An Operator Splitting Method for Pricing American Options

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Summary. Pricing American options using partial (integro-)differential equation based methods leads to linear complementarity problems (LCPs). The numerical solution of these problems resulting from the Black–Scholes model, Kou's jump-diffusion model, and Heston's stochastic volatility model are considered. The finite difference discretization is described. The solutions of the discrete LCPs are approximated using an operator splitting method which separates the linear problem and the early exercise constraint to two fractional steps. The numerical experiments demonstrate that the prices of options can be computed in a few milliseconds on a PC.

1 Introduction

Since 1973 Black, Scholes, and Merton developed models for pricing options in [BS73, Mer73] and, on the other hand, the Chicago Board Options Exchange started to operate, the trading of options has grown to tremendous scale. Basic options give either the right to sell (put) or buy (call) the underlying asset with the strike price. European options can be exercised only at the expiry time while American options can be exercised anytime before the expiry. The Black–Scholes partial differential equation (PDE) describes the evolution of the option price in time for European options. In order to avoid arbitrage opportunities with an American option, the so-called early exercise constraint has to be posed on its value. Combining this constraint with the PDE leads to a linear complementarity problem (LCP). For European options it is generally possible to derive formulas for their price, but American options usually need to be priced numerically. This paper considers the solution of these pricing problems. For the general discussion on these topics, we refer to the books [AP05, CT04, TR00, Wil98].

The Black–Scholes model [BS73] assumes a constant volatility for all options with different strike prices and expiry times on the same underlying asset. In practice, this does not hold in the markets. One possibility to make

the prices consistent with the markets is to define the volatility as a function of time and the value of the underlying asset, and then calibrate this function; see [Dup94], for example. In 1976, Merton suggested to add jumps to the model of the underlying asset in [Mer73]. This jump-diffusion model helps to explain a good part of difference between the market prices and the ones given by the Black–Scholes model with a constant volatility. Since then there has been growing activity to incorporate jumps to the model; see [CT04] and references therein. One of the models used in this paper is Kou's jump-diffusion model. Another generalization is to make the volatility a stochastic process. Several such multifactor models have been proposed; see [FPS00], for example. Here Heston's stochastic volatility model [Hes93] is used. One can also combine stochastic volatility and jump models like in [Bat96, DPS00], for example.

Several ways to solve the discretized LCPs resulting from pricing American options have been described in the literature. Maybe the most common method is the project SOR iteration proposed in [Cry71]. This method is fairly generic and easy to implement, but its convergence rate degrades as grids are refined. For one-dimensional PDE models the resulting LCPs can be solved very efficiently using the direct algorithm in [BS77] if the matrix is a tridiagonal M-matrix and the solution has suitable form. The full matrices resulting from jump-diffusion models require special techniques in order to obtain efficient algorithms. The papers [AO05, AA00, CV05, MSW05] study the numerical pricing of European options, and in [dFL04, dFV05, Toi06] the pricing of American options is considered. For higher-dimensional problems like the ones resulting from Heston's model multigrid methods have been considered in [BC83, CP99, Oos03, RW04], for example. An alternative way is to approximate the LCPs using a penalty method [FV02, ZFV98]. This leads to a sequence system of linear equations with varying matrices. With this approach the constraints are always slightly violated. With a fairly similar Lagrange multiplier method [AP05, HIK03, IK06, IT06b] it can be guaranteed that the constraints are satisfied.

This paper considers an operator splitting method proposed for the Black–Scholes model in [IT04a]. The method was applied to Heston's model and analyzed in [IT04b], and for Kou's model it was applied in [Toi06]. The basic idea of this method is to split a time step with a LCP to two fractional time steps. The first fractional step requires a system of linear equations to be solved and the second one enforces the early exercise constraint. The update to satisfy the constraint is simple and, thus, the main computational burden will be the solution linear systems. A similar approach is commonly used to treat the incompressibility condition in the computational fluid dynamics; see [Glo03], for example. The operator splitting method has two obvious benefits. There are several efficient methods available for solving resulting systems of linear equations while only a few methods are available for the original LCPs and they usually cannot compete in the efficiency. Secondly the operator splitting method is easier to implement than an efficient LCP solver. This

paper demonstrates that the operator splitting method is suitable for pricing American options with different models and that the computation of a sufficiently accurate price for most purposes requires only a few milliseconds on a contemporary PC.

