# Discontinuous Galerkin Methods

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**Summary.** In this article, we describe some simple and commonly used discontinuous Galerkin methods for elliptic, Stokes and convection-diffusion problems. We illustrate these methods by numerical experiments.

## 1 Introduction and Preliminaries

Discontinuous Galerkin (DG) methods use discontinuous piece-wise polynomial spaces to approximate the solution of PDE's in variational form. The concept of discontinuous space approximations was introduced in the early 70's, probably starting with the work of Nitsche [Nit71] in 1971 on domain decomposition and followed by a number of important contributions such as the work of Babuška and Zlamal [BZ73], Crouzeix and Raviart [CR73], Rachford and Wheeler [RW74], Oden and Wellford [OW75], Douglas and Dupont [DD76], Baker [Bak77], Wheeler [Whe78], Arnold [Arn79, Arn82] and Wheeler and Darlow [WD80]. Afterward, interest in DG methods for elliptic problems declined probably because computing facilities at that time were not sufficient to solve efficiently such schemes. By the end of the 90's, the thesis of Baumann [Bau97] and the spectacular increase in computing power, triggered a renewal of interest in discontinuous Galerkin methods for elliptic and parabolic problems. The work of Baumann was followed by numerous publications such as Oden, Babuška and Baumann [OBB98], Baumann and Oden [BO99], Rivière et al. [RWG99, RWG01], Rivière [Riv00], Arnold et al. [ABCM02], among many others. Research on DG methods is now a very active field.

In the meantime, discontinuous methods were applied extensively to hyperbolic problems [Bey94, BOP96]. One of the first is the upwind scheme introduced by Reed and Hill in their report [RH73] on neutron transport in 1973. The first numerical analysis was done by Lesaint and Raviart [LR74] in 1974 for the transport equation and by Girault and Raviart [GR79] in 1982 for the Navier–Stokes equations. We refer to the books by Pironneau [Pir89] and by Girault and Raviart [GR86] for a thorough study of this upwind scheme.

DG methods have many advantages over continuous methods. The discontinuity of their functions allow the use of non-conforming grids and variable degree of polynomials on adjacent elements. They are locally mass conservative on each element. Their mass matrix in time-dependent problems is block diagonal. They are particularly well-adapted to problems with discontinuous coefficients and can effectively capture discontinuities in the solution. They can impose essential boundary conditions weakly without the use of a multiplier and thus can be applied to domain decomposition without involving multipliers. They can be applied to incompressible elasticity problems. They can be easily coupled with continuous methods.

On the negative side, they are expensive, because they require many degrees of freedom and for this reason, efficient solvers using DG methods for elliptic or parabolic problems are still the object of research.

In this article, we present a survey on some simple DG methods for elliptic, flow and transport problems. We concentrate essentially on IIPG, SIPG, NIPG, OBB-DG and the upwind DG of Lesaint and Raviart. There is no space to present all DG methods and for this reason, we have left out the more sophisticated schemes such as Local Discontinuous Galerkin (LDG) methods for which we refer to Arnold et al. [ABCM02].

This article is organized as follows. In Section 2, we derive the equations on which number of DG methods are based when applied to simple model problems. Section 3 is devoted to the approximation of a Darcy flow. In Section 4, we describe some DG methods for an incompressible Stokes flow. A convection-diffusion equation is approximated in Section 5. Section 6 is devoted to numerical experiments performed at the Institute for Computational Engineering and Sciences, UT Austin.

In the sequel, we shall use the following functional notation. Let $\Omega$ be a domain in $\mathbb{R}^d$, where $d$ is the dimension. For an integer $m \geq 1$, $H^m(\Omega)$ denotes the Sobolev space defined recursively by

$$H^m(\Omega) = \{v \in H^{m-1}(\Omega);\ \nabla v \in H^{m-1}(\Omega)^d\},$$

and we set

$$H^0(\Omega) = L^2(\Omega),$$

equipped with the norm

$$\|v\|_{L^2(\Omega)} = \left(\int_\Omega |v|^2\, d\mathbf{x}\right)^{\frac{1}{2}}.$$

For fluid pressure and other variables defined up to an additive constant, it is useful in theory to fix the constant by imposing the zero mean value and, therefore, we use the space

$$L^2_0(\Omega) = \left\{v \in L^2(\Omega);\ \int_\Omega v\, d\mathbf{x} = 0\right\}.$$

# 2 An Elementary Derivation of Some Simple DG Methods

In this section, we use very simple examples to derive the equations that are at the basis of IIPG, SIPG, NIPG, OBB-DG methods and the upwind DG method of Lesaint–Raviart. In each example, we work out the equations on a plane domain $\Omega$, with boundary $\partial\Omega$, partitioned into two non-overlapping subdomains $\Omega_1$ and $\Omega_2$ with interface $\Gamma_{12}$, and to fix ideas we assume that each subdomain has part of its boundary on $\partial\Omega$.

## 2.1 The General Idea for Elliptic Problems

Consider the Laplace equation with a homogeneous Dirichlet boundary condition in $\Omega$ and with data in $L^2(\Omega)$:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Let $v$ be a test function that is sufficiently smooth in each $\Omega_i$, but does not belong necessarily to $H^1(\Omega)$. If we multiply both sides of the first equation in (1) by $v$, apply Green's formula in each $\Omega_i$, and assume that the solution $u$ is smooth enough, we obtain:

$$\sum_{i=1}^{2}\left(\int_{\Omega_i} \nabla u \cdot \nabla v \, d\mathbf{x} - \int_{\partial\Omega_i} (\nabla u \cdot \mathbf{n}_i)|_{\Omega_i} \, v|_{\Omega_i} \, d\sigma\right) = \int_{\Omega} f v \, d\mathbf{x}, \tag{2}$$

where $\mathbf{n}_i$ denotes the unit normal to $\partial\Omega_i$, exterior to $\Omega_i$. If $u$ has sufficient smoothness, then the trace of $\nabla u \cdot \mathbf{n}_i$ on the interface has the same absolute value, but opposite signs, on $\Gamma_{12}$ when coming either from $\Omega_1$ or from $\Omega_2$. As the change in sign comes from the normal vector, we choose once and for all the normal's orientation on $\Gamma_{12}$; for example, we choose the orientation of $\mathbf{n}_1$. Therefore, setting $\mathbf{n}_e = \mathbf{n}_1$, denoting by $\mathbf{n}_\Omega$ the exterior normal to $\partial\Omega$, denoting by $[v]_e$ and $\{v\}_e$ the jump and average of the trace of $v$ across $\Gamma_{12}$:

$$[v]_e = v|_{\Omega_1} - v|_{\Omega_2}, \quad \{v\}_e = \frac{1}{2}\left(v|_{\Omega_1} + v|_{\Omega_2}\right),$$

and using the identity

$$\forall a_1, a_2, b_1, b_2 \in \mathbb{R}, \quad a_1 b_1 - a_2 b_2 = \frac{1}{2}\left[(a_1 + a_2)(b_1 - b_2) + (a_1 - a_2)(b_1 + b_2)\right],$$

(2) becomes

$$\sum_{i=1}^{2}\left(\int_{\Omega_i} \nabla u \cdot \nabla v \, d\mathbf{x} - \int_{\partial\Omega_i \setminus \Gamma_{12}} (\nabla u \cdot \mathbf{n}_\Omega) v \, d\sigma\right) - \int_{\Gamma_{12}} \{\nabla u \cdot \mathbf{n}_e\}_e [v]_e \, d\sigma = \int_{\Omega} f v \, d\mathbf{x}. \tag{3}$$

The discontinuous Galerkin method called IIPG is based on (3). It uses the regularity of the normal derivative of $u$. If, in addition, we want to use the regularity of $u$ and its zero boundary value, then we can add or subtract the following terms to the left-hand side of (3):

$$\int_{\Gamma_{12}} \{\nabla v \cdot \mathbf{n}_e\}_e [u]_e \, d\sigma, \quad \int_{\partial\Omega_i \setminus \Gamma_{12}} (\nabla v \cdot \mathbf{n}_\Omega) u \, d\sigma, \quad i = 1, 2.$$

Since these terms are zero, the resulting equation is equivalent to (3). The discontinuous Galerkin method called SIPG is based on subtraction of these terms:

$$\sum_{i=1}^{2} \Big( \int_{\Omega_i} \nabla u \cdot \nabla v \, d\mathbf{x} - \int_{\partial\Omega_i \setminus \Gamma_{12}} \big( (\nabla u \cdot \mathbf{n}_\Omega) v + (\nabla v \cdot \mathbf{n}_\Omega) u \big) \, d\sigma \Big) \\ - \int_{\Gamma_{12}} \big( \{\nabla u \cdot \mathbf{n}_e\}_e [v]_e + \{\nabla v \cdot \mathbf{n}_e\}_e [u]_e \big) \, d\sigma = \int_\Omega f v \, d\mathbf{x}, \quad (4)$$

and the discontinuous Galerkin methods called NIPG and OBB-DG are based on addition of this term:

$$\sum_{i=1}^{2} \Big( \int_{\Omega_i} \nabla u \cdot \nabla v \, d\mathbf{x} - \int_{\partial\Omega_i \setminus \Gamma_{12}} \big( (\nabla u \cdot \mathbf{n}_\Omega) v - (\nabla v \cdot \mathbf{n}_\Omega) u \big) \, d\sigma \Big) \\ - \int_{\Gamma_{12}} \big( \{\nabla u \cdot \mathbf{n}_e\}_e [v]_e - \{\nabla v \cdot \mathbf{n}_e\}_e [u]_e \big) \, d\sigma = \int_\Omega f v \, d\mathbf{x}. \quad (5)$$

In fact, the OBB-DG formulation is precisely (5).

