A Lagrange Multiplier Based Domain Decomposition Method for the Solution of a Wave Problem with Discontinuous Coefficients

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Summary. In this paper we consider the numerical solution of a linear wave equation with discontinuous coefficients. We divide the computational domain into two subdomains and use explicit time difference scheme along with piecewise linear finite element approximations on semimatching grids. We apply boundary supported Lagrange multiplier method to match the solution on the interface between subdomains. The resulting system of linear equations of the "saddle-point" type is solved efficiently by a conjugate gradient method.

1 Problem Formulation

Let $\Omega \subset \mathbb{R}^2$ be a rectangular domain with sides parallel to the coordinate axes and boundary Γ_{ext} (see Fig. 1). Now let $\Omega_2 \subset \Omega$ be a proper subdomain of Ω with a curvilinear boundary and $\Omega_1 = \Omega \setminus \bar{\Omega}_2$.

We consider the following linear wave problem:

$$\begin{cases} \varepsilon \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot (\mu^{-1} \nabla u) = f & \text{in } \Omega \times (0,T), \\ \sqrt{\varepsilon\, \mu^{-1}} \dfrac{\partial u}{\partial t} + \mu^{-1} \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_{\text{ext}} \times (0,T), \\ \quad u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0. \end{cases} \tag{1}$$

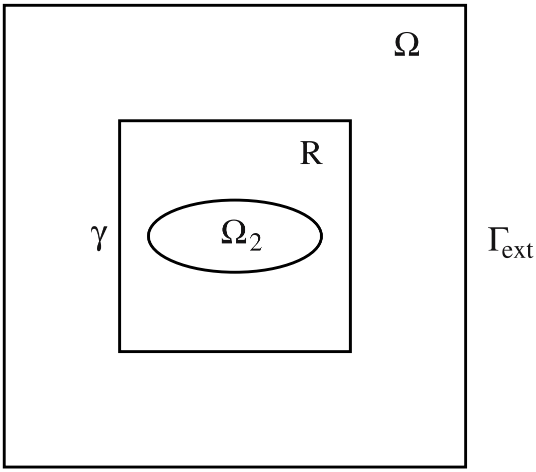

Fig. 1. Computational domain.

Here $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2})$, $\mathbf{n}$ is the unit outward normal vector on Γ_{ext}. We suppose that $\mu_i = \mu|_{\Omega_i}$, $\varepsilon_i = \varepsilon|_{\Omega_i}$ are positive constants for all $i = 1, 2$ and $f_i = f|_{\Omega_i} \in C(\bar{\Omega}_i \times [0, T])$.

Let

$$\varepsilon(x) = \begin{cases} \varepsilon_1 & \text{if } x \in \Omega_1, \\ , \varepsilon_2 & \text{if } x \in \Omega_2, \end{cases} \quad \text{and} \quad \mu(x) = \begin{cases} \mu_1 & \text{if } x \in \Omega_1, \\ \mu_2 & \text{if } x \in \Omega_2. \end{cases}$$

We define a weak solution of problem (1) as a function u such that

$$u \in L^\infty(0, T; H^1(\Omega)), \ \frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)), \ \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Gamma_{\text{ext}})) \quad (2)$$

for a.a. $t \in (0, T)$ and for all $w \in H^1(\Omega)$ satisfying the equation

$$\int_\Omega \varepsilon(x) \frac{\partial^2 u}{\partial t^2} w dx + \int_\Omega \mu^{-1}(x) \nabla u \cdot \nabla w dx + \sqrt{\varepsilon_1 \mu_1^{-1}} \int_{\Gamma_{\text{ext}}} \frac{\partial u}{\partial t} w d\Gamma = \int_\Omega f w dx \quad (3)$$

with the initial conditions

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0.$$

Note that the first term in (3) means the duality between $(H^1(\Omega))^*$ and $H^1(\Omega)$.

Now, using the Faedo–Galerkin method (as in [DL92]), one can prove the following:

Theorem 1. *Under the assumptions (2) there exists a unique weak solution of problem (1).*

Let

$$E(t) = \frac{1}{2}\int_{\Omega} \varepsilon(x) \left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{2}\int_{\Omega} \mu^{-1}(x)|\nabla u|^2 dx$$

be the energy of the system. We take $w = \frac{\partial u}{\partial t}$ in (3) and obtain:

$$\frac{dE(t)}{dt} + \sqrt{\varepsilon_1\mu_1^{-1}} \int_{\Gamma_{\text{ext}}} (\frac{\partial u}{\partial t})^2 d\Gamma = \int_{\Omega} f\frac{\partial u}{\partial t} dx \leq \|f\|_{L^2(\Omega)}\|\frac{\partial u}{\partial t}\|_{L^2(\Omega)},$$

since $E(0) = 0$, the following stability inequality holds:

$$E(t) \leq \text{const}\, T\|f\|_{L^2(\Omega\times(0,T))}, \quad \forall t \in (0, T).$$

In order to use a structured grid in a part of the domain Ω, we introduce a rectangular domain R with sides parallel to the coordinate axes, such that $\Omega_2 \subset R \subset \Omega$ with γ the boundary of R (Fig. 1).

Define $\tilde{\Omega} = \Omega \setminus \bar{R}$ and let the subscript 1 of a function v_1 mean that this function is defined over $\tilde{\Omega} \times (0, T)$, while v_2 is a function defined over $R \times (0, T)$.

