Chapter 5

Logic and Rough Sets: An Overview

"Any specific object has a specific logic" K. Marx.

Since the present Part has a certain complexity, it is worth introducing, with some details, the intuitive motivations of the entire picture and their connections with the mathematical machinery which will be used.

5.1 Foreword

Thus, let us sum up what we have discussed and discovered as far as now.

In Rough Set Theory, the starting point is a collection of observations which are stored in an Information System **I** and which induces an indiscernibility space $\langle U, E \rangle$. We denote the family of all basic categories by $IND(\mathbf{I})$.

We have seen that from any Information System **I** one can compute the extension $[\![\mathcal{D}]\!]$ on the *universe* U of a *basic property* $\mathcal{D}$ which we call a **I**-*basic property*, because it can be formulated using the linguistic material from **I**.[1]

I-*basic properties* make it possible to classify the objects from U into different disjoint equivalence classes which are to be intended as

[1] For instance, if **I** is an Attribute Systems, a deterministic property is a conjunction of sentences of the form "$a_i = v_j$", where a_i ranges on the *set of parameters* At, v_j ranges on the *set of values* V.

the gnoseological co-ordinates interpreting the universe U, the basis of the organization of U from a conceptual point of view. In a literal mathematical sense, $IND(\mathbf{I})$ is a basis for a topological space that provides the "gnoseological geometry" of our world. Objects belonging to the same class are *indiscernible* by means of our system of information **I**. Moreover, from the hypotheses about *P-systems* and *A-systems*, stated in Part I, every object from U will belong to the extension of some **I**-*basic property*. So we obtain the first important characteristic for our analysis:

Axiom 5.1.1. *The set* $IND(\mathbf{I}) = \{[\![\mathcal{D}]\!] : \mathcal{D}$ *is a* $\mathbf{I}-$basic property$\}$ *is a partition of* U.

These classes, or blocks, are the atoms of more complex conceptual constructions. In Rough Set Theory, they are called "elementary" (or "basic") "classes" (or "categories") and we adopt this use.

As noted at the very beginning of the Introduction, this construction is fundamental. Call it "*grouping*", "*association*", "*categorization*", we hardly can find an analysis of human knowledge leaving it out of consideration: "*When you learn a concept, you learn how to treat different things as instances of the same category. Without this classification procedure, thinking would be impossible because each event or entity would be unique*" (Johnson-Laird [1988], page 132).

Because $IND(\mathbf{I})$ coincides with the family of the equivalence classes modulo the equivalence relation induced by the Information System **I**, on a more abstract level we can start from any generic Indiscernibility Space $\langle U, E \rangle$, where $E \subseteq U \times U$ is an arbitrary equivalence relation.

The topological space for which the family U/E forms a basis, is called an *Approximation Space*. Nonetheless, in the present book we also use this term to denote the *frame* (complete distributive lattice) of its open subsets, denoted by $\mathbf{AS}(U/E)$ (the context shall avoid any confusion). Any open set is the union of basic sets. Therefore, they are extensions of *disjunctions* of basic properties, called thereafter "**I**-*properties*". So, given an Information System **I**, the *Approximation Space* $\mathbf{AS}(U/E)$ induced by **I**, represents in fact this kind of linguistic description of concepts. This intuitively explains why an Approximation Space is defined as the set of all the unions of elementary classes plus the empty-set $\emptyset$ (an arbitrary **I** -property could have an empty extension).

Thus, from an algebraic point of view, we have:

Axiom 5.1.2. *For any Indiscernibility Space $\langle U, E\rangle$, the Approximation Space* $\mathbf{AS}(U/E)$ *is the Boolean algebra of sets for which U/E is the set of atoms.*

Now, as we know, the second basic maneuver is to contrast $\mathbf{AS}(U/E)$ with the result of another categorization, say the Indiscernibility Space $\langle U, E'\rangle$. However, for the sake of maximal generalization, we shall assume that any arbitrary subset of U can be brought in contrast with $\mathbf{AS}(U/E)$, so that the second categorization will be assumed to be the discrete one ($\langle x, y\rangle \in E'$ if and only if $x = y$), if not otherwise stated. In other terms, any subset of U can play the role of a pre-figure. Thus we assume that the foreground Approximation Space will always be the powerset $\wp(U)$.[2]

For this reason the structure of $\mathbf{AS}(U/E')$ does not count and we shall reserve by now the name "Approximation Space" to the background space, and the term "datum" to the elements of the foreground space.

With respect to this structure we have the following fact:

Axiom 5.1.3. *For any Indiscernibility Space $\langle U, E\rangle$, the Approximation Space* $\mathbf{AS}(U/E)$ *is a subalgebra of the Boolean algebra of sets* $\mathbf{B}(U)$ *defined on the powerset $\wp(U)$ of the universe U.*

In accordance with these assumptions, Approximation Spaces are given a more general interpretation. In fact, if we assume that a generic subset of elements of U is the extension of a generic "datum" virtually definable on U, then the fact that an Approximation Space $\mathbf{AS}(U/E)$ is generally a strict subalgebra of $\wp(U)$, displays the popular observation that usually we do not have a complete information about any situation we face with, in a pretty "concrete" and "tangible" manner. In other words, the granularity of the knowledge represented by our properties, generally does not allow the exact representation of arbitrary concepts,

[2]For the sake of generalisation, but also because we are working within the monological approach. From this point of view, the foreground space is always subordinated to the background. Otherwise stated, the foreground space is inert, so that any subset of this space may be conceived of as a "crude" datum to be analysed, but not to be questioned. In the dialogical approach we do not have "crude data" any longer, but interacting categories.

but just an approximation depending on the gnoseological material at our disposal. Hence the term "Approximation Space".

Let us then consider an arbitrary set $X \subseteq U$. Obviously, either $X \in \mathbf{AS}(U/E)$ or not. In the first case X can be exactly described by means of an **I**-property, which can be named a *background property*, in view of the discussion in the Introduction. In the second case we cannot use **I**-properties for a direct description of X, which can be approximated by means of an *upper approximation*, $(uE)(X)$, and a *lower approximation*, $(lE)(X)$. If $\langle U, \mathbf{AS}(U/E)\rangle$ is intended as a topological space, we know that $(uE)(X)$ is the closure $\mathbb{C}(X)$ and $(lE)(X)$ is the interior $\mathbb{I}(X)$.

However, in general in between $(lE)(X)$ and $(uE)(X)$ we have the topological boundary of X: $\mathbb{B}(X) = \mathbb{C}(X) \cap -\mathbb{I}(X) = (uE)(X) \cap -(lE)(X)$. Notice that the boundary of X is the set of points which are neither in the lower approximation, nor in the complement of the upper approximation of X: $\mathbb{B}(X) = \mathbb{C}(X) \cap -\mathbb{I}(X) = -(-\mathbb{C}(X) \cup \mathbb{I}(X)) = -(-(uE)(X) \cup (lE)(X))$. From the point of view of Approximation Spaces, two sets that have exactly the same upper and lower approximations can be considered equivalent, and one obtains:

Definition 5.1.1. *A* rough set *of $\langle U, E\rangle$ is an equivalence class of subsets of U modulo the equality of their upper and lower approximations. Such an equivalence relation is called a* rough equality.

