Chapter 12

Modalities and Relations

12.1 Modal Systems and Binary Relations

Definition 12.1.1. *Let U be a set and R a binary relation on U, a map $f : \wp(U) \longmapsto \wp(U)$ is said to be* connected with R *if for any $X \in \wp(U)$, $f(X) = R(X) = \{y \in U : \exists x(x \in X \ \&\ \langle x, y\rangle \in R)\}$.*

In view of this definition we can say that finite distributive modal systems can be made into isomorphic k-modal systems and can be represented by k-modal systems where the standard knowledge map k^* is connected with a preorder relation. In fact, $k^*(X) = \uparrow_{\preceq} X = \preceq (X)$.

In this way we have, partially, answered the questions:

(A) "Given an abstract modal system $\langle \mathbf{S}, \mathbf{S}' \rangle$ is there a k-modal system $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ isomorphic to it?"

(B) "Given a k-modal system $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ is there a representation $\langle \mathbf{A}, k^*(\mathbf{A}) \rangle$ such that $\mathbf{A}$ is an algebra of subsets of a universe U and the knowledge map k^* is connected with a binary relation R on U?"

Now we reverse the starting point:

(A') "Given an algebra $\mathbf{A}$ of subsets of a universe U, a binary relation R on U, and a function f connected with R, is the pair $\langle \mathbf{A}, f(\mathbf{A}) \rangle$ a k-modal system? If yes, how do its modal properties vary, depending upon the properties of R?".

(B') "Given a k-modal system $\langle \mathbf{A}, k(\mathbf{A}) \rangle$ such that $\mathbf{A}$ is an algebra of subsets of a universe U, is there a relation R on U such that the knowledge map k is connected with R?".

In other terms, we want to know (i) what relationships exist between binary relations and knowledge maps, (ii) what relationships exist between the properties of a binary relation and the k-modal system connected with it (if any).

In view of our main topic, Approximation Spaces, we shall solve limited instances of the above problems, namely when $\mathbf{A}$ is a Boolean algebra of sets.

Therefore, henceforth, if not otherwise stated, the role of $\mathbf{A}$ will be played by the Boolean algebra of sets $\mathbf{B}(U) = \langle \wp(U), \cap, \cup, -, U, \emptyset \rangle$ for some universe U and R will denote a binary relation on $U : R \subseteq U \times U$.

Terminology and Notation. From now on the entities that populate the elements of an algebra of sets will be called "points" (or "elements" when they appear in sentences mentioning the set they belong to). The set of all points will be denoted by means of our familiar notation U (for "universe of discourse"; indeed, U plays the role of G in Part I. Here instead of the set of "Gegenstände", we prefer the more abstract notion of a "universe").

Now it is worthwhile recalling some properties of R-neighborhoods, $R(\ )$ i.e. $\langle R^{\smile} \rangle$, from Part I:

Proposition 12.1.1.

1. *Given a binary relation $R \subseteq U \times U$, the R-neighborhood $R(\ ) = \langle R^{\smile} \rangle$ is lower adjoint of $[R]$ with respect to the structure (small category) $\langle \wp(U), \subseteq \rangle$. Hence,*
2. *$R(\ )$ is continuous: $R(X) \cup R(Y) = R(X \cup Y)$,*
3. *R is normal: $R(\emptyset) = \emptyset$,*
4. *R is isotonic: $X \subseteq Y$ implies $R(X) \subseteq R(Y)$. Moreover,*
5. *R is co-discontinuous: $R(X \cap Y) \subseteq R(X) \cap R(Y)$.*

Point 2 is a direct consequence of *Proposition* 1.4.8.(2), because $R(\ )$ is a lower adjoint.

Obviously, the same holds for $R^{\smile}-$neighborhoods, i.e. $\langle R \rangle$.

In modal contexts, points are usually called *possible worlds*, *information states* or *states of affairs*.[1] The binary relation R, generally has the following meaning: if $\langle x, x' \rangle \in R$, then x' is a possible evolution of

[1] Or, sometimes, "knowledge states" (of a subject). However we use the term "knowledge" to denote a particular pattern of information states.

the state x (or x' represents a world that is conceivable from x, or an enrichment of the information in x). Usually if $\langle x, x' \rangle \in R$, then we say that x' is *accessible* from x. However, we shall see interpretations which are more perspicuous for our context. We shall develop this point later on. By now, consider that R induces a particular *geometry* on the set of points U, which is well represented by the space $\langle U, R \rangle$, which, for historical reasons, is called a *Kripke frame* (cf. Frame 4.13 of Part I).[2]

We hardly can deny, at this point, that at the end of this story we shall find not a generic relation R, but our familiar equivalence relation E. This is obvious. However it is better we reach that point gradually, passing through some preliminary steps illustrating how an equivalence relation is only one among a number of other interesting possibilities.

From the definitions of R-neighborhoods and $R^{\smile}$-neighborhoods, it is clear that for any $X \subseteq U$ there is only one X' such that $X' = R(X)$ and only one X'' such that $X'' = R^{\smile}(X)$. So we can define two functions from $\wp(U)$, *qua* carrier of $\mathbf{B}(U)$, to $\wp(U)$, *qua* range of the relation R, as follows:

- $f : \wp(U) \longmapsto \wp(U), f(X) = R(X)$ – that is, f is connected with R.
- $h : \wp(U) \longmapsto \wp(U), g(X) = R^{\smile}(X)$ – that is, h is connected with $R^{\smile}$.

Now, the first three points of *Proposition 12.1.1* tell us that any function connected with a binary relation is a knowledge map. In view of this fact, we can restate the definitions of the modal operators L_k and M_k using $R-$neighborhoods to provide these operators with a specific meaning based on the properties exhibited by the binary relation which is connected with k.[3]

Lemma 12.1.1. *Let* $\langle \mathbf{B}(U), k(\mathbf{B}(U)) \rangle$ *be a k-modal system such that k is connected with some relation* $R \subseteq U \times U$*, i.e.* $k(\mathbf{B}(U)) = \{R(X) :$

[2]For some specific purposes, also ternary relations are used (cf. [Allwein-Dunn 1993] as to Kripke models for Linear Logic or [Anderson et al. 1992] as to Kripke models for Relevant Logics). In these cases, the sentence "$\langle x, y, z \rangle \in R$" usually reads: "the information in x combined with the information in y, outputs the information in z". That is, $z = x \circ y$ where "$\circ$" is a monoidal operator. Also, this approach is connected with Phase Semantics for Linear Logic (see [Abrusci 1991]).

[3]If we do not assume $\mathbf{S} = \mathbf{B}(U)$, but we let $\mathbf{S}$ be a sublattice of $\mathbf{B}(U)$, then $f : S \longmapsto \wp(U)$ and it may happen, for some $X \subseteq U$, that $R(X) \notin S$ or $R^{\smile}(X) \notin S$ (we recall that S is the carrier of $\mathbf{S}$). So these definitions must be generalised. For instance, we can adopt (i) $f(X) = \bigcup\{X' \in S : X' \subseteq R(X)\}$ and (ii) $g(X) = \bigcup\{X' \in S : X' \subseteq R^{\smile}(X)\}$.

$X \subseteq U\}$, and g is connected with $R^{\smile}$. Then, $\forall X, Y, Z \in \mathbf{B}(U), \forall x, y \in U$*:*

1. $L_k(X) = \bigcup\{Z : R(Z) \subseteq X\}$.
2. $x \in L_k(X)$ *iff* $\forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \in X)$.
3. $M_k(X) = \bigcup\{\{x\} : R(\{x\}) \cap X \neq \emptyset\}$.
4. $x \in M_k(X)$ *iff* $\exists y \in U(\langle x, y\rangle \in R \;\&\; y \in X)$.

Proof. (**1**): From *Definition* 11.5.2.(1), by substituting X for a, Z for p and $\subseteq$ for $\leq$.
(**2**): From (1) we obtain $L_k(X) = \bigcup\{Z : \forall x \in Z, \forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \in X)\}$. Since we have no restrictions on Z, we get $L_k(X) = \{x : \forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \in X)\}$. Hence the thesis. (**3**) and (**4**): From *Definition* 11.5.2.(2), $M_k(X) = \bigcup\{Z : \exists Z'(g(Z') \supseteq Z \;\&\; Z' \subseteq X)\} = \bigcup\{Z : \exists Z'(R^{\smile}(Z') \supseteq Z \;\&\; Z' \subseteq X)\}$. But if $Z' \subseteq X$, from monotonicity $R^{\smile}(Z') \subseteq R^{\smile}(X)$, so that if $Z \subseteq R^{\smile}(Z')$ then $Z \subseteq R^{\smile}(X)$. Therefore, $M_k(X) = \bigcup\{z : \forall z(z \in Z \Rrightarrow \exists x(\langle z, x\rangle \in R \;\&\; x \in X))\}$. But the condition on z is equivalent to $\forall z(z \in Z \Rrightarrow R(z) \cap X \neq \emptyset)$. Henceforth we have: $M_k(X) = \bigcup\{\{x\} : R(\{x\}) \cap X \neq \emptyset\}$ and $x \in M_k(X)$ iff $\exists y \in U(\langle x, y\rangle \in U \;\&\; y \in X)$.[4] QED

Corollary 12.1.1. $\forall X \in \mathbf{B}(U)$*:*
(a) $L_k(X) = [R](X)$*;* (b) $M_k(X) = \langle R\rangle(X)$ *(where* $[R]$ *and* $\langle R\rangle$ *are the operators defined in Section 2.1.2 of Chapter 2).*

Definition 12.1.2. *Given a modal system connected with a relation R, the modal operators L_k and M_k will be denoted by L_R and M_R or $[R]$ and, respectively, $\langle R\rangle$. Moreover, the modal operators will be denoted by L and M when any reference to the relation R is understood or irrelevant.*

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[4]In the general case, that is, when we deal with a system $\langle \mathbf{S}, k(\mathbf{S})\rangle$ where $\mathbf{S}$ is a sublattice of $\mathbf{B}(U)$, not every subset of U is an element of $\mathbf{S}$. Hence we can have elements Z such that although $R(Z)$ is included in X, Z is not an element of $\mathbf{S}$. Therefore points *1* and *3* of the *Lemma* are valid only with the additional constraint: $\bigcup\{Z : Z \in \mathbf{S} \;\& \ldots\}$, and so on. Moreover, in this case points *2* and *4* are valid only from left to right:
(2') If $x \in L_k(X)$ then $\forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \in X)$.
(4') If $x \in M_k(X)$ then $\exists y \in U(\langle x, y\rangle \in R \;\&\; y \in X)$.

From the results of Part I and *Corollary* 12.1.1 we have:

Corollary 12.1.2. *For any* $X \in \mathbf{B}(U), -M_R(-X) = L_R(X)$*;* $-L_R(-X) = M_R(X)$.

Now we translate the forcing clauses over algebraic structures into forcing clauses over Kripke frames. Since we are dealing with modal systems where the algebraic operations form a Boolean algebra (of sets) $\mathbf{B}(U)$, we shall use *Proposition* 11.3.1.(4) in order to understand what happens to forcing at a point level. Therefore, given an evaluation ϕ from a modal language $\mathcal{L}$ to $\mathbf{B}(U)$, the translation will be lead by the obvious idea that a point x forces a formula α, in symbols $x \Vdash \alpha$ if x belongs to an element X of $\mathbf{B}(U)$ that algebraically forces α, $X \vDash \alpha$.

The translation will be accomplished through two Lemmata: the first will link the algebraic operations induced by an evaluation ϕ with the forcing relation $\Vdash$ between points and formulas, the second Lemma will use this link to list the forcing clauses of $\Vdash$ for any logical constants. The result will be summed up in *Window* 12.1.

Terminology and Notation. From now on, by $\mathcal{L}$ we shall intend a propositional language with Boolean constants $\wedge, \vee, \neg, \rightarrow, 0, 1$ and modal constants L and M, while $\alpha, \alpha', \beta, \beta'$ and so on, will vary over well-formed formulas. Notice that results on the material implication $\rightarrow$ will be sometimes omitted since it fulfills the definition $\alpha \rightarrow \beta =_{def} \neg\alpha \vee \beta$.

Lemma 12.1.2. *Let* $\langle U, R\rangle$ *be a Kripke frame,* ϕ *an evaluation map from a modal language* $\mathcal{L}$ *to a k-modal system* $\langle \mathbf{B}(U), k(\mathbf{B}(U))\rangle$*, and let* k *be connected with* R*. For any element* $x \in U$*, for any formula* $\alpha \in \mathcal{L}$*, let us set:* $x \Vdash \alpha$ *if and only if there is an element* X *of* $\mathbf{B}(U)$ *such that* $x \in X$ *and* $X \vDash \alpha$*. Then, for any formula* $\alpha, \alpha' \in \mathcal{L}$*, for all* $x \in U$*:*

1. $x \Vdash \alpha \wedge \alpha'$ *iff* $x \in \phi(\alpha) \cap \phi(\alpha')$.
2. $x \Vdash \alpha \vee \alpha'$ *iff* $x \in \phi(\alpha) \cup \phi(\alpha')$.
3. $x \Vdash \neg\alpha$ *iff* $x \in -\phi(\alpha)$.
4. $x \Vdash \alpha \rightarrow \alpha'$ *iff* $x \in -\phi(\alpha) \cup \phi(\alpha')$.
5. $x \Vdash L_R(\alpha)$ *iff* $x \in \{x' : \forall y \in U(\langle x', y\rangle \in R \Rrightarrow y \in \phi(\alpha)\}$.
6. $x \Vdash M_R(\alpha)$ *iff* $x \in \{x' : \exists y \in U(\langle x', y\rangle \in R \;\&\; y \in \phi(\alpha)\}$.

Proof.

(A) Boolean part of the proof:
by means of *Proposition* 11.3.1.(4) and the definition of function ϕ (see *Window* 11.1), we obtain the result straightforwardly. The detailed proof is left to the reader. (*Hints*: first, notice that the thesis' assumption reads $x \Vdash \alpha$ iff $x \in \bigcup\{X : X \subseteq \phi(\alpha)\}$, so that from *Proposition* 11.3.1.(4), after substituting $\subseteq$ for $\leq$ we obtain $x \Vdash \alpha$ iff $x \in \phi(\alpha)$. Therefore, for instance, from the definition of function ϕ in *Window* 11.1, $x \Vdash \alpha \wedge \alpha'$ iff $x \in \phi(\alpha \wedge \alpha')$, iff $x \in \phi(\alpha) \cap \phi(\alpha')$, and so on. The reader must only pay attention that in $\mathbf{B}(U)$ "$\neg$" is the set-theoretical complementation).

(B) Modal part of the proof (actually a corollary of *Lemma* 12.1.1): $x \Vdash L_R(\alpha)$ iff $x \in \bigcup\{X : k(X) \vDash \alpha\}$, iff $x \in \bigcup\{X : k(X) \subseteq \phi(\alpha)\}$, iff $x \in \bigcup\{\{X : R(X) \subseteq \phi(\alpha)\}$, iff $\forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \in \phi(\alpha))$; $x \Vdash M_R(\alpha)$ iff $x \in \bigcup\{X : \exists X'(g(X') \supseteq X \;\&\; X' \subseteq \phi(\alpha)\}$, iff $x \in \bigcup\{X : \exists X'(R^{\smile}(X') \supseteq X \;\&\; X' \subseteq \phi(\alpha)\}$, iff $\exists y \in U(\langle x, y\rangle \in R \;\&\; y \in \phi(\alpha))$. QED

Proposition 12.1.2. *Under the assumptions of Lemma 12.1.2, for any formula* $\alpha, \alpha' \in \mathcal{L}$*, for all* $x \in U$*:*

1. $x \Vdash \alpha \wedge \alpha'$ *iff* $x \Vdash \alpha \;\&\; x \Vdash \alpha'$.
2. $x \Vdash \alpha \vee \alpha'$ *iff* $x \Vdash \alpha$ *or* $x \Vdash \alpha'$.
3. $x \Vdash \neg\alpha$ *iff* $x \nVdash \alpha$.
4. $x \Vdash \alpha \to \alpha'$ *iff* $x \Vdash \neg\alpha$ *or* $x \Vdash \alpha'$.
5. $x \Vdash L_R(\alpha)$ *iff* $\forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \Vdash \alpha)$.
6. $x \Vdash M_R(\alpha)$ *iff* $\exists y \in U(\langle x, y\rangle \in R \;\&\; y \Vdash \alpha)$.
7. $x \Vdash L_R(\alpha)$ *iff* $x \Vdash \neg M_R(\neg\alpha)$*;* $x \Vdash M_R(\alpha)$ *iff* $x \Vdash \neg L_R(\neg\alpha)$.