Now we formulate the problem (3) variationally as follows: Let

$$W_1 = \left\{v \in L^\infty(0, T; H^1(\tilde{\Omega})),\ \frac{\partial v}{\partial t} \in L^\infty(0, T; L^2(\tilde{\Omega})),\ \frac{\partial v}{\partial t} \in L^2(0, T; L^2(\Gamma_{\text{ext}}))\right\},$$

$$W_2 = \left\{v \in L^\infty(0, T; H^1(R)),\ \frac{\partial v}{\partial t} \in L^\infty(0, T; L^2(R))\right\},$$

Find a pair $(u_1, u_2) \in W_1 \times W_2$, such that $u_1 = u_2$ on $\gamma \times (0, T)$ and for a.a. $t \in (0, T)$

$$\begin{cases} \int_{\tilde{\Omega}} \varepsilon_1 \frac{\partial^2 u_1}{\partial t^2} w_1 dx + \int_{\tilde{\Omega}} \mu_1^{-1}\nabla u_1 \cdot \nabla w_1 dx + \int_R \varepsilon(x)\frac{\partial^2 u_2}{\partial t^2} w_2 dx \\ +\int_R \mu^{-1}(x)\nabla u_2 \cdot \nabla w_2 dx + \sqrt{\varepsilon_1\mu_1^{-1}} \int_{\Gamma_{\text{ext}}} \frac{\partial u_1}{\partial t} w_1 d\Gamma = \int_{\tilde{\Omega}} f_1 w_1 dx + \int_R f_2 w_2 dx, \\ \text{for all } (w_1, w_2) \in H^1(\tilde{\Omega}) \times H^1(R) \text{ such that } w_1 = w_2 \text{ on } \gamma, \\ u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0. \end{cases} \tag{4}$$

Now, introducing the interface supported Lagrange multiplier λ (a function defined over $\gamma \times (0, T)$), the problem (4) can be written in the following way:

Find a triple $(u_1, u_2, \lambda) \in W_1 \times W_2 \times L^\infty(0, T; H^{-1/2}(\gamma))$, which for a.a. $t \in (0, T)$ satisfies

$$\int_{\tilde{\Omega}} \varepsilon_1 \frac{\partial^2 u_1}{\partial t^2} w_1 dx + \int_{\tilde{\Omega}} \mu_1^{-1} \nabla u_1 \cdot \nabla w_1 dx + \int_R \varepsilon(x) \frac{\partial^2 u_2}{\partial t^2} w_2 dx$$
$$+ \int_R \mu^{-1}(x) \nabla u_2 \cdot \nabla w_2 dx + \sqrt{\varepsilon_1 \mu_1^{-1}} \int_{\Gamma_{\text{ext}}} \frac{\partial u_1}{\partial t} w_1 d\Gamma + \int_\gamma \lambda (w_2 - w_1) d\gamma$$
$$= \int_{\tilde{\Omega}} f_1 w_1 dx + \int_R f_2 w_2 dx \quad \text{for all } w_1 \in H^1(\tilde{\Omega}),\ w_2 \in H^1(R), \tag{5}$$
$$\int_\gamma \zeta (u_2 - u_1) d\gamma = 0 \quad \text{for all } \zeta \in H^{-1/2}(\gamma), \tag{6}$$

and the initial conditions from (1).

Remark 1. We selected the time dependent approach to capture harmonic solutions since it substantially simplifies the linear algebra of the solution process. Furthermore, there exist various techniques to speed up the convergence of transient solutions to periodic ones (see, e.g., [BDG$^+$97]).

2 Time Discretization

In order to construct a finite difference approximation in time of the problem (5), (6), we partition the segment $[0, T]$ into N intervals using a uniform discretization step $\Delta t = T/N$. Let $u_i^n \approx u_i(n\,\Delta t)$ for $i = 1, 2$, $\lambda^n \approx \lambda(n\,\Delta t)$. The explicit in time semidiscrete approximation to the problem (5), (6) reads as follows:

$$u_i^0 = u_i^1 = 0$$

for $n = 1, 2, \ldots, N-1$. Find $u_1^{n+1} \in H^1(\tilde{\Omega})$, $u_2^{n+1} \in H^1(R)$ and $\lambda^{n+1} \in H^{-1/2}(\gamma)$ such that

$$\int_{\tilde{\Omega}} \varepsilon_1 \frac{u_1^{n+1} - 2u_1^n + u_1^{n-1}}{\Delta t^2} w_1 dx + \int_{\tilde{\Omega}} \mu_1^{-1} \nabla u_1^n \cdot \nabla w_1 dx +$$
$$+ \int_R \varepsilon(x) \frac{u_2^{n+1} - 2u_2^n + u_2^{n-1}}{\Delta t^2} w_2 dx + \int_R \mu^{-1}(x) \nabla u_2^n \cdot \nabla w_2 dx +$$
$$+ \sqrt{\varepsilon_1 \mu_1^{-1}} \int_{\Gamma_{\text{ext}}} \frac{u_1^{n+1} - u_1^{n-1}}{2\Delta t} w_1 d\Gamma + \int_\gamma \lambda^{n+1} (w_2 - w_1) d\gamma =$$
$$= \int_{\tilde{\Omega}} f_1^n w_1 dx + \int_R f_2^n w_2 dx \quad \text{for all } w_1 \in H^1(\tilde{\Omega}), w_2 \in H^1(R), \tag{7}$$
$$\int_\gamma \zeta (u_2^{n+1} - u_1^{n+1}) d\gamma = 0 \quad \text{for all } \zeta \in H^{-1/2}(\gamma). \tag{8}$$

Remark 2. The integral over γ is written formally; the exact formulation requires the use of the duality pairing $\langle \cdot, \cdot \rangle$ between $H^{-1/2}(\gamma)$ and $H^{1/2}(\gamma)$.

