Chapter 13

Modalities, Topologies and Algebras

13.1 Topological Boolean Algebras

We pack the above properties in the following definition:

Definition 13.1.1. *Let* $\langle U, \Omega(U)\rangle$ *be a topological space. Then the pair* $\langle \mathbf{B}(U), \mathbb{I}\rangle$ *is said to be a* topological Boolean algebra of sets.

Corollary 13.1.1. *Let* $\langle \mathbf{B}(U), L_R\rangle$ *be a pre-monadic Boolean algebra of the powersets of a set* U*, such that* R *is a preorder. Then* $\langle \mathbf{B}(U), L_R\rangle$ *is a topological Boolean algebra of sets.*

In a more abstract setting, we can easily observe that taking into account the formal properties of the operators $\mathbb{I}$ and $\mathbb{C}$ in a topological space $\langle U, \Omega(U)\rangle$, we obtain the following definition:

Definition 13.1.2. *Let* $\mathbf{B}$ *be a Boolean algebra and* $\mathbb{I}$ *a monadic operator on* $\mathbf{B}$ *such that, for any* $a, b \in \mathbf{B}$

1. $\mathbb{I}(1) = 1$.
2. $\mathbb{I}(a) \leq a$.
3. $\mathbb{I}(\mathbb{I}(a)) = \mathbb{I}(a)$.
4. $\mathbb{I}(a \wedge b) = \mathbb{I}(a) \wedge \mathbb{I}(b)$.

then the pair $\langle \mathbf{B}, \mathbb{I}\rangle$ *is called a* "topological Boolean algebra, tBa"*.*

From *Definition* 13.1.2, it follows immediately that $\mathbb{I}(a \vee b) \geq \mathbb{I}(a) \vee \mathbb{I}(b)$. Therefore, any *tBa* is a pre-monadic Boolean algebra with additional features (namely (2) and (3) – cf. *Definition* 12.1.3).

Proposition 13.1.1. *Let* $\langle \mathbf{B}, \mathbb{I} \rangle$ *be a* tBa *and* $\mathbb{C}$ *a monadic operator such that for any* $a \in \mathbf{B}, \mathbb{C}(a) = \neg\mathbb{I}(\neg a)$. *Then, for any* $a, b \in \mathbf{B}$,

1. $\mathbb{C}(0) = 0$.
2. $\mathbb{C}(a) \geq a$.
3. $\mathbb{C}(\mathbb{C}(a)) = \mathbb{C}(a)$.
4. $\mathbb{C}(a \vee b) = \mathbb{C}(a) \vee \mathbb{C}(b)$.

The above abstraction is adequate in that the following proposition holds:

Proposition 13.1.2. *Let* $\langle U, \Omega(U) \rangle$ *be a topological space, then* $\langle \mathbf{B}(U), \mathbb{I} \rangle$ *is a* tBa.

Proof. straightforward. QED

Proposition 13.1.3. *Any* tBa *is a model for the modal system* **S***4*.

For the complete proof see, for instance, Rasiowa [1974], Chapter XIII, where **S**4 is called $\mathcal{S}_{\lambda 4}$. By going back from *Corollary* 12.8.4 through our preceding discussion, we can easily obtain that **S**4 modal systems are characterised by reflexive and transitive binary relations, i.e. preorders (see Section 12.2).[1]

13.2 Monadic Topological Boolean Algebras

The equations stated in *Corollary* 12.8.5 give a partial answer to the problem risen at the end of Section 12.1. To completely solve it we have to understand when $M_R(X) = \bigcap\{R(Z) : R(Z) \supseteq X\}$.

Immediately we observe that the second equation holds whenever $R = R^{\smile}$. So, let us specialize the above results for the case when the binary relation at hand is an equivalence relation.

[1]However, if we are confined to finite partial orders we characterise the logic S4**GRZ** – see above.

Proposition 13.2.1. *Let $\langle U, \Omega_R(U)\rangle$ be a topological space associated with a relation $R \subseteq U \times U$. Then $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) = \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$ if and only if R is an equivalence relation.*