Proof. From the preceding *Lemma*: (1) $x \Vdash \alpha \wedge \alpha'$ iff $x \in \phi(\alpha) \cap \phi(\alpha')$, iff $x \in \phi(\alpha)$ and $x \in \phi(\alpha')$, iff for some $X, X' \in \mathbf{B}(U)$ such that $X \vDash \alpha$ and $X' \vDash \alpha'$, $x \in X$ and $x \in X'$, iff $x \Vdash \alpha \;\&\; x \Vdash \alpha'$. (2) dually, by substituting $\cup$ for $\cap$ and $\vee$ for $\wedge$. (3) $x \Vdash \neg\alpha$ iff $x \in -\phi(\alpha)$, iff $x \notin \phi(\alpha)$, iff $x \nVdash \alpha$. (4) straightforward from (2) and (3) and the fact that $\phi(\alpha \to \alpha') = -\phi(\alpha) \cup \phi(\alpha')$. (5) $x \Vdash L(\alpha)$ iff $\forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \in \phi(\alpha))$, iff $\forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \Vdash \alpha)$. (6) $x \Vdash M_R(\alpha)$ iff

$\exists y \in U(\langle x,y\rangle \in R \ \&\ y \in \phi(\alpha))$, iff $\exists y \in U(\langle x,y\rangle \in R \ \&\ y \Vdash \alpha)$. (7) The proof is left to the reader [Hints: use the first order equivalences $\forall \equiv \neg\exists\neg$ and $\neg\forall\neg \equiv \exists$]. QED

Therefore, thanks to the above *Proposition* 12.1.2, we have the following set of forcing clauses over Kripke frames:

Let $\mathcal{L}$ be a propositional modal language and let $\langle U, R\rangle$ be a Kripke frame. Let ϕ be a set-up: $\hat{\phi} : \mathcal{L} \longmapsto \wp(U)$. For any point $x \in U$, for any formula $\alpha, \alpha' \in \mathcal{L}$ we set the following forcing clauses: 1. $x \Vdash \alpha$ iff $x \in \hat{\phi}(\alpha)$, for α atomic. 2. $x \Vdash \alpha \wedge \alpha'$ iff $x \Vdash \alpha \ \&\ x \Vdash \alpha'$. 3. $x \Vdash \alpha \vee \alpha'$ iff $x \Vdash \alpha$ or $x \Vdash \alpha'$. 4. $x \Vdash \neg\alpha$ iff $x \nVdash \alpha$. 5. $x \Vdash L_R(\alpha)$ iff $\forall y \in U(\langle x,y\rangle \in R \Rrightarrow y \Vdash \alpha)$. 6. $x \Vdash M_R(\alpha)$ iff $\exists y \in U(\langle x,y\rangle \in R \ \&\ y \Vdash \alpha)$. The triple $\langle U, R, \Vdash\rangle$, with the above clauses for $\Vdash$, is called a *Kripke model for modal logic*

Window 12.1. **Forcing over Kripke frames**

From *Lemmata* 12.1.1 and 12.1.2, it follows that once again we can confine our attention to the Boolean set-theoretical operations and define two monadic operators L_R and M_R ranging on subsets of U. In this way, we avoid any reference to the language $\mathcal{L}$ and its formulae.

Otherwise stated, we can associate to Kripke models *Boolean algebras of sets with additional monadic operators*:

Definition 12.1.3. *Let* $\mathbf{B}(U)$ *be the Boolean algebra of* $\wp(U)$*. Let* $R \subseteq U \times U$*. Then* $\langle \mathbf{B}(U), L_R, M_R\rangle$ *is called a Pre-monadic Boolean algebra of sets.*

Strictly speaking, in order to denote a Pre-monadic Boolean algebra of sets, $\langle \mathbf{B}(U), L_R\rangle$ (or $\langle \mathbf{B}(U), M_R\rangle$) suffices, since the two monadic operators are dual *via* the Boolean complementation.

Remarks. Pay attention that in general L_R (i.e. $[R]$) and M_R (i.e. $\langle R\rangle$) are not adjoint to each other, because $[R]$ is adjoint to $\langle R^{\smile}\rangle$ and $[R^{\smile}]$ is adjoint to $\langle R\rangle$

From the above results we have the following statement, linking algebraic forcing and point-based forcing:

Proposition 12.1.3. *Let* $\langle \mathbf{B}(U), L_R \rangle$ *be a Pre-monadic Boolean algebra of the powerset of a set* U. *Let* ϕ *be an evaluation map from a modal language* $\mathcal{L}$ *to* $\mathbf{B}(U)$. *Then for any formula* $\alpha \in \mathcal{L}$, $\phi(\alpha) = U$ *if and only if* $\forall x \in U, x \Vdash \alpha$.

It is not difficult to derive the abstract (i.e. algebraic) properties of the modal operators, thanks to the following results:

Proposition 12.1.4. *Let* $\langle \mathbf{B}(U), L_R \rangle$ *be a Pre-monadic Boolean algebra of sets. Then, for any* $X, Y \subseteq U$,

L1. $L_R(U) = U$.

L2. $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$.

L3. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$.

L4. *if* $X \subseteq Y$ *then* $L_R(X) \subseteq L_R(Y)$.

M1. $M_R(\emptyset) = \emptyset$.

M2. $M_R(X \cap Y) \subseteq M_R(X) \cap M_R(Y)$.

M3. $M_R(X \cup Y) = M_R(X) \cup M_R(Y)$.

M4. *if* $X \subseteq Y$ *then* $M_R(X) \subseteq M_R(Y)$.

Proof. In view of *Corollary* 12.1.1, from the adjunction relations $M_{R^\smile} \dashv L_R$ and $M_R \dashv L_{R^\smile}$ that can be derived from *Proposition* 2.1.1 of Chapter 2. QED

Therefore, in a more abstract framework we shall set:

Definition 12.1.4. *Let* $\mathbf{A}$ *be a Boolean algebra. Let* L *be a monadic operator on* $\mathbf{A}$ *such that:*

1. $L(1) = 1$ – *L-conormality.*
2. $L(a \wedge b) = L(a) \wedge L(b)$ – *L-cocontinuity (or multiplicativity).*
3. $L(a) \vee L(b) \leq L(a \vee b)$ – *L-discontinuity.*

Then the structure $\langle \mathbf{A}, L \rangle$ *is called a* Pre-monadic Boolean algebra.

Proposition 12.1.5. *Let* $\langle \mathbf{A}, L \rangle$ *be a Pre-monadic Boolean algebra. Let* M *be a monadic operator defined, for any* $a \in \mathbf{A}$*, by* $M(a) = \neg L(\neg a)$*. Then for any* $a, b \in \mathbf{A}$*:*

1. $M(0) = 0$ *– M-normality.*
2. $M(a \vee b) = M(a) \vee M(b)$ *– M-continuity (or additivity).*
3. $M(a \wedge b) \leq M(a) \wedge M(b)$ *– M-codiscontinuity.*
4. $a \leq b$ *implies* $M(a) \leq M(b)$ *and* $L(a) \leq L(b)$ *– monotonicity.*

Thus L and M are comodal and, respectively, modal operators in the sense of *Definition* 1.4.3 of Chapter 1.

Definition 12.1.5. *Given a Pre-monadic Boolean algebra* $\langle \mathbf{A}, L \rangle$*, set* $L(A) = \{L(a) : a \in A\}$*,* $\wedge_L = \wedge \upharpoonright L(A)$ *and* $\vee_L = \vee \upharpoonright L(A)$*. Then we set* $\mathbf{L}(\mathbf{A}) = \langle L(A), \wedge_L, \vee_L, 1, 0 \rangle$*.*

Now we can notice that the sublattice $\mathbf{L}(\mathbf{A})$ is not necessarily distributive.

At this point we add stronger properties to the monadic operator, obtaining the notion of a *Monadic Boolean algebra*, that will be of central importance in our story:

Definition 12.1.6. *Let* $\langle \mathbf{A}, L \rangle$ *be a Pre-monadic Boolean algebra such that:*

1. $L(a \vee L(b)) = L(a) \vee L(b)$ *– monadic L-continuity.*
2. $L(a) \wedge a = L(a)$ *– L-deflationary property.*

Then $\langle \mathbf{A}, L \rangle$ *is called a* Monadic Boolean algebra*.*

We have to notice that *Property* 12.1.4.(2) is now derivable from the others.

Let us list other important properties of Monadic Boolean algebras:

Proposition 12.1.6. *Let* $\langle \mathbf{A}, L \rangle$ *be a Monadic Boolean algebra. Define, for all* $a \in A$ $M(a)$ *as* $-L(-a)$*. Then:*

1. $M(a \wedge M(b)) = M(a) \wedge M(b)$ *– monadic M cocontinuity.*
2. $a \wedge M(a) = a$ *– M-inflationary property.*

Example 12.1.1. Modal operators induced by R-neighborhoods

As an example of a structure equipped with a binary relation consider the set $U = \{x, y, z\}$ and the following relation $R \subseteq U \times U$:

R	x	y	z
x	0	1	1
y	0	1	0
z	0	0	1

Just by inspecting rows, we can see examples of some definitions and properties:

(a) R-neighborhoods: $R(x) = \{y, z\}, R(\{x, z\}) = \{y, z\}$.
(b) Monotonicity: $\{y\} \subseteq \{x, y\}$. $R(\{y\}) = \{y\} \subseteq \{y, z\} = R(\{x, y\})$.
(c) Continuity: $R(\{x\}) \cup R(\{y\}) = \{y, z\} \cup \{y\} = \{y, z\} = R(\{x, y\}) = R(\{x\} \cup \{y\})$.
(d) 0-preservation: $R(\emptyset) = \emptyset$ (on the contrary, $R(U) = \{y, z\} \neq U$).
Let us now compute some applications of the operators L_R and M_R:

(*)$L_R(\{x, z\}) = \{z\}$. Indeed: $R(\{z\}) = \{z\} \subseteq \{x, z\}, R(\{y\}) = \{y\} \nsubseteq \{x, z\}$, $R(\{x\}) = \{y, z\} \nsubseteq \{x, z\}$.

Notice that thanks to the continuity property, we obtain the result by gathering all the element of U whose R-neighborhoods are included in $\{y, z\}$. A better way in order to compute L_R is based on the duality $L_R(X) = -M_R(-X)$, any X.

(**) $M_R(\{x\}) = \emptyset, M_R(\{z\}) = \{x, z\}, -M_R(-\{x, z\}) = -M_R(\{y\}) = -\{x, y\} = L_R(\{x, z\})$.

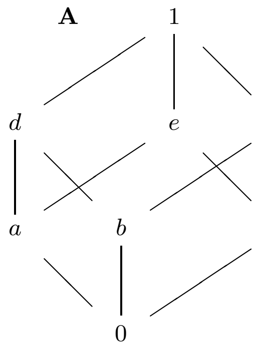
Now, consider the following set-up $\widehat{\phi}(A) = \{x, z\}$, $\widehat{\phi}(B) = \{z, y\}$, $\widehat{\phi}(C) = \{x, y\}$. From it we have:

- $z, y \models B$, $z, x \models A$, $x, y \models C$;
- $z \models A \wedge B$ (because $z \in \widehat{\phi}(A)$ and $z \in \widehat{\phi}(B)$):
- $y \models L_R(C)$ (because $R(\{y\}) = \{y\}$ and $\{y\} \subseteq \{x, y\} = \hat{\phi}(C)$. Otherwise stated, all the elements R-accessible from y force C. In this case the only element accessible from y is y itself). On the contrary, although $x \models C$, $x \not\models L_R(C)$ because $R(x) = \{y, z\} \nsubseteq \hat{\phi}(C)$ (indeed, $z \not\models C$).
- $x \models M_R(A \wedge B)$ (because $\langle x, z\rangle \in R$ and $z \in \phi(A) \cap \phi(B)$, so that $z \models A \wedge B$. Otherwise stated, there is an element R-accessible from x that forces $A \wedge B$).

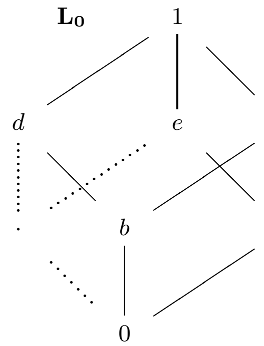
Exercise 12.1. *Let $\langle U, R\rangle$ be a relational structure. Without using the adjunction properties of $R()$, but pure logical deductions, prove isotonicity, normality, continuity and co-discontinuity of $R()$.*

Example 12.1.2. Example of a Boolean algebra with operator which is not pre-monadic

Consider the Boolean algebra $\mathbf{A}$ depicted below in the diagram on the left. Let us suppose that we are given the following table for an operator L_0. Then on the right we draw the resulting substructure $\mathbf{L_0} = \mathbf{L}(\mathbf{A})$, embedded in $\mathbf{A}$:



x	$L_0(x)$
0	0
a	b
b	0
c	c
d	d
e	f
f	e
1	1



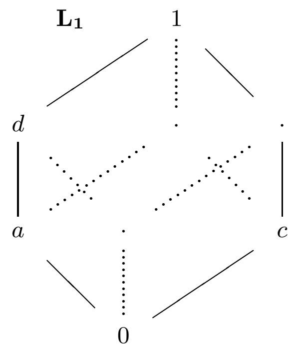
The structure $\langle \mathbf{A}, L_0 \rangle$ is a not a pre-monadic Boolean algebra. In fact, from the above table we can easily verify that $L_0(1) = 1$ and that the monotonicity property holds. But $L-$co-continuity fails: $L_0(d \wedge e) = 0 \neq b = d \wedge f = L_0(d) \wedge L_0(e)$.

Exercise 12.2. *(a) Compute $M_0(x)$ for any $\in \mathbf{A}$. (b) Find an example of $M-$codiscontinuity. (c) Can you find a Boolean algebra of sets $\mathbf{A}'$ such that $\mathbf{L}_R(\mathbf{A}')$ isomorphic to $\mathbf{L}_0$ for some binary relation $R \subseteq A \times A$?*

Example 12.1.3. Example of a pre-monadic Boolean algebra

Consider the Boolean algebra $\mathbf{A}$ depicted in Example 12.1.2. Let us suppose that we are given the following table for an operator L_1. Then on the right we draw the resulting substructure $\mathbf{L_1} = \mathbf{L}(\mathbf{A})$:

x	$L_1(x)$
0	0
a	0
b	a
c	0
d	d
e	c
f	a
1	1



It is evident that $\mathbf{L_1}$ is not distributive. By easy inspection we see that L is monotonic (for instance, $b \leq f$ and $L(b) = L(f) = a$).

Let us verify a case of $L-$cocontinuity and a case of $L-$discontinuity:
$L_1(d \wedge f) = L_1(b) = a = d \wedge a = L_1(d) \wedge L_1(f)$.
$L_1(d \vee f) = L_1(1) = 1 \geq d \vee a = L_1(d) \vee L_1(f)$.

However, $\langle \mathbf{A}, L_1 \rangle$ is not a Monadic Boolean algebra.

- Let us verify that $L(x) \leq x$ is not uniformly valid: $L_1(b) = a \nleq b$.
- Let us verify that the equality $L(x \vee L(y)) = L(x) \vee L(y)$ is not uniformly valid: $L_1(a \vee L_1(f)) = L_1(a \vee a) = L_1(a) = 0 \neq a = 0 \vee a = L_1(a) \vee L_1(f)$.

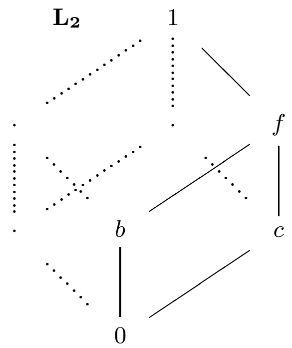
Exercise 12.3.
(a) Compute the table of M_1.
(b) Find a case of $M-$co-discontinuity.

(c) Find a case which invalidates the monadic $M-$co-continuity Property.
(d) Find a Boolean algebra of sets $\mathbf{A}'$ with top element a set U, such that $\mathbf{A}'$ is isomorphic to the above Boolean algebra $\mathbf{A}$ and a relation $R \subseteq U \times U$ such that $\mathbf{L}_R(\mathbf{A}')$ is isomorphic to $\mathbf{L}_1$.
(e) Classify R according to the following properties: reflexivity, transitivity, symmetry.

Example 12.1.4. Example of a pre-monadic operator L inducing a sublattice

Consider the Boolean algebra $\mathbf{A}$ of Example 12.1.2. Consider the following table for L (on the right we draw the resulting sublattice $\mathbf{L_2} = \mathbf{L}(\mathbf{A})$):

x	$L_2(x)$
0	0
a	0
b	b
c	c
d	b
e	c
f	f
1	1

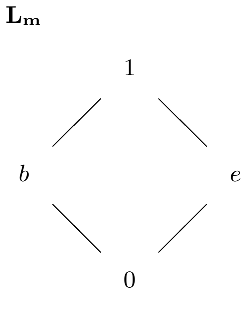


Exercise 12.4.
(a) Verify that $\langle \mathbf{A}, L_2 \rangle$ is a Pre-monadic Boolean algebra.
(b) Verify that $\mathbf{L_2}$ is distributive by computing a representation $\langle \mathbf{L_A}, k^*(\mathbf{L_A}) \rangle$ of $\langle \mathbf{A}, \mathbf{L_2} \rangle$ by means of the Representation Procedure.
(c) Classify the specialization preorder that you find during the Representation Procedure according to the following properties: reflexivity, transitivity, symmetry.
(d) Is $\langle \mathbf{A}, L_2 \rangle$ a Monadic Boolean algebra?