3 Fully Discrete Scheme

To construct a fully discrete space-time approximation to the problem (5), (6), we will use a lowest order finite element method on two grids semimatching on γ (Fig. 2) for the space discretization. Namely, let $\mathcal{T}_{1h}$ and $\mathcal{T}_{2h}$ be triangulations of $\tilde{\Omega}$ and R, respectively. Further we suppose that both triangulations are regular in the sense that

$$\frac{r(e)}{h(e)} \leq q = \text{const}$$

for all $e \in \mathcal{T}_{1h}$ and $e \in \mathcal{T}_{2h}$, where q does not depend on e; $r(e)$ is the radius of the circle inscribed in e, while $h(e)$ is the diameter of e.

We denote by $\mathcal{T}_{1h}$ a coarse triangulation and by $\mathcal{T}_{2h}$ a fine one. Every edge $\partial e \subset \gamma$ of a triangle $e \in \mathcal{T}_{1h}$ is supposed to consist of m_e edges of triangles from $\mathcal{T}_{2h}$, $1 \leq m_e \leq m$ for all $e \in \mathcal{T}_{1h}$

Moreover, let a triangulation $\mathcal{T}_{2h}$ be such that the curvilinear boundary $\partial\Omega_2$ is approximated by a polygonal line consisting of the edges of triangles from $\mathcal{T}_{2h}$ whose vertices belong to $\partial\Omega_2$. Further, we say that a triangle $e \in \mathcal{T}_{2h}$ lies in Ω_2 if its larger part lies in Ω_2, i.e. $\text{meas}(e \cap \Omega_2) > \text{meas}(e \cap (R \setminus \Omega_2))$, otherwise this triangle lies in $R \setminus \Omega_2$.

Let $V_{1h} \subset H^1(\tilde{\Omega})$ be the space of the functions globally continuous, and affine on each $e \in \mathcal{T}_{1h}$, i.e. $V_{1h} = \{u_h \in H^1(\tilde{\Omega}) \mid u_h \in P_1(e) \quad \forall e \in \mathcal{T}_{1h}\}$. Similarly, $V_{2h} \subset H^1(R)$ is the space of the functions globally continuous, and affine on each $e \in \mathcal{T}_{2h}$.

For approximating the Lagrange multipliers space $\Lambda = H^{-1/2}(\gamma)$ we proceed as follows. Assume that on γ, $\mathcal{T}_{1h}$ is two times coarser than $\mathcal{T}_{2h}$. Then let us divide every edge ∂e of a triangle e from the coarse grid $\mathcal{T}_{1h}$, which is located on γ ($\partial e \subset \gamma$), into two parts using its midpoint. Now, we consider the space of the piecewise constant functions, which are constant on every union of half-edges with a common vertex (see Fig. 3).

Further, we use quadrature formulas for approximating the integrals over the triangles from $\mathcal{T}_{1h}$ and $\mathcal{T}_{2h}$, as well as over Γ_{ext}. For a triangle e we set

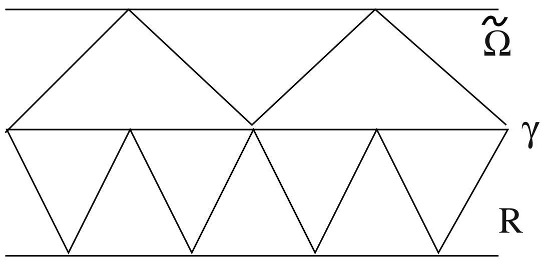

Fig. 2. Semimatching mesh on γ.

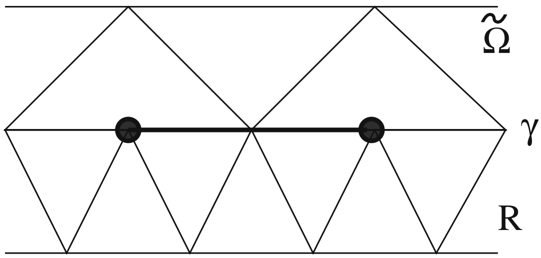

Fig. 3. Space Λ is the space of the piecewise constant functions defined on every union of half-edges with common vertex.

$$\int_e \phi(x)dx \approx \frac{1}{3}\,\mathrm{meas}(e)\sum_{i=1}^{3}\phi(a_i) \equiv S_e(\phi),$$

where the a_i's are the vertices of e and $\phi(x)$ is a continuous function on e. Similarly,

$$\int_{\partial e} \phi(x)dx \approx \frac{1}{2}\,\mathrm{meas}(\partial e)\sum_{i=1}^{2}\phi(a_i) \equiv S_{\partial e}(\phi),$$

where a_i's are the endpoints of the segment ∂e and $\phi(x)$ is a continuous function on this segment.

We use the notations:

$$S_i(\phi) = \sum_{e\in\mathcal{T}_{ih}} S_e(\phi),\ \ i=1,2, \quad \text{and} \quad S_{\Gamma_{\mathrm{ext}}}(\phi) = \sum_{\partial e\subset\Gamma_{\mathrm{ext}}} S_{\partial e}(\phi).$$