Proof.
(A) $\Rrightarrow$: Let R be an equivalence relation. Since $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) \supseteq \mathbb{I}_R(X) \cup \mathbb{I}_R(\mathbb{I}_R(Y))$, from idempotence of $\mathbb{I}_R$ we have $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) \supseteq \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$. Therefore we have to prove the reverse inclusion. So, let $a \in \mathbb{I}_R(X \cup \mathbb{I}_R(Y))$; we prove that $a \in \mathbb{I}_R(X)$ or $a \in \mathbb{I}_R(Y)$. We recall that $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) = \{x : R(x) \subseteq X \cup \mathbb{I}_R(Y)\} = \{x : R(x) \subseteq X \cup \{y : R(y) \subseteq Y\}\}$. Thus, $R(a) \subseteq X \cup \{y : R(y) \subseteq Y\}$, so that for any $a' \in R(a)$, $a' \in X \cup \{y : R(y) \subseteq Y\}$. Therefore, let $a' \in \{y : R(y) \subseteq Y\}$, then $R(a') \subseteq Y$. But $R(a') = R(a)$, because R is an equivalence relation. Hence, in this case, $a \in \mathbb{I}_R(Y)$. Otherwise $R(a') \cap Y = \emptyset$. But in this case we must have $R(a) \subseteq X$ and $a \in \mathbb{I}_R(X)$.
(B) $\Lleftarrow$: Assume now that $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) \subseteq \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$. We have to prove that R is an equivalence relation. Suppose $\mathbb{I}_R(X \cup \mathbb{I}_R(Y))$ is not included in $\mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$. We show that in this case R cannot be an equivalence relation. So, assume (i) $a \in \mathbb{I}_R(X \cup \mathbb{I}_R(Y))$ and (ii) $a \notin \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$. Therefore, $a \in \{x : R(x) \subseteq X \cup \mathbb{I}_R(Y)\}$. However, $R(a)$ cannot be included in X, otherwise $a \in \mathbb{I}_R(X)$. It follows that there is an $a' \in R(a)$ such that $a' \in \mathbb{I}_R(Y)$. This means that $R(a') \subseteq Y$. Suppose $R(a') = R(a)$. In this case $R(a) \subseteq Y$ and $a \in \mathbb{I}_R(Y)$, which contradicts our assumption (ii). Henceforth, $R(a) \neq R(a')$ (if R is transitive, $R(a') \nsubseteq R(a)$). It follows immediately that R is not an equivalence relation. QED

Proposition 13.2.2. *Let $\langle U, \Omega_E(U)\rangle$ be a topological space associated with an equivalence relation $E \subseteq U \times U$. Then,*

1. $\Omega_E(U) = \Gamma_E(U)$.
2. $\Omega_E(U)$, *is a Boolean algebra.*
3. $\langle U, \Omega_E\rangle = \langle U, \mathbf{AS}(U/E)\rangle$.

Proof. (**1**) $X \in \Omega_E(U)$ if and only if $X = \mathbb{I}_E(X) = \varkappa^E(X)$ if and only if $X = E(X)$, if and only if $X = E^{\smile}(X)$, if and only if $X = \varepsilon^E(X) = \mathbb{C}_E(X)$, if and only if $X \in \Gamma_E(U)$. (**2**) Since $\mathbb{I}_E$ and $\mathbb{C}_E$ are dual, from point (1) we have that if $X \in \Omega_E(U)$, then $X = \mathbb{I}_E(X)$, so that $-X = -\mathbb{I}_E(X) = \mathbb{C}_E(-X)$. Therefore, $-X \in \Gamma_E(U) = \Omega_E(U)$. So,

$\Omega_E(U)$ is closed under complementation. Moreover, from *Proposition* 12.5.5 and point (1), if $X, Y \in \Omega_E(U)$, then both $X \cup Y$ and $X \cap Y$ belong to $\Omega_E(U)$. Moreover, $\varkappa^E(U) = U$ and $\varkappa^E(\emptyset) = \emptyset$. Therefore $\Omega_E(U)$ is a Boolean algebra of sets. (**3**) From the definitions of Ω_E and $\mathbf{AS}(U/E)$. QED

So far, we have distilled the topological features of Approximation Spaces. Now we have enough material in order to understand why the pair $\langle \mathbf{B}(U), L_E \rangle$, where L_E is induced by an Approximation Space $\langle U, \Omega_E \rangle = \langle U, \mathbf{AS}(U/E) \rangle$, is a particular kind of *topological Boolean algebra of sets*.

As we have seen, this term applies, more in general, to any pair $\langle \mathbf{B}(U),\ L \rangle$ where L is the interior operator of any topology on U. In particular, it applies to the pair $\langle \mathbf{B}(U),\ L_R \rangle$ where L_R is induced by the topology $\{\varkappa^R(X) : X \subseteq U\}$ for some transitive and reflexive relation R. The distinguishing properties of Approximation Spaces, *qua* topological Boolean algebra of sets, are consequences of the fact that Approximation Spaces are induced not just by generic preorders, but by equivalence relations:

Corollary 13.2.1. *Let $\langle \mathbf{B}(U), L_R \rangle$ be a pre-monadic Boolean algebra of the powersets of a set U. Let R be an equivalence relation and $X \subseteq U$. Then,*

1. $L_R(X) = \bigcup \{R(Z) : R(Z) \subseteq X\}$.
2. $M_R(X) = \bigcap \{R(Z) : R(Z) \supseteq X\}$.