Outline of the paper is the following. We begin by describing the three models and the resulting P(I)DEs for European options. After this we formulate linear complementarity problems for the value of American options. Next we sketch finite difference discretizations for the partial differential operators. Then the operator splitting method is described and after this methods for solving the resulting systems of linear equations are discussed. The paper ends with numerical examples with all of the considered models and conclusions.

2 Models

2.1 Black–Scholes Model

By assuming that the value of the underlying asset denoted by x follows a geometric Brownian motion with a drift, the Black–Scholes PDE [BS73]

$$v_t = A_{BS}v = -\frac{1}{2}(\sigma x)^2 v_{xx} - rxv_x + rv \tag{1}$$

can be derived for the value of an option denoted by v, where σ is the volatility of the value of the asset and r is the risk free interest rate. In practice, the market prices of options do not satisfy (1). One possible way to make the model to match the markets is to use a volatility function σ which depends on the value of the underlying asset and time; see [AP05, Dup94], for example. In this case, the volatility function has to be calibrated with the market data.

2.2 Jump-Diffusion Models

When there is a high market stress like the crash of 1987 the value of assets can move faster than a geometric Brownian motion would predict. Partly due to this, models which allow also jumps for the value of asset have become more common; see [CT04] and references therein. Already in 1976 Merton considered such a model in [Mer76]. With independent and identically distributed jumps a partial integro-differential equation (PIDE)

$$v_t = A_{JD}v = -\frac{1}{2}(\sigma x)^2 v_{xx} - (r-\mu\zeta)xv_x + (r+\mu)v - \mu\int_{\mathbb{R}_+} v(t,xy)f(y)\,dy \tag{2}$$

can be derived for the value of an option, where μ is the rate of jumps, the function f defines the distributions of jumps, and ζ is the mean jump amplitude.

Merton used a Gaussian distribution for jumps in [Mer76]. Kou considered in [Kou02] a log-double-exponential distribution for jumps which leads a more flexible and tractable model. In this case, the density is

$$f(y) = \begin{cases} q\alpha_2 y^{\alpha_2 - 1}, & y < 1, \\ p\alpha_1 y^{-\alpha_1 - 1}, & y \geq 1, \end{cases} \tag{3}$$

where p, q, $\alpha_1 > 1$, and α_2 are positive constants such that $p + q = 1$. The mean jump amplitude is $\zeta = \frac{p\alpha_1}{\alpha_1 - 1} + \frac{q\alpha_2}{\alpha_2 + 1} - 1$. We will employ this model in the numerical experiments. Also in this case one possible way to calibrate the model is to let the volatility σ be a function of time and asset value like in [AA00].

2.3 Stochastic Volatility Models

In practice, the volatility of the value of an asset is not a constant over time. Several models have been also developed for the behavior of the volatility. Among several stochastic volatility models probably the one developed by Heston in [Hes93] is the most popular. It assumes the volatility to be a mean-reverting process. Under the assumption that the market price of risk is zero Heston's model leads to the two-dimensional PDE

$$v_t = A_{SV} v = -\frac{1}{2} y x^2 v_{xx} - \rho\gamma y x v_{xy} - \frac{1}{2}\gamma^2 y v_{yy} - r x v_x - \alpha(\beta - y) v_y + r v, \tag{4}$$

where y is the variance, that is, the square of the volatility, β is the mean level of the variance, α is the rate of reversion on the mean level, and γ is the volatility of the variance. The correlation between the price of the underlying asset and its variance is ρ.