Now, the fully discrete problem reads as follows: Let $u_{ih}^0 = u_{ih}^1 = 0$, $i=1,2$. For $n=1,2,\ldots,N-1$, find $(u_{1h}^{n+1}, u_{2h}^{n+1}, \lambda_h^{n+1}) \in V_{1h}\times V_{2h}\times \Lambda_h$ such that

$$\begin{aligned}
&\frac{\varepsilon_1}{\Delta t^2}S_1((u_{1h}^{n+1}-2u_{1h}^n+u_{1h}^{n-1})w_{1h}) + S_1(\mu_1^{-1}\nabla u_{1h}^n\cdot\nabla w_{1h}) + \\
&+\frac{1}{\Delta t^2}S_2(\varepsilon(x)(u_{2h}^{n+1}-2u_{2h}^n+u_{2h}^{n-1})w_{2h}) + S_2(\mu^{-1}(x)\nabla u_{2h}^n\cdot\nabla w_{2h}) + \\
&+\frac{\sqrt{\varepsilon_1\mu_1^{-1}}}{2\Delta t}S_{\Gamma_{\mathrm{ext}}}((u_{1h}^{n+1}-u_{1h}^{n-1})w_{1h}) + \int_\gamma \lambda_h^{n+1}(w_{2h}-w_{1h})d\gamma = \\
&= S_1(f_1^n w_{1h}) + S_2(f_2^n w_{2h}) \quad \text{for all } w_{1h}\in V_{1h}, w_{2h}\in V_{2h}, \qquad (9)
\end{aligned}$$

$$\int_\gamma \zeta_h(u_{2h}^{n+1}-u_{1h}^{n+1})d\gamma = 0 \quad \text{for all } \zeta_h\in\Lambda_h. \qquad (10)$$

Note that in $S_2(\varepsilon(x)(u_{2h}^{n+1}-2u_{2h}^n+u_{2h}^{n-1})w_{2h})$ we take $\varepsilon(x)=\varepsilon_2$ if a triangle $e\in\mathcal{T}_{2h}$ lies in Ω_2 and $\varepsilon(x)=\varepsilon_1$ if it lies in $R\setminus\Omega_2$, and similarly for $S_2(\mu^{-1}(x)\nabla u_{2h}^n\nabla w_{2h})$.

Denote by $\mathbf{u}_1$, $\mathbf{u}_2$ and $\boldsymbol{\lambda}$ the vectors of the nodal values of the corresponding functions u_{1h}, u_{2h} and λ_h. Then, in order to find $\mathbf{u}_1^{n+1}$, $\mathbf{u}_2^{n+1}$ and $\boldsymbol{\lambda}^{n+1}$ for a fixed time t^{n+1}, we have to solve a system of linear equations such as

$$\mathbf{A}\mathbf{u} + \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{F}, \tag{11}$$

$$\mathbf{B}\mathbf{u} = 0, \tag{12}$$

where matrix $\mathbf{A}$ is diagonal, positive definite and defined by

$$(\mathbf{A}\mathbf{u}, \mathbf{w}) = \frac{\varepsilon_1}{\Delta t^2} S_1(u_{1h}w_{1h}) + \frac{1}{\Delta t^2} S_2(\varepsilon(x)u_{2h}w_{2h}) + \frac{\sqrt{\varepsilon_1 \mu_1^{-1}}}{2\Delta t} S_{\Gamma_{\text{ext}}}(u_{1h}w_{1h}),$$

and where the rectangular matrix $\mathbf{B}$ is defined by

$$(\mathbf{B}\mathbf{u}, \boldsymbol{\lambda}) = \int_\gamma \lambda_h(u_{2h} - u_{1h})d\Gamma,$$

and vector $\mathbf{F}$ depends on the nodal values of the known functions u_{1h}^n, u_{2h}^n, u_{1h}^{n-1} and u_{2h}^{n-1}.

Eliminating $\mathbf{u}$ from the equation (11), we obtain

$$\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T\boldsymbol{\lambda} = \mathbf{B}\mathbf{A}^{-1}\mathbf{F}, \tag{13}$$

with a symmetric matrix $\mathbf{C} \equiv \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T$. Let us prove that $\mathbf{C}$ is positive definite. Obviously, $\ker \mathbf{C} = \ker \mathbf{B}^T$. Suppose, that $\mathbf{B}^T\boldsymbol{\lambda} = 0$, then a function $\lambda_h \in \Lambda_h$ corresponding to vector $\boldsymbol{\lambda}$ satisfies

$$I \equiv \int_\gamma \lambda_h u_h d\gamma = 0$$

for all $u_h \in V_{1h}$. Choose u_h equal to λ_h in the nodes of $\mathcal{T}_{1h}$ located on γ. Direct calculations give

$$I = \frac{1}{2}\sum_{i=1}^{N_\lambda}\left[\frac{h_i + h_{i+1}}{2}\lambda_i^2 + h_{i+1}\frac{(\lambda_i + \lambda_{i+1})^2}{2}\right],$$

where N_λ is the number of edges of $\mathcal{T}_{1h}$ on γ, h_i is the length of i-th edge and $h_{N_\lambda+1} \equiv h_1$, $\lambda_{N_\lambda+1} \equiv \lambda_1$. Thus, the equality $I = 0$ implies that $\boldsymbol{\lambda} = \mathbf{0}$, i.e. $\ker \mathbf{B}^T = \{\mathbf{0}\}$.

As a consequence we have

Theorem 2. *The problem (9), (10) has a unique solution* (u_h, λ_h).

Remark 3. A closely related domain decomposition method applied to the solution of linear parabolic equations is discussed in [Glo03].