Proof. Straightforward, from *Corollary* 12.8.5. QED

The above result completes the answer to the problem risen at the end of Section 12.1.

We summarize the properties of pre-monadic Boolean algebras induced by topological spaces associated with equivalence relations, in the following corollary:

Corollary 13.2.2. *Let $\langle \mathbf{B}(U), L_R \rangle$ be a pre-monadic Boolean algebra of the powersets of a set U, such that R is an equivalence relation. Then $\langle \mathbf{B}(U), L_R \rangle$ is a monadic topological Boolean algebra of sets.*

Proof. Straightforwardly from *Definition* 12.1.6 and *Proposition* 13.2.1. QED

In abstraction we set:

Definition 13.2.1. *If* $\langle \mathbf{B}, \mathbb{I}\rangle$ *is a* tBa *such that, for any* $a, b \in \mathbf{B}$, $\mathbb{I}(a \vee \mathbb{I}(b)) = \mathbb{I}(a) \vee \mathbb{I}(b)$*, then it is called a* "monadic topological Boolean algebra (mtBa)*".*

Corollary 13.2.3. *Any monadic Boolean algebra of sets* $\langle \mathbf{B}(U), L_R\rangle$ *is a mtBa of sets.*

Proof. From *Definition* 12.1.6.(2) and *Proposition* 13.2.1 the proof follows. QED

Proposition 13.2.3. *For all tBa* $\langle \mathbf{A}, L\rangle$, $\mathbf{L}(\mathbf{A})$ *is a sublattice of* $\mathbf{A}$.

Proof. Let $x, y \in \mathbf{L}(\mathbf{A})$. Then $x \wedge y = L(x) \wedge L(y)$ (from idempotence of L. Thus $x \wedge y = L(x \wedge y)$. Since for all $a, b \in \mathbf{A}$, $L(a) \vee L(b) \leq L(a \vee b)$, if $x, y \in \mathbf{L}(\mathbf{A})$, $x \vee y \leq L(x \vee y)$. But $L(x \vee y) \leq x \vee y$ (because L is deflationary). It follows that $x \vee y = L(x \vee y)$. QED

Proposition 13.2.4. *For all mtBa* $\langle \mathbf{A}, L\rangle$, $\mathbf{L}(\mathbf{A})$ *is a Boolean algebra.*

Proof. We have to prove that $\mathbf{L}(\mathbf{A})$ is closed under complementation. Let $x \in \mathbf{L}(\mathbf{A})$. Since $x \wedge -x = 0$ we have $L(x \wedge -x) = 0$ (from the deflationary property). Thus, $L(x \wedge -x) = L(x) \wedge L(-x) = 0$. Moreover, $x \vee -x = 1$ so that $L(L(x) \vee -x) = 1$ (because $x = L(x)$). From monadicity, $L(x) \vee L(-x) = x \vee L(-x) = 1$. Thus $L(-x)$ is the complement of x, because $\mathbf{L}(\mathbf{A})$ is a sublattice of $\mathbf{A}$. We conclude that $L(-x) = x$ and $\mathbf{L}(\mathbf{A})$ is closed under complementation. QED

Corollary 13.2.4. *In any mtBa* $\langle \mathbf{A}, L\rangle$, $\mathbf{M}(\mathbf{A}) = \{M(x) : x \in \mathbf{A}\}$ *coincides with* $\mathbf{L}(\mathbf{A})$.

Proof. For all $x \in \mathbf{A}$, $M(x) \in \mathbf{L}(\mathbf{A})$. In fact, $M(x) = -L(-x)$. But from *Proposition* 13.2.4 $-L(-x) \in \mathbf{L}(\mathbf{A})$. For all $y \in \mathbf{A}$, $L(x) \in \mathbf{L}(\mathbf{A})$: dually. QED

Corollary 13.2.5. *In any mtBa* $\langle \mathbf{A}, L\rangle$*, (a)* $ML(x) = L(x)$*; (b)* $LM(x) = M(x)$*, any* $x \in \mathbf{A}$.

Proof. (a) From *Proposition* 13.2.4, for all $x \in \mathbf{A}$, $-L(x) \in \mathbf{L}(\mathbf{A})$. Therefore, $-L(x) = L - L(x) = -ML(x)$. It follows that $L(x) = ML(x)$. (b) By duality. QED

Proposition 13.2.5. *Any* mtBa *is a model for the modal system* **S***5.*

As to the proof see, for instance, Rasiowa [1974], Chapter XIII, where **S**5 is called $\mathcal{S}_{\lambda 5}$. This result confirms what we have discussed in Section 12.2: **S**5 modal systems are characterised by symmetric, reflexive and transitive binary relations, i.e., equivalence relations.