The family of all rough sets induced by an Approximation Space $\mathbf{AS}(U/E)$ is called a *Rough Set System* and is denoted by $RS(U/E)$.

5.2 Rough Set Systems and Three-Valued Logics

As we have mentioned in the Introduction, one can give a logical interpretation to this machinery. The upper approximation $(uE)(X)$ is the set of elements that *possibly* belong to X since they share the same **I**-*properties* with some element actually in X. In other words, if $x \in (uE)(X)$, then we *can* associate it to X, even if it does not actually belong to this set, since some "twin" of x belongs to X already. On the other hand $(lE)(X)$ is the set of elements of X that *necessarily* belong to X since there are not elements outside of X which are describable by means of the same **I**-*properties*. In negative terms: if $x \in X$ but there is an x' belonging to $-X$ which is indiscernible from x, then in order to obtain the lower approximation of X, we discard x too, since its membership is *accidental*, according to the conceptual background represented by $\mathbf{AS}(U/E)$.

Example 5.2.1. Possibility and necessity in an information system
Consider the information system of Example 2.4 in the Introduction. Let X be the set $\{d, e, f, g\}$, which in $\mathbf{AS}(U/A')$ is characterized by the property "*Comfort = medium*". We can notice that in $\mathbf{AS}(U/A)$ the patient c may be associated to X since patient d belongs to X and is indiscernible from c which is as like as any patient having *Temperature = normal*, *Hemoglobin = good*, *Blood Pressure = low* and *Oxygen Saturation = good*. So we can assume that it is not impossible for these patients to have *Comfort = medium* because we have examples of patients with the same attribute-values that have this rate. In fact $(uE_A)(X) = \{c, d, e, f, g\}$. On the other hand, all the patients with *Temperature = low*, *Hemoglobin = good*, *Blood Pressure = normal* and *Oxygen Saturation = fair*, have *Comfort = medium*. This means, for instance, that e necessarily belongs to X, since we do not have counterexamples of patients with the same characteristics but with *Comfort* $\neq$ *medium*. This fact is reflected by the equation $(lE_A)(X) = \{e, f, g\}$.

From this point of view, for any set X we have two *definite* or *certain* states: the lower approximation (interior of X, necessary part of X), which means "definitely yes", and the complement of the upper approximation, $-(uE)(X) = -\mathbb{C}(X)$, (exterior of X, impossible part of X) which means "definitely no". Since $-\mathbb{C}(X) = \mathbb{I}(-X)$, this element coincides with the complementary figure $\neg X$. Everything that is neither in $(lE)(X)$ nor in the complementary figure, $\neg X$, is in the boundary of X, $\mathbb{B}(X)$. Indeed a boundary is a region of doubt: if $x \in \mathbb{B}(X)$, then we can say nothing certain about the membership of x in X. We cannot say either that x is certainly (or necessarily) in X, or that x has certainly nothing to do with X: in fact it *could* belong to X, since it is indiscernible from some element of X; but it could belong to $-X$, too, because it is indiscernible from some element of $-X$.

It follows that, generally, between $(lE)(X)$ and its complementary figure $\neg X$, there is not an empty region and that the union of $(lE)(X)$ and $\neg X$ does not give the unit universe U. In other words, the law of Excluded Middle is not uniformly valid for rough sets.

So we begin to see that the classical two-valued characteristic function must be generalized by a three-valued one, if we want to grasp this situation. It follows that in general Rough Set Systems are likely to have strict relationships with some three-valued logico-algebraic system. Actually, more than one of these systems are related to this information analysis and the reason depends on the deeper meaning of our construction.

In fact, the topological space $\langle U, \mathbf{AS}(U/E)\rangle$ may fulfill different separation properties depending on the granularity of our knowledge. In turn, this depends on the level of accuracy of the attributes. We have the best separation properties when $\mathbf{AS}(U/E) = \wp(U)$. In this case our topology is the *discrete* topology (which is Hausdorff and completely disconnected) and one can single out each element of U. Otherwise stated, in a sense we have enough properties for "*naming*" any single element of U. On the contrary, when no object can be discerned from the others, we have the *trivial* topology: $\mathbf{AS}(U/E) = \{\emptyset, U\}$. Using the famous sentence of the German philosopher G. W. F. Hegel, this is like the "*night in which all cows are black*". Indeed in this case we have the weakest separation property.

However, usually we shall have intermediate cases in which some elements can be singularly "*named*", while others cannot be individualised by the information at our disposal: in general in $\langle U, E\rangle$ some equivalence classes are *singletons* while others are not.

5.3 Exact and Inexact Local Behaviours in Rough Set Systems

Let us denote by B^* the family of the equivalence classes that are singletons, and by P^* the family of the equivalence classes that have cardinality strictly greater than 1. As mentioned in the introduction. B^* and P^* do not have the same logical role in the construction of a Rough Set System. In fact the elements in B^* are *exact* in nature, because they do not have any boundary, any region of doubt, so that they should enjoy the principle that in Classical Logic reflects completeness and exactness: Excluded Middle.[3] Indeed, given a set X and an open singleton $\{s\}$, either $\{s\}$ is included in $(lE)(X)$ or it is included in $-(uE)(X)$.[4] On the contrary any basic open set with at least two elements may be included in the boundaries of at least two different sets. This means that if there is no singleton in $\mathbf{AS}(U/E)$, then there are at least two sets X such that $(uE)(X) = U$ and $(lE)(X) = \emptyset$. In other words, there are at least two *undefinable* sets.

[3] If $\{x\}$ is a singleton, then x is an isolated point, in topological terms.

[4] In topological terms: an isolated element cannot be a member of any boundary.

Therefore, it is not difficult to understand why the class of the undefinable sets can play the role of intermediate value: this class represents situations in which everything could be true, or everything could be false. Thus, the rough set of all the undefinable sets is another "night in which all cows are black".

Example 5.3.1. Exact and inexact information

In the Approximation Space induced by the set A of attributes in the Information System of Example 2.4 discussed in the Introduction, we have two non singleton atoms, $\{c, d\}$ and $\{e, f\}$, and five singletons, $\{a\}, \{b\}, \{g\}, \{h\}, \{i\}$. The singleton $\{a\}$, for instance, is uniquely defined by the property "*Temperature = low, Hemoglobine = fair, Blood Pressure = low, Oxygen Saturation = fair*". This property applies only to the element a so that we have complete and unique information about a, because the attribute-values we are dealing with make it possible to distinguish a from all the other elements of U.

On the contrary, the element c fulfills the same property as the element d, so that we do not have enough information in At to distinguish c (or d).

Clearly, as far as we deal with the set of attributes A, we do not have subsets of U that are undefinable, because for instance if $\{a\}$ is included in the upper approximation of a subset of U, then it is included in its lower approximation, too. It follows that there are not sets X such that $(uE_A)(X) = U$ and $(lE_A) = \emptyset$.

