A Fixed Domain Approach in Shape Optimization Problems with Neumann Boundary Conditions

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Summary. Fixed domain methods have well-known advantages in the solution of variable domain problems, but are mainly applied in the case of Dirichlet boundary conditions. This paper examines a way to extend this class of methods to the more difficult case of Neumann boundary conditions.

1 Introduction

Starting with the well-known monograph of Pironneau [Pir84], shape optimization problems are subject to very intensive research investigations. They concentrate several major mathematical difficulties: unknown and possibly non-smooth character of optimal geometries, lack of convexity of the functional to be minimized, high complexity and stiff character of the equations to be solved numerically, etc. Accordingly, the relevant scientific literature is huge and we quote here just the books of Mohammadi and Pironneau [MP01] and of Neittaanmäki, Sprekels and Tiba [NST06] for an introduction to this domain of mathematics.

In this paper, we study the model optimal design problem

$$\text{Min} \int_{\Omega} j(x, y(x))\, dx \tag{1}$$

subject to the Neumann boundary value problem

$$\int_{\Omega} \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 y v \right] dx = \int_{\Omega} f v \tag{2}$$

for any $v \in H^1(\Omega)$.

Here, $\Omega \subset D \subset \mathbb{R}^d$ is an unknown domain (the minimization parameter), while D is a fixed smooth open set in the Euclidean space $\mathbb{R}^d$. The functions a_0 and a_{ij} are in $L^\infty(D)$ and $f \in L^2(D)$, that is (2) makes sense for any Ω admissible and defines, as it is well known, the unique weak solution $y = y_\Omega \in H^1(\Omega)$ of the second order elliptic equation

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial y}{\partial x_i}\right) + a_0 y = f \quad \text{in } \Omega \tag{3}$$

with Neumann boundary conditions for the conormal derivative

$$\frac{\partial y}{\partial n_A} = \sum_{i,j=1}^{d} a_{ij}\frac{\partial y}{\partial x_j}\cos(\bar{n}, x_i) = 0 \quad \text{on } \partial\Omega. \tag{4}$$

In the classical formulation (3), (4), $\partial\Omega$ has to be assumed smooth and $\bar{n}$ is the (outward) normal to $\partial\Omega$ in the considered points $x = (x_1, x_2, ..., x_d)$. Non-homogeneous Neumann problems (i.e. with the right-hand side non-zero in (4)) may be considered as well by a simple translation argument reducing everything to the homogeneous case.

The functional $j : D \times \mathbb{R} \to \mathbb{R}$ is a general convex integrand in the sense of Rockafellar [Roc70] – more assumptions will be added when necessary.

The open set Ω will be “parametrized” by some continuous function $g : D \to \mathbb{R}$ by

$$\Omega = \Omega_g = \text{int}\{x \in D \mid g(x) \geq 0\} \tag{5}$$

and $g \in C(\bar{D})$ will be the true unknown of the optimization problem (1), (2). The parametrization is, of course, non-unique, but this does not affect the argument. Arbitrary Caratheodory open sets $\Omega \subset D$ may be expressed in the form Ω_g if g is the signed distance function (at some power). Further constraints on $\Omega = \Omega_g$ (beside $\Omega \subset D$) may be imposed in the abstract form

$$g \in C, \tag{6}$$

where $C \subset C(\bar{D})$ is some convex closed subset. For instance, if $E \subset D$ is a given subset and $C = \{g \in C(\bar{D}) \mid g(x) \geq 0,\ x \in E\}$, then the constraint $g \in C$ is equivalent with the condition $E \subset \Omega$. Other cost functionals may be studied as well:

$$\int_E j(x, y(x))\, dx$$

(if the constraint $E \subset \Omega$ is imposed) or

$$\int_\Gamma j(x, y(x))\, dx,$$

where $\Gamma \subset D$ is a smooth given manifold and $\Omega \supset \Gamma$ for all admissible Ω. Robin boundary conditions (instead of (4)) may be also discussed by our method. In the case of Dirichlet boundary conditions other approaches may be used [NPT07, NT95, Tib92].

In Section 2 we recall some geometric controllability properties that are at the core of our approach, while Section 3 contains the basic arguments. The paper ends with some brief Conclusions.