4 Energy Inequality

Theorem 3. *Let $h_{\min}$ denote the minimal diameter of the triangles from $\mathcal{T}_{1h} \cup \mathcal{T}_{2h}$. There exists a positive number c such that the condition*

$$\Delta t \leq c \min\{\sqrt{\varepsilon_1 \mu_1}, \sqrt{\varepsilon_2 \mu_2}\} h_{\min} \tag{14}$$

ensures the positive definiteness of the quadratic form

$$\begin{aligned} \mathcal{E}^{n+1} &= \frac{1}{2}\varepsilon_1 S_1 \left(\left(\frac{u_{1h}^{n+1} - u_{1h}^n}{\Delta t} \right)^2 \right) + \frac{1}{2} S_2 \left(\varepsilon \left(\frac{u_{2h}^{n+1} - u_{2h}^n}{\Delta t} \right)^2 \right) + \\ &+ \frac{1}{2} S_1 \left(\mu_1^{-1} \left| \nabla \left(\frac{u_{1h}^{n+1} + u_{1h}^n}{2} \right) \right|^2 \right) + \frac{1}{2} S_2 \left(\mu^{-1} \left| \nabla \left(\frac{u_{2h}^{n+1} + u_{2h}^n}{2} \right) \right|^2 \right) - \\ &- \frac{\Delta t^2}{8} S_1 \left(\mu_1^{-1} \left| \nabla \left(\frac{u_{1h}^{n+1} - u_{1h}^n}{\Delta t} \right) \right|^2 \right) - \frac{\Delta t^2}{8} S_2 \left(\mu^{-1} \left| \nabla \left(\frac{u_{2h}^{n+1} - u_{2h}^n}{\Delta t} \right) \right|^2 \right), \end{aligned} \tag{15}$$

which we call the discrete energy.

The system (9), (10) satisfies the energy identity

$$\begin{aligned} \mathcal{E}^{n+1} - \mathcal{E}^n + \frac{\sqrt{\varepsilon_1 \mu_1^{-1}}}{4\Delta t} S_{\Gamma_{\text{ext}}}((u_{1h}^{n+1} - u_{1h}^{n-1})^2) = \\ = \frac{1}{2} S_1 (f_1^n (u_{1h}^{n+1} - u_{1h}^{n-1})) + \frac{1}{2} S_2 (f_2^n (u_{2h}^{n+1} - u_{2h}^{n-1})) \end{aligned} \tag{16}$$

and the numerical scheme is stable: There exists a positive number $M = M(T)$ such that

$$\mathcal{E}^n \leq M \Delta t \sum_{k=1}^{n-1} (S_1((f_1^k)^2) + S_2((f_2^k)^2)), \quad \forall n. \tag{17}$$

Proof. Let $n \geq 1$. From the equation (10) written for t_{n+1} and t_{n-1} we obtain

$$\int_\gamma \zeta_h ((u_{2h}^{n+1} - u_{2h}^{n-1}) - (u_{1h}^{n+1} - u_{1h}^{n-1})) d\gamma = 0 \quad \text{for all } \zeta_h \in \Lambda_h. \tag{18}$$

Choosing

$$w_{1h} = \frac{u_{1h}^{n+1} - u_{1h}^{n-1}}{2}, \quad w_{2h} = \frac{u_{2h}^{n+1} - u_{2h}^{n-1}}{2}$$

in (9) and

$$\zeta_h = -\frac{\lambda_h^{n+1}}{2}$$

in (18), we add these equalities. Using the identities

$$(u_{ih}^{n+1} - 2u_{ih}^{n} + u_{ih}^{n-1})(u_{ih}^{n+1} - u_{ih}^{n-1}) = (u_{ih}^{n+1} - u_{ih}^{n})^2 - (u_{ih}^{n} - u_{ih}^{n-1})^2$$

and

$$u_{ih}^{n} u_{ih}^{n+1} = \frac{1}{4}((u_{ih}^{n+1} + u_{ih}^{n})^2 - (u_{ih}^{n+1} - u_{ih}^{n})^2),$$

after several technical transformations we obtain

$$\mathcal{E}^{n+1} - \mathcal{E}^{n} + \frac{\sqrt{\varepsilon_1 \mu_1^{-1}}}{4\Delta t} S_{\Gamma_{\text{ext}}}((u_{1h}^{n+1} - u_{1h}^{n-1})^2) = \frac{1}{2} S_1(f_1^n(u_{1h}^{n+1} - u_{1h}^{n-1})) + \frac{1}{2} S_2(f_2^n(u_{2h}^{n+1} - u_{2h}^{n-1})).$$

Therefore,

$$\mathcal{E}^{n+1} \leq \mathcal{E}^{n} + \frac{1}{2}\Delta t S_1^{1/2}\left((f_1^n)^2\right)\left[S_1^{1/2}\left(\left(\frac{u_{1h}^{n+1} - u_{1h}^{n}}{\Delta t}\right)^2\right) + \right.$$
$$\left. + S_1^{1/2}\left(\left(\frac{u_{1h}^{n+1} - u_{1h}^{n}}{\Delta t}\right)^2\right)\right] + \frac{1}{2}\Delta t S_2^{1/2}\left((f_2^n)^2\right)\left[S_2^{1/2}\left(\left(\frac{u_{2h}^{n+1} - u_{2h}^{n}}{\Delta t}\right)^2\right) + \right.$$
$$\left. + S_2^{1/2}\left(\left(\frac{u_{2h}^{n+1} - u_{2h}^{n}}{\Delta t}\right)^2\right)\right]. \quad (19)$$

Now, we will show that under the condition (14) the quadratic form $\mathcal{E}^n$ is positive definite; more precisely, that there exists a positive constant δ such that

$$\mathcal{E}^{n} \geq \delta\left(S_1\left(\left(\frac{u_{1h}^{n+1} - u_{1h}^{n}}{\Delta t}\right)^2\right) + S_2\left(\left(\frac{u_{2h}^{n+1} - u_{2h}^{n}}{\Delta t}\right)^2\right)\right). \quad (20)$$

Obviously, it is sufficient to prove the inequality

$$4\varepsilon_e \mu_e S_e(v_h^2) \geq \Delta t^2 S_e(|\nabla v_h|^2) \quad \forall e \in \mathcal{T}_{1h} \cup \mathcal{T}_{2h},\ \forall v_h \in P_1(e), \quad (21)$$

where ε_e and μ_e are defined by $\varepsilon_e = \varepsilon_1$ or $\varepsilon_e = \varepsilon_2$ (respectively, $\mu_e = \mu_1$ or $\mu_e = \mu_2$). It is known that for a regular triangulation

$$S_e(|\nabla v_h|^2) \leq 1/c_1^2 h_e^{-2} S_e(v_h^2) \quad (22)$$

with a positive constant c_1, universal for all triangles e, where h_e is the minimal length of the sides of e. Combining (21) and (22), we observe that the time step Δt should satisfy the inequality

$$\Delta t \leq c\sqrt{\varepsilon_e \mu_e}\, h_e, \quad (c = \sqrt{2}c_1), \quad (23)$$

for all $e \in \mathcal{T}_{1h} \cup \mathcal{T}_{2h}$. Evidently, (14) ensures the validity of (23).