Example 12.1.5. Example of a monadic Boolean algebra

Consider on the Boolean algebra $\mathbf{A}$ the following monadic operator L_m on $\mathbf{A}$ (as usual, on the right we draw the resulting substructure $\mathbf{L_m} = \mathbf{L}(\mathbf{A})$, which is a sublattice, in this case):

x	$L_m(x)$
0	0
a	0
b	0
c	0
d	b
e	e
f	b
1	1



The system $\langle \mathbf{A}, L_m \rangle$ is a monadic Boolean algebra. It is worth noticing that $\mathbf{L_m}$ is a Boolean algebra, too.

Exercise 12.5.
(a) Compute a representation $\langle \mathbf{L_A}, k^*(\mathbf{L_A})\rangle$ *of* $\langle \mathbf{A}, \mathbf{L_m}\rangle$ *by means of the Representation Procedure.*
(b) Classify the specialization preorder that you find during the Representation Procedure according to the following properties: reflexivity, transitivity, symmetry.

If $\langle \mathbf{A}, L_R\rangle$ happens to be a Monadic Boolean algebra, where the operator L_R is induced by a binary relation R, we can ask if R enjoys some particular property. The answer is positive and will be given at the end of the present Section. Indeed we are going to see that the notions of a Pre-monadic Boolean algebra and Monadic Boolean algebra are the two extremes of a path that leads from operators associated with arbitrarily generic relations to operators connected with relations exhibiting the strongest properties, passing through intermediate cases.

For the reader's convenience, let us resume and align the definitions introduced so far in Table 12.1.

Table 12.1: Modalities, relations, forcing and algebraic structures

Operator	Definition	Context
$a \vDash L_k(\alpha)$	$\forall a'(a' \leq k(a) \Rrightarrow a' \vDash \alpha)$	Forcing on algebraic structures
$X \subseteq \phi(L_k(\alpha))$	$\forall X'(X' \subseteq R(X) \Rrightarrow X' \subseteq \phi(\alpha))$	Lattice of sets with $k(X) = R(X)$
$x \Vdash L_R(\alpha)$	$\forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \Vdash \alpha)$	Forcing on Kripke frames
$L_R(X)$	$\{x : \forall y \in U(\langle x, y\rangle \in R \Rrightarrow y \in X)\}$	Pre Monadic Boolean algebras of sets
$a \vDash M_k(\alpha)$	$\exists a'(g(a') \geq a \ \& \ a' \vDash \alpha)$	Forcing on algebraic structures
$X \subseteq \phi(M_k(\alpha))$	$\exists X'(R^{\smile}(X') \supseteq X \ \& \ X' \subseteq \phi(\alpha))$	Lattice of sets with $g(X) = R^{\smile}(X)$
$x \Vdash M_R(\alpha)$	$\exists y \in U(\langle x, y\rangle \in R \ \& \ y \Vdash \alpha)$	Forcing on Kripke frames
$M_R(X)$	$\{x : \exists y \in U(\langle x, y\rangle \in R \ \& \ y \in X)\}$	Pre Monadic Boolean algebras of sets

We know that, $L_R(X)$ and $M_R(X)$ equal $\{x : R(x) \subseteq X\}$ and, respectively, $\{x : R(x) \cap X \neq \emptyset\}$. Hence, using the distributivity property of R-neighborhoods, we obtain $L_R(X) = \bigcup\{Z : R(Z) \subseteq X\}$ and,

dually, $M_R(X) = \bigcap\{-Z : X \subseteq -R(Z)\}$ (the duality of the two equations will be proved in Frame 15.1). Therefore, if we compare the last definitions with the definitions of lower and, respectively, upper approximations, by substituting $[x]_R$ for $R(x)$, for R an equivalence relation, we observe that they differ slightly but in a significant way. We underline this difference by adding in Table 12.2 the intermediate definition of two hypothetical operators L^*_R and M^*_R.
We can notice that the passage from L_R and M_R to L^*_R and, respectively, M^*_R surely requires some extra features, as well as that from L^*_R and M^*_R to (lR) and, respectively, (uR). In what follows we analyse these extra features and their contexts of application.

Table 12.2: Three degrees of $R-$modal operators

R-modal operators	Necessity	Possibility
normal	$L_R(X) = \bigcup\{Z : R(Z) \subseteq X\}$	$M_R(X) = \bigcap\{-Z : X \subseteq -R(Z)\}$
with extra features	$L^*_R(X) = \bigcup\{R(Z) : R(Z) \subseteq X\}$	$M^*_R(X) = \bigcap\{R^{\smile}(Z) : X \subseteq R^{\smile}(Z)\}$
approximation	$(lR)(X) = \bigcup\{[x]_R : [x]_R \subseteq X\}$	$(uR)(X) = \bigcap\{[x]_R : X \subseteq [x]_R\}$

12.2 From Loosely Structured Spaces to Structured Spaces: A Variety of Modal Properties

Now we analyse the properties of the monadic operators L_R and M_R as dependent on the properties of the relation R.

If we do not impose any particular property on R, we cannot predict interesting uniform relationships between X and $L_R(X)$ – or $M_R(X)$ – nor special nice behaviours of the two modal operators.

What we can predict derives just from the fact that our operators happen to be Diodorean modalities, as one can see from *Proposition* 12.1.2.(6) above (*viz* it is valid to assert the possibility of α at point x if there is some state of affair accessible from x in which α is true).

What we can say without adding extra hypothesis is listed in Proposition 12.1.4, and we denote this basic set of properties with the symbol **K** (after the fact that they characterise a modal system usually denoted by this symbol). A modal logic with at least the same properties as **K**, is called *normal*.

Indeed a relation between elements of U without any specific constraints, reflects, in a obvious sense, empirical and variable relationships between pieces of information. But we can impose particular constraints to R according to theoretical intuitions or, as it happens for Approximation Spaces, according to a particular organisation of data. In Table 12.3 one can see how do specific constraints transform the properties of the modelled logic.

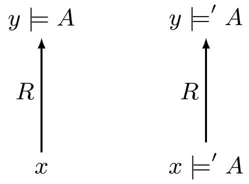
Table 12.3: Relational properties and modal properties

Properties of R and derived set-theoretical characteristics	Modelled modal properties on top of **K**	Label
Reflexivity $\forall x(\langle x, x\rangle \in R)$ $X \subseteq R(X)$	$L(\alpha) \rightarrow \alpha$ $\alpha \rightarrow M(\alpha)$	T
Seriality $\forall x, \exists y(\langle x, y\rangle \in R)$ $X \neq \emptyset$ implies $R(X) \neq \emptyset$	$L(\alpha) \rightarrow M(\alpha)$	D
Symmetricity $\forall x, y(\langle x, y\rangle \in R \Rrightarrow$ $\langle y, x\rangle \in R); Y \subseteq R(X)$ iff $X \subseteq R(Y); R(X) = R^{\smile}(X)$	$\alpha \rightarrow L(M(\alpha))$ $M(L(\alpha)) \rightarrow \alpha$	B
Transitivity $\forall x, y, z(\langle x, y\rangle \in R\ \&\ \langle y, z\rangle$ $\in R \Rrightarrow \langle x, z\rangle \in R)$; $R(R(X))$ $\subseteq R(X); Y \subseteq R(X)$ implies $R(Y) \subseteq R(X)$	$L(\alpha) \rightarrow L(L(\alpha))$ $M(M(\alpha)) \rightarrow M(\alpha)$	4
Euclidean property $\forall x, y, z(\langle x, y\rangle \in R\ \&\ \langle x, z\rangle$ $\in R \Rrightarrow \langle y, z\rangle \in R); Y \subseteq R(X)$ implies $R(X) \subseteq R(Y)$	$M(L(\alpha)) \rightarrow L(\alpha)$ $M(\alpha) \rightarrow L(M(\alpha))$	5

Example 12.2.1. Property D implies $L(\alpha) \to M(\alpha)$; $L(\alpha) \longrightarrow \alpha$ implies T

Suppose that (a) there is at least an element y such that xRy and (b) for all y', if xRy' then $y' \models \alpha$. Then $[\![L(\alpha)]\!] \subseteq [\![M(\alpha)]\!]$, because there is at least an element accessible form x that forces A, so that x forces $[\![M(\alpha)]\!]$ whenever x forces $[\![L(\alpha)]\!]$. So, $L(\alpha) \to M(\alpha)$. But if we drop hypothesis (a), that is, if we drop seriality, hypothesis (b) is vacuously true if there is no element accessible form x that forces α. In this case $x \models [\![L(\alpha)]\!]$ but $x \not\models \alpha$. Hence either $[\![M(\alpha)]\!] \subset [\![L(\alpha)]\!]$ or $[\![M(\alpha)]\!]$ and $[\![L(\alpha)]\!]$ are incomparable. Example:

R	x	y
x	0	1
y	0	0



According to the model with forcing $\models$, $[\![M(A)]\!] = \{x\}$, while $[\![L(A)]\!] = \{y\}$ (since no element is accessible from y). Since R is not reflexive (i.e. $x \not\leq x$), this prove that if $[\![L(\alpha)]\!] \leq [\![L\alpha]\!]$ then T (reflexivity) must hold.

Example 12.2.2. Example of a non symmetric relation where $\alpha \to L(M(\alpha))$ fails

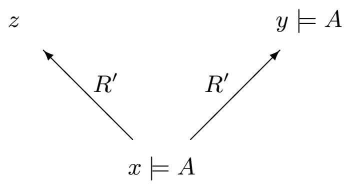
According to the model with forcing $\models'$ of Example 12.2.1, we have:

	$\models' A$	$\models' M(A)$	$\models' L(M(A))$
y	*yes (set-up)*	*no (no accessible element forces A)*	*yes (void precondition "$\forall y'(yRy' \ldots)$")*
x	*yes (set-up)*	*yes (because $y \models' A$)*	*no (because $y \not\models' M(a)$)*

Therefore, $[\![A]\!] = \{x, y\}$ and $[\![L(M(A))]\!] = \{y\}$.

Next we verify that adding seriality to a non-symmetric relation does not change the effect:

R'	x	y	z
x	0	1	1
y	0	1	0
z	0	0	1



Therefore in this model we have $[\![A]\!] = \{x, y\}$, $[\![M(A)]\!] = \{x, y\}$ and $[\![L(M(A))]\!] = \{y\}$.

Example 12.2.3. $[\![L(A)]\!] \longrightarrow [\![LL(A)]\!]$ implies property 4
Consider the relation

R''	x	y	z
x	1	1	0
y	1	1	1
z	0	1	1

R'' is reflexive and symmetric. However it is not transitive. We leave to the reader the verification of instances of property B. We show that property 4 does not hold. Let $z, y \models A$. Then $[\![L(A)]\!] = \{z\}$ but $[\![LL(A)]\!] = \emptyset$, because $\langle z, y\rangle \in R''$ and $y \not\models L(A)$.

Exercise 12.6.
(a) Prove that property T (reflexivity) implies $L(\alpha) \rightarrow \alpha$.
(b) Prove that property 4 (transitivity) implies $L(\alpha) \rightarrow L(L(\alpha))$.

Some combinations of the above properties are equivalent. For instance, **K***T5*, **K***TB4*, **K***DB4*, **K***DB5* are equivalent (the reader should try and prove it – in Frame 15.2 it is possible to find some hints).

Indeed, the following result is folklore in Modal Logic:

Proposition 12.2.1. *For any relation* $R \subseteq U \times U$*, the following are all the possible distinct combinations of the properties D, T, B, 4, 5, on top of* **K***:*
K*,* **K***D,* **K***T,* **K***B,* **K***4,* **K***5,* **K***DB,* **K***D4,* **K***D5,* **K***45,* **K***TB,* **K***T4,* **K***D45,* **K***B4,* **K***T5.*

Some of the above combinations have received a particular attention in modal logic literature, because of their philosophical and/or mathematical importance.[5] As such they are known by means of traditional names: **K***T*=**T**, **K***TB*=**B**, **K***T4*=**S**4, **K***T5*=**S**5 A number

[5]Nonetheless, in many cases, properties are adopted not because they reflect specific intuitions about the way states of affairs are organised, but only in view of the formal properties that the modelled Logical system must feature. For instance, if L has to model a doxastic operator (i.e. "subject S believes that ..."), then since an opinion is not guaranteed to be true, the reflexive property cannot be adopted, otherwise we should have $L(A) \rightarrow A$, that is read "If subject S believes A, then A is true". On the contrary, this property is required for modelling epistemic operators, such as "Subject S knows that ..", according to the classical definition advocating that "knowledge" is *true* and justified belief (cf. [Halpern, Moses 1985] for a technical overview. Cf. [Ellis 76] for a philosophical introduction and *Box* "Logico-philosophical remarks. 1" of Section 9.2 of Chapter 9).

of coarser/finer relationships between these systems are well-known in logical literature, as well as some intermediate systems. We address the reader to the *References*, for details.

However, the reader has surely noticed that, as a matter of fact, the properties of system **S**4 have been analysed in Part I, because any IQRS is a Kripke frame with reflexive and transitive accessibility relation.[6] Here we want to mention that Proximity Spaces are models for system **B**: in fact, Proximity Spaces are relational spaces $\langle U, R\rangle$ where R is reflexive and symmetric.[7] And it is clear now, that **S**5 is about to be adopted as the referent modal logic for Approximation Spaces, because **S**5 models are characterised by reflexive, transitive and symmetric relations. Actually, this will be the starting point for understanding the modal features of Rough Set Systems.

For the time being, we shall investigate some further formal properties of relational spaces connected with pre-topological and topological spaces.

12.3 Relations, Pre-Topologies and Topologies

Our interest in studying relations is the fact that the main concern in Rough Set Analysis is the way "perceptions" are connected in order to form conceptually meaningful patterns. Henceforth, a single element of the domain of concern is not interesting by its own ("an sich"), but to the extent it is connected (or not) with other elements. Otherwise stated, we are interested in the geometry that relations impose on a

[6] More precisely, since IQRS are finite, hence they fulfill the so-called McKinsey condition $\forall x \exists y(\langle x,y\rangle \in R \ \& \ \forall z(\langle y,z\rangle \in R \Rrightarrow y = z))$, they are Kripke frames adequate to the system **S**4.1, which is obtained by adding to **S**4 the axiom $L(M(a)) \rightarrow M(L(A))$.

[7] The symbol "**B**" is after the name of L. E. J. Brouwer, founder of the Intuitionistic school (cf. Introduction). This traditional use is justified by a translation of the intuitionistic negation $\neg$ as $L\sim$ (here "$\sim$" is the Boolean complementation of the modal system). In accordance with it, the intuitionistically admissible low $a \Longrightarrow \neg\neg a$ becomes $a \rightarrow L({\sim}L({\sim}a))$, i.e. $a \rightarrow L(M(a))$ (which characterises modal operators modelled by symmetric relations) which is the characteristic axiom of the "Brouwerian" system. On the contrary, the intuitionistically invalid law $\neg\neg a \Longrightarrow a$ becomes $L(M(a)) \rightarrow a$, which is invalidated by models with relations fulfilling *TB*. However, the "real" modal system connected with Intuitionistic Logic is $\mathbf{S}4 + \mathbf{Grz}$, where **Grz** is Segerberg's translation $L(L(p \rightarrow L(p)) \rightarrow p) \rightarrow p$ of the principle introduced by Andrzej Grzegorczyk for a modal interpretation of Heyting's logic (cf. [Grzegorczyk 1967] and [Segerberg 1971]).

universe of possible perceptions/stimulations (or "empirical results", "uninterpreted data" and the like).

Of course, we shall not remain at the abstraction level of a point-like geometry for ever. We are more interested in the general, universal properties of a "perception system". Therefore the abstraction level shall be lifted to a sort of pointless geometry. This more abstract level was discussed at the end of the Introduction and constituted already the playground of our algebraic analysis of Rough Set Systems. Here we are going to reach the same abstraction level for the modal interpretation.