2 A Controllability-Like Result

In the classical book of Lions [Lio68], it is shown that, when $u \in L^2(\Gamma_1)$ is arbitrary and y_u is the unique solution (in the transposition sense) of

$$-\Delta y = 0 \quad \text{in } G,$$

$$y = u \quad \text{on } \Gamma_1, \quad y = 0 \quad \text{on } \Gamma_2,$$

then the set of normal traces $\{\frac{\partial y_u}{\partial n} \mid u \in L^2(\Gamma_1)\}$ is linear and dense in the space $H^{-1}(\Gamma_2)$. Notice that $\frac{\partial y_u}{\partial n} \in H^{-1}(\Gamma_2)$ due to some special regularity results, Lions [Lio68]. Here $G \subset \mathbb{R}^d$ is an open connected set such that its boundary $\partial G = \Gamma_1 \cup \Gamma_2$ and $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$. This density result may be interpreted as an approximate controllability property in the sense that the "attainable" set of normal derivatives $\frac{\partial y_u}{\partial n}$ (when u ranges in $L^2(\Gamma_1)$) may approximate any element in the "image" space $H^{-1}(\Gamma_2)$. Constructive approaches, results involving constraints on the boundary control u are reported in [NST06, Ch. 5.2].

We continue with a distributed approximate controllability property, which is a constructive variant of Theorem 5.2.21 in [NST06]. We consider the equation (2) in D and with a modified right-hand side:

$$\int_D \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial \tilde{y}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_j} + a_0 \tilde{y} \tilde{v} \right] dx = \int_D \chi_0 u \tilde{v} \, dx \quad \forall \tilde{v} \in H^1(D), \tag{7}$$

where $u \in L^2(D)$ is a distributed control and χ_0 is the characteristic function of some smooth open set $\Omega_0 \subset D$ such that $\partial D \subset \bar{\Omega}_0$. That is, Ω_0 is a relative neighborhood of ∂D and we denote $\Gamma = \partial \Omega_0 \setminus \partial D$. Clearly, $\bar{\Gamma} \cap \partial D = \emptyset$.

Theorem 1. *Let $w \in H^{1/2}(\Gamma)$ be given and let $[u_\varepsilon, y_\varepsilon]$ be the unique optimal pair of the control problem:*

$$\underset{u \in L^2(\Omega_0)}{\text{Min}} \left\{ \frac{1}{2} |y - w|_{H^{1/2}(\Gamma)} + \frac{\varepsilon}{2} |u|^2_{L^2(\Omega_0)} \right\}, \quad \varepsilon > 0, \tag{8}$$

$$\int_\Omega \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} + a_0 y z \right] dx = \int_{\Omega_0} u z \, dx \quad \forall z \in H^1(\Omega_0). \tag{9}$$

Then, we have

$$y_\varepsilon|_\Gamma \underset{\varepsilon \to 0}{\longrightarrow} w \quad \textit{strongly in } H^{1/2}(\Gamma). \tag{10}$$

Proof. The existence and the uniqueness of the optimal pair $[u_\varepsilon, y_\varepsilon] \in L^2(\Omega_0) \times H^1(\Omega_0)$ of the control problem (8), (9) is obvious. The pair [0,0] is clearly admissible and, for any $\varepsilon > 0$, we obtain

$$\frac{1}{2}|y_\varepsilon - w|^2_{H^{1/2}(\Gamma)} + \frac{\varepsilon}{2}|u_\varepsilon|^2_{L^2(\Omega_0)} \leq \frac{1}{2}|w|^2_{H^{1/2}(\Gamma)}.$$

Therefore, $\{y_\varepsilon\}$ and $\{\varepsilon^{1/2}u_\varepsilon\}$ are bounded respectively in $H^{1/2}(\Gamma)$, $L^2(\Omega_0)$. We denote by $l \in H^{1/2}(\Gamma)$ the weak limit (on a subsequence) of $\{y_\varepsilon - w\}$.

Let us define the adjoint system by:

$$\int_{\Omega_0} \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial z}{\partial x_i} \frac{\partial p_\varepsilon}{\partial x_j} + a_0 z p_\varepsilon \right] dx = \int_\Gamma (y_\varepsilon - w) z \, d\sigma \quad \forall z \in H^1(\Omega_0), \quad (11)$$

which is a non-homogeneous Neumann problem and $p_\varepsilon \in H^1(\Omega_0)$. We also introduce the equation in variations

$$\int_{\Omega_0} \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial \mu}{\partial x_i} \frac{\partial z}{\partial x_j} + a_0 \mu z \right] dx = \int_{\Omega_0} \nu z \, dx \quad \forall z \in H^1(\Omega_0), \quad (12)$$

which defines the variations $y_\varepsilon + \lambda\mu$, $u_\varepsilon + \lambda\nu$ for any $\nu \in L^2(\Omega_0)$ and $\lambda \in \mathbb{R}$.