3 Linear Complementarity Problems

The value of an option at the expiry time T is given by

$$v(T, x) = g(x), \tag{5}$$

where the payoff function g depends on the type of the option. For example, for a put option with a strike price K it is

$$g(x) = \max\{K - x, 0\}. \tag{6}$$

The value v of an American option satisfies a linear complementarity problem (LCP)

$$\begin{cases} (v_t - Av) \geq 0, & v \geq g, \\ (v_t - Av)(v - g) = 0, \end{cases} \tag{7}$$

where A is one of the operators A_{BS}, A_{JD}, or A_{SV} defined by (1), (2), and (4), respectively.

The operator splitting method is derived from a formulation with a Lagrange multiplier λ after a temporal discretization. In the continuous level, the formulation with the Lagrange multiplier reads

$$\begin{cases} (v_t - Av) = \lambda, & \lambda \geq 0,\ v \geq g, \\ \lambda(v - g) = 0. \end{cases} \tag{8}$$

4 Discretizations

4.1 Spatial Discretizations

The LCPs are posed on infinite domain as there is no upper limit for the value of the asset and also for variance in the case of Heston's stochastic volatility model. In order to use finite difference discretizations for the spatial derivatives, the domain is truncated from sufficiently large values of x and y which are denoted by X and Y, respectively. The choice of X for the Black–Scholes model is considered in [KN00], for example. On the truncation boundaries a suitable boundary condition needs to be posed. For the one-dimensional models for put options, we use homogeneous Dirichlet boundary condition $v = 0$ at $x = X$. For Heston's model homogeneous Neumann boundary conditions are posed. While these are fairly typical choices for boundary conditions there are also other choices.

For the interval $[0, X]$, we define subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, where x_is satisfy $0 = x_0 < x_1 < \cdots < x_m = X$. For Heston's model, the interval $[0, Y]$ is similarly divided by the points $0 = y_0 < y_1 < \cdots < x_n = Y$. Finite difference discretizations seek approximations for the value of v at the grid points x_is for one-dimensional models and (x_i, y_j) for Heston's model. The spatial partial derivatives appearing in (7) and (8) needs to be approximated using the grid point values. For the second-order derivative with respect to x, we use a finite difference approximation

$$v_{xx}(t, x_i) \approx \frac{2}{\Delta x_{i-1}(\Delta x_{i-1} + \Delta x_i)} v(t, x_{i-1}) - \frac{2}{\Delta x_{i-1} \Delta x_i} v(t, x_i) + \frac{2}{\Delta x_i(\Delta x_{i-1} + \Delta x_i)} v(t, x_{i+1}), \tag{9}$$

where $\Delta x_{i-1} = x_i - x_{i-1}$ and $\Delta x_i = x_{i+1} - x_i$. For the first-order derivative, one possible approximation is

$$v_x(t, x_i) \approx -\frac{\Delta x_i}{\Delta x_{i-1}(\Delta x_{i-1} + \Delta x_i)} v(t, x_{i-1}) + \frac{\Delta x_i - \Delta x_{i-1}}{\Delta x_{i-1} \Delta x_i} v(t, x_i) + \frac{\Delta x_{i-1}}{\Delta x_i(\Delta x_{i-1} + \Delta x_i)} v(t, x_{i+1}). \tag{10}$$

For Heston's model the approximations for the partial derivatives with respect to y can be defined analogously. The approximations (9) and (10) can be shown to be second-order accurate with respect to the grid step size when the step size varies smoothly; see [MW86], for example.

When the coefficient for the first-order derivative is large compared to the coefficient of the second-order derivative, the above discretizations lead to matrices with positive off-diagonal entries. In this case the matrix cannot have the M-matrix property and the resulting numerical solutions can have oscillations. This situation can be avoided by using locally one-sided differences for the first-order derivative. The drawback of this approach is that it reduces the order of accuracy to be first-order with respect to the grid step size. Nevertheless we will use this choice to ensure that the spatial discretizations lead to M-matrices and, thus, stable discretizations.