Clearly, the contribution of the surface integrals to the left-hand side of (5) is anti-symmetric and hence the left-hand side of (5) is non-negative when $v = u$. The left-hand side of (4) is symmetric, but there is no reason why it should be non-negative and the left-hand side of (3) has no symmetry and no positivity. The left-hand side of (5) can be made positive when $v = u$ by adding to it the jump terms

$$\frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} [u]_e [v]_e \, d\sigma + \sum_{i=1}^{2} \frac{1}{|\partial\Omega_i \setminus \Gamma_{12}|} \int_{\partial\Omega_i \setminus \Gamma_{12}} u v \, d\sigma,$$

where for any set $S$, $|S|$ denotes the measure of $S$. But, of course, this will not do for (3) and (4). However, considering that all these formulations will be applied to functions in finite-dimensional spaces, we expect to make (3) and (4) positive by incorporating into the jump terms adequate parameters. Thus we add

$$J_0(u, v) = \frac{\sigma_{12}}{|\Gamma_{12}|} \int_{\Gamma_{12}} [u]_e [v]_e \, d\sigma + \sum_{i=1}^{2} \frac{\sigma_i}{|\partial\Omega_i \setminus \Gamma_{12}|} \int_{\partial\Omega_i \setminus \Gamma_{12}} u v \, d\sigma, \quad (6)$$

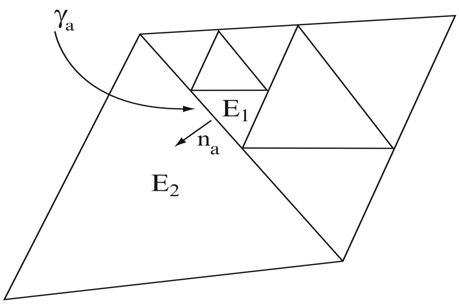


**Fig. 1.** Jumps and averages: the jump on an interior edge is given by $[v] = v|_{E_1} - v|_{E_2}$ and on a boundary edge by $[v] = v|_{E_1}$; the averages are respectively given by $v = \frac{1}{2}(v|_{E_1} + v|_{E_2})$ and $v = v|_{E_1}$. The unit normal to $\gamma_a$ is $\mathbf{n}_a$

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where $\sigma_{12}$ and $\sigma_i$ are suitable non-negative parameters. Summing up, the IIPG, SIPG, NIPG and OBB-DG formulations read:

$$\sum_{i=1}^{2}\left(\int_{\Omega_i}\nabla u\cdot\nabla v\,d\mathbf{x}-\int_{\partial\Omega_i\setminus\Gamma_{12}}\big((\nabla u\cdot\mathbf{n}_\Omega)v+\varepsilon(\nabla v\cdot\mathbf{n}_\Omega)u\big)\,d\sigma\right)$$
$$-\int_{\Gamma_{12}}\big(\{\nabla u\cdot\mathbf{n}_e\}_e[v]_e+\varepsilon\{\nabla v\cdot\mathbf{n}_e\}_e[u]_e\big)\,d\sigma+J_0(u,v)=\int_\Omega fv\,d\mathbf{x},\quad (7)$$

with $\varepsilon = 0$ for IIPG, $\varepsilon = 1$ for SIPG and $\varepsilon = -1$ for NIPG and OBB-DG, $\sigma_i = \sigma_{12} = 1$ for NIPG, $\sigma_i = \sigma_{12} = 0$ for OBB-DG and $\sigma_i$ and $\sigma_{12}$ are well chosen positive parameters for IIPG and SIPG. An example of jumps and average for a non-conforming mesh are shown in Figure 1.

*Remark 1.* The NIPG and OBB-DG formulations differ only on the presence or absence of jump terms. It turns out that in several cases, such as in Section 3, the jump terms are not necessary, but they can be added to enhance convergence. However, there are cases, such as in Section 4, where OBB-DG seems sub-optimal without jumps.

*Remark 2.* As the normal derivative of the solution has no jumps, it is also possible to add jumps involving this normal derivative (cf. [Dar80, WD80]):

$$|\Gamma_{12}|\int_{\Gamma_{12}}[\nabla u\cdot\mathbf{n}]_e[\nabla v\cdot\mathbf{n}]_e\,d\sigma.$$

The resulting equation is still equivalent to (3).

Finally, let us examine a Laplace equation with mixed non-homogeneous Dirichlet–Neumann boundary conditions. As an example, we replace (1) by

$$-\Delta u = f \ \text{ in } \Omega, \quad u = g_1 \ \text{ on } \partial\Omega_1\setminus\Gamma_{12}, \quad \nabla u\cdot\mathbf{n}_\Omega = g_2 \ \text{ on } \partial\Omega_2\setminus\Gamma_{12}. \quad (8)$$

In this case, we suppress from $J_0$ the boundary term on $\partial\Omega_2 \setminus \Gamma_{12}$:

$$J_0(u,v) = \frac{\sigma_{12}}{|\Gamma_{12}|}\int_{\Gamma_{12}} [u]_e[v]_e \, d\sigma + \frac{\sigma_1}{|\partial\Omega_1 \setminus \Gamma_{12}|}\int_{\partial\Omega_1\setminus\Gamma_{12}} uv \, d\sigma, \tag{9}$$

and the IIPG, SIPG, NIPG and OBB-DG formulations become:

$$\begin{aligned}\sum_{i=1}^{2}\int_{\Omega_i} \nabla u\cdot\nabla v \, d\mathbf{x} - \int_{\partial\Omega_1\setminus\Gamma_{12}} \big((\nabla u\cdot\mathbf{n}_\Omega)v + \varepsilon(\nabla v\cdot\mathbf{n}_\Omega)u\big)\, d\sigma \\ - \int_{\Gamma_{12}} \big(\{\nabla u\cdot\mathbf{n}_e\}_e[v]_e + \varepsilon\{\nabla v\cdot\mathbf{n}_e\}_e[u]_e\big)\, d\sigma + J_0(u,v) \\ = \int_\Omega f v\, d\mathbf{x} + \int_{\partial\Omega_2\setminus\Gamma_{12}} g_2 v\, d\sigma - \varepsilon\int_{\partial\Omega_1\setminus\Gamma_{12}} g_1(\nabla v\cdot\mathbf{n}_\Omega)\, d\sigma \\ + \frac{\sigma_1}{|\partial\Omega_1 \setminus \Gamma_{12}|}\int_{\partial\Omega_1\setminus\Gamma_{12}} g_1 v\, d\sigma,\end{aligned} \tag{10}$$

with the same values of $\varepsilon$, $\sigma_1$ and $\sigma_{12}$ as in (7).

### 2.2 The General Idea for the Stokes Problem

Consider the incompressible Stokes problem in $\Omega$ with data $\mathbf{f}$ in $L^2(\Omega)^2$:

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega, \tag{11}$$

where the viscosity parameter $\mu$ is a given positive constant. This is a typical problem with a linear constraint (the zero divergence) and a Lagrange multiplier (the pressure $p$).

For treating the pressure term and divergence constraint, we take again a test function $\mathbf{v}$ that is not necessarily globally smooth, but has smooth components in each $\Omega_i$, and assuming the pressure $p$ is sufficiently smooth, we apply Green's formula in each $\Omega_i$:

$$\begin{aligned}\int_\Omega (\nabla p)\cdot\mathbf{v}\, d\mathbf{x} = \sum_{i=1}^{2}\left(-\int_{\Omega_i} p \operatorname{div}\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_i\setminus\Gamma_{12}} p(\mathbf{v}\cdot\mathbf{n}_\Omega)\, d\sigma\right) \\ + \int_{\Gamma_{12}} \{p\}_e[\mathbf{v}]_e\cdot\mathbf{n}_e\, d\sigma.\end{aligned} \tag{12}$$

We apply the same formula to the divergence constraint. Thus combining (12) with (7), we have the following IIPG, SIPG, NIPG and OBB-DG formulations for the Stokes problem (11):

$$\sum_{i=1}^{2}\mu\left(\int_{\Omega_i}\nabla\mathbf{u}:\nabla\mathbf{v}\,d\mathbf{x}-\int_{\partial\Omega_i\setminus\Gamma_{12}}\big((\nabla\mathbf{u}\cdot\mathbf{n}_\Omega)\mathbf{v}+\varepsilon(\nabla\mathbf{v}\cdot\mathbf{n}_\Omega)\mathbf{u}\big)\,d\sigma\right)$$
$$-\int_{\Gamma_{12}}\mu\big(\{\nabla\mathbf{u}\cdot\mathbf{n}_e\}_e[\mathbf{v}]_e+\varepsilon\{\nabla\mathbf{v}\cdot\mathbf{n}_e\}_e[\mathbf{u}]_e\big)\,d\sigma+\mu J_0(\mathbf{u},\mathbf{v})$$
$$+\sum_{i=1}^{2}\left(-\int_{\Omega_i}p\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x}+\int_{\partial\Omega_i\setminus\Gamma_{12}}p(\mathbf{v}\cdot\mathbf{n}_\Omega)\,d\sigma\right)+\int_{\Gamma_{12}}\{p\}_e[\mathbf{v}]_e\cdot\mathbf{n}_e\,d\sigma$$
$$=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}, \tag{13}$$

$$\sum_{i=1}^{2}\left(\int_{\Omega_i}q\,\mathrm{div}\,\mathbf{u}\,d\mathbf{x}-\int_{\partial\Omega_i\setminus\Gamma_{12}}q(\mathbf{u}\cdot\mathbf{n}_\Omega)\,d\sigma\right)-\int_{\Gamma_{12}}\{q\}_e[\mathbf{u}]_e\cdot\mathbf{n}_e\,d\sigma=0, \tag{14}$$

with the interpretation for the parameters $\varepsilon$ and $\sigma$ of the formula (7).