A standard computation using (11), (12) and the optimality of $[u_\varepsilon, y_\varepsilon]$ gives

$$\begin{aligned} 0 &= \varepsilon(u_\varepsilon, \nu)_{L^2(\Omega_0)} + (y_\varepsilon - w, \mu)_{H^{1/2}(\Gamma)} \\ &= \varepsilon(u_\varepsilon, \nu)_{L^2(\Omega_0)} + \int_{\Omega_0} \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial \mu}{\partial x_i} \frac{\partial p_\varepsilon}{\partial x_j} + a_0 \mu p_\varepsilon \right] dx \\ &= \varepsilon(u_\varepsilon, \nu)_{L^2(\Omega_0)} + (p_\varepsilon, \nu)_{L^2(\Omega_0)}. \end{aligned} \quad (13)$$

Due to the convergence properties of the right-hand side in (11), $\{p_\varepsilon\}$ is bounded in $H^1(\Omega_0)$ and we can pass to the limit (on a subsequence) $p_\varepsilon \to p$ weakly in $H^1(\Omega_0)$, to obtain

$$\int_{\Omega_0} \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial z}{\partial x_i} \frac{\partial p}{\partial x_j} + a_0 z p \right] dx = \int_\Gamma l z \, d\sigma \quad \forall z \in H^1(\Omega_0). \quad (14)$$

The passage to the limit in (13), as $\{\varepsilon^{1/2}u_\varepsilon\}$ is bounded, gives that $p \equiv 0$ in Ω_0 and (14) shows that $l = 0$ in Γ.

We have proved (10) in the weak topology of $H^{1/2}(\Gamma)$. The strong convergence is a consequence of the Mazur theorem [Yos80] and of a variational argument.

Remark 1. The Mazur theorem alone and the linearity of (9) produces a sequence $\tilde{u}_\varepsilon$ (of convex combinations of u_ε) such that the corresponding sequence of states $\tilde{y}_\varepsilon$ satisfies (10). Theorem 1 gives a constructive answer to the approximate controllability property.

If Ω_0 is smooth enough and $w \in H^{3/2}(\Gamma)$, then the trace theorem ensures the existence of $\hat{y} \in H^2(\Omega_0)$ such that $\frac{\partial \hat{y}}{\partial n_A} = 0$ (null conormal derivative) and $\hat{y}|_\Gamma = w$. That is, the control

$$\hat{u} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial \hat{y}}{\partial x_i}\right) + a_0 \hat{y}$$

ensures the exact controllability property. Notice that $\hat{u}$ is not unique since any element in $H_0^2(\Omega_0)$ may be added to $\hat{y}$ with all the properties being preserved.

3 A Variational Fixed Domain Formulation

We assume that $\Omega = \Omega_g$, where $g \in C(\bar{D})$, is as in (5). Motivated by the result in the previous section, we consider the following homogeneous Neumann problem in D:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial \tilde{y}}{\partial x_i}\right) + a_0 \tilde{y} = f + (1 - H(g))u \quad \text{in } D, \tag{15}$$

$$\frac{\partial y}{\partial n_A} = 0 \qquad \text{on } \partial D. \tag{16}$$

Here $H(\cdot)$ is the Heaviside function in $\mathbb{R}$ and $H(g)$ is, consequently, the characteristic function of Ω_g. Under conditions of Theorem 1, the restriction $y = \tilde{y}|_{\Omega_g}$ is the solution of (2) in $\Omega = \Omega_g$. Moreover, since $g = 0$ on $\partial\Omega_g$, under smoothness conditions, ∇g is parallel to $\bar{n}$, the normal to $\partial\Omega_g$. Then, we can rewrite (4) as

$$\sum_{i,j=1}^{d} a_{ij}\frac{\partial y}{\partial x_j}\nabla g \cdot e_i = 0 \quad \text{on } \partial\Omega_g, \tag{17}$$

where we use that $\cos(\bar{n}, x_i) = \cos(\nabla g, x_i)$ and e_i is the vector of the axis x_i.