Special care must be taken when discretizing the cross derivative v_{xy} in Heston's model if M-matrices are sought. In [IT05], a seven-point stencil leading an M-matrix is described. With strong correlation between the value of asset and its volatility there can be severe restrictions on grid step sizes in order to obtain M-matrices and accurate discretizations.

The discretization of the integral term in the jump-diffusion model (2) leads to a full matrix; see [AO05, dFL04, MSW05], for example. Computationally it is expensive to operate with the full matrix and, due to this, different fast ways have been proposed for operating with it in the above mentioned articles. Fortunately, with Kou's log-double-exponential f in (2) is possible to derive recursive formulas with optimal computational complexity for evaluating quadratures for the integrals. This has been described in [Toi06] and we will employ this approach with our numerical experiments.

The grid point values of v are collected to a vector $\mathbf{v}$. Similarly we define a vector $\mathbf{g}$ containing the grid point values of the payoff function g. The spatial discretization leads to a semi-discrete form of the LCP (7) given by

$$\begin{cases} (\mathbf{v}_t - \mathbf{A}\mathbf{v}) \geq \mathbf{0}, & \mathbf{v} \geq \mathbf{g}, \\ (\mathbf{v}_t - \mathbf{A}\mathbf{v})^T(\mathbf{v} - \mathbf{g}) = 0, \end{cases} \tag{11}$$

where the matrix $\mathbf{A}$ is defined by the used finite differences and the inequalities of vectors are componentwise. The semi-discrete form with the Lagrange multiplier λ corresponding to (8) reads

$$\begin{cases} (\mathbf{v}_t - \mathbf{A}\mathbf{v}) = \boldsymbol{\lambda}, & \boldsymbol{\lambda} \geq \mathbf{0},\ \mathbf{v} \geq \mathbf{g}, \\ \boldsymbol{\lambda}^T(\mathbf{v} - \mathbf{g}) = 0, \end{cases} \tag{12}$$

where the vector $\boldsymbol{\lambda}$ contains the grid point values of the Lagrange multiplier.

4.2 Temporal Discretization

For the temporal discretization the time interval $[0, T]$ is divided into subintervals which are defined by the times $0 = t_0 < t_1 < \cdots < t_l = T$. The vector containing the grid point values of v at t_k is denoted by $\mathbf{v}^{(k)}$.

Usually in option pricing problems the backward time stepping is started from a non-smooth final value. Due to this, the time stepping scheme should have good damping properties in order to avoid oscillations. For example, the popular Crank–Nicolson method does not have good damping properties and it can lead to approximations with excessive oscillations. Instead of it we employ the Rannacher time-stepping scheme [Ran84]. In the option pricing context it has been analyzed recently in [GC06].

In the Rannacher time-stepping scheme a few first time steps are performed with the implicit Euler method and then the Crank–Nicolson method is used. This leads to second-order accuracy and good damping properties. For the semi-discrete LCP (11) the scheme reads

$$\begin{cases} \mathbf{B}^{(k)}\mathbf{v}^{(k)} - \mathbf{C}^{(k)}\mathbf{v}^{(k+1)} - \mathbf{f}^{(k)} \geq \mathbf{0}, \qquad \mathbf{v}^{(k)} \geq \mathbf{g}, \\ \left(\mathbf{B}^{(k)}\mathbf{v}^{(k)} - \mathbf{C}^{(k)}\mathbf{v}^{(k+1)} - \mathbf{f}^{(k)}\right)^T \left(\mathbf{v}^{(k)} - \mathbf{g}\right) = 0, \end{cases} \tag{13}$$

for $k = l-1, \ldots, 0$, where

$$\mathbf{B}^{(k)} = \mathbf{I} + \theta_k \Delta t_k \mathbf{A}, \qquad \mathbf{C}^{(k)} = \mathbf{I} - (1-\theta_k)\Delta t_k \mathbf{A}, \tag{14}$$

and $\mathbf{f}^{(k)}$ is due to possible non-homogeneous Dirichlet boundary conditions. When the first four time steps are performed with the implicit Euler method the parameter θ_k is defined by

$$\theta_k = \begin{cases} 1, & k = l-1, \ldots, l-4, \\ \frac{1}{2}, & k = l-5, \ldots, 0. \end{cases} \tag{15}$$