### 2.3 Upwinding in a Transport Problem: General Idea

Consider the simple transport problem in $\Omega$:

$$c+\mathbf{u}\cdot\nabla c=f\quad\text{in }\Omega, \tag{15}$$

where $f$ belongs to $L^2(\Omega)$ and $\mathbf{u}$ is a sufficiently smooth vector-valued function that satisfies

$$\mathrm{div}\,\mathbf{u}=0\ \text{ in }\Omega,\quad \mathbf{u}\cdot\mathbf{n}_\Omega=0\ \text{ on }\partial\Omega. \tag{16}$$

Recall the notation

$$\mathbf{u}\cdot\nabla c=\sum_{i=1}^{2}u_i\frac{\partial c}{\partial x_i},$$

and note that when the functions involved are sufficiently smooth, Green's formula and (16) yield

$$\int_\Omega(\mathbf{u}\cdot\nabla c)c\,d\mathbf{x}=0. \tag{17}$$

For the applications we have in mind, let us assume that $c$ is sufficiently smooth in each $\Omega_i$, but is not necessarily in $H^1(\Omega)$. Then, we must give a meaning to the product $\mathbf{u}\cdot\nabla c$. From the following identity and the fact that the divergence of $\mathbf{u}$ is zero:

$$\mathrm{div}(c\mathbf{u})=c(\mathrm{div}\,\mathbf{u})+\mathbf{u}\cdot\nabla c=\mathbf{u}\cdot\nabla c,$$

and we derive for any smooth function $\varphi$ with compact support in $\Omega$

$$\langle\mathbf{u}\cdot\nabla c,\varphi\rangle=\langle\mathrm{div}(c\mathbf{u}),\varphi\rangle=-\langle c\mathbf{u},\nabla\varphi\rangle=-\int_\Omega(c\mathbf{u})\cdot\nabla\varphi\,d\mathbf{x}$$
$$=-\sum_{i=1}^{2}\int_{\Omega_i}(c\mathbf{u})\cdot\nabla\varphi\,d\mathbf{x}. \tag{18}$$

We use the last equality to define $\mathbf{u}\cdot\nabla c$ in the sense of distributions.

Now, we wish to extend this definition to functions $\mathbf{u}$ and $\varphi$ that are not necessarily smooth. Then, we take again a test function $v$ that is sufficiently smooth in each $\Omega_i$, but may not be in $H^1(\Omega)$. Applying Green's formula to the last equality in (18) in each $\Omega_i$ and using the fact that $\mathbf{u}$ has zero divergence, we define:

$$\int_\Omega (\mathbf{u}\cdot\nabla c)v\,d\mathbf{x} := \sum_{i=1}^{2}\left(\int_{\Omega_i}(\mathbf{u}\cdot\nabla c)v\,d\mathbf{x} - \int_{\partial\Omega_i} c(\mathbf{u}\cdot\mathbf{n})v\,d\sigma\right). \tag{19}$$

In order to introduce an upwinding into this formula, we consider each $\Omega_i$ and the portion of its boundary where the flow driven by $\mathbf{u}$ enters $\Omega_i$, i.e., where $\{\mathbf{u}\}\cdot\mathbf{n}_i < 0$. We set

$$(\partial\Omega_i)_- = \{\mathbf{x}\in\partial\Omega_i;\ \{\mathbf{u}\}\cdot\mathbf{n}_i(\mathbf{x}) < 0\}. \tag{20}$$

Then we replace (19) by

$$\int_\Omega (\mathbf{u}\cdot\nabla c)v\,d\mathbf{x} := \sum_{i=1}^{2}\left(\int_{\Omega_i}(\mathbf{u}\cdot\nabla c)v\,d\mathbf{x} - \int_{(\partial\Omega_i)_-}\{\mathbf{u}\}\cdot\mathbf{n}_i(c^{\text{int}}-c^{\text{ext}})v^{\text{int}}\,d\sigma\right), \tag{21}$$

where the superscript int (resp. ext) refers to the interior (resp. exterior) trace of the function in $\Omega_i$, and on the part of $(\partial\Omega_i)_-$ that lies on $\partial\Omega$, $c^{\text{ext}} = 0$ and $\{\mathbf{u}\} = \mathbf{u}$. This is a straightforward extension of the Lesaint–Raviart upwind scheme.

Finally, we wish to extend (21) to the case where $\mathbf{u}$ satisfies (14) instead of (16), while preserving some property analogous to (17). Keeping in mind the identity:

$$\int_\Omega (\mathbf{u}\cdot\nabla c)c\,d\mathbf{x} + \frac{1}{2}\int_\Omega (\operatorname{div}\mathbf{u})c^2\,d\mathbf{x} - \frac{1}{2}\int_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})c^2\,d\sigma = 0, \tag{22}$$

that holds if $c$ and $\mathbf{u}$ are sufficiently smooth, we replace (21) by:

$$\begin{aligned}\int_\Omega (\mathbf{u}\cdot\nabla c)v\,d\mathbf{x} := \sum_{i=1}^{2}\Bigg(&\int_{\Omega_i}\left(\mathbf{u}\cdot\nabla c + \frac{1}{2}(\operatorname{div}\mathbf{u})c\right)v\,d\mathbf{x}\\ &-\frac{1}{2}\int_{\partial\Omega_i\setminus\Gamma_{12}}(\mathbf{u}\cdot\mathbf{n}_\Omega)cv\,d\sigma - \int_{(\partial\Omega_i)_-}\{\mathbf{u}\}\cdot\mathbf{n}_i(c^{\text{int}}-c^{\text{ext}})v^{\text{int}}\,d\sigma\Bigg)\\ &-\frac{1}{2}\int_{\Gamma_{12}}[\mathbf{u}]_e\cdot\mathbf{n}_e\{cv\}_e\,d\sigma.\end{aligned} \tag{23}$$

This is the upwind formulation proposed and analyzed by Rivière et al. [GRW05].

# 3 DG Approximation of an Elliptic Problem

Let $\Omega$ be a polygon in dimension $d = 2$ or a Lipschitz polyhedron in dimension $d = 3$, with boundary $\partial\Omega$ partitioned into two disjoint parts: $\partial\Omega = \Gamma_D \cup \Gamma_N$, with polygonal boundaries if $d = 3$. For simplicity, we assume that $|\Gamma_D|$ is positive. Consider the continuity equation for Darcy flow in pressure form in $\Omega$:

$$-\operatorname{div}(\mathbf{K}\nabla p) = f, \quad \text{in } \Omega, \tag{24}$$

$$p = g_1, \quad \text{on } \Gamma_D, \tag{25}$$

$$\mathbf{K}\nabla p \cdot \mathbf{n}_\Omega = g_2, \quad \text{on } \Gamma_N, \tag{26}$$

where $\mathbf{n}_\Omega$ is the unit normal vector to $\partial\Omega$, exterior to $\Omega$, and the permeability $\mathbf{K}$ is a uniformly bounded, positive definite symmetric tensor, that is allowed to vary in space. For $f \in L^2(\Omega)$, $g_1 \in H^{1/2}(\Gamma_D)$ and $g_2 \in L^2(\Gamma_N)$, system (24)–(26) has a unique solution $p \in H^1(\Omega)$ and we assume that $p$ is sufficiently regular to guarantee the consistency of the schemes below.

Let $\mathcal{E}_h$ be a regular family of triangulations of $\overline{\Omega}$ consisting of triangles (or tetrahedra if $d = 3$) $E$ of maximum diameter $h$, and such that no face or side of $\partial E$ intersects both $\Gamma_D$ and $\Gamma_N$. It is regular in the sense of Ciarlet [Cia91]: There exists a constant $\gamma > 0$, independent of $h$, such that

$$\forall E \in \mathcal{E}_h, \quad \frac{h_E}{\varrho_E} = \gamma_E \leq \gamma, \tag{27}$$

where $h_E$ denotes the diameter of $E$ (bounded above by $h$) and $\varrho_E$ denotes the diameter of the ball inscribed in $E$.

To simplify the discussion, we assume that $\mathcal{E}_h$ is conforming, but most results in this section remain valid for non-conforming grids as well as for quadrilateral (or hexahedral if $d = 3$) grids. We denote by $\Gamma_h$ the set of all interior edges (or faces if $d = 3$) of $\mathcal{E}_h$ and by $\Gamma_{h,D}$ (resp. $\Gamma_{h,N}$) the set of all edges or faces of $\mathcal{E}_h$ that lie on $\Gamma_D$ (resp. $\Gamma_N$). The elements $E$ of $\mathcal{E}_h$ are numbered and denoted by $E_i$, say for $1 \leq i \leq P_h$. With any edge or face $e$ of $\Gamma_h$ shared by $E_i$ and $E_j$ with $i < j$, we associate once and for all the unit normal vector $\mathbf{n}_e$ directed from $E_i$ to $E_j$ and we define the jump $[\varphi]_e$ and average $\{\varphi\}_e$ of a function $\varphi$ by:

$$[\varphi]_e = \varphi|_{E_i} - \varphi|_{E_j}, \quad \{\varphi\}_e = \frac{1}{2}(\varphi|_{E_i} + \varphi|_{E_j}).$$

If $e \subset \partial\Omega$, then $\mathbf{n}_e = \mathbf{n}_\Omega$ and the jump and average of $\varphi$ coincide with the trace of $\varphi$.