If the elliptic operator is the Laplace operator, then (17) becomes simply

$$\nabla g \cdot \nabla y = 0 \quad \text{on } \partial\Omega_g.$$

In order to fix a unique $u \in L^2(D)$ satisfying to (15), (16), (17), we define the following optimal control problem with state constraints:

$$\underset{u \in L^2(D)}{\text{Min}} \left\{ \frac{1}{2}\int_D u^2\, dx \right\}, \tag{18}$$

governed by the state system (15), (16) and subject to the state constraint (17).

The discussion in Section 2 shows the existence of infinitely many admissible pairs $[u, y]$ for the constrained control problem (15)–(18). (Here g is fixed satisfying the necessary smoothness properties.)

In case g and $\Omega_g \subset D$ are variable and unknown, we say that (15)–(18) is the variational fixed domain (in D!) formulation of the Neumann boundary value problem. One can write the optimality conditions that give a system of equations equivalent with (15)–(18) and extend the Neumann problem from Ω_g to D.

We introduce the penalized control problem, for $\varepsilon > 0$, as follows (here $[g \equiv 0]$ denotes $\partial\Omega_g$):

$$\underset{u \in L^2(D)}{\mathrm{Min}} \left\{ \frac{1}{2} \int_D u^2 \, dx + \frac{1}{2\varepsilon} \int_{[g\equiv 0]} F(y_\varepsilon)^2 \, d\sigma \right\} \tag{19}$$

subject to

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial y_\varepsilon}{\partial x_i} \right) + a_0 y_\varepsilon = f + (1 - H(g))u \quad \text{in } D, \tag{20}$$

$$\frac{\partial y_\varepsilon}{\partial n_A} = 0 \qquad \text{on } \partial D. \tag{21}$$

Above,

$$F(y) = \sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_j} \nabla g \cdot e_i$$

and the problem (19)–(21), which is unconstrained, remains a coercive and strictly convex control problem. That is, we have the existence and the uniqueness of the approximating optimal pair $[u_\varepsilon, y_\varepsilon] \in L^2(D) \times H^2(D)$ (if ∂D is smooth enough).

Proposition 1. *We have*

$$|F(y_\varepsilon)|_{L^2(\partial\Omega_g)} \le C\varepsilon^{\frac{1}{2}}, \tag{22}$$

$$u_\varepsilon \to \hat{u} \quad \textit{strongly in } L^2(D), \tag{23}$$

$$y_\varepsilon \to \hat{y} \quad \textit{strongly in } H^2(D), \tag{24}$$

where C is a constant independent of $\varepsilon > 0$ and $[\hat{u}, \hat{y}] \in L^2(D) \times H^2(D)$ is the unique optimal pair of (15)–(18).

Proof. As in Section 2, by the trace theorem, we may choose $\tilde{y} \in H^2(D \setminus \Omega_g)$ with the property that $\frac{\partial \tilde{y}}{\partial n_A} = 0$ in $\partial(D \setminus \Omega_g)$ and $\tilde{y}$ may be extended to the solution of (2) inside Ω_g. We can compute $\tilde{u} \in L^2(D \setminus \Omega_g)$ by (20) and extend it by 0 inside Ω_g. Then $[\tilde{u}, \tilde{y}]$ is an admissible pair for the control problem (19)–(21) and, by the optimality of $[u_\varepsilon, y_\varepsilon]$, we get

$$\frac{1}{2}\int_D u_\varepsilon^2\,dx + \frac{1}{2\varepsilon}\int_{[g\equiv 0]} F(y_\varepsilon)^2\,d\sigma \le \frac{1}{2}\int_D \tilde{u}^2\,dx \tag{25}$$

since $F(\tilde{y}) = 0$ in $\partial\Omega_g$.

The inequality (25) gives (22) and $\{u_\varepsilon\}$ bounded in $L^2(D)$. By (20), (21), $\{y_\varepsilon\}$ is bounded in $H^2(D)$ and, on a subsequence, we have $y_\varepsilon \to \hat{y}$, $u_\varepsilon \to \hat{u}$ weakly in $H^2(D)$, respectively in $L^2(D)$, where $[\hat{u}, \hat{y}]$ again satisfy (20), (21). Moreover, one can pass to the limit in (22) with $\varepsilon \to 0$, to see that $F(\hat{y}) = 0$ in $\partial\Omega_g$. This shows that $[\hat{u}, \hat{y}]$ is an admissible pair for the original state constrained control problem (15)–(18). For any admissible pair $[\mu, z] \in L^2(D) \times H^2(D)$ of (15)–(18), we have $F(z) = 0$ on $\partial\Omega_g$ and the inequality (25) is valid with $\tilde{u}$ replaced by μ and we infer