Now consider, instead, the following sub-table of the same Information System, where $U^* = \{a, b, c, d, e, f\}$ and $A^* = \{Temperature, Hemoglobin\}$:

v	$Temperature$	$Hemoglobin$
a	low	fair
b	low	fair
c	normal	good
d	normal	good
e	low	good
f	low	good

Clearly $E/_{A^*} = \{\{a, b\}, \{c, d\}, \{e, f\}\}$. So the induced Approximation Space has three atoms and none of them is a singleton. If we contrast the set $X = \{a, c, e\}$ against $E/_{A^*}$ then we find $(uE_{A^*})(X) = U^*$ and $(lE_{A^*})(X) = \emptyset$. In fact it is impossible to find a disjunction of basic properties exactly describing some member of X but not all the members of U. Hence X is an undefinable set. Indeed, the process of peaking up an element out of each (non singleton) equivalence class gives us a combination of eight undefinable sets:

$$\{\{a, c, e\}, \{a, c, f\}, \{a, d, e\}, \{a, d, f\}, \{b, c, e\}, \{b, c, f\}, \{b, d, e\}, \{b, d, f\}\}$$

This collection is therefore the rough set of all sets X such that $(uE_{A^*})(X) = U^*$ and $(lE_{A^*}) = \emptyset$. Therefore, it represents all the undefinable sets.

Therefore, we may suppose that for any rough set there are two distinct *local* logical behaviours: one is classical and localized on $B = \bigcup B^*$, whereas the other one, localized on $P = \bigcup P^*$, is purely three-valued. It is the combination of these local behaviours that defines the overall logical features of rough sets. It follows that the construction of $RS(U/E)$ will depend on the parameter B (or, equivalently, P).

Moreover, in $RS(U/E)$, any rough set induced by an element of the Approximation Space $\mathbf{AS}(U/E)$ has a particular logical behaviour too: such an element corresponds to an exactly definable subset of U, hence, again, it should fulfill Classical Logic, but within the logical environment determined by the overall Rough Set System. And, as just seen, this environment might be three-valued.

Thus we have two levels of local logical behaviours: one is related to the internal definition of rough sets, the other deals with the global logical properties of Rough Set Systems.

The first level completely depends on the parameter B (or P). These parameters cannot be recovered from the "*geometrical*" shape of the Approximation Space $\mathbf{AS}(U/E)$, except for trivial cases. It follows that in general an inspection of the atoms is unavoidable in order to define $RS(U/E)$. Because the information provided by this inspection does not have any theoretic content, we call B and P *external parameters* or *empirical parameters* and we say that they are able to distinguish the classical local behaviour within an Approximation Space.

On the contrary, we can analyse the lattice structure of $RS(U/E)$ from a pure abstract point of view. In fact, also in this case we have to use a particular parameter, but curiously enough, though it is induced by the empirical parameter B, nevertheless it is definable in $RS(U/E)$ by means of a mere lattice-theoretic definition. For this reason we call it an *internal parameter* or *theoretical parameter* and we shall see that it distinguishes a classical local behaviour within a Rough Set System.

It follows that Rough Set Systems should be analysed using some notion able to manage the concept of "**it is locally the case that**".

For this purpose we shall exploit the mathematical notions of a "*Grothendieck topology*" and a "*Lawvere-Tierney operator*" which have been introduced to deal with local properties.

5.4 Representing Rough Sets

A *rough set* is an equivalence class of sets modulo the equality of their approximations. Thus a rough set from U belongs to $\wp(\wp(U))$.

However a rough set is naturally and more comfortably representable by a pair $\langle X_1, X_2\rangle$ of elements of $\mathbf{AS}(U/E)$, where X_1 and X_2, are the two approximations.

So, consider the (by now informal) family

Definition 5.4.1. $RS(U/E) = \{\langle X_1, X_2\rangle \in \mathbf{AS}(U/E) \times \mathbf{AS}(U/E)$: $\langle X_1, X_2\rangle$ *is a Rough Set in* $\mathbf{AS}(U/E)\}$.

We immediately have the problem of the formal and abstract characterization of the sentence "is a Rough Set in $\mathbf{AS}(U/E)$". A first sub-problem is:

Problem 5.4.1. *For any Approximation Space* $\mathbf{AS}(U/E)$*, determine the* internal formal characteristics *that must be satisfied by an ordered pair to represent a rough set.*

The answer depends on the intuitive motivations that drive our reading of the nature of rough sets. A first, and in a sense the most immediate and "naive", solution is considering pairs of the form

$$\langle (uE)(X), (lE)(X)\rangle \tag{5.4.1}$$

This ordered pair uniquely describes the equivalence class in question.

From this point of view, the "internal property" to be fulfilled by a pair $\langle X_1, X_2\rangle$ in order to belong to $RS(U/E)$ is necessarily:

$$X_2 \subseteq X_1 \tag{5.4.2}$$

because the first element X_1 stands for the upper approximation of a set X and X_2 stands for its lower approximation. Thereafter we call such a representation the *decreasing representation* of a rough set.

A second reading, probably less "naive" but still intuitive, is suggested by the application of Rough Set Theory to some semantics for Logics of Knowledge and Learning (see the Frame section of Part III and is connected with the following intuition: any rough set represents what definitely is known to satisfy a concept and what definitely is known not to satisfy it. Between the two areas, eventually, there is a *doubtful region* which is due to the incompleteness of our knowledge.

Thereafter, from this point of view the “internal property” of a pair $\langle X_1, X_2 \rangle$ is necessarily:

$$X_1 \cap X_2 = \emptyset \tag{5.4.3}$$

and we call it the *disjoint representation* of a rough set.

We have already seen that in a more logical setting, the upper approximation $(uE)(X)$ corresponds to the modal application $M(X)$ – “what *possibly* belongs to X” – and the lower approximation $(lE)(X)$ corresponds to the modal application $L(X)$ – “what *necessarily* belongs to X”. According to this reading, the decreasing representation of a rough set is of the type

$$\langle M(X), L(X) \rangle \tag{5.4.4}$$

However, consider $-M(X)$. Since $-M(X)$ means “it is *impossible* to belong to X”, we have that $L(X)$ and $-M(X)$ are the only statements expressing “certainty”. Thus a definite knowledge about a specific phenomenon will be expressed by a pair

$$\langle L(X), -M(X) \rangle \tag{5.4.5}$$

that is to say, ⟨*maximal internal body, complementary body*⟩. In order to make rough sets reflect the above intuition, one must represent them as a pair

$$\langle (lE)(X), -(uE)(X) \rangle \tag{5.4.6}$$

that is exactly the *disjoint representation* of a rough set.