Considering the differential operator in (24), we define the "discontinuous" space:

$$H^1(\mathcal{E}_h) = \{v \in L^2(\Omega); \forall E \in \mathcal{E}_h, v|_E \in H^1(E)\},$$

equipped with the "broken" semi-norm

$$\|\mathbf{K}^{\frac{1}{2}}\nabla v\|_{L^2(\mathcal{E}_h)} = \left[\sum_{E\in\mathcal{E}_h} \|\mathbf{K}^{\frac{1}{2}}\nabla v\|^2_{L^2(E)}\right]^{\frac{1}{2}}, \tag{28}$$

and norm (for which it is a Hilbert space)

$$\|v\|_{H^1(\mathcal{E}_h)} = \left(\|v\|^2_{L^2(\Omega)} + \|\mathbf{K}^{\frac{1}{2}}\nabla v\|^2_{L^2(\mathcal{E}_h)}\right)^{\frac{1}{2}}.$$

In view of (9), we define the jump bilinear form

$$J_0(u,v) = \sum_{e\in\Gamma_h\cup\Gamma_{h,D}} \frac{\sigma_e}{h_e}\int_e [u]_e[v]_e\,d\sigma, \tag{29}$$

where $h_e$ denotes the diameter of $e$, and each $\sigma_e$ is a suitable non-negative parameter. It is convenient to define also the mesh-dependent semi-norm

$$|v|_{H^1(\mathcal{E}_h)} = \left(\|\mathbf{K}^{\frac{1}{2}}\nabla v\|^2_{L^2(\mathcal{E}_h)} + J_0(v,v)\right)^{\frac{1}{2}}. \tag{30}$$

Now, we choose an integer $k \geq 1$ and we discretize $H^1(\mathcal{E}_h)$ with the finite element space

$$X_h = \{v \in L^2(\Omega) : \forall E \in \mathcal{E}_h,\ v|_E \in \mathbb{P}_k(E)\}. \tag{31}$$

It is possible to let $k$ vary from one element to the next, but for simplicity we keep the same $k$. Then, keeping in mind (10), we discretize (24)–(26) by the following discrete system: Find $p_h \in X_h$ such that for all $q_h \in X_h$,

$$\begin{aligned}
&\sum_{E\in\mathcal{E}_h}\int_E \mathbf{K}\nabla p_h\cdot\nabla q_h\,d\mathbf{x}\\
&-\sum_{e\in\Gamma_h\cup\Gamma_{h,D}}\int_e \big(\{\mathbf{K}\nabla p_h\cdot\mathbf{n}_e\}_e[q_h]_e + \varepsilon\{\mathbf{K}\nabla q_h\cdot\mathbf{n}_e\}_e[p_h]_e\big)\,d\sigma + J_0(p_h,q_h)\\
&= \int_\Omega f q_h\,d\mathbf{x} + \int_{\Gamma_N} g_2 q_h\,d\sigma - \varepsilon\sum_{e\in\Gamma_{h,D}}\int_e g_1(\mathbf{K}\nabla q_h\cdot\mathbf{n}_\Omega)\,d\sigma\\
&\qquad + \sum_{e\in\Gamma_{h,D}}\frac{\sigma_e}{h_e}\int_e g_1 q_h\,d\sigma,
\end{aligned} \tag{32}$$

with $\varepsilon = 1$ for SIPG, $\varepsilon = 0$ for IIPG and $\varepsilon = -1$ for NIPG and OBB-DG; and for each $e$, $\sigma_e = 1$ for NIPG, $\sigma_e = 0$ for OBB-DG and again $\sigma_e$ is a well chosen positive parameter for IIPG and SIPG.

*Remark 3.* Let $E$ be an element of $\mathcal{E}_h$ with no edge (or face) $e$ on $\partial\Omega$. Taking $q_h = \chi_E$, the characteristic function of $E$ in (32), we easily derive the discrete mass balance relation where $\mathbf{n}_E$ denotes the unit normal exterior to $E$:

$$-\sum_{e\in\partial E}\int_e \{\mathbf{K}\nabla p_h\}\cdot\mathbf{n}_E\,d\sigma + \sum_{e\in\partial E}\frac{\sigma_e}{h_e}\int_e (p_h^{\text{int}} - p_h^{\text{ext}})\,d\sigma = \int_E f\,d\mathbf{x}.$$

### 3.1 Numerical Analysis

To simplify the discussion, we introduce the bilinear form defined for any pair of functions $p$ and $q$ in $X_h + H^s(\Omega)$ with $s > \frac{3}{2}$ (so that the integrals over $e$ are well-defined):

$$a_h(p,q) = \sum_{E\in\mathcal{E}_h} \int_E \mathbf{K}\nabla p \cdot \nabla q \, d\mathbf{x} \\ - \sum_{e\in\Gamma_h\cup\Gamma_{h,D}} \int_e \big(\{\mathbf{K}\nabla p \cdot \mathbf{n}_e\}_e [q]_e + \varepsilon \{\mathbf{K}\nabla q \cdot \mathbf{n}_e\}_e [p]_e\big)\, d\sigma. \quad (33)$$

Clearly, for NIPG,

$$a_h(q_h, q_h) + J_0(q_h, q_h) = \|q_h\|^2_{H^1(\mathcal{E}_h)}, \quad (34)$$

and, therefore, (32) has a unique solution. For IIPG and SIPG [Whe78, DSW04], an argument on finite-dimensional spaces (cf. [GSWY]) shows that for each $e$ there exists a constant $c_e$, independent of $h$, but depending on $k$, the regularity constant $\gamma$ of (27) and the maximum and minimum eigenvalues of $\mathbf{K}$ on the elements adjacent to $e$, such that for all $p_h$ and $q_h$ in $X_h$

$$\left| \sum_{e\in\Gamma_h\cup\Gamma_{h,D}} \int_e \{\mathbf{K}\nabla p_h \cdot \mathbf{n}_e\}_e [q_h]_e \, d\sigma \right| \\ \leq \|\mathbf{K}^{\frac{1}{2}} \nabla p_h\|_{L^2(\mathcal{E}_h)} \left( \sum_{e\in\Gamma_h\cup\Gamma_{h,D}} \frac{c_e}{h_e} \|[q_h]\|^2_{L^2(e)} \right)^{\frac{1}{2}}. \quad (35)$$

The assumptions on $\mathbf{K}$ imply that the constants $c_e$ can be bounded above independently of $h$ and $e$ and, therefore, applying Young's inequality, we can choose constants $\sigma_e$, uniformly bounded above and below with respect to $h$:

$$\forall e \in \Gamma_h \cup \Gamma_{h,D}, \quad 1 \leq \sigma_0 \leq \sigma_e \leq \sigma_m, \quad (36)$$

such that (for instance)

$$\left| \sum_{e\in\Gamma_h\cup\Gamma_{h,D}} \int_e \{\mathbf{K}\nabla q_h \cdot \mathbf{n}_e\}_e [q_h]_e \, d\sigma \right| \leq \frac{1}{4} \|q_h\|^2_{H^1(\mathcal{E}_h)}. \quad (37)$$

With this choice of penalty parameters $\sigma_e$, the system (32) for IIPG and SIPG has a unique solution. Furthermore, there exist two positive constants $\alpha$ and $M$, independent of $h$ such that for all $p_h$ and $q_h$ in $X_h$

$$|a_h(p_h, q_h)| + |J_0(p_h, q_h)| \leq M \|p_h\|_{H^1(\mathcal{E}_h)} \|q_h\|_{H^1(\mathcal{E}_h)}, \\ a_h(q_h, q_h) + J_0(q_h, q_h) \geq \alpha \|q_h\|^2_{H^1(\mathcal{E}_h)}. \quad (38)$$

This analysis cannot be applied to establish the solvability of OBB-DG, because the term $J_0$ is missing. If $k \geq 2$, one can show directly for OBB-DG that (32) has a unique solution cf. [RWG01], but the second part of (38) does not hold. When $k = 1$, there is a counter-example that shows that (32) is not well-posed (cf. [OBB98]). For this reason, OBB-DG is only applied when $k \geq 2$.

With the above choice of penalty parameters $\sigma_e$, a standard error analysis allows to prove optimal a priori error estimates in the norm $\|\cdot\|_{H^1(\mathcal{E}_h)}$ for IIPG, SIPG and NIPG: if the exact solution $p$ of (24)–(26) belongs to $H^{k+1}(\Omega)$, then for the three methods

$$\|p_h - p\|_{H^1(\mathcal{E}_h)} = \mathrm{O}(h^k).$$

The same result holds for OBB-DG, but the proof is more subtle. The difficulty lies in estimating the term

$$T = \sum_{e \in \Gamma_h \cup \Gamma_{h,D}} \int_e \{\mathbf{K}\nabla(p - R_h p) \cdot \mathbf{n}_e\}_e [q_h]_e \, d\sigma,$$

where $R_h$ is an interpolation operator in $X_h$ and $q_h \in X_h$ is an arbitrary test function. If we had jumps, we would write as in the cases of IIPG, SIPG and NIPG:

$$|T| \leq \sum_{e \in \Gamma_h \cup \Gamma_{h,D}} \left(\frac{h_e}{\sigma_e}\right)^{\frac{1}{2}} \|\{\mathbf{K}\nabla(p - R_h p) \cdot \mathbf{n}_e\}_e\|_{L^2(e)} \left(\frac{\sigma_e}{h_e}\right)^{\frac{1}{2}} \|[q_h]_e\|_{L^2(e)}.$$

With a standard interpolation operator, owing to the factor $h_e^{\frac{1}{2}}$, the term

$$\left(\frac{h_e}{\sigma_e}\right)^{\frac{1}{2}} \|\{\mathbf{K}\nabla(p - R_h p) \cdot \mathbf{n}_e\}_e\|_{L^2(e)} = \mathrm{O}(h^k).$$

Here we have no jumps and the only way in which we can recover the factor $h_e^{\frac{1}{2}}$ is by constructing an interpolation operator $R_h$ such that

$$\int_e \{\mathbf{K}\nabla(p - R_h p) \cdot \mathbf{n}_e\}_e \, d\sigma = 0.$$

If this is the case, then we can write

$$T = \sum_{e \in \Gamma_h \cup \Gamma_{h,D}} \int_e \{\mathbf{K}\nabla(p - R_h p) \cdot \mathbf{n}_e\}_e ([q_h]_e - c_e) \, d\sigma,$$

where the number $c_e$ is chosen so that

$$\|[q_h]_e - c_e\|_{L^2(e)} \leq C\big(h_{E_i}^{\frac{1}{2}} \|\nabla q_h\|_{L^2(E_i)} + h_{E_j}^{\frac{1}{2}} \|\nabla q_h\|_{L^2(E_j)}\big),$$

and $E_i$ and $E_j$ are the elements adjacent to $e$. This interpolation operator is constructed in [RWG01], for $k \geq 2$. When $k = 1$, there are not enough degrees of freedom for its construction.