Further, using the relation (20), $\mathcal{E}^1 = 0$ and summing the inequalities (19), one obtains the stability inequality (17):

$$\mathcal{E}^{n} \leq M\Delta t \sum_{k=1}^{n-1}(S_1((f_1^k)^2) + S_2((f_2^k)^2)), \quad \forall n.$$

5 Numerical Experiments

In order to solve the system of linear equations (11)–(12) at each time step we use a Conjugate Gradient Algorithm in the form given by Glowinski and LeTallec [GL89]:

Step 1. $\boldsymbol{\lambda}^0$ given.
Step 2. $\mathbf{A}\mathbf{u}^0 = \mathbf{F} - \mathbf{B}\boldsymbol{\lambda}^0$.
Step 3. $\mathbf{g}^0 = -\mathbf{B}^T\mathbf{u}^0$.
Step 4. If $\|\mathbf{g}^0\| \leq \varepsilon_0$ take $\boldsymbol{\lambda} = \boldsymbol{\lambda}^0$,
else $\mathbf{w}^0 = \mathbf{g}^0$.
Step 5. For $m \geq 0$, assuming that $\boldsymbol{\lambda}^m$, $\mathbf{g}^m$, $\mathbf{w}^m$ are known,

$$\mathbf{A}\bar{\mathbf{u}}^m = \mathbf{B}\mathbf{w}^m.$$
$$\bar{\mathbf{g}}^m = \mathbf{B}^T\bar{\mathbf{u}}^m.$$
$$\rho_m = \frac{|\mathbf{g}^m|^2}{(\bar{\mathbf{g}}^m, \bar{\mathbf{w}}^m)}.$$
$$\boldsymbol{\lambda}^{m+1} = \boldsymbol{\lambda}^m - \rho_m\mathbf{w}^m.$$
$$\mathbf{u}^{m+1} = \mathbf{u}^m + \rho_m\bar{\mathbf{v}}^m.$$
$$\mathbf{g}^{m+1} = \mathbf{g}^m - \rho_m\bar{\mathbf{g}}^m.$$

Step 6. If $\dfrac{\mathbf{g}^{m+1}\cdot\mathbf{g}^{m+1}}{\mathbf{g}^0\cdot\mathbf{g}^0} \leq \varepsilon$ then take $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{m+1}$,
else $\gamma_m = \dfrac{\mathbf{g}^{m+1}\cdot\mathbf{g}^{m+1}}{\mathbf{g}^m\cdot\mathbf{g}^m}$.
Step 7. $\mathbf{w}^{m+1} = \mathbf{g}^{m+1} + \gamma_m\mathbf{w}^m$.
Step 8. Do $m = m + 1$ and go to Step 5.

We consider the problem (9)–(10) with a source term given by the harmonic planar wave

$$u^{inc} = -e^{ik(t-\boldsymbol{\alpha}\cdot\mathbf{x})}, \tag{24}$$

where $\{x_j\}_{j=1}^2$, $\{\alpha_j\}_{j=1}^2$, k is the angular frequency and $|\boldsymbol{\alpha}| = 1$.

For our numerical simulation we consider two cases: the first with the frequency of the incident wave $f = 0.6$ GHz and the second with $f = 1.2$ GHz, which gives us wavelengths $L = 0.5$ meters and $L = 0.25$ meters, respectively.

We performed a series of numerical experiments: scattering by a perfectly reflecting obstacle, wave propagation through a domain with an obstacle completely consisting of a coating material and scattering by an obstacle with coating.

First, we consider the scattering by a perfectly reflecting obstacle. For the experiment we have chosen Ω_2 to be in a form of a perfectly reflecting airfoil, and Ω is a 2 meter $\times$ 2 meter rectangle. We used a finite element mesh with 8019 nodes and 15324 elements in the case of $f = 0.6$ GHz (Fig. 4) and 19246 nodes and 37376 elements for $f = 1.2$ GHz.

Figure 5 shows the contour plot for the case when the incident wave is coming from the left and Figure 6 shows the case when the incident wave is coming from the lower left corner with an angle of 45°. For all the experiments

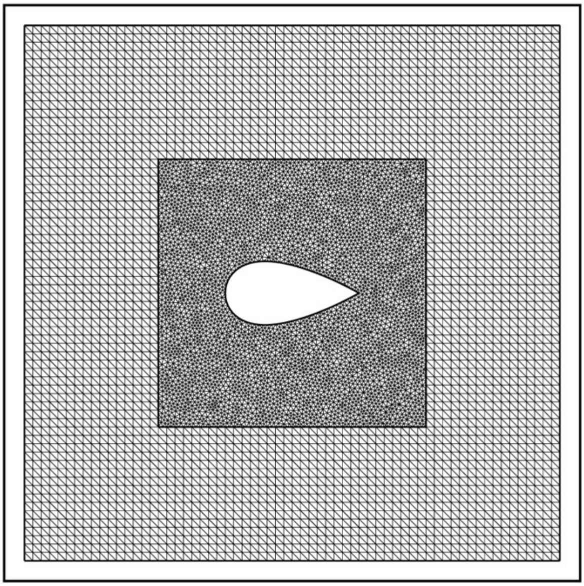

Fig. 4. Example of a finite element mesh.