So we shall set:

Definition 5.4.2 (Decreasing representation of rough sets)**.** *For any Approximation Spaces* $\mathbf{AS}(U/E)$ *and* $X \subseteq U$*:* $rs(X) = \langle (uE)(X), (lE)(X) \rangle$*.*

Definition 5.4.3 (Disjoint representation of rough sets)**.** *For any Approximation Spaces* $\mathbf{AS}(U/E)$ *and* $X \subseteq U$*:* $rs'(X) = \langle (lE)(X), -(uE)(X) \rangle$*.*

The application rs will be called a “rough set map”.

From the involution property of “$-$”, one easily shows that the two representations are interchangeable.

Although a choice between the two representations is somewhat arbitrary, since we prefer to deal with the two standard modalities (Necessity and Possibility) we adopt the decreasing representation.[5]

Therefore, we assume by default the decreasing representation until the disjoint representation is explicitly considered. Moreover, when the context is clear, with the term "rough set" we shall denote the decreasing (disjoint) representations of a rough set (which, actually, is an equivalence class).

Example 5.4.1. Representing rough sets
Let us represent some rough sets induced by the Approximation Space $X \in \mathbf{AS}(U/E_A)$ of Example 1.2.3 (cf. also Example 1.2.5).

Disjoint representation:

$$\begin{array}{llll} \{\{a\},\{c\}\} & \longrightarrow \langle\emptyset,\{b,d\}\rangle; & \{\{b,a\},\{b,c\}\} & \longrightarrow \langle\{b\},\{d\}\rangle; \\ \{\{d,a\},\{d,c\}\} & \longrightarrow \langle\{d\},\{b\}\rangle; & \{\{b,d,a\},\{b,d,c\}\} & \longrightarrow \langle\{b,d\},\emptyset\rangle. \end{array}$$

and any $X \in \mathbf{AS}(U/E_A)$ is represented by $\langle X, -X\rangle$.

Decreasing representation:

$$\begin{array}{llll} \{\{a\},\{c\}\} & \longrightarrow \langle\{a,c\},\emptyset\rangle; & \{\{b,a\},\{b,c\}\} & \longrightarrow \langle\{a,b,c\},\{b\}\rangle; \\ \{\{d,a\},\{d,c\}\} & \longrightarrow \langle\{a,c,d\},\{d\}\rangle; & \{\{b,d,a\},\{b,d,c\}\} & \longrightarrow \langle U,\{d,b\}\rangle. \end{array}$$

and any $X \in \mathbf{AS}(U/E_A t)$ is represented by $\langle X, X\rangle$.

A second sub-problem is:

Problem 5.4.2. *For any Approximation Space* $\mathbf{AS}(U/E)$*, determine the* internal empirical characteristics *of the ordered pairs representing a rough set.*

More precisely, this problem is related to the previous discussion about singleton and non singleton basic categories. If we assume the decreasing representation we have to notice that not every pairs of elements fulfilling the formal property (5.4.2) are legal. In other words, (5.4.2) is a necessary but not sufficient condition for a pair to represent a rough set of an Information System **I**.

[5] Other reasons supporting this choice can be found in the Frame section of Part II. However, from a strictly mathematical point of view the disjoint representation is to be preferred because it has more general applications – see Example 9.6.1 of Section 9.6.

In fact, as we already know, if an elementary class S is a singleton then for any $X \subseteq U$, S is included either in $(lE)(X)$ or in $-(uE)(X)$. Thus S belongs to $(lE)(X)$ whenever S is included in $(uE)(X)$. This is the required general characteristic. It follows, for instance, that the pair $\langle S, \emptyset \rangle$ it is not a legal one although it fulfills property 5.4.2, while, for example, $\langle S, S \rangle$ is.

In the same case, if we assume the disjoint representation, we have to discard, for instance, the pair $\langle \emptyset, \emptyset \rangle$: indeed it enjoys property 5.4.3 but it is clear that the singleton S must necessarily be included either in X_1 or in X_2, for any pair $\langle X_1, X_2 \rangle$. Again 5.4.3 is only a necessary condition.

The problem becomes thereafter:

Problem 5.4.3. *For any Approximation Space* $\mathbf{AS}(U/E)$ *characterize the set* $RS(U/E)$ *within the family of elements of the Cartesian product* $\mathbf{AS}(U/E) \times \mathbf{AS}(U/E)$ *which fulfill property (5.4.2) (or (5.4.3) if we prefer the disjoint representation).*

The solution of this problem will be given within the following more general:

Problem 5.4.4. *Determine if there is some logico-algebraic structure behind Rough Sets Systems.*

Example 5.4.2. Local validity in Rough Set Systems – 1

Let us consider the information system of Example 2.3. According to it, the pair $\langle \{a, b, c, d\}, \{a, c, d\} \rangle$ is not a legal rough set (in decreasing representation). In fact if $b \in (uE_A)(X)$, for some $X \subseteq U$, then for some $x \in X$, $\langle x, b \rangle \in E_A$ But since $\{b\}$ is a singleton we have $\langle b, x \rangle \in E_A$ if and only if $x = b$, so that $b \in X$ too. It follows that $\{b\} \subseteq X$. Hence, $\{b\} \subseteq (lE_{At})(X)$ and $b \in (lE_{At})(X)$.

The union B of all the singletons is $\{b, d\}$ and we have that for any $x \in B$, either $x \in (lE_{At})(X)$ or $x \in -(uE_{At})(X)$. Otherwise stated, $x \in (uE_{At})(X)$ if and only if $x \in (lE_{At})(X)$. This means that for any rough set $\langle X_1, X_2 \rangle$, $X_1 \cap B = X_2 \cap B$.

In disjoint representation, the above considerations lead to the fact that for any $X \subseteq U$, for any rough set $\langle X_1, X_2 \rangle$, $X_1 \cup X_2 \supseteq B$, i.e. $(lE_{At})(X) \cup -(uE_{At})(X) \supseteq \{b, d\}$.

Let us depict the situation in Figures 5.1 and 5.2 below.

Given the universe U, a usual set X has a complement $-X$ such that $X \cup -X = U$ (Figure 5.1 left). In an Approximation Space, on the contrary we have $(lE)(X) \cup -(uE)(X) \subseteq U$ (Figure 5.1 right). The intermediate area is the boundary of X.

But if the union B of all the singletons is not void (Figure 5.2 left), we have a different situation: any subset B' of B is a sub-body with it own complement $-B'$ as complementary body. Indeed, $(lE)(B') \cup -(uE)(B') = B \cup B' = B$ (Figure 5.2 right).

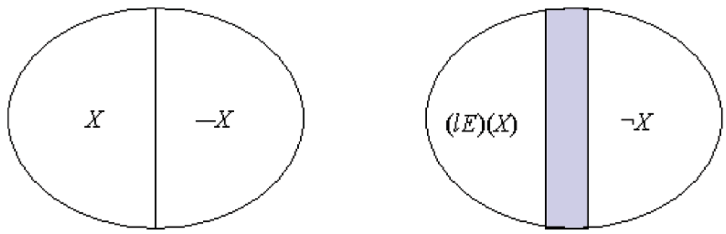

Figure 5.1: An empty union B of singletons

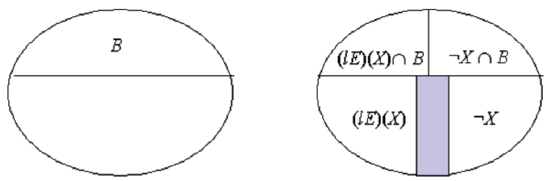

Figure 5.2: A non-empty union B of singletons – subsets of B behave as usual

5.5 Rough Set Systems, Local Validity, and Logico-Algebraic Structures

Notwithstanding its "practical" motivations, Rough Set Theory happens to be able to model a number of logical systems.