When the solution of (24)–(26) belongs to $H^2(\Omega)$ for all sufficiently smooth data (this holds, for example, when $\mathbf{K}$ and $g_1$ are sufficiently smooth and $\Gamma_D$ is the whole boundary), then a duality argument shows that the error for SIPG in the $L^2$ norm has a higher order:

$$\|p_h - p\|_{L^2(\Omega)} = \mathrm{O}(h^{k+1}). \tag{39}$$

More generally, if there exists $s \in \left]\frac{3}{2}, 1\right]$ such that the solution of (24)–(26) belongs to $H^{1+s}(\Omega)$ for all correspondingly smooth data then (cf. [RWG01])

$$\|p_h - p\|_{L^2(\Omega)} = \mathrm{O}(h^{k+s}).$$

This result follows from the symmetry of $a_h$. For the other methods, which are not symmetric, the same duality argument (cf. [RWG01]) does not yield any increase in order, namely all we have is

$$\|p_h - p\|_{L^2(\Omega)} = \mathrm{O}(h^{k}). \tag{40}$$

Nevertheless, numerical results for NIPG and OBB-DG tend to prove that (39) holds if $k$ is an odd integer, but so far we have no proof of this result.

*Remark 4.* The choice of penalty parameters for IIPG and SIPG is not straightforward. If chosen too small, the stability properties in (38) may be lost. But if chosen too large, the matrix of system (32) may become ill-conditioned.

*Remark 5.* One cannot prove basic inequalities on the functions of $X_h$, such as Poincaré's Inequality, without adding jumps to the broken norm; i.e., the gradients in each element are not sufficient to control the $L^2$ norm. With jumps, one can prove Poincaré–Friedrich's inequalities, Sobolev inequalities, Korn's inequalities and trace inequalities. For Poincaré–Friedrich's inequalities and Korn's inequalities, we refer to the very good contributions of Brenner [Bre03, Bre04]. The Sobolev and trace inequalities can be derived by using similar arguments (cf. [GRW05]). Note that, by virtue of Poincaré's Inequality, (40) can be established directly for IIPG, SIPG and NIPG without having to assume that the solution of (24)–(26) has extra smoothness for all smooth data.

## 4 DG Approximation of an Incompressible Stokes Problem

Let us revert to the problem (11) on a connected polygonal or polyhedral domain:

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega.$$

For a given force $\mathbf{f} \in L^2(\Omega)^d$, this problem has a unique solution $\mathbf{u} \in H_0^1(\Omega)^d$ and $p \in L_0^2(\Omega)$ (cf., for instance, [Tem79, GR86]). In fact, the solution is more regular and the scheme below is consistent (cf. [Gri85, Dau89]).

In view of the operator and boundary condition in (11), the relevant spaces here are $H^1(\mathcal{E}_h)^d$ and $L_0^2(\Omega)$, and the set $\Gamma_{h,N}$ is empty. The definition of $J_0$ is extended straightforwardly to vectors and the permeability tensor is replaced by the identity multiplied by the viscosity. Thus, the semi-norms (28) and (30) are replaced by

$$\|\!|\nabla \mathbf{v}\|\!|_{L^2(\mathcal{E}_h)} = \left[ \sum_{E \in \mathcal{E}_h} \|\nabla \mathbf{v}\|^2_{L^2(E)} \right]^{\frac{1}{2}}, \tag{41}$$

$$\|\!|\mathbf{v}\|\!|_{H^1(\mathcal{E}_h)} = \mu^{\frac{1}{2}} \left( \|\!|\nabla \mathbf{v}\|\!|^2_{L^2(\mathcal{E}_h)} + J_0(\mathbf{v}, \mathbf{v}) \right)^{\frac{1}{2}}. \tag{42}$$

Again, we choose an integer $k \geq 1$ and we discretize $H^1(\mathcal{E}_h)^d$ and $L_0^2(\Omega)$ with the finite element spaces

$$\mathbf{X}_h = \{\mathbf{v} \in L^2(\Omega)^d : \forall E \in \mathcal{E}_h,\ \mathbf{v}|_E \in \mathbb{P}_k(E)^d\}, \tag{43}$$

$$M_h = \{q \in L_0^2(\Omega) : \forall E \in \mathcal{E}_h,\ q|_E \in \mathbb{P}_{k-1}(E)\}. \tag{44}$$

The choice $\mathbb{P}_{k-1}$ for the discrete pressure, one degree less than the velocity, is suggested by the fact that $L^2$ is the natural norm for the pressure. Keeping in mind (13) and (14), we discretize (11) by the following discrete system: Find $\mathbf{u}_h \in \mathbf{X}_h$ and $p_h \in M_h$ satisfying for all $\mathbf{v}_h \in \mathbf{X}_h$ and $q_h \in M_h$:

$$\begin{aligned} &\mu \sum_{E \in \mathcal{E}_h} \int_E \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \, d\mathbf{x} \\ &- \mu \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \left( \{\nabla \mathbf{u}_h \cdot \mathbf{n}_e\}_e [\mathbf{v}_h]_e + \varepsilon \{\nabla \mathbf{v}_h \cdot \mathbf{n}_e\}_e [\mathbf{u}_h]_e \right) d\sigma + \mu J_0(\mathbf{u}_h, \mathbf{v}_h) \\ &- \sum_{E \in \mathcal{E}_h} \int_E p_h \operatorname{div} \mathbf{v}_h \, d\mathbf{x} + \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{p_h\}_e [\mathbf{v}_h]_e \cdot \mathbf{n}_e \, d\sigma = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, d\mathbf{x}, \end{aligned} \tag{45}$$

$$\sum_{E \in \mathcal{E}_h} \int_E q_h \operatorname{div} \mathbf{u}_h \, d\mathbf{x} - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{q_h\}_e [\mathbf{u}_h]_e \cdot \mathbf{n}_e \, d\sigma = 0, \tag{46}$$

with the interpretation for the parameters $\varepsilon$ and $\sigma$ of formula (7).

Let $a_h$ and $b_h$ denote the bilinear forms

$$a_h(\mathbf{u},\mathbf{v}) = \mu \sum_{E\in\mathcal{E}_h} \int_E \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x} \\ - \mu \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \big(\{\nabla\mathbf{u}\cdot\mathbf{n}_e\}_e[\mathbf{v}]_e + \varepsilon\{\nabla\mathbf{v}\cdot\mathbf{n}_e\}_e[\mathbf{u}]_e\big)\, d\sigma, \tag{47}$$

$$b_h(\mathbf{v},q) = \sum_{E\in\mathcal{E}_h} \int_E q \operatorname{div}\mathbf{v}\, d\mathbf{x} - \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{q\}_e[\mathbf{v}]_e\cdot\mathbf{n}_e\, d\sigma. \tag{48}$$

Clearly, the properties of $a_h$ listed in the previous section are valid here and, therefore, existence and uniqueness of $\mathbf{u}_h$ hold for IIPG and SIPG if the penalty parameters $\sigma_e$ are well-chosen; they hold unconditionally for NIPG and they hold for OBB-DG if $k \geq 2$. But existence and uniqueness of $p_h$ is not straightforward because it is the consequence of the uniform "inf-sup" condition, that is now a standard tool in studying problems with a linear constraint (cf. [Bab73, Bre74]): There is a constant $\beta^* > 0$ independent of $h$ such that

$$\inf_{q_h\in M_h} \sup_{\mathbf{v}_h\in\mathbf{X}_h} \frac{b_h(\mathbf{v}_h,q_h)}{|\!|\!|\mathbf{v}_h|\!|\!|_{H^1(\mathcal{E}_h)}\|q_h\|_{L^2(\Omega)}} \geq \beta^*. \tag{49}$$

By using the Raviart–Thomas interpolation operator (cf. [RT75, GR86]), we can readily show that (49) holds for IIPG, SIPG, NIPG and OBB-DG (cf., for instance, [SST03]). Hence the four schemes have a unique solution. However, in order to derive optimal error estimates, we have to bound the term $b_h(\mathbf{v}_h, p-\rho_h p)$, where $\rho_h$ is a suitable approximation operator, for instance, a local $L^2$ projection on each $E$, and $\mathbf{v}_h$ is an arbitrary test function in $\mathbf{X}_h$. It is easy to prove that if $p \in H^k(\mathcal{E}_h)$ then

$$|b_h(\mathbf{v}_h, p-\rho_h p)| \leq Ch^k \Bigg(\sum_{e\in\Gamma_h\cup\partial\Omega} \frac{1}{h_e}\|[\mathbf{v}_h]\|^2_{L^2(e)} + |\!|\!|\nabla\mathbf{v}_h|\!|\!|^2_{L^2(\mathcal{E}_h)}\Bigg)^{\frac{1}{2}}.$$

As $J_0$ is zero for OBB-DG, we cannot obtain a good estimate for this method: it does not seem to be well-adapted to this formulation of the Stokes problem.