Indeed, in case of the algebraic analysis, first we started noticing that a concrete Approximation Space on U is induced by a subalgebra of the Boolean algebra $\mathbf{B}(U)$, so that it was possible to define the notion of an abstract Approximation System as a pair $\langle \mathbf{B}, \mathbf{B}' \rangle$ made up of a Boolean subalgebra $\mathbf{B}'$ of a given Boolean algebra $\mathbf{B}$. Secondly, $\mathbf{B}$ was transformed into a new algebraic structure (namely, a Rough Set System), embedding the transformation of the elements of $\mathbf{B}$ induced by $\mathbf{B}'$. In the modal analysis, we shall follow the same strategy: the only difference is that we shall transform $\mathbf{B}$ into a modal system in accordance with the way its elements are modalised by means of an operator $L_{\mathbf{B}'}$ (or $M_{\mathbf{B}'}$), which is the abstract companion of L_R (of M_R).

This analysis will not mention the population of the elements of $\mathbf{B}$. However, we shall again reach this abstraction level starting from the intuitive ground of a "concrete" analysis of universes populated by "real" elements connected by "operating" relations.

12.4 Pre-Topological Spaces

We shall approach topological spaces from more general structures, called "pre-topological spaces". This choice is suggested by the fact that pre-topological spaces are widely (although often implicitly) used in Rough Set Theory in order to generalise the basic concepts of lower and upper approximation (as one is able to verify in the Frame section). Intuitively, whereas in Kripke frames any single world is linked with a set of accessible world, in pre-topologies any point x is associated with a family of sets, its neighborhood system $n(x)$. Each element of $n(x)$ may be intended as representing a collection of points that are relevant to x. Or, from another perspective, $n(x)$ is the family of observable phenomena connected with x. Therefore a formula α is necessarily valid at point x if the set of points validating α is relevant to x, i.e. $x \Vdash L(\alpha)$ *iff* $[\![\alpha]\!] \in n(x)$. This is the basic intuition leading to the definition of a *core map* (see below).

Clearly if $[\![\neg\alpha]\!] \in n(x)$, then α is unnecessary at point x. So, since α is possible at point x if it is not unnecessary at x, we can define $x \Vdash M(\alpha)$ *iff* $[\![\neg\alpha]\!] \notin n(x)$. This leads to the notion of a *vicinity map* dual of the core map.

Obviously, a vicinity map (a core map) is a generalisation of the usual notion of a closure map (interior map). The main difference, intuitively, is that vicinity maps reflect the notion of "x is close to a set X" under one or more possible points of view, while closure operators account for single cumulative points of view, by gluing all the elements of $n(x)$ through the imposition for $n(x)$ to be a filter.

Moreover, neighborhoods of points of U are not required to be subsets of U. Indeed, in a more general setting, we can think of situations in which $n(x) \subseteq \wp(U')$ for $x \in U$ and $U' \neq U$. Hence U' acts as a "medium", via a map $f : U \longmapsto U'$, in the evaluation of a closeness relation between a point x from U and another point y of U. In fact, a certain closeness criterion might not be applicable directly on the elements of U, but can be applicable on their f-images in U' (for instance we cannot understand if professor Smith's and professor Brown's scientific interests are similar by looking at the list of the pure names of the academic body of San Jose University. However, this is possible when we map Smith and Brown onto the set of academic disciplines).

In this case x will belong to the core of a subset $X \subseteq U$ if $f(X)$ belongs to $n(x)$. Below we illustrate this more general situation:

In Figure 12.1, x is related to y, w, w', from the point of view of a criterion α acting between their f-images on U'. On the other hand, x is related to z, q, z', z'' through a different criterion β. The collection of these aggregations forms $n(x)$. It follows that x belongs to the core of the set $\{y, w, w', x\}$ and of the set $\{z, q, z', z''\}$ because both $f(\{y, w, w', x\})$ and $f(\{z, q, z', z''\})$ belong to $n(x)$. Moreover, x belongs, for instance, to the vicinity of $\{y, z\}$ because $-\{y, z\}$ does not belong to $n(x)$. Notice that $f(x)$ does not belong to $f^{\rightarrow}(\{z, q, z', z''\})$.

So, let U, U' be sets. We can consider that the elements of U are connected (classified, characterised, labeled, perceived, ...) by means of

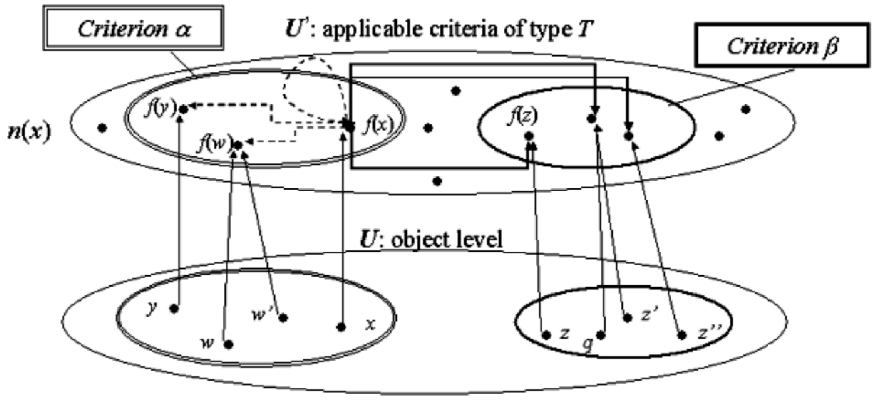


Figure 12.1: Observations and pre-topological spaces

the relationships that occur between the elements of another set U'. Therefore, according to this connection, any element p of U can be associated with one ore more elements of $\wp(U')$, obtaining thereby a family of subsets of U', denoted by $n(p)$, so that each $N \in n(p)$, links p with other elements of U under a specific respect.

We summarize these intuitions in the following definitions:

Definition 12.4.1. *Let U, U' be sets, $X' \subseteq U'$, $u \in U$ and $f : U \longmapsto U'$ a total function. Then,*

1. *A* neighborhood map *is a total function $n : U \longmapsto \wp(\wp(U'))$, such that $f(x) = f(y)$ implies $n(x) = n(y)$.*

2.
   - *$n(u)$ is called a* concrete neighborhood family of *u.*
   - *If $N \in n(u)$, then N is called a* concrete neighborhood *of u.*
   - *If $u' \in N \in n(u)$, then u' is called a* concrete neighbor *of u.*
   - *The family $\mathcal{N}(U) = \{n(x) : x \in U\}$ is called a* concrete neighborhood system.
   - *The pair $\langle U, \mathcal{N}(U)\rangle$ is called a* concrete neighborhood space.

3. *If $G(X') = \{x : X' \in n(x)\}$, then G is called the* core map induced by *$\mathcal{N}(U)$.*

4. *If $F(X') = -G(-X') = \{x : -X' \notin n(x)\}$, then F is called the* vicinity map induced by *$\mathcal{N}(U)$.*

5. *The set* $F(X') \cap -G(X') = \{x : \forall N \in n(x)(N \cap X' \neq \emptyset \neq N \cap -X')\}$ *is called the* boundary *of* X'*, denoted by* $\partial(X')$.

We can notice the reason why the notion of a core map (a vicinity map) is a generalisation of the notion of an interior (closure) operator. Indeed, if $U = U$ and f is the identity map then, as we shall prove in *Lemma* 12.4.1, $x \in G(X)$ if and only if $X \in n(x)$, that is, if and only if X itself is a neighborhood of x, whereas in topological spaces $x \in \mathbb{I}(X)$ (the interior of X) if and only if there is a neighborhood of x included in X. We shall see that the two definitions coincide just under some specific assumptions. Under the same assumptions we shall prove that $x \in F(X)$ if and only if X has no void intersection with all of the neighborhoods of x.

Example 12.4.1. A simple neighborhood system

Let $U = \{x, y, z, w\}, U' = \{a, b, c\}, f(x) = a, f(y) = b, f(z) = f(w) = c$. Consider the following neighborhood system: $n(x) = \{\{a, c\}, \{a, b, c\}\}, n(y) = \{\{b\}, \{a, b\}\}$, $n(z) = n(w) = \{\{c\}\}$. Then, $G(\{b\}) = \{u : \{b\} \in n(u)\} = \{y\}, G(\{b, c\}) = \emptyset$, and so on;

$F(\{b\}) = \{u : -\{b\} \notin n(u)\} = \{u : \{a, c\} \notin n(u)\} = \{y, z, w\}, F(\{a, b\}) = \{x, y\}$ and so on.

Notice that neither G nor F are isotonic.

Terminology and Notation. Given $p \in U$, from now on the image $n(p)$ of p along n will be usually denoted by $\mathcal{N}_p$.

Consider the following conditions on $\mathcal{N}(U)$, for any $x \in U$, $A, N, N' \subseteq U'$:

1. $U' \in \mathcal{N}_x$.
0. $\emptyset \notin \mathcal{N}_x$.
Id. if $x \in G(A)$ then $f^{\rightarrow}(G(A)) \in \mathcal{N}_x$.
N1. $f(x) \in N$, for all $N \in \mathcal{N}_x$.
N2. if $N \in \mathcal{N}_x$ and $N \subseteq N'$, then $N' \in \mathcal{N}_x$.
N3. if $N, N' \in \mathcal{N}_x$, then $N \cap N' \in \mathcal{N}_x$.
N4. there is an $N \neq \emptyset$ such that $\mathcal{N}_x = \uparrow_{\subseteq} N$.

Because function f occurs in the definitions of **Id** and **N1**, the two conditions will be said to be "*point-dependent*".

From a practical point of view the distinction between U and U' is relevant (think, for instance, of the different attributes in relational databases). However, on a theoretical side, dealing with a single universe is more comfortable and does not cause any information shortcoming, because instead of $x' \in N \in \mathcal{N}_x$ we can consider the inverse image $f^{\leftarrow}(\{x\})$ (this is what we usually do in relational databases when we move from a set of attribute-values V to the entities identified by V).

Definition 12.4.2. *If $U = U'$ and f is the identity map, then the pair $\langle U, \mathcal{N}(U)\rangle$ is called a* Fréchet space*.*

REMARKS. From now on we shall deal only with Fréchet spaces. In a Fréchet space, **N1** reads: "$x \in N$, for all $N \in \mathcal{N}_x$" and **Id** turns into "$\forall A \subseteq U, \forall x \in G(A), G(A) \in \mathcal{N}_x$". A neighborhood system $\mathcal{N}(U)$ will be denoted also by $\mathcal{N}$ if the set U is understood.

Example 12.4.2. A simple Fréchet space
Consider the universe $U = \{a, b, c\}$. The following is a Fréchet neighborhood system: $\mathcal{N}_a = \{\{b\}, \{a, c\}, U\}, \mathcal{N}_b = \{\{a, b\}, \{b, c\}, U\}, \mathcal{N}_c = \{\{b\}, \{a, c\}, \{a, b\}, U\}$. Clearly in $\mathcal{N}(U)$ **0** and **1** hold. On the contrary, **N1** does not hold because, for instance, $a \notin \{b\} \in n(a)$.

The above conditions carry particular properties that reflect on the operators G and F:

Lemma 12.4.1. *Let $\mathcal{N}(U)$ be a neighborhood system. Then, for any $X, Y \subseteq U$, $x \in U$:*

(**G1**) $x \in G(X)$ *iff* $X \in \mathcal{N}_x$; (**G2**) $G_x =_{def} \{X : x \in G(X)\} = \mathcal{N}_x$.

Condition	Equivalent properties of G	Equivalent properties of F
1	$G(U) = U$	$F(\emptyset) = \emptyset$
0	$G(\emptyset) = \emptyset$	$F(U) = U$
Id	$G(X) \subseteq G(G(X))$	$F(F(X)) \subseteq F(X)$
N1	$G(X) \subseteq X$	$X \subseteq F(X)$
N2	$X \subseteq Y \Rrightarrow G(X) \subseteq G(Y)$ $G(X \cap Y) \subseteq G(X) \cap G(Y)$	$X \subseteq Y \Rrightarrow F(X) \subseteq F(Y)$ $F(X \cup Y) \supseteq F(X) \cup F(Y)$
N3	$G(X \cap Y) \supseteq G(X) \cap G(Y)$	$F(X \cup Y) \subseteq F(X) \cup F(Y)$

Proof. (**G2**) $\{X : x \in G(X)\} = \{X : x \in \{y : X \in \mathcal{N}_y\}\} = \{X : X \in \mathcal{N}_x\} = \mathcal{N}_x$. From (G2) we straightforwardly obtain (**G1**). (**1**) Trivial. (**0**) Trivial. (**Id**) Assume **Id** holds and $x \in G(X)$. From **Id**, $G(X) \in \mathcal{N}_x$, so from (G1) $x \in G(G(X))$. Conversely, if $G(X) \subseteq G(G(X))$ then from (G1) $X \in \mathcal{N}_x$ implies $G(X) \in \mathcal{N}_x$, so that **Id** holds. (**N1**) Assume **N1** holds. If $x \in G(X)$, from (G1) $X \in \mathcal{N}_x$ and from **N1** $x \in X$. *Vice-versa*, assume $G(X) \subseteq X$. But $G(X) = \{x : X \in \mathcal{N}_x\}$; thus from (G1) $x \in G(X)$, hence $x \in X$. Henceforth $x \in X$. Thus **N1** holds. (**N2**) (a) Assume **N2**. If $X \in \mathcal{N}_x$ and $X \subseteq Y$ then $Y \in \mathcal{N}_x$. From (G1) we deduce that if $x \in G(X)$ and $X \subseteq Y$ then $x \in G(Y)$, that is, $G(X) \subseteq G(Y)$. Conversely, if $X \subseteq Y$ implies $G(X) \subseteq G(Y)$, then from (G1) $X \in \mathcal{N}_x$ implies $Y \in \mathcal{N}_x$. Hence **N2** holds. (b) Assume **N2**. If $x \in G(X \cap Y)$ then $X \cap Y \in \mathcal{N}_x$. But $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. Thus, from **N2** $X \in \mathcal{N}_x$ and $Y \in \mathcal{N}_x$, so that $x \in G(X)$ and $x \in G(Y)$. Conversely, assume $G(X \cap Y) \subseteq G(X) \cap G(Y)$, $X \in \mathcal{N}_x$ and $X \subseteq Y$. Then $G(X \cap Y) = G(X) \subseteq G(X) \cap G(Y)$. This means that $G(X) \subseteq G(Y)$, from (G1), so that $Y \in \mathcal{N}_x$, and **N2** holds. (**N3**) Assume **N3** and $x \in G(X) \cap G(Y)$. From (G1) we obtain $X \in \mathcal{N}_x$ and $Y \in \mathcal{N}_x$. Therefore in view of **N3**, $X \cap Y \in \mathcal{N}_x$, and again from (G1) $x \in G(X \cap Y)$. Henceforth, $G(X) \cap G(Y) \subseteq G(X \cap Y)$. Conversely, assume $X \in \mathcal{N}_x$, $Y \in \mathcal{N}_x$ and $G(X) \cap G(Y) \subseteq G(X \cap Y)$. From the latter assumption if $x \in G(X) \cap G(Y)$ then $x \in G(X \cap Y)$. Therefore, from (G1), if X and $Y \in \mathcal{N}_x$, then $X \cap Y \in \mathcal{N}_x$, so that **N3** holds. As to F we obtain the results by duality. Here we prove only (i) $G(X \cap Y) \subseteq G(X) \cap G(Y) \Rrightarrow F(X) \cup F(Y) \subseteq F(X \cup Y)$ and (ii) $G(X) \subseteq G(G(X)) \Rrightarrow F(F(X)) \subseteq F(X)$. (i) Indeed $G(X \cap Y) \subseteq G(X) \cap G(Y)$ iff $-(G(X) \cap G(Y)) \subseteq -G(X \cap Y)$, iff $-(G(-X) \cap G(-Y)) \subseteq -G(-X \cap -Y)$, iff $-G(-X) \cup -G(-Y)) \subseteq -G-(X \cup Y)$, iff $F(X) \cup F(Y) \subseteq F(X \cup Y)$. (ii) $G(X) \subseteq G(G(X))$ iff $-G(G(X)) \subseteq -G(X)$ iff $-G(G(-X)) \subseteq -G(-X)$ iff $-G-(-G(-X)) \subseteq -G(-X)$ iff $F(F(X)) \subseteq F(X)$. QED[8]

REMARKS. One should not confuse **G1** with the principle "$X \in \mathcal{N}_x \Rrightarrow x \in X$" which holds if $\mathcal{N}(U)$ fulfills **N1**.

[8] Note that if $U \neq U'$, then property **1** turns into $G(U') = U$, property **0** turns into $F(U') = U$ and, finally, **N1** turns into $G(X) \subseteq f^{\leftarrow}(X)$ [in the proof of **N1** substitute "$f(x) \in X$" for "$x \in X$" and "$G(X) \subseteq f^{\leftarrow}(X)$" for "$G(X) \subseteq X$", and notice that $f(X) \in X$ iff $x \in f^{\leftarrow}(X)$].