Indeed, Rough Set Systems have many connections with Heyting and bi-Heyting algebras, Łukasiewicz algebras, Post algebras, Stone algebras, Chain Based Lattices and P-algebras. In what follows we provide the overall picture of these connections.

First of all we have to show that the language-oriented operations provided by Logic are meaningful in Rough Set Systems. As a matter of fact, this is partially true on the part of the operations "and" and "or".

Indeed, let X, Y, Z be subsets of U. We have:

- Interpretation of the operation $\wedge$: if $rs(X) \wedge rs(Y) = rs(Z)$, then Z is a maximal set in the class $\{X' \cap Y' : rs(X') = rs(X) \ \ \textit{and} \ \ rs(Y') = rs(Y)\}$.

- Interpretation of the operation $\vee$: if $rs(X) \vee rs(Y) = rs(Z)$, then Z is a minimal set in the class $\{X' \cup Y' : rs(X') = rs(X) \ \ \textit{and} \ \ rs(Y') = rs(Y)\}$.

It follows that rs distributes over $\vee$ just with respect to the upper approximations and, dually, it distributes over $\wedge$ just with respect to the lower approximations (details later in the text). Hence, the two binary connectives $\vee$ and $\wedge$ make sense in defining a Rough Set Logic, under the limitations of the above proviso.

Now, it is well known that the set $\mathbf{B}^{[n]} = \{\langle a_1, \ldots, a_{n-1}\rangle \in \mathbf{B}^{n-1} : a_i \geq a_j \text{ for } i \leq j\}$, where $\mathbf{B}$ is a Boolean algebra, is an example of n-valued Łukasiewicz algebra (see Boicescu et al. [1991]).

Thus $\mathbf{AS}(U/E)^{[3]}$ is a three-valued Łukasiewicz algebra.

From this consideration it follows that $RS(U/E)$ is a *substructure* of $\mathbf{AS}(U/E)^{[3]}$ if we assume the decreasing representation.

On the side of the disjoint representation, if $\mathbf{D}$ is a finite distributive lattice with least element $\perp$, then the set $K(\mathbf{D}) = \{\langle a_1, a_2\rangle \in \mathbf{D}^2 : a_1 \wedge a_2 = \perp\}$ is an example of De Morgan algebra. In particular if $\mathbf{D}$ is a finite Boolean algebra, then $K(\mathbf{D})$ is a Post algebra of order three. Since $\mathbf{AS}(U/E)$ is a Boolean algebra, from the above considerations it follows that if we assume the disjoint representation, then $RS(U/E)$ is a *substructure* of the Post algebra of order three, $K(\mathbf{AS}(U/E))$.

Our last problem can now be restated in the following way:

Problem 5.5.1. *For any Approximation Space $\mathbf{AS}(U/E)$, characterize within $\mathbf{AS}(U/E)^{[3]}$ and within $K(\mathbf{AS}(U/E))$ the logical status of the substructure $RS(U/E)$ using only information-oriented parameters depending on $\mathbf{AS}(U/E)$.*

In this Part we shall start answering these questions by representing $RS(U/E)$ as a *semi-simple Nelson algebra*. We decide to start with this interpretation for a couple of reasons. First, although David Nelson introduced his systems in order to circumvent some non constructivistic features of intuitionistic negation (in connection with Kleene's notion of "Recursive Realizability"), Nelson's deep intuitive motivations can be completely framed in our context:

"*In general, an experimental verification of a statement consists of an operation followed by an observation of a property.* [...] *However, if we have not observed the property, there remains an ambiguous situation insofar as the truth of a statement is concerned. The failure to observe the property may be significant of the falsity of the statement or may merely be an indication of some deficiency on the part of the observer.* [...] *In view of this ambiguity, it might be maintained that* *every significant observation must be an observation of some property and further that the absence of a property P if it may be established empirically at all, must be established by the observation of (another) property N which is taken as a token for the absence of P.*" (Nelson [1959], page 208).

On the basis of these intuitions, in the quoted paper David Nelson introduced a logical system named **S**, which makes it possible to distinguish concepts such as "from A a contradiction is provable" and "the negation of A is established", which are usually unified.

We call this difference the issue of "separation of concepts", and we record it by saying that in the former case the proof ends with a weak form of negation, $\lrcorner\alpha$, and in the latter with a strong form of negation, $\sim\alpha$. System **S** is strictly connected to semi-simple Nelson algebras, that constitute a subvariety of the class of Nelson algebras, which in its turn is connected with the system **N** introduced in Nelson [1949].[6]

The second reason to start with semi-simple Nelson algebras is the fact that the duality theory of these algebras provides us with the mathematical machinery that is needed in order to exhibit a rigorous characterization of $RS(U/E)$. The main result of this approach is that for any Approximation Space $\mathbf{AS}(U/E)$ the Rough Set System $RS(U/E)$ can be made into a finite semi-simple Nelson algebra, which is precisely definable by means of the parameter B (viz. the union of all the singleton elementary classes) that was discussed in the previous subsections. We shall use B for filtering $RS(U/E)$ out of $\mathbf{AS}(U/E)^{[3]}$ and $K(\mathbf{AS}(U/E))$.

This use of B will be completely framed within the theory of Grothendieck Topologies, because it will be based on the notion of "local validity", as has been anticipated.

These Nelson algebras will be proved to be equipped with a pseudo-complementation, $\neg$, which, in fact, can be defined by means of the weak negation $\lrcorner$ and the strong negation $\sim$.

[6] One of the principal differences between **N** and **S** is that in **S** we have just a restricted form of *thinning*, namely $\frac{\alpha,\alpha,\alpha\vdash\beta}{\alpha,\alpha\vdash\beta}$. As always happens, restrictions on structural rules make formerly unified logical meanings split. The above restricted form of thinning is shared also by three-valued Łukasiewicz logic which may be defined by consistently extending **S** by means of the axiom $\alpha \equiv\sim \alpha$, for a suitable formula α (cf. the discussion below about the connection between semi-simple Nelson algebras and three-valued Łukasiewicz algebras, and about central elements).

What is the rough set interpretation of these negations?

- Strong negation "$\sim$": we have $\sim rs(X) = rs(-X)$, so that the strong negation of a rough set equals the rough set of its set-theoretical complement. In other words, the strong negation faithfully represents the set-theoretical complement at the rough set level.
- Pseudo-complementation "$\neg$": if $\neg rs(X) = rs(Y)$, then Y is the greatest definable set disjoint from $(uE)(X)$.
- Weak negation "$\lrcorner$": if $\lrcorner rs(X) = rs(Y)$, then Y is the greatest definable set disjoint from $(lE)(X)$.