On the other hand, we can obtain optimal error estimates for IIPG, SIPG, NIPG: if the exact solution $(\mathbf{u}, p)$ of the problem (11) belongs to $H^{k+1}(\Omega)^d \times H^k(\Omega)$, then for the three methods

$$|\!|\!|\mathbf{u}_h - \mathbf{u}|\!|\!|_{H^1(\mathcal{E}_h)} + \|p_h - p\|_{L^2(\Omega)} = \mathrm{O}(h^k). \tag{50}$$

*Remark 6.* Let $E$ be an element as in Remark 3. Taking first $q_h = \chi_E$ in (46) and next the $i$-th component of $\mathbf{v}_h$, $v_{h,i} = \chi_E$ in (45), we obtain the discrete mass balance relations:

$$\int_E \operatorname{div}\mathbf{u}_h\, d\mathbf{x} - \frac{1}{2}\sum_{e\in\partial E}\int_e (\mathbf{u}_h^{\text{int}} - \mathbf{u}_h^{\text{ext}})\cdot\mathbf{n}_E\, d\sigma = 0,$$

$$-\mu\sum_{e\in\partial E}\int_e \{\nabla u_{h,i}\}\cdot\mathbf{n}_E\, d\sigma + \mu\sum_{e\in\partial E}\frac{\sigma_e}{h_e}\int_e (u_{h,i}^{\text{int}} - u_{h,i}^{\text{ext}})\, d\sigma = \int_E f_i\, d\mathbf{x}.$$

## 5 DG Approximation of a Convection-Diffusion Equation

Consider the convection-diffusion equation combining (24) and (15) in the domain $\Omega$ of the previous sections:

$$-\operatorname{div}(\mathbf{K}\nabla c) + \mathbf{u}\cdot\nabla c = f, \quad \text{in } \Omega, \tag{51}$$

$$\mathbf{K}\nabla c\cdot \mathbf{n}_\Omega = 0, \quad \text{on } \partial\Omega, \tag{52}$$

where $f$ belongs to $L^2_0(\Omega)$, the tensor $\mathbf{K}$ satisfies the assumptions listed in Section 3 and $\mathbf{u}$ satisfies (16):

$$\operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u}\cdot\mathbf{n}_\Omega = 0 \text{ on } \partial\Omega.$$

This problem has a solution $c \in H^1(\Omega)$, unique up to an additive constant under mild restrictions on the velocity $\mathbf{u}$, for instance, when $\mathbf{u}$ belongs to $H^1(\Omega)^d$. We propose to discretize it with a DG method when $\mathbf{u}$ is replaced by the solution $\mathbf{u}_h \in \mathbf{X}_h$ of a flow problem that satisfies $b_h(\mathbf{u}_h, q_h) = 0$ for all $q_h \in M_h$:

$$\sum_{E\in\mathcal{E}_h}\int_E q_h \operatorname{div}\mathbf{u}_h\, d\mathbf{x} - \sum_{e\in\Gamma_h\cup\partial\Omega}\int_e \{q_h\}_e[\mathbf{u}_h]_e\cdot\mathbf{n}_e\, d\sigma = 0.$$

For an integer $\ell \geq 1$, we define

$$Y_h = \{c \in L^2(\Omega) : \forall E\in\mathcal{E}_h,\ c|_E \in \mathbb{P}_\ell(E)\}. \tag{53}$$

In view of (23) and (32), we discretize (51)–(52) by: Find $c_h \in Y_h$ such that for all $v_h \in Y_h$:

$$\begin{aligned}
&\sum_{E\in\mathcal{E}_h}\int_E \mathbf{K}\nabla c_h\cdot\nabla v_h\, d\mathbf{x}\\
&\quad - \sum_{e\in\Gamma_h\cup\partial\Omega}\int_e \big(\{\mathbf{K}\nabla c_h\cdot\mathbf{n}_e\}_e[v_h]_e + \varepsilon\{\mathbf{K}\nabla v_h\cdot\mathbf{n}_e\}_e[c_h]_e\big)\, d\sigma + J_0(c_h, v_h)\\
&+ \sum_{E\in\mathcal{E}_h}\int_E \big(\mathbf{u}_h\cdot\nabla c_h + \frac{1}{2}(\operatorname{div}\mathbf{u}_h)c_h\big)v_h\, d\mathbf{x} - \frac{1}{2}\sum_{e\in\Gamma_h\cup\partial\Omega}\int_e [\mathbf{u}_h]_e\cdot\mathbf{n}_e\{c_h v_h\}_e\, d\sigma\\
&\qquad - \sum_{E\in\mathcal{E}_h}\int_{(\partial E)_-}\{\mathbf{u}_h\}\cdot\mathbf{n}_E(c_h^{\text{int}} - c_h^{\text{ext}})v_h^{\text{int}}\, d\sigma = \int_\Omega f v_h\, d\mathbf{x},
\end{aligned} \tag{54}$$

where $(\partial E)_-$ is defined by (20)

$$(\partial E)_- = \{\mathbf{x}\in\partial E : \{\mathbf{u}_h\}\cdot\mathbf{n}_E(\mathbf{x}) < 0\},$$

and the parameters $\varepsilon$ and $\sigma_e$ are the same as previously.

To simplify, we introduce the form $t_h$ with the upwind approximation of the transport term in (54):

$$t_h(\mathbf{u}_h; v_h, w_h) = \sum_{E\in\mathcal{E}_h} \int_E \left(\mathbf{u}_h \cdot \nabla v_h + \frac{1}{2}(\operatorname{div}\mathbf{u}_h) v_h\right) w_h \, d\mathbf{x}$$
$$- \sum_{E\in\mathcal{E}_h} \int_{(\partial E)_-} \{\mathbf{u}_h\}\cdot\mathbf{n}_E (v_h^{\text{int}} - v_h^{\text{ext}}) w_h^{\text{int}} \, d\sigma - \frac{1}{2} \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e [\mathbf{u}_h]_e \cdot \mathbf{n}_e \{v_h w_h\}_e \, d\sigma. \tag{55}$$

This form is positive in the following sense (cf. [GRW05]): for all $v_h \in Y_h$

$$t_h(\mathbf{u}_h; v_h, v_h) = \frac{1}{2} \sum_{E\in\mathcal{E}_h} \| |\{\mathbf{u}_h\} \cdot \mathbf{n}_E|^{\frac{1}{2}} (v_h^{\text{int}} - v_h^{\text{ext}}) \|^2_{L^2((\partial E)_- \setminus \partial\Omega)}$$
$$+ \| |\mathbf{u}_h \cdot \mathbf{n}_\Omega|^{\frac{1}{2}} v_h \|^2_{L^2((\partial\Omega)_-)}, \tag{56}$$

where

$$(\partial\Omega)_- = \{\mathbf{x} \in \partial\Omega : \mathbf{u}_h \cdot \mathbf{n}_\Omega(\mathbf{x}) < 0\}.$$

Therefore, if the penalty parameters $\sigma_e$ are chosen as in Section 3, we see that system (54) has a solution $t_h$ in $Y_h$, unique up to an additive constant. In particular, this means that (54) is *compatible* with (51)–(52) and this is an important property, cf. [DSW04].

However, proving a priori error estimates is more delicate, considering that $\mathbf{u}_h$ proceeds from a previous computation. If the error in computing $\mathbf{u}_h$ is measured in the norm (42), then the contribution of $t_h(\mathbf{u}_h; c_h, v_h)$ to the error is estimated as in the Navier–Stokes equations. This requires discrete Sobolev inequalities, and as mentioned in Remark 5, this does not seem to be possible for OBB-DG schemes. On the other hand, for IIPG, SIPG and NIPG, the analysis in [GRW05] carries over here and yields, when $\mathbf{u}$ and $c$ are sufficiently smooth:

$$\| c_h - c \|_{H^1(\mathcal{E}_h)} = \mathrm{O}(h^{\min(k,\ell)}),$$

where $k$ is the exponent in (50).

*Remark 7.* Let $E$ be an element as in Remark 3. Taking $v_h = \chi_E$ in (54), we obtain the discrete mass balance relation:

$$- \sum_{e\in\partial E} \int_e \{\mathbf{K}\nabla c_h\} \cdot \mathbf{n}_E \, d\sigma + \sum_{e\in\partial E} \frac{\sigma_e}{h_e} \int_e (c_h^{\text{int}} - c_h^{\text{ext}}) \, d\sigma$$
$$+ \frac{1}{2}\left( \int_E (\operatorname{div}\mathbf{u}_h) c_h \, d\mathbf{x} - \frac{1}{2} \sum_{e\in\partial E} \int_e (\mathbf{u}_h^{\text{int}} - \mathbf{u}_h^{\text{ext}}) \cdot \mathbf{n}_E c_h^{\text{int}} \, d\sigma \right)$$
$$+ \sum_{e\in(\partial E)_-} \int_e |\{\mathbf{u}_h\} \cdot \mathbf{n}_E| (c_h^{\text{int}} - c_h^{\text{ext}}) \, d\sigma = \int_E f \, d\mathbf{x}.$$

## 6 Some Darcy Flow in Porous Media: Numerical Examples

In recent years DG methods have been investigated and applied to a wide collection of fluid and solid mechanics problems arising in many engineering and scientific fields such as aerospace, petroleum, environmental, chemical and biomedical engineering, and earth and life sciences. Since the list of publications is substantial and continues to grow, we include only a few references to illustrate the diversity of applications, [CKS00]. We do provide some numerical examples arising in modeling Darcy flow and transport in porous media in which DG algorithms offer major advantages over traditional conforming finite element and finite difference methods.

Geological media such as aquifers and petroleum reservoirs exhibit a high level of spatial variability at a multiplicity of scales, from the size of individual grains or pores, to facies, stratigraphic and hydrologic units, up to sizes of formations. These problems are of great importance to a number of scientific disciplines that include the management and protection of groundwater resources, the deposition of nuclear wastes, the recovery of hydrocarbons, and the sequestration of excessive carbon dioxide. Numerical simulation of physical flows and chemical reactions in heterogeneous geological media and their interplay is required for understanding as well as designing mitigation strategies for environmental cleanup or optimizing oil and gas production.