The temporal discretization of the semi-discrete form with the Lagrange multiplier (12) leads to

$$\begin{cases} \mathbf{B}^{(k)}\mathbf{v}^{(k)} - \mathbf{C}^{(k)}\mathbf{v}^{(k+1)} - \mathbf{f}^{(k)} = \Delta t_k \boldsymbol{\lambda}^{(k)}, \qquad \boldsymbol{\lambda}^{(k)} \geq \mathbf{0},\ \mathbf{v}^{(k)} \geq \mathbf{g}, \\ \left(\boldsymbol{\lambda}^{(k)}\right)^T \left(\mathbf{v}^{(k)} - \mathbf{g}\right) = 0, \end{cases} \tag{16}$$

for $k = l-1, \ldots, 0$.

5 Operator Splitting Method

Here we describe an operator splitting method [IT04a] which approximates the solution of the LCP in (16) by two fractional time steps. The first step requires the solution of a system of linear equations and the second step updates the solution and Lagrange multiplier to satisfy the linear complementarity conditions. The advantage of this approach is that it simplifies the solution procedure and allows to use any efficient method for solving linear systems. More precisely, the steps in the operator splitting method are

$$\mathbf{B}^{(k)}\tilde{\mathbf{v}}^{(k)} = \mathbf{C}^{(k)}\mathbf{v}^{(k+1)} + \mathbf{f}^{(k)} + \Delta t_k \boldsymbol{\lambda}^{(k+1)} \tag{17}$$

and

$$\begin{cases} \mathbf{v}^{(k)} - \tilde{\mathbf{v}}^{(k)} - \Delta t_k(\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}^{(k+1)}) = \mathbf{0}, & \boldsymbol{\lambda}^{(k)} \geq \mathbf{0},\ \mathbf{v}^{(k)} \geq \mathbf{g}, \\ \left(\boldsymbol{\lambda}^{(k)}\right)^T \left(\mathbf{v}^{(k)} - \mathbf{g}\right) = 0. \end{cases} \tag{18}$$

The first step (17) uses the Lagrange multiplier vector $\boldsymbol{\lambda}^{(k+1)}$ from the previous step and not $\boldsymbol{\lambda}^{(k)}$ which leads to the decoupling of the linear system and the constraints. The second step does not have any spatial couplings and the update can be made quickly by going through components of the vectors $\mathbf{v}^{(k)}$ and $\boldsymbol{\lambda}^{(k)}$ one by one. Due to this, the main computational cost is the solution of the linear system in the first step (17). Under reasonable assumptions it can be shown that the difference between the solutions of the original time stepping and the operator splitting time stepping is second-order with respect to the time step size [IT04b]. Hence, it does not reduce the order of accuracy compared to second-order accurate time stepping method like the Rannacher scheme.

6 Solution of Linear Systems

In each time step with the operator splitting method it is necessary to solve a system of linear equations with the matrix $\mathbf{B}$ defined in (14). Here and in the following we have omitted the subscript (k) in order to simplify the notations. The Black–Scholes PDE leads to a tridiagonal $\mathbf{B}$ with the above finite difference discretization. In this case the linear systems can be solved efficiently using the $\mathbf{LU}$ decomposition.