The following results link the definitions of lower and upper approximations to the fact that $\langle \mathbf{B}(U), \Omega_E(U)\rangle$ is a modal system:

Corollary 13.2.6. *Let* $\langle U, \Omega_E(U)\rangle$ *be a topological space associated with an equivalence relation* $E \subseteq U \times U$ *and let* $\langle U, E\rangle$ *be the Indiscernibility Space based on* E*. Then for any* $X \in \mathbf{B}(U)$*,*

1. $L_E(X) = \mathbb{I}_E(X) = \bigcup\{Y : Y \in \Omega_E \ \& \ Y \subseteq X\} = \bigcup\{[x]_E : [x]_E \subseteq X\} = (lE)(X)$.
2. $M_E(X) = \mathbb{C}_E(X) = \bigcap\{Y : Y \in \Omega_E \ \& \ Y \supseteq X\} = \bigcap\{[x]_E : X \subseteq [x]_E\} = (uE)(X)$.
3. (i) $L_E(M_E(X)) = M_E(X)$; (ii) $M_E(L_E(X)) = L_E(X)$.

Proof. (**1**) and (**2**) come straightforward from the above results. (**3**) $L_E(X)$ is an open set. Thus it has the form $E(Y)$ for some $Y \subseteq U$ (namely $Y = \bigcup\{E(x) : E(x) \subseteq X\}$. Therefore, $M_E(L_E(X)) = M_E(E(Y)) = E^{\smile}(E(Y)) = E(E(Y)) = E(Y) = L_E(X)$. Dually for the first equation. QED

Corollary 13.2.7. *Let* $\langle U, \mathbf{AS}(U)\rangle$ *be an Approximation Space. Then* $\langle \mathbf{B}(U), (lE)\rangle$ *(or* $\langle \mathbf{B}(U), (uE)\rangle$*) is a* mtBa *of sets.*

The two axioms which characterise **S**5, that is $L(M(\alpha)) \longleftrightarrow M(\alpha)$ and $M(L(\alpha)) \longleftrightarrow L(\alpha)$ say, in logical terms, that any string $(m_1, ..., m_n)$ of nested modal operators $m_i \in \{[R], \langle R\rangle\}$, collapses into the one-term string m_n. In our Rough Set reading, this collapse says that any single approximation of a subset of the universe of discourse provides an *exact set*, that is a set invariant under further approximations.

At this point we can list a series of connections between some fundamental results we have proved so far.

- From a topological point of view, in any mtBa, for all $x \in \mathbf{L}(\mathbf{A})$, $x = \mathbb{I}C(x)$ (from *Corollary* 13.2.5). Hence any $x \in \mathbf{L}(\mathbf{A})$ is a regular element [cf. Subsection 7.3.1 of Chapter 7].

- Any Approximation Space $\mathbf{AS}(U/E)$ is a Boolean subalattice of the Boolean algebra of $\wp(U)$.
- In **S**5 modal systems, $ML(\alpha) \longleftrightarrow L(a)$ and $LM(\alpha) \longleftrightarrow M(a)$, any formula α) [cf. Table 12.3 of Section 12.2].
- In any Approximation Space $\mathbf{AS}(U/E)$, for any $X \subseteq U, (uE)(X)$ and $(lE)(X)$ are exact elements. Hence, $(lE)(uE)(X) = (uR)(X)$ and $(uR)(lE)(X) = (lE)(X)$.
- $M_R \dashv L_R$ (because $R = R^{\smile}$, from *Proposition* 13.2.1 and *Proposition* 13.2.1) [cf. *Corollary* 8.2.1 of Chapter 8].

Example 13.2.1. A topological Boolean algebra and a monadic topological Boolean algebra

The pre-monadic Boolean algebra $\langle \mathbf{A}, L_2 \rangle$ of Example 12.1.4 is a topological Boolean algebra. The structure $\langle \mathbf{A}, L' \rangle$ with the operator L' below, is a monadic topological Boolean algebra:

x	0	a	b	c	d	e	f	1
$L'(x)$	0	0	b	0	b	e	b	1

The sublattices $\mathbf{L}'(\mathbf{A})$ of the images of the operator L' coincides with that of the monadic Boolean algebra $\langle \mathbf{A}, L_m \rangle$ of Example 12.1.5. However, $\langle \mathbf{A}, L_m \rangle$ is not topological because, for instance, $L_m L_m(d) = 0 \neq b = L_m(d)$ (i.e. L_m is not idempotent).