DG methods are effective in treating complex geological heterogeneities such as impermeable boundaries or flow faults occuring in the interior of a reservoir. Because of the flexibility of DG, these boundaries do no require special meshing. Instead the face between two internal elements is simply switched to a no flow boundary condition for both neighboring elements. In Figure 2 we show an example of a mesh with 1683 triangular elements, in which the dark

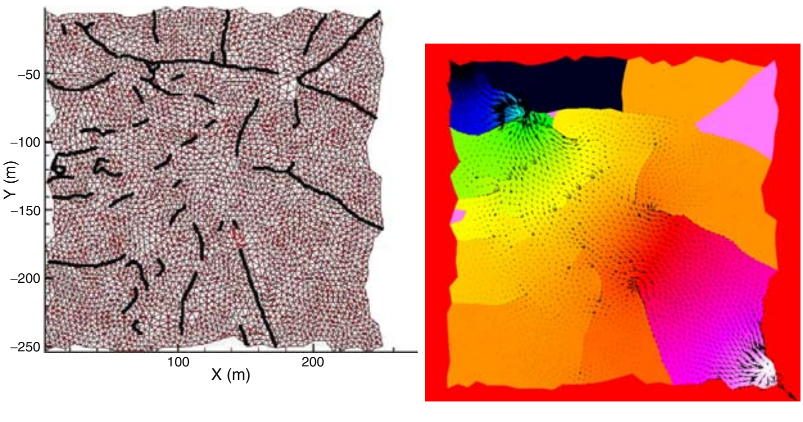


**Fig. 2.** Mesh with internal boundary conditions (left) and pressure and flux solutions (right)

lines are impermeable boundaries. Also shown is the corresponding pressure and flux solution and the impact of these boundaries is clearly observed.

Another important porous media application where DG could prove to be extremely important is reactive transport. When dealing with general chemistry and transport, it is imperative that the transport operators be monotone and conservative. While a number of monotone finite difference methods have been proposed for structured grids, many of these approaches have not been extended to unstructured grids. With the use of appropriate numerical fluxes, approximate Riemann solvers and stability post-processing (slope-limiting), DG methods can be used to construct discretizations which are conservative and monotone.

A benchmark case in reactive transport is a simulation of a far field nuclear waste management problem [cpl01, cpl]. The problem is characterized by large discontinuous jumps in permeability, effective porosity, and diffusivity, and by the need to model small levels of concentration of the radioactive constituents. The permeability field layers of the subsurface are shown in Figure 3.

For this example the magnitude of the velocity varies greatly in the different layers due to the discontinuities in the permeability of the layers. In addition, in the clay and marl layers, where permeability is small, transport is dominated by molecular diffusion. In the limestone and dogger limestone layers, where permeability is large, transport is dominated by advection and dispersion. This example demonstrates the ability of DG to handle both

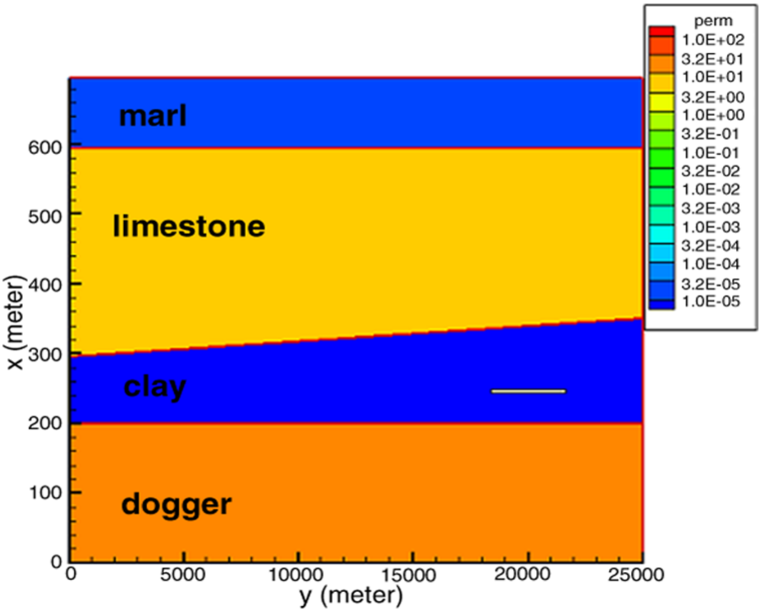


**Fig. 3.** Permeability field layers in the reactive transport problem

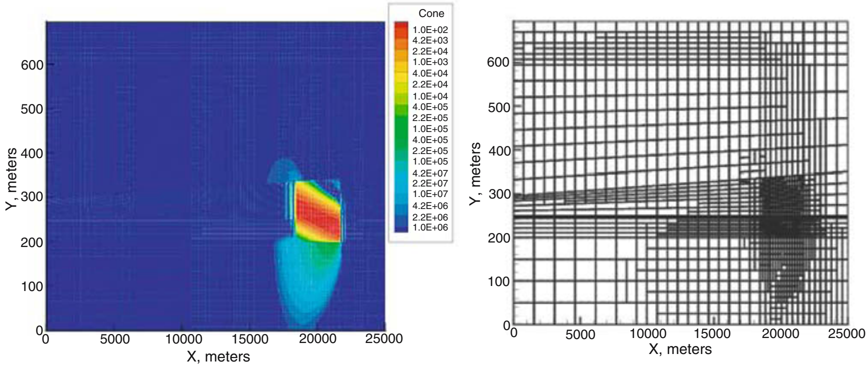


**Fig. 4.** Simulation of nuclear reactive transport using DG - 1

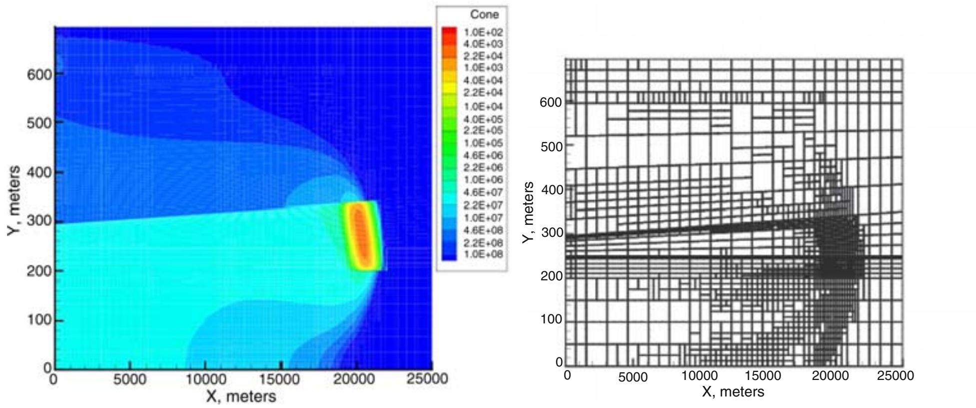


**Fig. 5.** Simulation of nuclear reactive transport using DG - 2

advection-dominated and diffusion-dominated problems. Figure 4 shows Iodine concentration at 200K years and Figure 5 at 2 million years. The low numerical diffusion of the DG method was also found to be important in this benchmark problem because of the long simulation time, cf. [WESR03]. Details regarding this simulation and several mesh adaptation strategies are discussed in [SW06a, SW06b]. The latter demonstrated that by employing dynamic adaptivity, time-dependent transport could be resolved without slope limiting for both long-term and short-term simulations. Moreover, mass conservation was retained locally during dynamic mesh modification.

The theoretical and computational results obtained for primal DG methods for transport and flow are summarized in Table 1. Two rows provide a comparison of the methods for treating flow problems with highly varying

**Table 1.** Primal DG for transport

| | OBB-DG | NIPG | SIPG | IIPG |
|---|---|---|---|---|
| Penalty Term | 0 | $\geq 0$ | $> \sigma_0 > 0$ | $> 0$ and $\ll \sigma_0$ |
| Optimality in $L^2(H^1)$ or $H^1$ | Yes | Yes | Yes | Yes |
| Optimality in $L^2(L^2)$ or $L^2$ | No | No | Yes | No |
| Robust probs. with highly var. coeffs. | Yes | Yes | No | Yes |
| Scalar primary interest(transp.) | No | No | Yes | No |
| Compatibility Flow Condition | No | No | No | Yes |

coefficients and for transport problems in which the scalar variable is of primary interest. These results were obtained from an extensive set of numerical experiments. The studies indicate that the non-symmetric DG formulations are more robust in handling rough coefficients. The symmetric form performs better for treating diffusion/advection/reaction problems since the SIPG form yield optimal $L^2$ and non-negative norm estimates. The last row summarizes a compatibility condition formulated in [DSW04] in which the objective is to choose a flow field that preserves positive concentrations in reactive transport. The IIPG method is the only primal DG for which this holds.

DG methods are currently being investigated for modeling multiphase flow in porous media, e.g., see [BR04, KR06] for two-phase incompressible and for two and three phases compressible systems see [HF06, Esl05, SW]. While much progress has been made in modeling transport a major disadvantage for DG has been the development of efficient parallel solvers for large linear and nonlinear systems, the pressure equation or a fully implicit formulation for multiphase flow respectively. The development of DG solvers is an active area of research and new domain decomposition approaches are currently being developed, e.g., see [Kan05, Joh05, AA07, Esl05, BR00].

## References

[AA07] P. F. Antonietti and B. Ayuso. Schwarz domain decomposition preconditioners for discontinuous Galerkin approximations of elliptic problems: non-overlapping case. *M2AN Math. Model. Numer. Anal.*, 41(1):21–54, 2007.

[ABCM02] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2002.