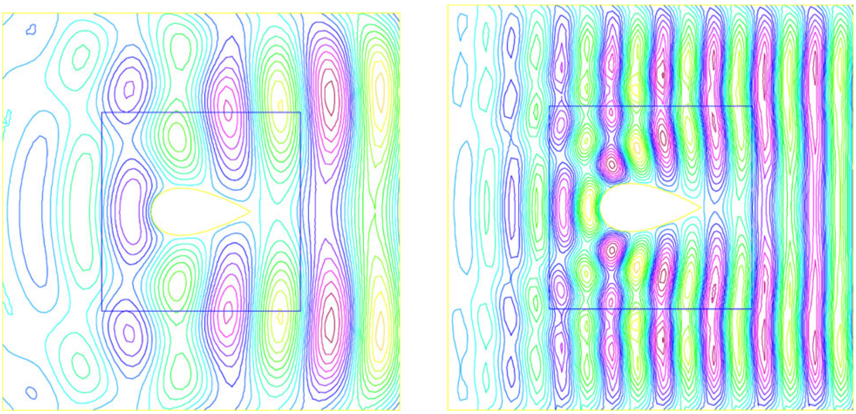

Fig. 5. Contour plot of the real part of the solution for $L = 0.5$ (left) and $L = 0.25$ (right) meters. Incident wave coming from the left.

we chose the time step to be $\Delta t = T/50$, where $T = 1/f = 1.66 \times 10^{-9}$ sec is a time period corresponding to $L = 0.5$ meters and $T = 1/f = 0.83 \times 10^{-9}$ sec for $L = 0.25$ meters.

The next set of numerical experiments contains the simulations of wave propagation through a domain with an obstacle completely consisting of a

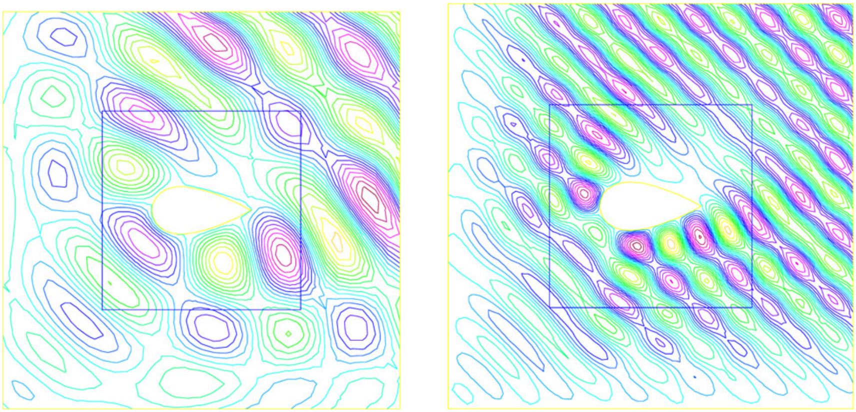

Fig. 6. Contour plot of the real part of the solution for $L = 0.5$ (left) and $L = 0.25$ (right) meters. Incident wave coming from the lower left corner with an angle of 45 degrees.

coating material. We have taken the coating material coefficients to be $\varepsilon_2 = 1$ and $\mu_2 = 9$, implying that the speed of propagation in the coating material is three times slower than in air. As before Ω is a 2 meter $\times$ 2 meter rectangle and Ω_2 has the shape of an airfoil.

For the solution of this problem for an incident frequency $f = 0.6$ GHz we have used a mesh with a total of 8435 nodes and 16228 elements. The time step was taken to be $\Delta t = T/50$, where $T = 1/f = 1.66 \times 10^{-9}$ sec is a time period. We used a mesh consisting of 20258 nodes (39514 elements) for solving the problem for an incident wave with the frequency $f = 1.2$ GHz. The time step was equal to $T/50$, $T = 1/f = 0.83 \times 10^{-9}$ sec.

In Figures 7 and 8 we present the contour plot of the real part of the solution for the incident frequency $L = 0.5$ and $L = 0.25$. We also performed numerical computations for the case when the obstacle is an airfoil with a coating (Figure 9). The coating region is moon shaped and, as before, $\varepsilon_2 = 1$ and $\mu_2 = 9$. We show in Figure 10 the contour plot of the real part of the solution for the incident frequency $L = 0.5$ meters and $L = 0.25$ meters for the case when the incident wave is coming from the left. Figure 11 presents the contour plot of the real part of the solution for incident frequency, $L = 0.5$ meters and $L = 0.25$ meters for the case when incident wave is coming from the lower left corner with angle equal to $45°$.

An important observation for all of the numerical experiments mentioned is that, despite the fact that a mesh discontinuity takes place over γ together with a weak forcing of the matching conditions, we do not observe a discontinuity of the computed fields.