With the jump-diffusion models $\mathbf{B}$ is a full matrix and the use of $\mathbf{LU}$ decomposition would be computationally too expensive. We adopt the approach proposed in [AO05, dFV05] which is an iterative method based on a regular splitting of $\mathbf{B}$. We use the splitting

$$\mathbf{B} = \mathbf{T} - \mathbf{R}, \tag{19}$$

where $\mathbf{R}$ is the full matrix resulting from the integral term and, thus, $\mathbf{T}$ is a tridiagonal matrix defined by other terms. Now the iterative method for a system $\mathbf{Bv} = \mathbf{b}$ reads

$$\mathbf{v}^{l+1} = \mathbf{T}^{-1}\left(\mathbf{b} + \mathbf{R}\mathbf{v}^l\right), \qquad l = 0, 1, \ldots, \tag{20}$$

where $\mathbf{v}^0$ is the initial guess taken to be the solution from the previous time step. The solutions with $\mathbf{T}$, that is, multiplications with $\mathbf{T}^{-1}$ can be computed efficiently using $\mathbf{LU}$ decomposition. The multiplications with $\mathbf{R}$ can be performed using the fast recursion formulas in [Toi06] when Kou's model is used. Furthermore, it has been shown in [dFV05] that the iteration (20) converges fast. As the numerical experiments will demonstrate, usually two or three iterations are enough to obtain the solution with sufficient accuracy.

With Heston's model **B** is a block tridiagonal matrix corresponding to a two-dimensional PDE. Furthermore, **B** is usually not well conditioned partly due to varying coefficient in the PDE. In order to obtain a method with optimal computational complexity, we will employ a multigrid method. The analysis in [Oos03] shows that a multigrid with an alternating direction smoother is robust with respect to all parameters in the problem and discretization. This smoother is computationally more expensive and complicated to implement than point smoothers, but we used it as it guarantees a fast multigrid convergence. The grid transfers are performed using full weighting restriction and bilinear prolongation.

7 Numerical Results

In our numerical examples we price American put options with the parameters

$$\sigma = 0.25, \quad r = 0.1, \quad T = 0.25, \quad \text{and} \quad K = 10. \tag{21}$$

The additional parameters for Kou's and Heston's models are defined in the subsequent sections. In Table 1, we have collected reference option prices for three asset values. They are computed with very fine discretizations for the one-dimensional models on the interval $[0, 40]$ and the prices under Heston's model are from [IT06b] with $y = 0.0625$. Fig. 1 shows the price of the option as a function of x computed with the different models in the interval $8.5 \leq x \leq 12.5$.

In the following tables all CPU times are given in milliseconds on a PC with 3.8 GHz Intel Xeon processor and implementations have been made using Fortran.

7.1 Black–Scholes Model

Based on a few numerical experiments using the model parameters in (21) we observed that the truncation boundary can be chosen to be $X = 2K = 20$ with the truncation error being so small that it does not influence the first five decimals of the prices at $x = 9$, 10, and 11. We define the spatial grid as

$$x_i = \left(1 + \frac{\sinh(\beta(i/n - \gamma))}{\sinh(\beta\gamma)}\right) K, \qquad i = 0, 1, \ldots, m, \tag{22}$$

Table 1. Reference prices for options with the different models

model \ asset value	$x = 9$	$x = 10$	$x = 11$
Black–Scholes	1.030463	0.402425	0.120675
Kou	1.043796	0.429886	0.148625
Heston	1.107621	0.520030	0.213677

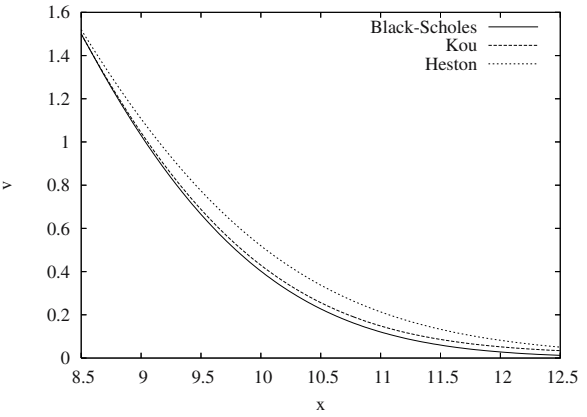

Fig. 1. The price of the option with respect to the value of the underlying asset for the three different models.