Thus negations have a straightforward meaning in Rough Set Systems.

Moreover, the above algebraic structures may also be regarded as bi-Heyting algebras. More precisely, on can show that the operator $\lrcorner$ is the pseudo-complementation in the co-Heyting algebra $RS(U/E)^{op}$ that is obtained by reversing the order, thus swapping $\wedge$ and $\vee$, 1 and 0 (and defining a dual relative pseudo-complementation). Therefore in $RS(U/E)$, if we set $1 = \langle U, U\rangle$ and $0 = \langle\emptyset, \emptyset\rangle$ we have, for any a:

$$a \vee \neg a \leq 1, a \wedge \neg a = 0;$$
$$a \vee \lrcorner a = 1, a \wedge \lrcorner a \geq 0;$$
$$a \vee \sim a \leq 1, a \wedge \sim a \geq 0.$$

These failures of the laws of contradiction and excluded middle, have an immediate informational interpretation, displayed by the following symmetries, for $a = \langle (uE)(X), (lE)(X)\rangle$:

$$a \vee \neg a = \langle U, -\mathbb{B}(X)\rangle = a \vee \sim a$$
$$a \wedge \lrcorner a = \langle \mathbb{B}(X), \emptyset\rangle = a \wedge \sim a.$$

So, it is absolutely evident that the lack of the classical principles is due to the presence of the doubtful boundary region: the excluded middle and the law of contradiction are valid up to the presence of a non-empty boundary. Indeed, if $\mathbb{B}(X) = \emptyset$, then $\langle U, -\mathbb{B}(X)\rangle = \langle U, U\rangle$, which is the top element, while $\langle \mathbb{B}(X), \emptyset\rangle = \langle\emptyset, \emptyset\rangle$, which is the bottom element.

Particular attention is given to the logico-algebraic characterization of definable sets. It is possible to define, by means of the weak negation and the pseudo-complementation, two operators $\lrcorner\lrcorner$ and $\neg\neg$. These operators project any rough set X onto particular exact elements, that is elements $\langle X_1, X_2\rangle$ such that $X_1 = X_2$ (assuming X to be in decreasing representation). More precisely, $\neg\neg$ is a possibility operator , while $\lrcorner\lrcorner$ is a necessity modalizator.

The interpretation in Rough Set Theory of these modalities is:

- "Possibility" operator: if $\neg\neg rs(X) = rs(Y)$, then Y is the least definable set containing X. That is, $\neg\neg rs(X) = rs((uE)(X))$;
- "Necessity" operator: if $\lrcorner\lrcorner rs(X) = rs(Y)$, then Y is the greatest definable set included in X. That is, $\lrcorner\lrcorner rs(X) = rs((lE)(X))$.

As it will more detailed in Frame 10.12.4, it is worth mentioning that the operation $\lrcorner$ was exploited in Lawvere [1982] to give a logical account for the notions of a "boundary", "essential core of a body" and "sub-body" or "body", in the context of Continuum Physics. If we compare our terminology with Lawvere's, we can observe that the notion of "essential core of a body" corresponds to our "maximal internal body". In our terminology, however, a "body" is so if it coincides with its own essential core, that is to say if it is a regular element.[7]

It is clear that, because of their atomicity, singleton elementary classes are sub-bodies that either belong to X or to its complementary figure $\neg X$, for *any* given subspace X of the universe of discourse. Otherwise stated, $B \subseteq X \cup \neg X$. There is no notion of a boundary involving B: any point which can be isolated by an elementary class, cannot belong to any boundary. It follows that for all $a \in RS(U/E)$ we have $a \vee \neg a = \langle U, -\mathbb{B}(X)\rangle \geq \langle U, B\rangle$ and $a \wedge \lrcorner a = \langle \mathbb{B}(X), \emptyset\rangle \leq \langle -B, \emptyset\rangle$, so that the law of excluded middle and the law of contradiction are valid with respect to the subspace B.

At this point, Grothendieck topologies display their power, as we shall see in Chapter 7. Indeed, our use of Grothendieck topologies has the objective to formally render, from a mathematical point of view, that in a part of our universe we have to apply Classical Logic, while in the remaining part we have to apply a three-valued Logic. Roughly speaking, given the family $\mathbf{G}$ of open sets of a Grothendieck topology over a universe U, we say that a property $\mathcal{P}$ is *locally valid* on a set $X \subseteq U$, if its domain of validity, $[\![\mathcal{P}]\!]$, has a *large enough intersection* with X, where the meaning of "large enough" is given by the topology $\mathbf{G}$;

[7] Therefore, we are not able to distinguish between a body and its essential core, while we can distinguish the maximal internal body within a generic set (or prefigure). As a matter of fact, our topology is coarser than Lawvere's.

namely, if $[\![\mathcal{P}]\!] \cap X \in \mathbf{G}$. So we shall define a suitable Grothendieck topology $\mathbf{G}_B$ on $\mathbf{AS}(U/E)^{[3]}$, depending on the parameter B, such that the disjunction $a \vee \lrcorner a$ is absolutely valid while $a \vee \neg a$ is greater than or equal to the *local top element* $\langle U, B\rangle$ (i.e. the transformation *via* $\mathbf{G}_B$ of the absolute top element $\langle U, U\rangle$) and the conjunction $a \wedge \neg a$ is absolutely invalid, while $a \wedge \lrcorner a$ is less than or equal to the *local bottom element* $\langle -B, \emptyset\rangle$ (i.e. the transformation *via* $\mathbf{G}_B$ of the absolute bottom element $\langle \emptyset, \emptyset\rangle$).

Hence Grothendieck topologies will code the fact that both excluded middle and the law of contradiction are locally valid with respect to the sub-universe B, according to the picture of Figure 5.3.

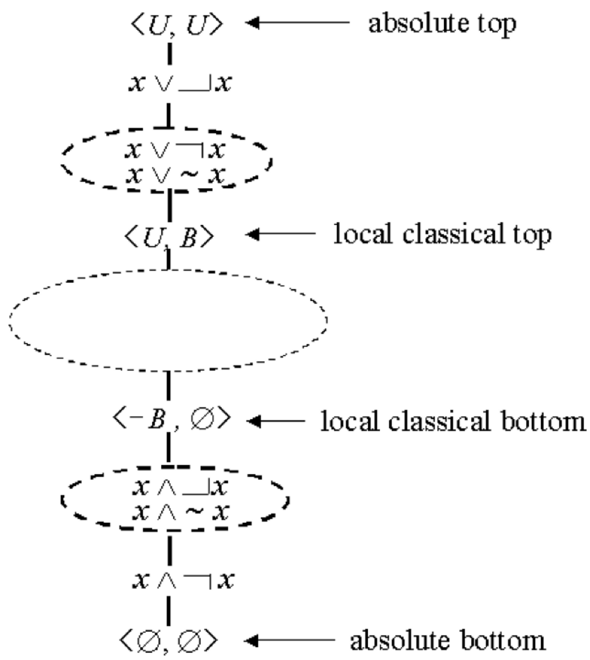

Figure 5.3: Local and global elements

Example 5.5.1. Local validity in Rough Set Systems – 2
Negations and boundaries:
Given a rough set $x = \langle X_1, X_2\rangle$ (in decr. repr.), $\sim x = \langle -X_2, -X_1\rangle = \langle -(lE)(X), -(uE)(X)\rangle$ that is the rough set of $-X$. $\lrcorner x = \langle -X_2, -X_2\rangle = \langle -(uE)(X), -(uE)(X)\rangle$ that is the rough set of $(uE)(-X)$ (of $-(lE)(X)$). $\neg x = \langle -X_1, -X_1\rangle = \langle -(lE)(X), -(lE)(X)\rangle$, that is the rough set of $(lE)(-X)$ (of $-(uE)(X)$). If $y =$