$$\frac{1}{2}\int_D u_\varepsilon^2\,dx \le \frac{1}{2}\int_D \mu^2\,dx.$$

The weak lower semicontinuity of the norm gives

$$\frac{1}{2}\int_D (\hat{u})^2\,dx \le \frac{1}{2}\int_D \mu^2\,dx,$$

that is, the pair $[\hat{u}, \hat{y}]$ is, in fact, the unique optimal pair of (15)–(18) and we also have

$$\lim_{\varepsilon\to 0}\int_D u_\varepsilon^2\,dx = \int_D (\hat{u})^2\,dx.$$

Then $u_\varepsilon \to \hat{u}$ strongly in $L^2(D)$ and $y_\varepsilon \to \hat{y}$ strongly in $H^2(D)$ by the strong convergence criterion in uniformly convex spaces. The convergence is valid without taking subsequences due to the uniqueness of $[\hat{u}, \hat{y}]$.

Remark 2. One can further regularize H in (20), by replacing it with a mollification H^ε of the Yosida approximation H_ε of the maximal monotone extension of H.

Remark 3. One may take in D even null Dirichlet boundary conditions instead of (16). Similar distributed controllability properties (approximate or exact) may be established in very much the same way.

To write shortly, we consider the case of the Laplace operator. The penalized and regularized problem is the following:

$$\underset{u\in L^2(D)}{\mathrm{Min}} \left\{ \frac{1}{2}\int_D u^2\,dx + \frac{1}{2\varepsilon}\int_{[g\equiv 0]} [\nabla y \cdot \nabla g]^2\,d\sigma \right\},$$

$$-\Delta y + y = f + (1 - H^\varepsilon(g))u \qquad \text{in } D,$$

$$y = 0 \qquad \text{on } \partial D.$$

Here, the control u ensures the “transfer” from Dirichlet to Neumann (null) conditions on $\partial\Omega_g$ and all the results are similar as for the Neumann–Neumann case.

Theorem 2. *The gradient of the cost functional (19) with respect to $u \in L^2(D)$ is given by*

$$\nabla J(u_\varepsilon) = u_\varepsilon + (1 - H(g))p_\varepsilon \quad \text{in } D, \tag{26}$$

where $p_\varepsilon \in L^2(D)$ is the unique solution of the adjoint equation

$$\int_D p_\varepsilon \left[-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial z}{\partial x_i}\right) + a_0 z \right] dx = \frac{1}{\varepsilon}\int_{[g\equiv 0]} F(y_\varepsilon)F(z)\, d\sigma$$
$$\forall z \in H^2(D),\ \frac{\partial z}{\partial n_A} = 0 \text{ on } \partial D, \tag{27}$$

in the sense of transpositions.

Proof. We discuss first the existence of the unique transposition solution to (27).

The equation in variations corresponding to (20), (21) is

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial z}{\partial x_i}\right) + a_0 z = (1 - H(g))v \quad \text{in } D, \tag{28}$$

$$\frac{\partial z}{\partial n_A} = 0 \qquad \text{on } \partial D, \tag{29}$$

for any $v \in L^2(D)$. By regularity theory for differential equations, the unique solution of (28), (29) satisfies $z \in H^2(D)$.

We perturb this equation by adding δv, $\delta > 0$, in the right-hand side and we denote by z_δ the corresponding solution, $z_\delta \in H^2(D)$. The mapping $v \to z_\delta$, as constructed above, is an isomorphism $T_\delta : L^2(D) \to W = \{z \in H^2(D) \mid \frac{\partial z}{\partial n_A} = 0 \text{ on } \partial D\}$.