Example 12.4.3. A neighborhood system satisfying **Id** but not **N1**, whose core map G is not idempotent

Consider the universe $U = \{a, b, c\}$ and the neighborhood system $\mathcal{N}(U)$ given by:

x	a	b	c
$\mathcal{N}_x$	$\{\{a\}, \{a, b\}, \{b, c\}, U\}$	$\{\{b\}, \{a, b\}, \{b, c\}, U\}$	$\{\{a, b\}, U\}$

Let us check that in this neighborhood system property **Id** is satisfied. Indeed the core map G is given by:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U
$G(x)$	$\emptyset$	$\{a\}$	$\{b\}$	$\emptyset$	U	$\emptyset$	$\{a, b\}$	U

It is easy to verify that if $X \in \mathcal{N}_x$ then $G(X) \in \mathcal{N}_x$ (for instance $G(\{b, c\}) = \{a, b\}$ because $\{b, c\}$ belongs to $\mathcal{N}_a$ and $\mathcal{N}_b$. However $G(G(\{b, c\})) = U \neq G(\{b, c\})$. Also, we can observe that $\mathcal{N}(U)$ does not fulfill **N1** ($\{a, b\} \in \mathcal{N}_c$ but $c \notin \{a, b\}$). Actually, had $\mathcal{N}(U)$ fulfilled **N1**, G would have been idempotent (cf. *Proposition* 12.4.5 and Example 12.4.5 below).

Example 12.4.4. A neighborhood system satisfying **Id** but not **N1**, whose core map G is idempotent

Consider the universe $U = \{a, b, c\}$ and the neighborhood system $\mathcal{N}(U)$ given by:

x	a	b	c
$\mathcal{N}_x$	$\{\{a, b\}, \{a, c\}, U\}$	$\{\{a\}, \{b\}, \{a, b\}, \{b, c\}, U\}$	$\{\{c\}, \{a, c\}, U\}$

In this neighborhood system property **Id** is satisfied, but **N1** is not ($\{a\} \in \mathcal{N}_b$ but $b \notin \{a\}$). However the core map G is idempotent:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U
$G(x)$	$\emptyset$	$\{b\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b\}$	U

Example 12.4.5. A neighborhood system which satisfies **0**, **1**, **N1** and **Id**

Consider the universe $U = \{a, b, c\}$ and the neighborhood system $\mathcal{N}(U)$ given by:

x	a	b	c
$\mathcal{N}_x$	$\{\{a, b\}, \{a, c\}, U\}$	$\{\{b\}, \{a, b\}, \{b, c\}, U\}$	$\{\{c\}, \{a, c\}, U\}$

It is easy to check that in this neighborhood system **0**, **1**, **N1** and **Id** hold. The core map is idempotent:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U
$G(x)$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b\}$	U

However, **N2** does not hold. In fact, $\mathcal{N}_c$ is not an order filter (for instance, $\{c\} \in \mathcal{N}_c, \{c\} \subseteq \{b, c\}$ but $\{b, c\} \notin \mathcal{N}_c$).

The following is a very simple but useful statement:

Proposition 12.4.1. *Let $\mathcal{N}(U)$ be a neighborhood system. Then $\mathcal{N}(U)$ fulfills* **Id** *if and only if for all $X \subseteq U$ and $x \in U$, $X \in N_x$ implies $G(X) \in N_x$.*

Proof. Suppose **Id** holds and $X \in \mathcal{N}_x$. From **G1** $x \in G(X)$. Hence from **Id**, $G(X)$ must belong to $\mathcal{N}_x$, too. Conversely, suppose $X \in \mathcal{N}_x \Rrightarrow G(X) \in \mathcal{N}_x$ and $x \in G(X)$. Again from **G1**, $X \in \mathcal{N}_x$. Therefore $G(X) \in \mathcal{N}_x$. We conclude that **Id** holds. QED

Notice that if **N2** is assumed, then **N3** is equivalent to the following weaker condition:

$$if\ \ X \in \mathcal{N}_x\ \ and\ \ Y \in \mathcal{N}_x\ \ then\ \ \exists Z \in \mathcal{N}_x\ \ such\ that\ \ Z \subseteq X \cap Y \qquad (\mathbf{N3}-)$$

Proposition 12.4.2. *Assume* **N2** *and* **N3**$-$*. Then for any $X, Y \subseteq U$, $G(X \cap Y) \supseteq G(X) \cap G(Y)$.*

Proof. If $x \in G(X) \cap G(Y)$, then $x \in G(X)$ and $x \in G(Y)$. Thus, from **G1** $X, Y \in \mathcal{N}_x$. Therefore from **N3**$-$ there exists $Z \subseteq X \cap Y$ such that $Z \in \mathcal{N}_x$. But from **N2**, $X \cap Y$ must belong to $\mathcal{N}_x$, too. QED

(Notice that in literature even weaker conditions are studied, such as the so-called "connection condition": if $X \in \mathcal{N}_x$ *and* $Y \in \mathcal{N}_x$ *then* $X \cap Y \neq \emptyset$. An example of the use of this condition in modal logic can be found in Frame 15.13.3).

Moreover, if **N2** is assumed then **Id** is equivalent to the following weaker condition:

$$if\ N \in \mathcal{N}_x,\ then\ \exists N' \in \mathcal{N}_x\ such\ that\ for\ any\ y \in N', N \in \mathcal{N}_y \qquad (\tau)$$

This is the familiar topological property usually explained by the sentence: "if X is a neighborhood of a point x, then it is also a neighborhood of all those points that are sufficiently close to x".

Proposition 12.4.3. *Let $\mathcal{N}(U)$ be a neighborhood system. Then if $\mathcal{N}(U)$ satisfies* **Id***, it satisfies (τ), too.*

Proof. Suppose $p \in U$ and $N \in \mathcal{N}_p$. From **Id**, $G(N)$ belongs to $\mathcal{N}_p$. But from definition of $G(N)$, $N \in \mathcal{N}_x$ for any $x \in G(N)$. Hence (τ) holds. QED

The converse implication does not hold without **N2**, as is illustrated in Example 12.4.6 below.

Proposition 12.4.4. *Let $\mathcal{N}(U)$ be a neighborhood system satisfying* **N2**. *Then an element $\mathcal{N}_x$ of $\mathcal{N}(U)$ satisfies (τ) if and only if it satisfies* **Id**.

Proof. From *Proposition* 12.4.3, **Id** implies (τ). Conversely, let (τ) hold. Consider any neighbor $N \in \mathcal{N}_x$. From (τ) there is an element $N' \in \mathcal{N}_x$ such that for any $y \in N'$, $N \in \mathcal{N}_y$. Clearly the set $G(N) = \{z : N \in \mathcal{N}_z\}$ includes N' because it is the largest collection of elements z such that $N \in \mathcal{N}_z$. Therefore in view of **N2**, $G(N) \in \mathcal{N}_x$. QED

Condition **Id** alone does not guarantee the idempotence of G and F (for a counterexample see Example 12.4.3). We have idempotence by adding **N1** to **Id**:

Proposition 12.4.5. *Let $\mathcal{N}(U)$ be a neighborhood system satisfying* **N1** *and* **Id**. *Then for any $X \subseteq U$, $G(G(X)) = G(X)$ and $F(F(X)) = F(X)$.*

Proof. Immediate from *Lemma* 12.4.1. QED

Proposition 12.4.6. *If G is idempotent, then* **Id** *holds.*

Proof. Suppose **Id** does not hold. Then $\exists x \in U, X \subseteq U$ such that $X \in \mathcal{N}_x$ but $G(X) \notin \mathcal{N}_x$. Therefore, $x \notin G(G(X))$, although $x \in G(X)$. It follows that $G(G(X)) \neq G(X)$. QED

Example 12.4.6. A neighborhood system fulfilling **N1** and (τ) but neither **Id** nor **N2**

Let $U = \{a, b, c, d\}$. Let $\mathcal{N}(U)$ be given by $\mathcal{N}_a = \{\{a\}, \{a, b, c\}, U\}, \mathcal{N}_b = \{\{b\}, \{a, b\}, \{a, b, c\}, U\}$, and $\mathcal{N}_c = \{\{c\}, U\}$. Then property ($\tau$) is fulfilled by all the elements of $\mathcal{N}(U)$. However, $\{a, b, c\} \in \mathcal{N}_a$ but $G(\{a, b, c\}) = \{a, b\} \notin \mathcal{N}_a$. Hence $\mathcal{N}_a$ does not satisfy **Id**. According to *Corollary* 12.4.1 it follows that in the pre-topological space induced by $\mathcal{N}(U)$ the operator G is not idempotent ($G(\{a, b, c\}) = \{a, b\}$, but $G(\{a, b\}) = \{b\}$).

Notice that **N2** does not hold in $\mathcal{N}(U)$ (for instance $\{a, b\} \supseteq \{a\} \in \mathcal{N}_a$, but $\{a, b\} \notin \mathcal{N}_a$). Henceforth ($\tau$) plus **N1** does not imply **N2**.

In general idempotence of G does not imply **N1**. However we have,

Corollary 12.4.1. *In the presence of* **N1**, *G is idempotent if and only if* **Id** *holds.*

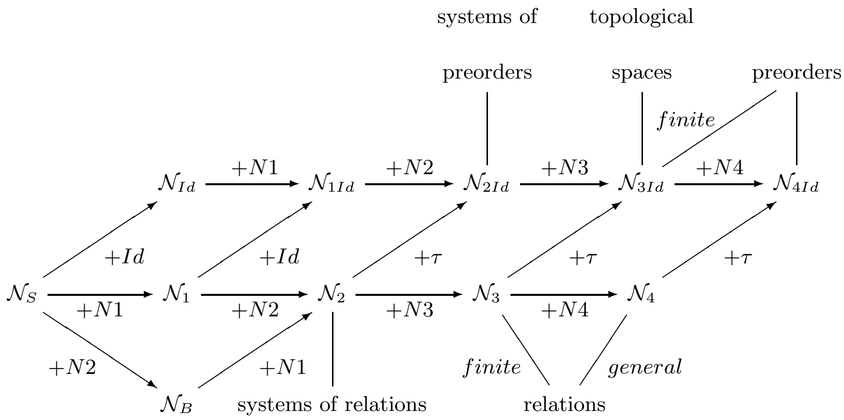
In general G and F are not required to be idempotent. Intuitively, the lack of idempotence reflects a sort of flowing situation in which boundaries are not fixed once for ever, so that by adding the boundary $\partial(X)$ to a subset X by means of the vicinity map F we do not gain a stable situation, since a new boundary could appear.
Now we list various combinations of the above properties that shall dealt with in this section. The right column displays their main relational characteristics, that will be proved in this Chapter:[9]

If all the elements of $\mathcal{N}(U)$ satisfy	$\mathcal{N}(U)$ *is said to be of type*	Elements of $\mathcal{N}(U)$	*Relational properties*
0, 1	$\mathcal{N}_S$		
0, Id	$\mathcal{N}_{Id}$		
0,1,N1	$\mathcal{N}_1$		
0,1,N1, Id	$\mathcal{N}_{1Id}$		
0,1,N2	$\mathcal{N}_B$	*proper order filter* w. r. t. $\subseteq$	Induced by systems of serial relations
0, 1, N1, N2	$\mathcal{N}_2$	*proper order filter* w. r. t. $\subseteq$	Induced by systems of reflexive relations
0, 1, N1, N2, τ	$\mathcal{N}_{2Id}$	*proper order filter* w. r. t. $\subseteq$	Induced by systems of preorders
0, 1, N1, N2, N3	$\mathcal{N}_3$	*proper filter*	
0, 1,N1, N2, N3, τ	$\mathcal{N}_{3Id}$	*proper filter*	
0, 1, N1, N2, N3, N4	$\mathcal{N}_4$	*principal filter*	Induced by single reflexive relations
0, 1, N1, N2, N3,N4, τ	$\mathcal{N}_{4Id}$	*principal filter*	Induced by single preorders

[9] In some papers these properties are denoted by different names (for instance in [Stadler & Stadler 2001] we have: **1** = K0, **N1** = K2, **N2** = K1, **N3** = K3). Also, the

Remarks. If $\mathcal{N}(U)$ is finite and of type $\mathcal{N}_3$ then it is also of type $\mathcal{N}_4$.

The resulting picture, that we shall justify throughout this Section, will be the following:



Terminology and Notation.

In what follows, we shall deal only with spaces of type at least $\mathcal{N}_1$. Therefore, by abuse of language we shall refer to a neighborhood system at least of type $\mathcal{N}_1$ as a "neighborhood system" tout-court. Moreover, since we shall typically deal with finite spaces, $\mathcal{N}_3$ systems will stand also for $\mathcal{N}_4$ systems. A distinct use will be generally adopted for systematic purposes (introduction of notions, an so on). In the Frame section we shall illustrate some applications of the most general form of neighborhood systems.

Now we shall see that vicinity maps in neighborhood systems of type $\mathcal{N}_1$ reflect, so to say, a process of *extension*. An extension is a process that applied to a set X collects all the elements of X plus those elements that, under some point of view, are connected with them

terms used to refer to types of pre-topological spaces may vary (in the quoted paper we have $\mathcal{N}_B$ = *Extended topology*, $\mathcal{N}_1$ = *Brissaud space*, $\mathcal{N}_2$ = *Neighborhood space*, $\mathcal{N}_3$ = *pre-topology*, $\mathcal{N}_{2Id}$ = *Convex closure space*). Other combinations have been studied. For instance, neighborhood systems satisfying **1** + **N2** + **N3**, which induce the so called "*Smith spaces*". Spaces induced by neighborhood systems satisfying **N2** + **Id** are called "*intersection spaces*", while $\mathcal{N}_{2Id}$ spaces are also called "*topped intersection structures*" or "*closure systems*". Notice that some authors call neighborhood systems satisfying **N1** and **N3**− "neighborhood basis" and neighborhood systems of type $\mathcal{N}_3$ "neighborhood filters". Neighborhood systems of type $\mathcal{N}_4$ are usually called "binary neighborhood systems", because they are univocally related to binary relations (as we shall widely see in this Chapter).

(if any). Therefore such a process is an increasing map f between subsets of U and we call it an "*expansion process*":

Definition 12.4.3. *Let U be a set. An* expansion process *is any map $f : \wp(U) \longmapsto \wp(U)$ such that for any $X \subseteq U$, $X \subseteq f(X)$.*

Dually, we can think of a process of erosion which cuts down some connections between elements of U, just leaving the elements from a subset X that are strictly connected each others. We call such a process a "*contraction*".

Definition 12.4.4. *Let U be a set. A* contraction process *is any map $g : \wp(U) \longmapsto \wp(U)$ such that for any $X \subseteq U$, $g(X) \subseteq X$.*

Proposition 12.4.7. *Let U be a set and f an expansion process. If for any $X \subseteq U, g(X) = -f(-X)$, then g is a contraction process, called the* dual *of f.*

Proof. For any $X \subseteq U, -X \subseteq f(-X)$. Hence $-f(-X) \subseteq --X = X$.
QED

From now on, by $\langle \varepsilon, \varkappa \rangle$ we shall indicate a pair of duals: expansion and, respective, contraction maps.

Definition 12.4.5. *A* pre-topological space *is a triple $\langle U, \varepsilon, \varkappa \rangle$ such that:* (i) *U is a set,* (ii) *$\varepsilon : \wp(U) \longmapsto \wp(U)$ is an expansion map such that $\varepsilon(\emptyset) = \emptyset$,* (iii) *$\varkappa : \wp(U) \longmapsto \wp(U)$ is a contraction map dual to ε.*

Proposition 12.4.8. *If $\langle U, \varepsilon, \varkappa \rangle$ is a pre-topological space, then $\varkappa(U) = U$.*

The proof is left to the reader.

Now we have to note that the notion of an expansion (contraction) cannot be immediately related with that of a R-neighborhood. In fact, given a generic relation $R \subseteq U \times U$, we do not have either $R(X) \subseteq X$ or $X \subseteq R(X)$, for any $X \subseteq U$ (the same happens for $R^{\smile}$, of course). Indeed, as we have seen in Section 12.2, $X \subseteq R(X)$ is valid only if R is reflexive. Moreover, both ε and $\varkappa$ lack the isotonicity law which, on the contrary, is valid for R-neighborhoods. Finally, differently from R-neighborhoods, neither the definition of ε, nor that of $\varkappa$ make any assumption about the distribution over disjunctions or conjunctions.

Also, notice that neither ε nor $\varkappa$ are required to be idempotent in a pre-topological space (anyway, the same happens for R-neighborhoods). As already pointed out, this reflects a floating situation.

