\gamma}_i = 1$, $i = 1,\ldots n$ and $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$. A closer look at this information gives a hint that this kind of source data setup is nothing other than a possibility distribution. This case of hierarchical models was described in [12] and [14].

The combination rule for this case follows:

$$\underline{b} = \sum_{i=1}^{n} (\gamma_i - \gamma_{i-1})\, b_1^i$$
$$\overline{b} = \sum_{i=1}^{n} (\gamma_i - \gamma_{i-1})\, b_2^i$$

In this rule, it is assumed that $\gamma_0 = 0$ and $\gamma_n = 1$.

A model for "equally credible" judgements could be differently constructed with the hierarchical model introduced. The modeler may choose to model equal credibility in the following way:

$[\underline{\gamma}_i, \overline{\gamma}_i] = [\gamma_i, 1]$ and $\gamma_1 = \gamma_2 = \ldots = \gamma_n = \gamma$ then the last rule of combination degenerates to

$$\underline{b} = \gamma b_1^1 + (1-\gamma) b_1^n$$
$$\overline{b} = \gamma b_2^1 + (1-\gamma) b_2^n$$

If γ tends to 1, then the results of this rule coincide with the results of the conjunction rule.

The conjunction rule can also be applied to consonant intervals as this rule is valid for a consistent collection of intervals, and it is clear that nested intervals are nonconflicting pieces of evidence. But it should be kept in mind that the conjunction rule presupposes that the analyst is 100% confident about all the judgments; i.e., $\underline{\gamma}_i = \overline{\gamma}_i = 1$.

If the collection of intervals is conflicting (there is at least one pair of nonoverlapping intervals), then one way of reconciling the conflict is to accept complete ignorance concerning the level of credibility in the judgments. That is, the analyst can assume $\underline{\gamma}_i = 0$, $\overline{\gamma}_i = 1$, $\forall i = 1,\ldots n$. Using this assumption we arrive at the unanimity rule

$$\underline{b} = \min_{i=1,\ldots,n} b_1^i \text{ and } \overline{b} = \max_{i=1,\ldots,n} b_2^i$$

These are simple combination rules that have been derived based on the hierarchical model, and the way they have been derived was fully predefined by the theoretical framework of coherent imprecise probabilities. This fact is worth stressing, since, in contrast, in the framework of purely Bayesian approach and point-valued probabilities only some ad-hoc combination rules are possible. An example is the linear opinion pool which is one of many others devised to combine evidence.

6. Precautionary Principle and Indecision

Determinacy and decisiveness in decision making are favored by the public and decision makers, while fuzziness and indecision in providing crisp answers are reckoned as signs of incompetence and meekness which are usually disliked. In this regard, Bayesian decision theory appears the right one as providing a clear-cut answer to what action is to be preferred.

In contrast, the approach to decision making based on imprecise (interval-valued) probabilistic criteria will reach results that, generally, do not yield an 'optimal' action that is preferred to all others. In effect, this means that there is a third alternative answer under decision making. It is indecision in saying neither 'yes' nor 'no.' The failure to determine a uniquely optimal action simply reflects the absence of information about the set of possible actions.

What would be a strategy which could be used to make a decision in case there is more than one reasonable action? One of them is to search for more information concerning the set of possible actions to make the probabilities and utilities more precise. The other is to postpone a decision until a later time, when more information may be available. For more strategies see [4], p. 239–240.

A small but growing number of authors have called for, and observed the development towards, a paradigm shift in environmental decision making. As uncertainty becomes an accepted fact by scientists on the one side and the public and politicians on the other:

> this requires a change of attitude on both sides: The politicians have to accept that fuzzy answers may be the best expression of expertise. The scientists have to learn that identification of the fuzzy borderline between knowledge and ignorance may be the sign of real competence. [25]

Imprecise statistical reasoning provides models to quantify scientific incertitude that is a result of a lack of relevant information or sizable uncertainty.

When there is little information on which to base our conclusions, we cannot expect reasoning (no matter how clever or thorough) to reveal a most probable hypothesis or a uniquely reasonable course of action. There are limits to the power of reasons [4]. An educated mind should provide answers consistent with the relevant knowledge and uncertainty.

One of the important novelties of imprecise statistical reasoning approach is that we now have a formal framework in which we can articulate uncertainty and indecision.

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