[Arn79] D. N. Arnold. *An interior penalty finite element method with discontinuous elements.* PhD thesis, University of Chicago, Chicago, IL, 1979.

[Arn82] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.

[Bab73] I. Babuška. The finite element method with Lagrangian multipliers. *Numer. Math.*, 20:179–192, 1973.

[Bak77] G. Baker. Finite element methods for elliptic equations using nonconforming elements. *Math. Comp.*, 31:45–59, 1977.

[Bau97] C. E. Baumann. *An hp-adaptive discontinuous finite element method for computational fluid dynamics.* PhD thesis, University of Texas at Austin, Austin, TX, 1997.

[Bey94] K. S. Bey. *An hp-adaptive discontinuous Galerkin method for hyperbolic conservative laws.* PhD thesis, University of Texas at Austin, Austin, TX, 1994.

[BO99] C. E. Baumann and J. T. Oden. A discontinuous *hp* finite element method for convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 175(3–4):311–341, 1999.

[BOP96] K. S. Bey, J. T. Oden, and A. Patra. *hp*-version discontinuous Galerkin methods for hyperbolic conservation laws. *Compt. Methods Appl. Mech. Engrg.*, 133:259–286, 1996.

[BR00] P. Bastian and V. Reichenberger. Multigrid for higher order discontinuous Galerkin finite elements applied to groundwater flow. Technical Report 2000-37, SFB 359, 2000.

[BR04] P. Bastian and B. Riviere. Discontinuous Galerkin for two-phase flow in porous media. Technical Report 2004-28, IWR(SFB 359), University of Heidelberg, 2004.

[Bre74] F. Brezzi. On the existence, uniqueness and approximation of the saddle-point problems arising from Lagrangian multipliers. *RAIRO Anal. Numér.*, 8:129–151, 1974.

[Bre03] S. Brenner. Poincaré–Friedrichs inequalities for piecewise $h^1$ functions. *SIAM J. Numer. Anal.*, 41:306–324, 2003.

[Bre04] S. Brenner. Korn's inequalities for piecewise $h^1$ vector fields. *Math. Comp.*, 73:1067–1087, 2004.

[BZ73] I. Babuška and M. Zlámal. Nonconforming elements in the finite element method with penalty. *SIAM J. Numer. Anal.*, 10:863–875, 1973.

[Cia91] P. G. Ciarlet. Basic error estimates for elliptic problems. In P. G. Ciarlet and J. L. Lions, editors, *Handbook of Numerical Analysis, Vol. II*, pages 17–351. North-Holland, Amsterdam, 1991.

[CKS00] B. Cockburn, G. E. Karniadakis, and C.-W. Shu, editors. *Discontinuous Galerkin methods. Theory, computation and applications (Newport, RI, 1999).* Number 11 in Lecture Notes in Computational Science and Engineering. Springer-Verlag, Berlin, 2000.

[cpl] Couplex1 test case, nuclear waste disposal far field simulation. ANDRA (the French National Radioactive Waste Management Agency), http://www.andra.fr/couplex/.

[cpl01] The couplex test cases. ANDRA (the French National Radioactive Waste Management Agency), http://www.andra.fr/couplex/, 2001.

[CR73] M. Crouzeix and P. A. Raviart. Conforming and non-conforming finite element methods for solving the stationary Stokes problem. *RAIRO Anal. Numér.*, 8:33–76, 1973.

[Dar80] B. L. Darlow. *An Penalty-Galerkin method for solving the miscible displacement problem.* PhD thesis, Rice University, Houston, TX, 1980.

[Dau89] M. Dauge. Stationary Stokes and Navier–Stokes systems on two or three-dimensional domains with corners. *SIAM J. Math. Anal.*, 20(1):74–97, 1989.

[DD76] J. Douglas, Jr. and T. Dupont. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing Methods in Applied Sciences (Second Internat. Sympos., Versailles, 1975)*, number 58 in Lecture Notes in Phys., pages 207–216. Springer-Verlag, Berlin, 1976.

[DSW04] C. Dawson, S. Sun, and M. F. Wheeler. Compatible algorithms for coupled flow and transport. *Comput. Methods Appl. Mech. Engrg.*, 194:2565–2580, 2004.

[Esl05] O. Eslinger. *Discontinuous Galerkin finite element methods applied to two-phase air-water flow problems.* PhD thesis, University of Texas at Austin, Austin, TX, 2005.

[GR79] V. Girault and P.-A. Raviart. An analysis of upwind schemes for the Navier–Stokes equations. *SIAM J. Numer. Anal.*, 19(2):312–333, 1979.

[GR86] V. Girault and P.-A. Raviart. *Finite Element Methods for the Navier–Stokes Equations. Theory and Algorithms.* Number 5 in Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 1986.

[Gri85] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Number 24 in Pitman Monographs and Studies in Mathematics. Pitman, Boston, MA, 1985.

[GRW05] V. Girault, B. Rivière, and M. Wheeler. A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier–Stokes problems. *Math. Comp.*, 74:53–84, 2005.

[GSWY] V. Girault, S. Sun, M. F. Wheeler, and I. Yotov. Coupling discontinuous Galerkin and mixed finite element discretizations using mortar finite elements. *SIAM J. Numer. Anal.* Submitted Oct. 2006.

[HF06] H. Hoteit and A. Firoozabadi. Compositional modeling by the combined discontinuous Galerkin and mixed methods. *SPE J.*, 11:19–34, 2006.

[Joh05] K. Johannsen. A symmetric smoother for the nonsymmetric interior penalty discontinuous Galerkin discretization. ICES Report 05-23, University of Texas at Austin, 2005.

[Kan05] G. Kanschat. Block preconditioners for LDG discretizations of linear incompressible flow problems. *J. Sci. Comput.*, 22(1–3):371–384, 2005.

[KR06] W. Klieber and B. Riviere. Adaptive simulations of two-phase flow by discontinuous Galerkin methods. *Comput. Methods Appl. Mech. Engrg.*, 196(1–3):404–419, 2006.

[LR74] P. Lesaint and P. A. Raviart. On a finite element method for solving the neutron transport equation. In C. deBoor, editor, *Mathematical Aspects of Finite Elements in Partial Differential Equations*, pages 89–123. Academic Press, 1974.

[Nit71] J. A. Nitsche. Über ein Variationsprinzip auf Lösung von Dirichlet-Problemen bei Verwendung von Teilsraumen, die keinen Randbedingungen unteworfen sind. *Math. Sem. Univ. Hamburg*, 36:9–15, 1971.

[OBB98] J. T. Oden, I. Babuška, and C. E. Baumann. A discontinuous *hp* finite element method for diffusion problems. *J. Comput. Phys.*, 146:491–516, 1998.

[OW75] J. T. Oden and L. C. Wellford, Jr. Discontinuous finite element approximations for the analysis of shock waves in nonlinearly elastic materials. *J. Comput. Phys.*, 19(2):179–210, 1975.

[Pir89] O. Pironneau. *Finite Element Methods for Fluids.* Wiley, Chichester, 1989.

[RH73] W. H. Reed and T. R. Hill. Triangular mesh methods for the neutron transport equation. Los Alamos Scientific Laboratory Report LA-UR-73-479, 1973.

[Riv00] B. Rivière. *Discontinuous Galerkin finite element methods for solving the miscible displacement problem in porous media.* PhD thesis, University of Texas at Austin, Austin, TX, 2000.

[RT75] P. A. Raviart and J. M. Thomas. A mixed finite element method for second order elliptic problems. In *Mathematical Aspects of Finite Element Methods*, number 606 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1975.

[RW74] H. Rachford and M. F. Wheeler. An H1-Galerkin procedure for the two-point boundary value problem. In C. deBoor, editor, *Mathematical Aspects of Finite Elements in Partial Differential Equations*, pages 353–382. Academic Press, 1974.

[RWG99] B. Riviere, M. F. Wheeler, and V. Girault. Part I: Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. *Comput. Geosci.*, 3:337–360, 1999.

[RWG01] B. Riviere, M. F. Wheeler, and V. Girault. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM J. Numer. Anal.*, 39(3):902–931, 2001.

[SST03] D. Shotzau, C. Schwab, and A. Toselli. Mixed *hp*-DGFEM for incompressible flows. *SIAM J. Numer. Anal.*, 40(319):2171–2194, 2003.

[SW] S. Sun and M. F. Wheeler. Discontinuous Galerkin methods for multiphase compressible flows. In preparation.

[SW06a] S. Sun and M. F. Wheeler. Anisotropic and dynamic mesh adaptation for discontinuous Galerkin methods applied to reactive transport. *Comput. Methods Appl. Mech. Engrg.*, 195(25–28):3382–3405, 2006.

[SW06b] S. Sun and M. F. Wheeler. A posteriori error estimation and dynamic adaptivity for symmetric discontinuous Galerkin approximations of reactive transport problems. *Comput. Methods Appl. Mech. Engrg.*, 195:632–652, 2006.

[Tem79] R. Temam. *Navier–Stokes equations. Theory and numerical analysis.* North-Holland, Amsterdam, 1979.

[WD80] M. F. Wheeler and B. L. Darlow. Interior penalty Galerkin procedures for miscible displacement problems in porous media. In *Computational methods in nonlinear mechanics (Proc. Second Internat. Conf., Univ. Texas, Austin, Tex., 1979)*, pages 485–506, Amsterdam, 1980. North-Holland.

[WESR03] M. F. Wheeler, O. Eslinger, S. Sun, and B. Rivière. Discontinuous Galerkin method for modeling flow and reactive transport porous media. In *Analysis and Simulation of Multifield Problems*, pages 37–58. Springer-Verlag, Berlin, 2003.

[Whe78] M. F. Wheeler. An elliptic collocation-finite element method with interior penalties. *SIAM J. Numer. Anal.*, 15(1):152–161, 1978.