Example 12.4.7. Expansions and contractions
In the following figure we depict an example of a floating boundary:

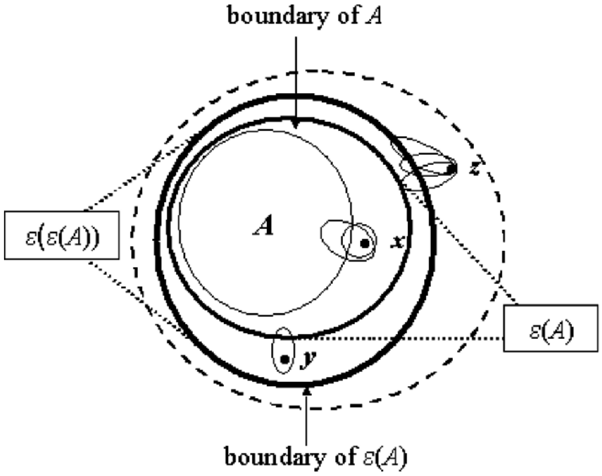


Figure 12.2: Example of a floating boundary

In Figure 12.2, point x belongs to the boundary of A, $\partial(A)$, while $y \notin \partial(A)$. Therefore $x \in \varepsilon(A)$, while $y \notin \varepsilon(A)$. However, $y \in \partial(\varepsilon(A))$. It follows that $\varepsilon(\varepsilon(A)) \supsetneq \varepsilon(A)$, $\varepsilon(\varepsilon(\varepsilon(A))) \supsetneq \varepsilon(\varepsilon(A))$, and so on up to an eventual fix point of the operator ε.

Suppose to process a set $A = \{x, y, z\}$. In A, the elements x and y are tightly linked, while x and z are loosely linked. Moreover, z is connected with the elements a and b, and y with the element c, that lies all outside of A. When we apply the expansion process ε to A, we gather together all the elements of A (x, y and z), plus the elements they are connected with, that is, a, b and c. When we contract A, we keep just the tight connected elements inside A, (x and y) and miss the elements which are loosely connected with these "core" elements of A. Therefore, $\varepsilon(A) = \{x, y, z, a, b, c\}$ and $\varkappa(A) = \{x, y\}$.

Definition 12.4.6. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space, $X \subseteq U$. Then,*

1. *X is said to be* "closed" *iff $\varepsilon(X) = X$.*
2. *X is said to be* "open" *iff $\varkappa(X) = X$.*

3. *The intersection of all closed sets containing X, whenever it is a closed set, is called "$\varepsilon-$*closure of *X" and denoted by $\mathbb{C}_\varepsilon(X)$.*

4. *The union of all open sets contained in X, whenever it is an open set, is called "$\varkappa-$*interior of *X" and denoted by $\mathbb{I}_\varkappa(X)$.*

Proposition 12.4.9. *Let $\langle U, \varepsilon, \varkappa\rangle$ be a pre-topological space. Then, for any $X, Y \subseteq U$, if $\mathbb{I}_\varkappa(X)$ and $\mathbb{I}_\varkappa(Y)$ exist, then $\mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y) \subseteq X \cap Y$.*

Proof. From the very definition of this operator, $\mathbb{I}_\varkappa(X) \subseteq X$ and $\mathbb{I}_\varkappa(Y) \subseteq Y$. Hence $\mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y) \subseteq X \cap Y$. QED

In general, the existence of the closure (of the interior) of a set X is not guaranteed, since it is not guaranteed, in a pre-topological space, that the intersection (union) of a family of closed (open) sets is a closed (open) set. In turn, this situation is related to the fact that in a generic pre-topological space, as we have seen, isotonicity fails for both ε and $\varkappa$.

In fact, assume that X and Y are open. By definition of a contraction map, $\varkappa(X \cup Y) \subseteq X \cup Y$, but although $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$ we have neither $X = \varkappa(X) \subseteq \varkappa(X \cup Y)$ nor $Y = \varkappa(Y) \subseteq \varkappa(X \cup Y)$. Therefore we cannot obtain the converse inclusion $X \cup Y = \varkappa(X) \cup \varkappa(Y) \subseteq \varkappa(X \cup Y)$. Hence, $X \cup Y$ may fail to be open since $\varkappa(X \cup Y)$ may be different from $X \cup Y$.[10] By duality we obtain that X and Y closed do not imply that $X \cap Y$ is closed.

Therefore, in pre-topological spaces, neighborhood systems are more important than open set systems.

Now we reveal the obvious fact that $\varkappa$ is the core map induced by a neighborhood system of type (at least) $\mathcal{N}_1$.

Definition 12.4.7. *Given a contraction $\varkappa : \wp(U) \longmapsto \wp(U)$, for any $x \in U$ the family*

$$\varkappa_x = \{Z \subseteq U : x \in \varkappa(Z)\}$$

*is called the family of $\varkappa$-*neighborhoods *of x. We set $\mathcal{N}^{\varkappa}(U) = \{\varkappa_x\}_{x \in U}$ and call $\mathcal{N}^{\varkappa}(U)$ a $\varkappa-$*neighborhood system.

[10] The reader is invited not to confuse the equation $\varkappa(X) \cup \varkappa(Y) = \varkappa(Y \cup Y)$, when both X and Y are open (hence $\varkappa(X) = X$ and $\varkappa(Y) = Y$), which is a situation that does not hold without the isotonicity law, with the same equation when X and Y are generic sets (not necessarily open), which may fail also in the presence of isotonicity.

Intuitively, by means of $\varkappa_x$ we obtain all the subsets Z such that x is strictly connected with some element of Z.

Proposition 12.4.10. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then,*

1. *The family $\mathcal{N}^{\varkappa}(U)$ is a neighborhood system of type $\mathcal{N}_1$.*
2. *$\varkappa$ is the core map induced by $\mathcal{N}^{\varkappa}(U)$.*

Proof. (**1**) If $Z \in \varkappa_x$, then $x \in \varkappa(Z) \subseteq Z$. Hence, $x \in Z$. Thus **N1** holds in $\mathcal{N}^{\varkappa}(U)$.
(**2**) $G(Z) = \{x : Z \in \varkappa_x\} = \{x : x \in \varkappa(Z)\} = \varkappa(Z)$. **0** and **1** follow from *Definition* 12.4.5. QED

Conversely, in view of *Lemma* 12.4.1.(N1), we have:

Proposition 12.4.11. *Given a neighborhood system $\mathcal{N}(U)$ of type $\mathcal{N}_1$, the core map G induced by $\mathcal{N}(U)$ is a contraction operator, which is said to be* induced by *$\mathcal{N}(U)$.*

TERMINOLOGY AND NOTATION. If in a pre-topological space $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ the operator $\varkappa$ is induced by a neighborhood system $\mathcal{N}(U)$, then $\mathbf{P}$ itself is said to be induced by $\mathcal{N}(U)$.

Now we shall prove that a neighborhood system of type (at least) $\mathcal{N}_1$ induces a pre-topological space and, viceversa, that a pre-topological space induces an $\mathcal{N}_1$ neighborhood system.

Corollary 12.4.2. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space induced by a neighborhood system $\mathcal{N}(U)$ of type $\mathcal{N}_1$, then $\mathcal{N}^{\varkappa}(U) = \mathcal{N}(U)$.*

Proof. Immediate, from *Proposition* 12.4.10, *Definition* 12.4.7 and *Lemma* 12.4.1.(G2).

Proposition 12.4.12. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then, for any $X \subseteq U$, X is open if and only if X belongs to $\varkappa_x$ for any $x \in X$.*

Proof. If X is open, then $X = \varkappa(X)$. Hence for any element $x \in X, x \in \varkappa(X)$. It follows that X belongs to $\varkappa_x$. The converse is trivial in view of **N1**. QED

Therefore, a set is open if and only if it is a neighborhood for all its own elements. But this is exactly what condition **Id** requires for $G(X)$

(alias $\varkappa(X)$), any $X \subseteq U$. Indeed, it is immediate to verify that a neighborhood system $\mathcal{N}(U)$ is of type $\mathcal{N}_{1Id}$ if and only if for any $X \subseteq U$, $G(X)$ is an open set.

REMARKS. The bi-implication of *Proposition* 12.4.12 holds because in every pre-topological space $\mathcal{N}^{\varkappa}(U)$ is a neighborhood system of at least type $\mathcal{N}_1$.

Example 12.4.8. A sample pre-topology

Consider the universe $U = \{a, b, c\}$. Suppose we are given the following contraction map:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varkappa(x)$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b\}$	U

If we compute the family $\mathcal{N}^{\varkappa}(U) = \{\varkappa_x\}_{x\in U}$, we obtain that $\mathcal{N}^{\varkappa}(U) = \mathcal{N}(U)$, where $\mathcal{N}(U)$ is the neighborhood system of *Example* 12.4.5 (for instance, $\varkappa_a = \{Z \subseteq U : a \in \varkappa(Z)\} = \{\{a,b\}, \{a,c\}, U\}$).

We know that this neighborhood system is of type $\mathcal{N}_{1Id}$ but not of type $\mathcal{N}_2$. Linked to this fact, we note that the contraction operator $\varkappa$ (i.e. G) is not isotone: $\{c\} \subseteq \{b,c\}$, but $\varkappa(\{c\}) = \{c\} \nsubseteq \{b\} = \varkappa(\{b,c\})$.

We note immediately that $\varkappa$ is a co-discontinuous contraction operator:

$\varkappa(\{a,b\}) \cap \varkappa(\{a,c\}) = \{a\} \neq \emptyset = \varkappa(\{a\}) = \varkappa(\{a,b\} \cap \{a,c\})$.

Given $\mathcal{N}^{\varkappa}(U)$ we can recover the contraction map $\varkappa$ using the equation $\varkappa(X) = \{z : X \in \varkappa_z\}$. Let us compute, for instance, $\varkappa(\{b,c\})$ and $\varkappa(\{a,b\})$:

$\varkappa(\{b,c\}) = \{x : \{b,c\} \in \varkappa_x\} = \{b\}$ (indeed, $\{b,c\}$ belongs only to $\varkappa_b$);

$\varkappa(\{a,b\}) = \{x : \{a,b\} \in \varkappa_x\} = \{a,b\}$ (indeed, $\{a,b\}$ belongs to $\varkappa_b$ and to $\varkappa_a$).

Example 12.4.9. Open sets

In the pre-topology $\langle U, \varkappa, \varepsilon\rangle$ of Example 12.4.8, the sets $\{b\}, \{c\}, \{a,b\}, \{a,c\}, U$ and $\emptyset$ are open, because they are fix points of the contraction map $\varkappa$. On the contrary, $\varkappa(\{b,c\}) = \{b\}$.

We have seen that a set X is open if it is a $\varkappa$-neighborhood of all its points; that is, if for any $x \in X, X \in \varkappa_x$. Therefore, we can verify also in this way that $\{b,c\}$ is not open: indeed, $\{b,c\} \notin \varkappa_c$. Moreover, the set $\{b,c\}$ does not have an interior: the set of all open subsets of $\{b,c\}$ is $\{\emptyset, \{b\}, \{c\}\}$ whose union is $\{b,c\}$ itself, which is not open.[11]

This example shows that in the above pre-topological space not every union of open sets is an open set: $\{b\}$ and $\{c\}$ are open; however, $\{b\} \cup \{c\} = \{b,c\}$ is not open.

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[11] One should not confuse the fact that a set X is a $\varkappa$-neighborhood of all its points, which is always true of open sets, with the existence of a subset Y of X such that X is a $\varkappa$-neighborhood of all the elements of Y, which is related to topological spaces – see further in the text.

Exercise 12.7. *Given a pre-topological space:*
(a) Is the family $\varkappa_x$ *closed with respect to intersections (unions), for any* x*?*
(b) Is the family $\varkappa_x$ *closed with respect to supersets, for any* x*?*
(c) Compute the table for the dual expansion map ε *of the pre-topological space of Example 12.4.8. (i) Is this map continuous?*
(ii) Find a subset of U *which does not have a closure.*

12.4.1 Excursus. Dynamics 1: The Failure of the Isotonicity Law

We want to recall again that the failure of the isotonicity law prevents us from using generic pre-topologies in order to provide the knowledge order embedded in a relation R over a universe U with a pre-topological interpretation, even if R is reflexive. In fact, for any $X \subseteq U$ we cannot coherently set $R(X) = \varepsilon(X)$ or $R(X) = \varkappa(X)$, because if $X \subseteq Y$ we have $R(X) \subseteq R(Y)$, but both $\varkappa(X) \subseteq \varkappa(Y)$ and $\varepsilon(X) \subseteq \varepsilon(Y)$ may fail.

Intuitively this difference reflects the fact that a single relation R on U is a static representation of the relationships between the elements of U, while ε and $\varkappa$ may account for a dynamic evaluation of these relationships. In fact, if $X \subseteq Y$ but $\varepsilon(X) \not\subseteq \varepsilon(Y)$ we can imagine a situation in which an element x of X fulfills a connection with some element x' as far as x is considered just within the set X (which is recorded by the fact $x' \in \varepsilon(X)$), but whenever x is associated, by expanding X to a superset Y by means of ε, with other elements outside of X, then the connection between x and x' is lost, because of a sort of incompatibility between x' and some new element in $Y \cap -X$.

Here is an example.

In Figure 12.3 we have a set $A = \{x, y, z\}$ and a superset of A, $B = \{x, y, z, w\}$. Assume that (i) y is connected with c, (ii) z is connected with b, (iii) w is connected with a, and (iv) that w and b are incompatible. If we expand A, we obtain $\varepsilon(A) = \{x, y, z, b, c\}$. But if we extend A to B, then b, which is incompatible with the new entry w, breaks its alliance with z. Therefore, the expansion of B will be $\varepsilon(B) = \{x, y, z, w, c, a\}$ which is not even comparable with $\varepsilon(A)$. Therefore, expansion is not a monotonic (isotonic) operator, in general.

Moreover, one might have the case in which z and w are incompatible. So that after enlarging A to B, the connection between z and the other elements of A is lost. Therefore, when we expand B to $\varepsilon(B)$ we

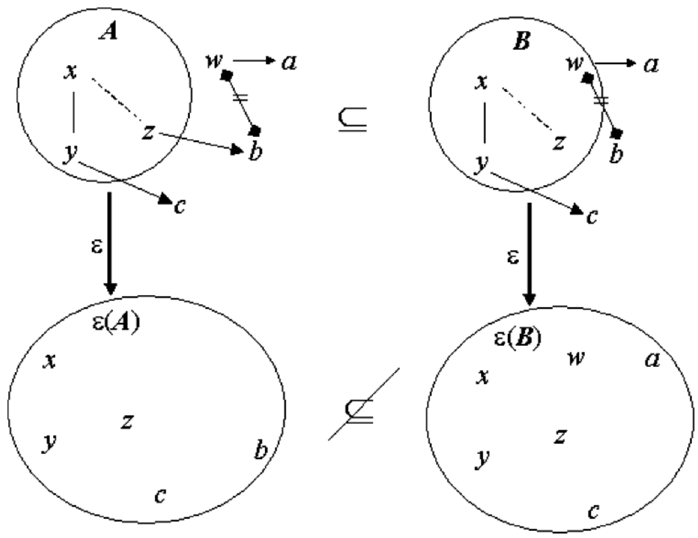


Figure 12.3: A non-isotonic expansion

obtain $\{x, y, z, c, w, a\}$, but if we contract $\varepsilon(B)$ we obtain a set in which z does not appear any longer. It follows that $\varkappa(\varepsilon(B))$ and B are not comparable, in this case. Differently, consider the fact that the relation $\mathbb{IC}(X) \supseteq X$ is always valid in topological spaces. For an extremely simple example of a pre-topology where $\varkappa(\varepsilon(X)) \not\supseteq X$, take $\langle U, \varepsilon, \varkappa\rangle$, where $U = \{a, \ldots\}, \varkappa(\{a\}) = \emptyset$ and $\varepsilon(\{a\}) = \{a\}$. In this pre-topology $\varkappa(\varepsilon(\{a\})) = \emptyset \not\supseteq \{a\}$.

As a concrete simple example, consider the following three binary tables:

R_1	a	b	c
a	1	1	0
b	0	1	0
c	0	0	1

R_2	a	b	c
a	1	0	1
b	0	1	0
c	0	0	1

R_3	a	b	c
a	1	0	1
b	1	1	0
c	0	0	1

Suppose this is the behaviour of the same relation R under different conditions. For instance $R_1(a)$ is the behaviour of R at point a when this element is taken alone and b and c are not considered together; $R_2(a)$ is the behaviour of R at point a when this element is taken alone and b and c are considered together, or when it is joined with b. $R_3(a)$ is the behaviour of R at point a when this element is taken jointly with c. Going on with this interpretation, we can see that for any x and $i \in \{1, 2, 3\}$, $R_i(x)$ is the behaviour of R at x for a certain context. Here we display a possible context-sensitive set of evaluations, distinguishing "internal contexts" containing the elements to which R applies, and "external contexts" otherwise:

Internal contexts	*External contexts*	*Element*	*Applicable version of R*
$\{a\}$	$\{\{c\},\{b\}\}$	a	R_1
$\{a\}$	$\{\{b,c\}\}$	a	R_2, R_3
$\{a,b\}$	$\{\{c\}\}$	a,b	R_2
$\{a,c\}$	$\{\{b\}\}$	a,c	R_3
$\{b\}$	$\{\{a\},\{c\}\}$	b	R_1
$\{b\}$	$\{\{a,c\}\}$	b	R_3
$\{b,c\}$	$\{\{a\}\}$	b,c	R_2
$\{c\}$	$\{\{a\},\{b\},\{a,b\}\}$	c	R_1, R_2, R_3
$\{a,b,c\}$	$\emptyset$	a,b,c	R_1

Obviously, the fact that the behaviour of R changes along the contexts, does not make R- neighborhood formation an isotonic process. For instance, although $\{b\} \subseteq \{b,c\}$, if the external context of the evaluation of $R(\{b\})$ is $\{a,c\}$, then we have $R(\{b\}) = R_3(\{b\}) = \{a,b\}$. But the external context of evaluation of $R(\{b,c\})$ is $\{a\}$ so that $R(\{b,c\}) = R_2(\{b,c\}) = \{b,c\}$. Hence $R(\{b\}) \nsubseteq R(\{b,c\})$.