Table 2. Results for different grids with Black–Scholes model

l	m	error	ratio	time
10	20	0.01056		0.02
18	40	0.00208	5.1	0.06
34	80	0.00058	3.6	0.21
66	160	0.00022	2.7	0.79
130	320	0.00007	3.3	3.08

where we have chosen $\beta = 6$ and $\gamma = 1/2$ which leads to some refinement near the strike price K. For the temporal discretization, we choose the approximation times to be

$$t_k = \left(\frac{a^{-k/(l-2)} - 1}{a^{-1} - 1} \right) T, \qquad k = 0, 1, \ldots, l-4, \tag{23}$$

and

$$t_k = \left(\frac{a^{-(k+l-4)/(2l-4)} - 1}{a^{-1} - 1} \right) T, \qquad k = l-3, \ldots, l. \tag{24}$$

The parameter a in (23) and (24) has been chosen to be $a = 2$ which leads to a mild refinement near the expiry.

Table 2 reports the l_2 errors computed using the reference prices in Table 1 at $x = 9$, 10, and 11 for five different space-time grids. The ratio column in the table gives the ratios between two successive l_2 errors. The time is the CPU time in milliseconds needed to price the options.

7.2 Kou's Jump-Diffusion Model

The parameters defining the jump probability and its distribution in Kou's model are chosen to be

Table 3. Results for different grids with Kou's model

l	m	error	ratio	iter	time
10	20	0.01050		3.1	0.10
18	40	0.00231	4.5	3.0	0.29
34	80	0.00056	4.1	3.0	0.97
66	160	0.00022	2.6	2.3	2.95
130	320	0.00006	3.7	2.0	10.17

Table 4. Results for different grids with Heston's model

l	m	n	error	ratio	iter	time
10	20	8	0.02576		1.0	0.7
18	40	16	0.00574	4.5	1.3	5.7
34	80	32	0.00420	1.4	2.0	59.4
66	160	64	0.00049	8.5	2.0	487.5
130	320	128	0.00012	4.1	2.0	4373.7

$$\alpha_1 = 3, \quad \alpha_2 = 3, \quad p = \frac{1}{3}, \quad \text{and} \quad \mu = 0.1. \tag{25}$$

We have used the same space-time grids as with the Black–Scholes model. Table 3 reports the errors, their ratios and CPU times in milliseconds. The column iter in the table gives the average number of the iterations (20). The stopping criterion for the iterations was that the norm of the residual vector is less than 10^{-11} times the norm of the right-hand side vector.

7.3 Heston's Stochastic Volatility Model

In Heston's model the behavior of the stochastic volatility and its correlation with the value of the asset are described by the parameters

$$\alpha = 5, \quad \beta = 0.16, \quad \gamma = 0.9, \quad \text{and} \quad \rho = 0.1. \tag{26}$$

The values of these parameters are the same as in many previous studies including [CP99, IT07, Oos03, ZFV98]. The computational domain is truncated at $X = 20$ and $Y = 1$ like also in [Oos03, IT07], for example. We use the same non-uniform grids as in [IT05] and the parameter w in the discretization of the cross derivative (not discussed in this paper) is chosen using the formula in [IT07]. For the time stepping we use uniform time steps.

Table 4 reports the errors, their ratios, the average number of multigrid iterations, and CPU times in milliseconds. The stopping criterion for the multigrid iterations was that the norm of the residual vector is less than 10^{-6} times the norm of the right-hand side vector.

8 Conclusions

We described an operator splitting method for solving linear complementarity problems (LCPs) resulting from American option pricing problems. We considered it in the case of the Black–Scholes model, Kou's jump-diffusion model, and Heston's stochastic volatility model for the value of the underlying asset. The numerical results demonstrated that with all these models the prices can be computed in a few milliseconds on a PC.

As future research one could consider the construction of adaptive discretization; see [AP05, LPvST07], for example. Also the robustness and accuracy of discretizations for Heston's model with higher correlations could be studied. A natural generalization would be to extent the methods for stochastic volatility models including jumps like the ones in [Bat96, DPS00].

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