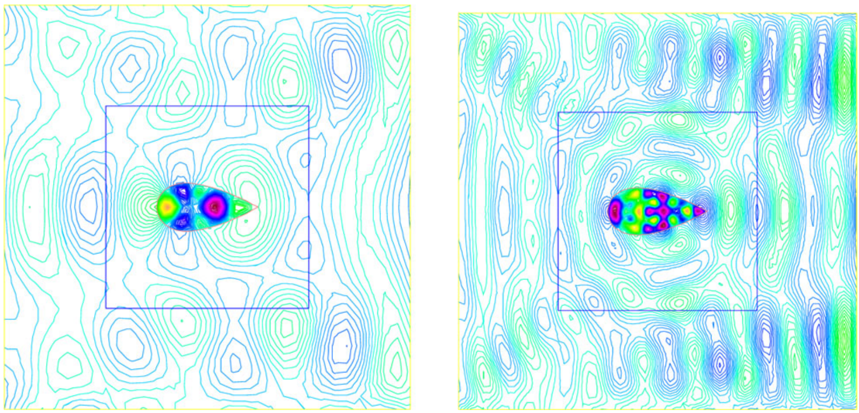

Fig. 7. Contour plot of the real part of the solution for $L = 0.5$ (left) and $L = 0.25$ (right). Incident wave coming from the left.

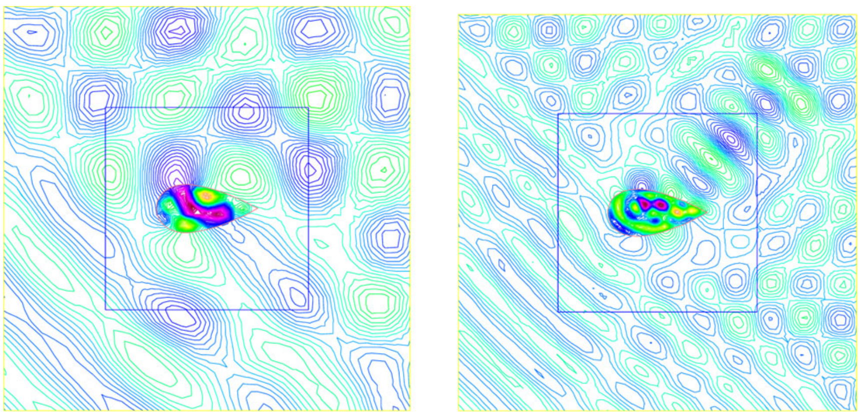

Fig. 8. Contour plot of the real part of the solution for $L = 0.5$ (left) and $L = 0.25$ (right). Incident wave coming from the lower left corner with an angle of 45 degrees.

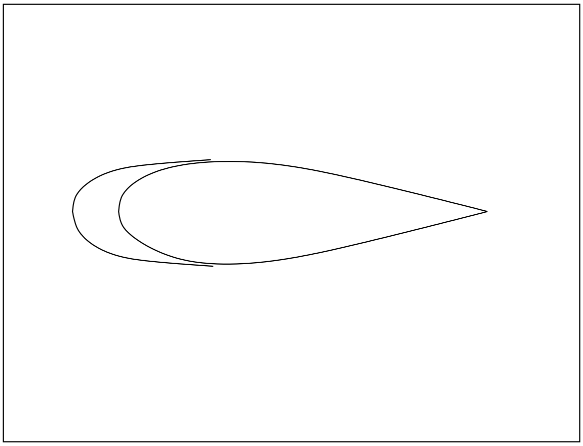

Fig. 9. Obstacle in a form of an airfoil with a coating.

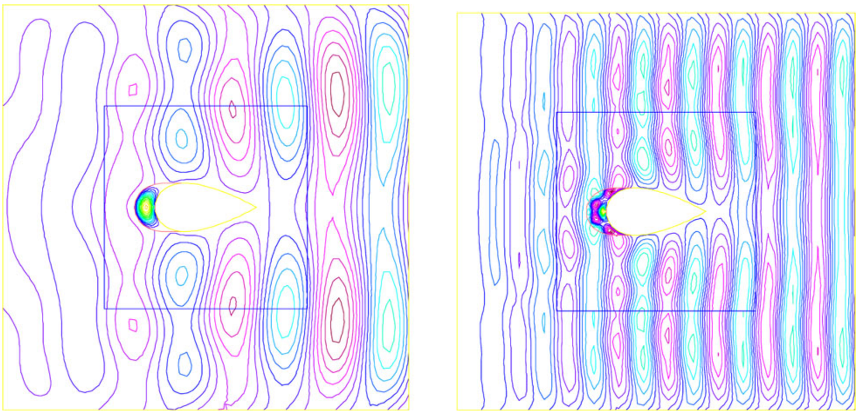

Fig. 10. Contour plot of the real part of the solution for $L = 0.5$ (left) and $L = 0.25$ (right). Incident wave coming from the left.

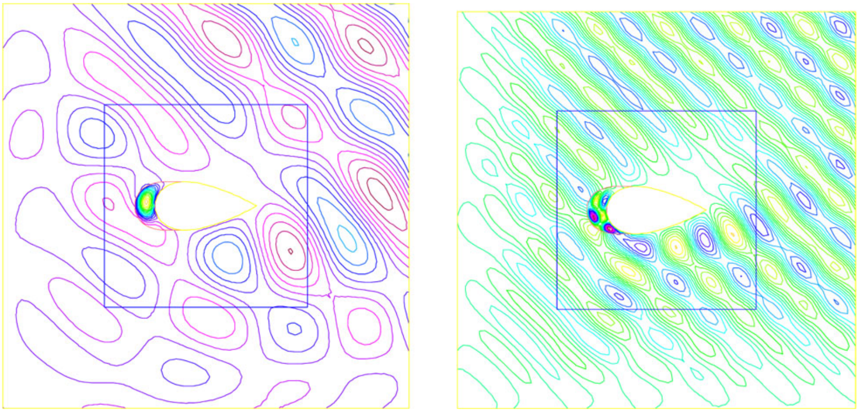

Fig. 11. Contour plot of the real part of the solution for $L = 0.5$ (left) and $L = 0.25$ (right). Incident wave coming from the left lower corner with a 45 degrees angle.

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