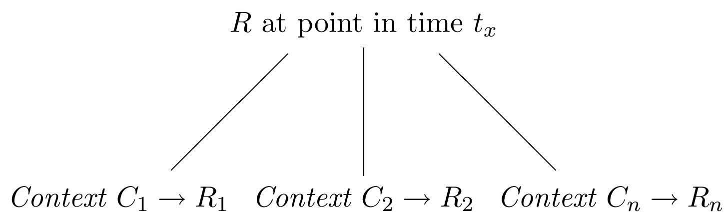
However, these three versions of R may represent other situations. For instance they could be the results of three surveys about the same relation R with respect to three different points in time t_1, t_2 and t_3.

Along this line of interpretation we shall develop interesting dynamic frameworks in information analysis, in which isotonicity is valid, although we have still to renounce other nice properties.

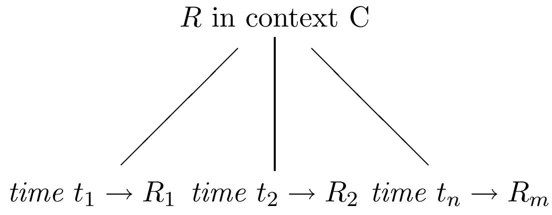
This point will be developed in Excursus 12.6.2 below. First, we have to introduce other kinds of pre-topologies.

To sum up, a dynamic analysis is required by two basic situations and a mixed one.

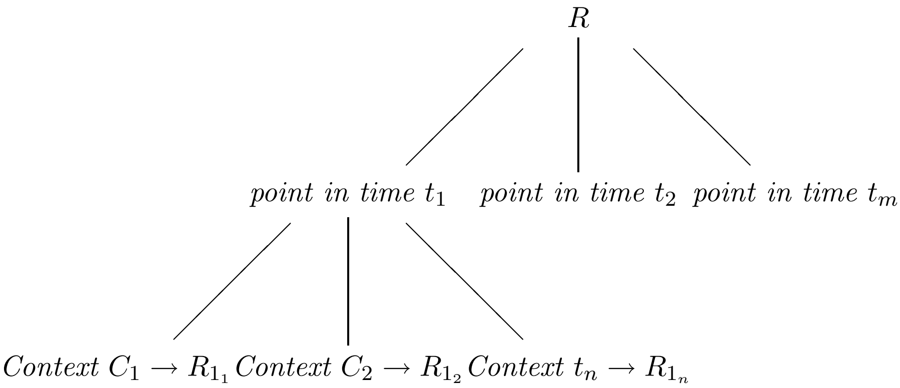
The first is when we fix the point in time and let the observation process depend on contexts:



The second happens when we fix the context and let the observation vary over time;



A third situation is given mixing the previous two:



Classical Rough Set Theory does not account for this kind of dynamic phenomena. Indeed, as far as we are confined to a single Information System, we can deal just with a picture taken at a particular point in time and at a particular point in space (meaning that the picture fixes a situation in space and time). In this picture, relations are static and definite.

Dynamics can be taken into account if we consider possible evolutions of Information Systems over time and/or evolutions of these behaviours of the analysed elements.

As we shall see, in all these cases pre-topologies are useful in order to synthesize and represent evolution. For instance, we can think of a collection of Approximation Spaces with operations which are able to synthesise their different information.

12.5 Towards Topology 1

In what follows, we shall progressively impose new properties to a pre-topological space in order to encompass the features required by our analysis, like isotonicity and distribution.

In this journey we still use examples taken from the dynamic approach illustrated above, so that the reader will be able to appreciate where approaches which use topological concepts are positioned in data analysis, like Approximation Space.

Definition 12.5.1. *A pre-topological space* $\langle U, \varepsilon, \varkappa \rangle$ *is said to be of type* $\mathfrak{V}_{Id}$ *if and only if for all* $X \subseteq U$ *the operator* $\varkappa$ *and* ε *are idempotent.*

It is easy to check that if one of the two conjugate operators is idempotent, so is the other.

Proposition 12.5.1. *A pre-topological space* $\langle U, \varepsilon, \varkappa \rangle$ *is of type* $\mathfrak{V}_{Id}$ *if* $\mathcal{N}^{\varkappa}(U)$ *is of type* $\mathcal{N}_{1Id}$.

Proof. Immediate, from *Proposition* 12.4.5 and *Proposition* 12.4.10. QED

Notice that a pre-topological space of type $\mathfrak{V}_{Id}$ is much weaker than a topological space, although $\varkappa$ is idempotent. For instance $\varkappa$ is not required to be isotonic.

Definition 12.5.2. *A pre-topological space* $\langle U, \varepsilon, \varkappa \rangle$ *is said to be of type* $\mathfrak{V}_I$ *if and only if for all* $X, Y \subseteq U, X \subseteq Y$ *implies* $\varepsilon(X) \subseteq \varepsilon(Y)$.

Proposition 12.5.2. *A pre-topological space* $\langle U, \varepsilon, \varkappa \rangle$ *is of type* $\mathfrak{V}_I$ *if and only if for all* $X, Y \subseteq U, X \subseteq Y$ *implies* $\varkappa(X) \subseteq \varkappa(Y)$.

Therefore, a pre-topological space is of *type* $\mathfrak{V}_I$ if and only if its expansion and contraction operators are isotonic. And this happens if every $\varkappa-$neighborhood system is a proper filter:

Proposition 12.5.3. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space. Then the following statements are equivalent:*

1. $\mathbf{P}$ *is of type* $\mathfrak{V}_I$.
2. *The family* $\mathcal{N}^{\varkappa}(U)$ *is a neighborhood system of type* $\mathcal{N}_2$.
3. $\mathbf{P}$ *is induced by a neighborhood system of type* $\mathcal{N}_2$.

Proof. Immediate, from *Lemma* 12.4.1 and *Proposition* 12.4.10. QED

Given a neighborhood system of type $\mathcal{N}_2$ we can define a pre-topological space of type $\mathfrak{V}_I$ in a manner that will be recognised to be very familiar. Let us indeed define two new operators g and f on $\wp(U)$.

Definition 12.5.3. *Let $\mathcal{N}(U)$ be a neighborhood system on U. Let us set:*

1. $g(X) = \{x \in U : \exists N(N \in \mathcal{N}_x \ \&\ N \subseteq X)\}$.
2. $f(X) = \{x \in U : \forall N(N \in \mathcal{N}_x \Rrightarrow N \cap X \neq \emptyset)\}$.

Clearly, the maps g and f are dual (the proof is left to the reader [*hints*: consider the set $-g(-X)$; so, as usual, apply in sequence the first order equivalences $\neg\forall x \equiv \exists x\neg$, $\neg(A \Rrightarrow B) \equiv A \ \&\ \neg B$ and, finally, $\neg(Y \cap -X \neq \emptyset) \equiv Y \subseteq X$]). Moreover, these maps are weaker than G and, respectively, F, because, obviously, for all $X \in \wp(U)$, $G(X) \subseteq g(X)$ (since $X \subseteq X$)

On the basis of this definition we have:

Proposition 12.5.4. *Let U be a set. Let $\mathcal{N}(U)$ be a neighborhood system. Then the following are equivalent:*

1. **N2** *holds in $\mathcal{N}(U)$.*
2. *For any subset X of the universe, $g(X) = G(X)$ and $f(X) = F(X)$.*

Proof. Since $G(X) \subseteq g(X)$, $g(X) = G(X)$ if and only if $\forall X \subseteq U$, $\forall x \in U((\exists N \in \mathcal{N}_x \ \&\ N \subseteq X) \Rrightarrow X \in \mathcal{N}_x)$, if and only if **N2** holds. Dually for F. QED

So, in case of neighborhood systems of type $\mathcal{N}_2$ the vicinity map (the expansion operator) is defined in the usual topological way: a point x is close to a set X if and only if all the elements of $\mathcal{N}_x$ have non null intersection with X. A well-known intuitive picture is that displayed by Figure 12.4.

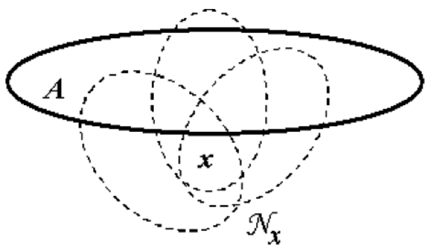


Figure 12.4: Point x is close to the set A because all of its neighborhoods have non empty intersections with A

The above definitions sound familiar to the reader, since it is obvious that as soon as we consider neighborhood systems $\mathcal{N}(U)$ such that for all x, $\mathcal{N}_x$ is a filter with a least element $l(x)$, then (1) and (2) of *Definition* 12.5.3 turn into: (1') $g(X) = \{x : l(x) \subseteq X\}$ and (2') $f(X) = \{x \in U : l(x) \cap X \neq \emptyset\}$.

Thus when this least element is an equivalence class $[x]_{\approx}$, we obtain the definitions of upper and, respectively, lower approximations.

However, from this discussion it follows that a pre-topological space must be equipped with additional structural properties in order to exhibit the characteristics of Approximation Spaces.

Proposition 12.5.5. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space of type $\mathfrak{V}_I$. Then,*

1. *If $\{O_i\}_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} \{O_i\}$ is open.*
2. *If $\{C_i\}_{i \in I}$ is a family of closed sets, then $\bigcap_{i \in I} \{C_i\}$ is closed.*
3. *U and $\emptyset$ are both closed and open.*

Proof. (**1**) Let $\{O_i\}_{i \in I}$ be a family of open sets. For any element $x \in \bigcup_{i \in I} O_i$ there is a $j \in I$ such that $x \in O_j$. But O_j is open, hence $O_j = x \in \varkappa(O_j)$. Since $O_j \subseteq \bigcup_{i \in I} O_i$, by isotonicity we have $\varkappa(O_j) \subseteq \varkappa(\bigcup_{i \in I} O_i)$ so that $x \in \varkappa(\bigcup_{i \in I} O_i)$. Therefore, for all $x \in \bigcup_{i \in I} O_i$ we have $\bigcup_{i \in I} O_i \in \varkappa_x$ and from *Proposition* 12.4.12 we obtain the result. (**2**) By duality. (**3**) Left to the reader. QED

Corollary 12.5.1. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space of type $\mathfrak{V}_I$. Then for any $X \subseteq U$, the closure $\mathbb{C}_\varepsilon(X)$ and the interior $\mathbb{I}_\varkappa(X)$ always exist.*

Another way to interpret the above result is that isotonicity implies a sort of fix-point property.

Although pre-topological spaces are spaces endowed with a rather rich structure, nevertheless, they are not able to completely account for the geometrical features of relational spaces. Let us consider again the cause of this limitation.

On the one hand, a pre-topological space provides a decreasing map, $\varkappa$, and an increasing map, ε, while $R-$neighboring is neither, for an arbitrary relation R. It is a decreasing map only if R is reflexive.

On the other hand, $R-$neighboring distributes both on unions and intersection, while this feature is not standard for generic pre-topological spaces.

So, let us now analyse the meaning of reflexivity and distribution.

12.5.1 Excursus: Reflexivity, Distribution and Perception

Since we are interested in how conceptual patterns are formed around the perception of a "point" x (item, object, stimulation, event, ...), we can assume that if a conceptual pattern is induced by a process π that gathers together all the "points" that are related with the given perceived "point" x, then x should belong to the result $\pi(x)$ of this process. Otherwise we should admit, rather metaphysically, that some *phenomena* appear to our consciousness by means of perceptions related with something which still remains a *noumenon* and not a part of the induced phenomena (see Figure 12.5).

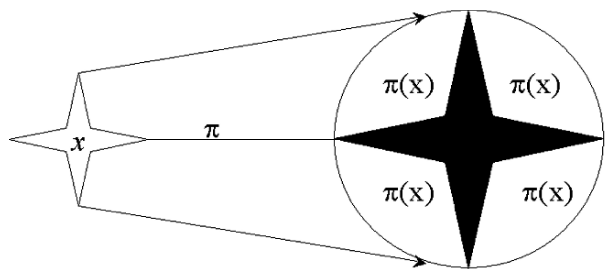


Figure 12.5: A non-reflexive phenomenological process may induce a phenomenon in which it partially or totally disappears

In order to avoid this metaphysical drawback, we can assume reflexivity on a quite intuitive basis.

As for distribution, we can have different attitudes. As a matter of facts, there is no evidence for claiming that if a phenomenon P_1 is the result of an inflationary (i.e. increasing) process π applied to "point" x, $P_1 = \pi(x)$, and P_2 is the result of the same process applied to "point" y, $P_1 = \pi(y)$, then the result of the application of π to both points, $\pi(x+y)$, is $P_1 + P_2$. Indeed, the two points taken together could carry more information than the sum of the two pieces of information carried by the two "points" singularly taken. Proximity Spaces and Concept Lattices are good examples of this situation (see Part I). On the contrary, the classical upper approximation in Rough Set Theory is additive. Additivity is a symptom of phenomena that fulfill some compositional property, in the sense that our ideal process π is additive: $\pi(x + y) = \pi(x) + \pi(y)$ (or, in a set-theoretical framework, $\pi(\{x\} \cup \{y\}) = \pi(\{x\}) \cup \pi(\{y\})$).

Moreover, one might wonder if it is possible to have an inflationary and distributive map avoiding isotonicity, i.e. monotonicity. We have already seen that this is not possible: we can have inflationary isotonic maps that are not additive. However, if a map is additive, then it is isotonic with respect to the lattice order.

12.6 Towards Topology 2

So, we have done a step towards the direction of relational spaces (Kripke frames) and Rough Sets by means of the concept of a pre-topological space of type $\mathfrak{V}_I$. Now we shall go further ahead, in order to grasp the distribution features.

Definition 12.6.1. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is said to be of type $\mathfrak{V}_D$ if and only if for all $X, Y \subseteq U, \varepsilon(X \cup Y) = \varepsilon(X) \cup \varepsilon(Y)$.*

Proposition 12.6.1. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is of type $\mathfrak{V}_D$ if and only if for all $X, Y \subseteq U, \varkappa(X \cap Y) = \varkappa(X) \cap \varkappa(Y)$.*

We have already seen that these distribution laws implies the isotonicity law. So any pre-topological space of type $\mathfrak{V}_D$ is also of type $\mathfrak{V}_I$. However the converse implication is not valid, as we can see in Example 12.6.2 below.