$\langle Y_1, Y_2\rangle$, we define $x \wedge y$ and $x \vee y$ point-wise:

$$x \wedge y = \langle X_1 \cap Y_1, X_2 \cap Y_2\rangle,\ x \vee y = \langle X_1 \cup Y_1, X_2 \cup Y_2\rangle$$

(for all these operations see Window 7.1).

In the information system of Example 2.3, we have for instance that if $a = \langle\{a,c,b\},$
$\{b\}\rangle$ then $a\wedge \sim a = \langle\{a,c,b\},\{b\}\rangle \wedge \langle\{a,c,d\},\{d\}\rangle = \langle\{a,c\},\emptyset\rangle$. Now, $\langle\{a,c,b\},\{b\}\rangle$ represents the rough set $\{\{a,b\},\{c,b\}\}$. Let us consider, for instance, $\{a,b\}$. The boundary $\mathbb{B}(\{a,b\})$ is $\{a,b,c\} \cap -\{b\} = \{a,c\}$. It follows that $a\wedge \sim a = \langle\mathbb{B}(\{a,b\}),\emptyset\rangle$. Hence $\neg(a\wedge \sim a) = \neg(a \wedge \neg a) = \langle -\mathbb{B}(\{a,b\}), -\mathbb{B}(\{a,b\})\rangle = \langle\{a,c\},\{a,c\}\rangle$.

On the contrary, $a\vee \sim a = \langle\{a,b,c,d\},\{b,d\}\rangle = \langle U, -\mathbb{B}(\{a,b\})\rangle$, so that $\neg(a\vee \sim a) = \neg(a \vee \neg a) = \langle - - \mathbb{B}(\{a,b\}), - - \mathbb{B}(\{a,b\})\rangle = \langle\mathbb{B}(\{a,b\}),\mathbb{B}(\{a,b\})\rangle$.

Local Validity:
Let us consider again the rough set $a = \langle\{a,c,b\},\{b\}\rangle$. Then $a\vee\neg a = \langle\{a,c,b\},\{b\}\rangle\vee \langle\{d\},\{d\}\rangle = \langle U,\{b,d\}\rangle = \langle U,B\rangle$. However, if we take the illegal pair $a' = \langle\{a,c,b\},$ $\emptyset\rangle$, then $a' \vee \neg a' = \langle U,\{d\}\rangle \lneqq \langle U,B\rangle$. So the property $x \vee \neg x \geqslant \langle U,B\rangle$ reflects our constraint on the admissible forms of rough sets with decreasing representation.

On the other side, $a \wedge \neg a = a \wedge \langle\{c,d\},\{c,d\}\rangle = \langle\{c\},\emptyset\rangle \leq \langle -B,\emptyset\rangle$. Again, $a' \wedge \neg a' = a' \wedge \langle U,U\rangle = a' \gneqq \langle -B,\emptyset\rangle$. Henceforth, also the property $x \wedge \neg x \leqslant \langle -B,\emptyset\rangle$ testifies for the legality of x.

Once we have accomplished this logico-algebraic interpretation of Rough Set Systems, we can exploit well-known relationships between the class of semi-simple Nelson algebras and the class of *three-valued Łukasiewicz algebras* in order to move from Nelson's philosophical issues concerning the separation of concepts to the standpoint of Multi-Valued Logics. It will be proved that for any Approximation Space $\mathbf{AS}(U/E)$ the Rough Set System $RS(U/E)$ is a finite three-valued Łukasiewicz algebra. In this framework the projection operators correspond to the two endomorphisms provided by these algebras.

The logical status of the intermediate value in these algebras is worth being discussed.

Generally, three-valued Łukasiewicz algebras lack the presence of a central element. An element x is called *central* if $\sim x = x$. One can prove that in $RS(U/E)$, *qua* three-valued Łukasiewicz algebra, there is at most one central element. Now we show that there is a central element *only if there are not singleton elementary categories*. In fact, we know that in this case we have at least two undefinable sets whose corresponding rough set is $\langle U,\emptyset\rangle$ (by definition of “undefinable set”, the closure of these sets is the entire universe, while their interior is empty). It happens that $\sim\langle U,\emptyset\rangle = \langle U,\emptyset\rangle$.

Moreover, in this specific case $RS(U/E)$ can be made into a *Post algebra of order three*, characterized by the three-element chain

$$\langle\emptyset,\emptyset\rangle \leq \langle U,\emptyset\rangle \leq \langle U,U\rangle.$$

However, in general we do not have such a central element because $B \neq \emptyset$. In this case is it impossible to define an algebraic structure exhibiting a three-element chain of values, in full generality? It is possible, if we conceive, once more, the concept of an intermediate value in a *relative* manner, not in an *absolute* one. This means that the property "to be an intermediate element" must be given a local, or relative, meaning exactly as the notion of a "rough set" was given, exploiting the Grothendieck topology $\mathbf{G}_B$, a meaning relative to the sub-universe B of the exact pieces of information. In this way we enter the realm of the generalizations of Post algebras called *Chain Based Lattices* investigated by Epstein and Horn.

Particularly, we shall see that for any Approximation Space $\mathbf{AS}(U/E)$, the Rough Set System $RS(U/E)$ is a $P_2 - lattice$ of order three characterized by means of the parameter B. Under this interpretation, the above local top element $\langle U,B\rangle$ and local bottom element $\langle -B,\emptyset\rangle$ play the role of intermediate and, respectively, co-intermediate elements.

If we compare the fact that $\langle U,\emptyset\rangle$ means "totally undefinable" with the local top and bottom elements, we find a meaningful reading for the intermediate value of a Rough Set Systems *qua* P_2-lattices: the worst informational situation is $\langle U,B\rangle$ which means "totally undefined up to B".