We define the linear continuous functional on $L^2(D)$ by

$$v \longrightarrow \frac{1}{\varepsilon}\int_{[g\equiv 0]} F(y_\varepsilon)F(T_\delta v)\, d\sigma \quad \forall v \in L^2(D). \tag{30}$$

The Riesz representation theorem applied to (30) ensures the existence of a unique $\tilde{p}_\delta \in L^2(D)$ such that

$$\int_D \tilde{p}_\delta v = \frac{1}{\varepsilon}\int_{[g\equiv 0]} F(y_\varepsilon)F(T_\delta v)\, d\sigma \quad \forall v \in L^2(D). \tag{31}$$

Choosing $v = T_\delta^{-1} z$, $z \in W$ arbitrary, the relation (31) gives

$$\int_D \tilde{p}_\delta (1 - H(g) + \delta)^{-1} \left(-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial z}{\partial x_i}\right) + a_0 z \right) dx$$
$$= \frac{1}{\varepsilon}\int_{[g\equiv 0]} F(y_\varepsilon)F(z)\, d\sigma \quad \forall z \in W. \tag{32}$$

By redenoting $p_\varepsilon = \tilde{p}_\delta(1 - H(g) + \delta)^{-1} \in L^2(D)$ (which conceptually may depend on $\delta > 0$) in (32) we have proved the existence for (27). The uniqueness of p_ε may be shown by contradiction, directly in (27), as the factor multiplying p_ε in the left-hand side of (27) "generates" the whole $L^2(D)$ when $z \in W$ is arbitrary.

Coming back to the equation in variations (28), (29) and to the definition of the control problem (19)–(21), the directional derivative of the cost functional (19) is given by

$$\lim_{\lambda \to 0} \frac{1}{\lambda}[J(u_\varepsilon + \lambda v) - J(u_\varepsilon)] = \int_D u_\varepsilon v \, dx + \frac{1}{\varepsilon} \int_{[g \equiv 0]} F(y_\varepsilon) F(z) \, d\sigma \tag{33}$$

and the Euler equation is

$$0 = \int_D u_\varepsilon v \, dx + \frac{1}{\varepsilon} \int_{[g \equiv 0]} F(y_\varepsilon) F(z) \, d\sigma \quad \forall v \in L^2(D) \tag{34}$$

with z defined by (28), (29). By using (27) in (34), since z given by (28), (29) is an admissible test function, we get

$$\begin{aligned} 0 = \int_D u_\varepsilon v \, dx + \int_D p_\varepsilon \left[- \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial z}{\partial x_i} \right) + a_0 z \right] dx \\ = \int_D u_\varepsilon v \, dx + \int_D p_\varepsilon (1 - H(g)) v \, dx. \end{aligned} \tag{35}$$

This proves (26) and ends the argument.

Remark 4. Theorem 2 may be applied for any control $u \in L^2(D)$. For the optimal control u_ε, the directional derivative (and the gradient) is null and we obtain $u_\varepsilon = -p_\varepsilon(1 - H(g))$, that is, u_ε has support in $D \setminus \Omega_g$. This relation is the maximum (Pontryagin) principle applied to the control problem (19)–(21). Moreover, one can eliminate u_ε and write the following system of two elliptic equations:

$$- \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial y_\varepsilon}{\partial x_i} \right) + a_0 y_\varepsilon = f - (1 - H(g))^2 p_\varepsilon \quad \text{in } D, \tag{36}$$

$$\frac{\partial y_\varepsilon}{\partial n_A} = 0 \qquad \text{on } \partial D,$$

$$\int_D p_\varepsilon \left[- \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial z}{\partial x_i} \right) + a_0 z \right] = \frac{1}{\varepsilon} \int_{[g \equiv 0]} F(y_\varepsilon) F(z) \, d\sigma \quad \forall z \in W, \tag{37}$$

which constructs in an explicit manner the extension of the Neumann boundary value problem from Ω_g to D, modulo the approximation discussed in Proposition 1.

4 Conclusions

The shape optimization problem (1), (2) is transformed in this way into the optimal control problem

$$\operatorname*{Min}_{g \in C} \int_D H(g) j(x, y(x))\, dx \tag{38}$$

subject to (15)–(17) which, in turn, may be approximated by (19)–(21) or, equivalently, by (36)–(37). To obtain good differentiability properties with respect to g in the optimization problem (38), one should replace H by H^ε, some regularization of H, as previously mentioned. Analyzing further approximation properties and the gradient for (38) is a nontrivial task. However, the application of evolutionary algorithms is possible since it involves just the values of the cost (38) and no computation of the gradient with respect to g.

As initial population of controls g for the genetic algorithm, corresponding to the finite element mesh in D, one may use the basis functions for the piecewise linear and continuous finite element basis. In case some supplementary information is available on the desired shape (for instance, coming from the constraints), this should be imposed on the initial population. Then, standard procedures specific to evolutionary algorithms [Hol75] are to be applied.

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