So one can pass from an extreme situation when $B = U$, to an opposite extreme situation when $B = \emptyset$, through an intermediate one when $U \neq B \neq \emptyset$. In the first case $\langle U,B\rangle = \langle U,U\rangle = \sim\langle\emptyset,\emptyset\rangle = \langle -B,\emptyset\rangle$. In the second $\langle U,U\rangle \neq \langle U,B\rangle = \langle -B,\emptyset\rangle \neq \langle\emptyset,\emptyset\rangle$. In the intermediate case $\langle U,U\rangle \neq \langle U,B\rangle \neq \langle -B,\emptyset\rangle \neq \langle\emptyset,\emptyset\rangle$.

We illustrate these situations in Figure 5.4 below.
Moreover we shall show that the pseudo-supplementation and the dual pseudo-supplementation which are definable in P_2-lattices play the same roles as the projection operators in semi-simple Nelson algebras and the two endomorphisms in Łukasiewicz three valued algebras.

It will also be proved that any finite semi-simple Nelson algebra, three-valued Łukasiewicz algebra, Post algebra of order-three or P_2-lattice of order three, is isomorphic to the rough sets system induced

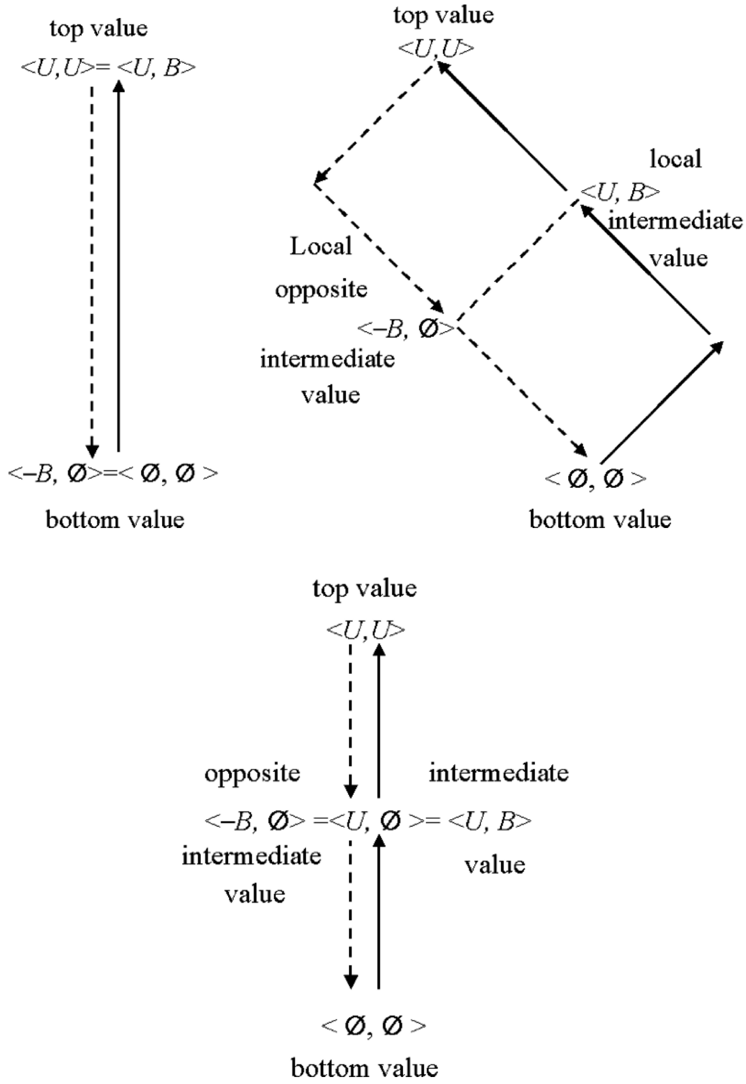

Figure 5.4: Three-valued lattices connected with Rough Set Systems

by some Approximation Space $\mathbf{AS}(U/E)$. More importantly, we shall exhibit a logico-algebraic decomposition of the structure of Rough Sets Systems (hence of semi-simple Nelson algebras, three-valued Łukasiewicz algebras, Post algebra or P_2-lattices of order three), based on the distinction between *locally exact* (or Boolean) part and *locally inexact* (or Postean) part of an Information System.

The first section of the Part will run as follows.

- We formally define the sets B and P and explain why they induce local logical behaviours in an Approximation Space.
- We introduce the mathematical notions of a "Grothendieck Topology" and a "Lawvere-Tierney operator", underlining their suitability for managing the notion "to be locally valid".
- The set B and its dual P will be exploited as information-dependent logico-topological parameters in order to define a Grothendieck topology for identifying $RS(U/E)$ within the set of all the ordered pairs of decreasing elements of $\mathbf{AS}(U/E)$.
- Via two Lawvere-Tierney operators, defined by means of B and P and inherited from the previous step, we shall define two modal operators M and L in $RS(U/E)$ that parallel the upper and the lower approximations, respectively. We shall see that M is an example of a closure operator induced by a well-known Grothendieck topology, namely the dense topology on the dual space of $RS(U/E)$ *qua* Heyting algebra while L is the closure operator induced by the dense topology on the dual space of the opposite Heyting algebra $RS(U/E)^{op}$.
- Using the above machinery we shall be able to show when and how a Rough Set System can be made into a Boolean algebra, a Łukasiewicz algebra, a Post algebra, a P_2-lattice, a P-algebra or a Nelson algebra. We shall see the roles played in these constructions by the notions of a "central element" and an "intermediate value", and the knowledge-oriented content that they are given in our setting.
- Finally by means of two additional Lawvere-Tierney operators based on the parameters P and B, we define a couple of new Grothendieck closure operators which make it possible to discover the double local logical nature of the above algebraic structures: the Post-like one (related to the inexact information of a knowledge system) and the Boolean one (related to its exact information).

In the second section of the Part the above results will be linked with an analysis of the notion of a "constructive logical system", by discussing the following points:

- The difference between the “truth-oriented” and the “knowledge-oriented” approaches in Logic.
- Why a knowledge-oriented approach leads us to the rejection of some classical principles and the assumption of new principles such as explicit definability (any derivable existential sentence must be explicitly instantiated by a closed term) and the disjunction principle (a disjunction is provable if at least one disjoint sentence is explicitly derivable). These principles define what are usually accepted as “constructive systems”.
- The limits of this understanding of a “constructive system” and their relations with the classical definition of the concept of “knowledge”.
- What is hidden in the knowledge-oriented approach. More precisely the difference between the logical status of atomic and non-atomic sentences.
- As a consequence the need to make classical and constructive systems coexist either by endowing constructive systems with well-suited classical principles or by adopting “context operators” which are able to identify the logical environment of a sentence, either classical or constructive, thus making the logical understanding of a sentence explicit.

To conclude, we shall record two notable conclusions:

R1 The “context operators” are the starting points of an approach to study maximal constructive logics, that is, constructive logics embedding a maximal amount of classical principles, in the sense that they cannot be augmented with any new principle without making them collapse into a non constructive system.

R2 The “context operators” are tightly connected with the Lawvere-Tierney operators which we use to formalise the notion of “local validity” and to define Rough Set Systems, both from a philosophical and a technical point of view.