It is possible to prove that in order for a pre-topology to be of type $\mathfrak{V}_D$, the structure of the $\varkappa$-neighborhoods of any element of U must be a filter and not only an order filter. That is, if X and Y belongs to $\varkappa_x$, any x, then $X \cap Y$ must belong to $\varkappa_x$, too:

Proposition 12.6.2. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then the following statements are equivalent:*

1. *$\mathbf{P}$ is of type $\mathfrak{V}_D$.*
2. *The family $\mathcal{N}^{\varkappa}(U)$ is a neighborhood system of type $\mathcal{N}_3$.*
3. *$\mathbf{P}$ is induced by a neighborhood system of type $\mathcal{N}_3$.*

Proof. Immediate, from *Lemma* 12.4.1, *Proposition* 12.4.10 and *Corollary* 12.4.2. QED

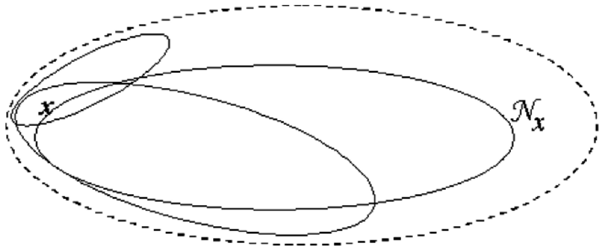


Figure 12.6: In a neighborhood system of type $\mathcal{N}_3$, the elements of the neighborhood family of any point x form a filter with respect to the relation $\subseteq$

Proposition 12.6.3. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space of type $\mathfrak{V}_D$. Then,*

1. *If $\{O_i\}_{i \in I}$ is a finite family of open sets, then $\bigcap_{i \in I} \{O_i\}$ is open.*
2. *If $\{C_i\}_{i \in I}$ is a finite family of closed sets, then $\bigcup_{i \in I} \{C_i\}$ is closed.*
3. *For any $X, Y \subseteq U$: $\mathbb{I}_{\varkappa}(X \cap Y) = \mathbb{I}_{\varkappa}(X) \cap \mathbb{I}_{\varkappa}(Y)$.*
4. *For any $X, Y \subseteq U$: $\mathbb{C}_{\varepsilon}(X \cup Y) = \mathbb{C}_{\varepsilon}(X) \cup \mathbb{C}_{\varepsilon}(Y)$.*

Proof. (**1**) If A and B are open sets, then $\varkappa(A) = A$ and $\varkappa(B) = B$. Since $\langle U, \varepsilon, \varkappa \rangle$ is of type $\mathfrak{V}_D$, $\varkappa(A \cap B) = \varkappa(A) \cap \varkappa(B) = A \cap B$. It follows that $A \cap B$ is an open set. (**2**) Dually. (**3**) From the first statement we have that for any $X, Y \subseteq U$, $\mathbb{I}_{\varkappa}(X) \cap \mathbb{I}_{\varkappa}(Y)$ is an open set; moreover, from *Proposition* 12.4.9 we obtain that $\mathbb{I}_{\varkappa}(X) \cap \mathbb{I}_{\varkappa}(Y)$ is an open set included in $X \cap Y$. On the other hand, since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we have (i): $\mathbb{I}_{\varkappa}(X \cap Y) \subseteq \mathbb{I}_{\varkappa}(X) \cap \mathbb{I}_{\varkappa}(Y)$. But $\mathbb{I}_{\varkappa}(X \cap Y)$ is the largest open set included in $X \cap Y$, so that we have (ii): $\mathbb{I}_{\varkappa}(X) \cap \mathbb{I}_{\varkappa}(Y) \subseteq \mathbb{I}_{\varkappa}(X \cap Y)$. We conclude from (i) and (ii) that $\mathbb{I}_{\varkappa}(X) \cap \mathbb{I}_{\varkappa}(Y) = \mathbb{I}_{\varkappa}(X \cap Y)$. (**4**) From duality. QED

Therefore, a pre-topological space of type $\mathfrak{V}_D$ features properties very close to those that characterise topological spaces. The remaining difference is that in a pre-topological space of type $\mathfrak{V}_D$ the two maps $\varkappa$ and ε are not required to be idempotent. Anyway, before adding the remaining clause and obtaining topological spaces, we have to introduce a new element to the taxonomy of pre-topological spaces, that will make it possible to associate reflexive relations with them.

Definition 12.6.2. *A pre-topological space* $\langle U, \varepsilon, \varkappa \rangle$ *is said to be of* type $\mathfrak{V}_S$*, or an* Alexandroff *pre-topological space, if and only if for all* $X \subseteq U, \varepsilon(X) = \bigcup_{x \in X} \varepsilon(\{x\})$.

A pre-topological space is of type $\mathfrak{V}_S$ only if any $\varkappa-$ neighborhood system is a principal filter.

Proposition 12.6.4. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space. Then the following are equivalent:*

1. $\mathbf{P}$ *is of type* $\mathfrak{V}_S$.
2. $\mathbf{P}$ *is of type* $\mathfrak{V}_I$ *and for any* x*, for any family* $\{X_i\}_{i \in I}$ *of elements of* $\varkappa_x$, $\bigcap_{i \in I} X_i \in \varkappa_x$.
3. *The family* $\mathcal{N}^{\varkappa}(U)$ *is a neighborhood system of type* $\mathcal{N}_4$.
4. $\mathbf{P}$ *is induced by a neighborhood system of type* $\mathcal{N}_4$.

Proof. Immediate, from *Lemma* 12.4.1, *Proposition* 12.4.10 and *Corollary* 12.4.2. QED

Therefore, if U is finite, then the notions of $\mathfrak{V}_S$ and $\mathfrak{V}_D$ pre-topological spaces coincide.

Example 12.6.1. A pre-topological space not of type $\mathfrak{V}_I$

We have seen that in the pre-topological space of *Example* 12.4.8, $\varkappa$ is idempotent but not isotonic.

Example 12.6.2. A pre-topological space in which $\varkappa$ is isotonic and idempotent but not multiplicative

We show that $\varkappa-$distributivity is independent of $\varkappa-$isotonicity and $\varkappa-$idempotence.

Consider the pre-topology $\mathbf{P}_1 = \langle U, \varepsilon, \varkappa \rangle$ such that $U = \{a, b, c\}$ and $\varkappa$ and ε are given by the following table:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varepsilon(x)$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	U	$\{a,c\}$	$\{b,c\}$	U
$\varkappa(x)$	$\emptyset$	$\{a\}$	$\{b\}$	$\emptyset$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U

By easy inspection we can verify that both $\varkappa$ and ε are isotonic. However,

- $\varepsilon(\{a\}) \cup \varepsilon(\{b\}) = \{a\} \cup \{b\} = \{a, b\} \neq U = \varepsilon(\{a, b\}) = \varepsilon(\{a\} \cup \{b\})$.
- $\varkappa(\{a, c\}) \cap \varkappa(\{b, c\}) = \{a, c\} \cap \{b, c\} = \{c\} \neq \emptyset = \varkappa(\{c\}) = \varkappa(\{a, c\} \cap \{b, c\})$.

Indeed, $\varkappa_c = \{\{a,c\},\{b,c\},U\}$ is an order filter but not a filter because $\{a,c\} \cap \{b,c\} \notin \varkappa_c$.

It should be noticed, moreover, that the family $\{\varkappa_x\}_{x\in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$. We can conclude that adding **Id** to **N2** does not say anything about **N3**.

Example 12.6.3. Contraction operators and order filters

We have seen that in the pre-topology $\mathbf{P}_1$ above, $\varkappa_c$ is an order filter but not a filter. This is the reason why the cocontinuity law fails when $\varkappa$ is applied to the intersection of $\{a,c\}$ and $\{b,c\}$, and the continuity law fails when ε is applied to their complements, $\{b\}$ and respectively $\{a\}$.

The family of $\varkappa$-neighborhoods is:

x	a	b	c
$\varkappa_x$	$\{\{a\},\{a,b\},\{a,c\},U\}$	$\{\{b\},\{a,b\},\{b,c\},U\}$	$\{\{a,c\},\{b,c\},U\}$

Notice that $\varkappa_a$ and $\varkappa_b$, incidentally, are filters. However, $\{\varkappa_x\}_{x\in U}$ is not a neighborhood system of type $\mathcal{N}_3$ because of $\varkappa_c$.

Example 12.6.4. A pre-topological space $\mathbf{P}$ where $\varkappa$ is idempotent but $\mathbf{P}$ is not of type $\mathfrak{V}_I$: $\varkappa$-isotonicity is independent of $\varkappa$-idempotence

A simple example is given by the pre-topological space of Example 12.4.8

Example 12.6.5. A pre-topological space $\mathbf{P}$ where any intersection of two open subsets is open, but not of type $\mathfrak{V}_D$

Consider the neighborhood system $\mathcal{N}(U)$

x	a	b	c
$\mathcal{N}_x$	$\{\{a,b\},\{a,b,c\}\}$	$\{\{b,c\},\{a,b\},\{a,b,c\}\}$	$\{\{a,b,c\}\}$

The open subsets are $\emptyset, \{a,b\}$ and $\{a,b,c\}$, and it is easy to check that they are closed under intersection. However, $\mathcal{N}_b$ is not a filter; thus $\mathcal{N}(U)$ is not of type $\mathfrak{V}_D$.

12.6.1 Bases

Definition 12.6.3. *Given a pre-topological space* $\mathbf{P} = \langle U, \varepsilon, \varkappa\rangle$*, the family* $\Omega_\varkappa(U) = \{\varkappa(A)\}_{A\subseteq U}$ *will be called the* pre-topology *of* U*.*

Terminology and Notation. From now on given a pre-topological space $\mathbf{P} = \langle U, \varepsilon, \varkappa\rangle$, with the symbol $\mathbf{P}$ we shall mean, whenever convenient and appropriate in the context, also its pre-topology $\Omega_\varkappa$.

Definition 12.6.4. *Let* $\mathbf{P_1} = \langle U, \varepsilon', \varkappa'\rangle$ *and* $\mathbf{P_2} = \langle U, \varepsilon'', \varkappa''\rangle$ *be two pre-topological spaces on the same universe* U*. Then we say that (the pre-topology of)* $\mathbf{P_1}$ *is finer than (the pre-topology of)* $\mathbf{P_2}$ *(or* $\mathbf{P_2}$ *is coarser than* $\mathbf{P_1}$*), in symbols* $\mathbf{P_2} \trianglelefteq \mathbf{P_1}$*, if for any* $X \subseteq U$*,* $\varkappa''(X) \subseteq \varkappa'(X)$*.*

Proposition 12.6.5. *Given two pre-topological spaces* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *on the same universe* U, $\mathbf{P_2} \trianglelefteq \mathbf{P_1}$ *if and only if* $\varkappa''_x \subseteq \varkappa'_x$, *any* $x \in U$.

Proof. Suppose $\varkappa''_x \subseteq \varkappa'_x$ and $x \in \varkappa''(Z)$. Hence $Z \in \varkappa''_x$ so that $Z \in \varkappa'_x$, too. It follows that $x \in \varkappa'(Z)$. Conversely, if $\varkappa''_x \nsubseteq \varkappa'_x$ there is an $F \in \varkappa''_x$ such that $F \notin \varkappa'_x$. Hence $x \in \varkappa''(F)$ but $x \notin \varkappa'(F)$. It follows that $\varkappa''(X) \nsubseteq \varkappa'(X)$. QED

Terminology and Notations. If $\mathcal{X}$ is a family of subsets of a given set U, then by $\Uparrow \mathcal{X}$ we shall denote the set $\{Y \subseteq U : \exists X(X \in \mathcal{X}\ \&\ X \subseteq Y)\}$ (the order filter generated by $\mathcal{X}$ in $\wp(U)$: $\Uparrow \mathcal{X} = \{\uparrow_{\subseteq} X : X \in \mathcal{X}\}$).

The following definition and properties will be useful.

Definition 12.6.5. *Let* U *be a set,* $\mathscr{F}$, $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *order filters or filters of elements of* $\wp(U)$. *Moreover let* $\mathcal{B}$, $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be families of subsets of* U. *Then:*

1. *If* $\mathscr{F} = \Uparrow \mathcal{B}$, *then* $\mathcal{B}$ *is called a* basis *for* $\mathscr{F}$ *and we say that* $\mathcal{B}$ induces $\mathscr{F}$. *We call a collection* $\mathfrak{B} = \{\mathcal{B}_i\}_{i \in I}$ *of bases, a* basis system.
2. *If* $\mathscr{F}_1 \subseteq \mathscr{F}_2$, *then* $\mathscr{F}_2$ *is said to be a* finer *filter than* $\mathscr{F}_1$.

Proposition 12.6.6. *Let* $\mathcal{B}_1, \mathcal{B}_2 \subseteq \wp(U)$, $\mathscr{F}_1 = \Uparrow \mathcal{B}_1, \mathscr{F}_2 = \Uparrow \mathcal{B}_2$ *and* $\mathcal{B}_2 \subseteq \mathcal{B}_1$. *Then* $\mathscr{F}_1$ *is finer than* $\mathscr{F}_2$.

The converse of the above *Proposition*, generally does not hold. Consider, indeed, $U = \{a, b, c\}$, $\mathcal{B}_1 = \{\{a\}\}$, $\mathcal{B}_2 = \{\{a, b\}\}$. Then, $\Uparrow \mathcal{B}_2 \subseteq \Uparrow \mathcal{B}_1$ although $\mathcal{B}_1 \nsubseteq \mathcal{B}_2$.

Corollary 12.6.1. *Let* $\mathcal{B} \subseteq \wp(U)$ *and* $\mathscr{F} = \Uparrow \mathcal{B}$. *Then,*
$\bigcap \mathscr{F} = \bigcap \mathcal{B} \in \mathcal{B}$ *(i.e.* $\bigcap \mathcal{B} \in \mathscr{F}$*).*

Proof. If $\mathscr{F}$ is a filter and $A, B \in \mathscr{F}$, then $A \cap B \in \mathscr{F}$. So $\bigcap \mathscr{F} \in \mathscr{F}$. Clearly, for any $X \in \mathscr{F}$, $\bigcap \mathscr{F} \subseteq X$. Therefore, if $\mathscr{F} = \Uparrow \mathcal{B}$, then $\bigcap \mathscr{F} \subseteq B$, for any $B \in \mathcal{B}$. It follows that $\bigcap \mathscr{F} = \bigcap \mathcal{B}$. QED

In view of *Definition* 12.5.4, if we are given a family of filters induced by a basis system, then in order to compute $\varepsilon(X)$ and $\varkappa(X)$ it is sufficient to consider the bases:

Proposition 12.6.7. *Let* U *be a set. Let* $\mathcal{N}(U)$ *be a neighborhood system of type (at least)* $\mathcal{N}_2$ *and* $\mathcal{N}'(U)$ *a neighborhood system of type*

$\mathcal{N}_4$. Assume that, for any x, $\mathcal{N}_x = \Uparrow \mathcal{B}_x$ for some $\mathcal{B}_x \subseteq \wp(U)$ and $\mathcal{N}'_x = \Uparrow \{Q_x\}$ for some $Q_x \subseteq U$. Then for any $X \subseteq U$ the following equations hold:

1. $\{x \in U : \exists N(N \in \mathcal{N}_x \ \& \ N \subseteq X)\} =$
   $\{x \in U : \exists A(A \in \mathcal{B}_x \ \& \ A \subseteq X)\}$.
2. $\{x \in U : \forall N(N \in \mathcal{N}_x \Rrightarrow N \cap X \neq \emptyset) =$
   $\{x \in U : \forall A(A \in \mathcal{B}_x \Rrightarrow A \cap X \neq \emptyset)\}$.
3. $\{x \in U : \forall N'(N' \in \mathcal{N}'_x \Rrightarrow N' \cap X \neq \emptyset)\} = \{x \in U : Q_x \cap X \neq \emptyset\}$.
4. $\{x \in U : \exists N'(N' \in \mathcal{N}'_x \ \& \ N' \subseteq X)\} = \{x \in U : Q_x \subseteq X\}$.

Proof. (**1**): Since $\mathscr{F}_x = \Uparrow \mathcal{B}_x$, if N is such that $N \in \mathcal{N}_x$ and $N \subseteq X$, then there is a $A \in \mathcal{B}_x$ such that $A \subseteq N \subseteq X$. Therefore, since $A \in \mathcal{N}_x$, A itself satisfies the condition of the right term of the equation. The converse is trivial. (**2**) If $X \cap A \neq \emptyset$ for $A \in \mathcal{B}_x$, then $X \cap F \neq \emptyset$ for any $F \supseteq A$. On the other hand, if $X \cap F \neq \emptyset$ for any $F \in \mathcal{N}_x$, then this holds of any $A \in \mathcal{B}_x$. (**3**) Trivially because the left part of the equation reduces to $\{x \in U : \bigcap \mathcal{N}_x \cap X \neq \emptyset)\}$ and $\bigcap \mathcal{N}_x = Q_x$, because $\mathcal{N}'_x = \Uparrow \{Q_x\}$. (**4**) Trivial, because Q_x is the least element of $\mathcal{N}'_x$. QED

Definition 12.6.6. *Let $\mathcal{N}(U)$ be a neighborhood system of type at least $\mathcal{N}_2$, $\mathfrak{B} = \{\mathcal{B}_x\}_{x \in U} \subseteq \wp(\wp(I))$ and $\mathcal{N}(U) = \{\Uparrow \mathcal{B}_x\}_{x \in U}$. If a pre-topological space* **P** *is* induced *by $\mathcal{N}(U)$, then we say that it is induced by $\mathfrak{B}$, too, and that $\mathfrak{B}$ is a* basis *for* **P**. *In this case to define $\varkappa$ and ε we shall also use the right side of the equations (1) and, respectively, (2) of Proposition 12.6.7 above.*

Trivially we have:

Proposition 12.6.8. *In any neighborhood system induced by a basis,* **1** *and* **N2** *hold.*

Example 12.6.6. From order filters to contraction operators
Given a neighborhood system of type $\mathcal{N}_2$, we can recover the contraction operator of a pre-topological space of type $\mathfrak{V}_I$ by means of the equations of *Proposition* 12.6.7.

Consider the family of neighborhood system $\mathcal{N}(U)$ on $U = \{a, b, c\}$ given by $\mathcal{N}_a = \{\{a\}, \{a, b\}, \{a, c\}, U\}$, $\mathcal{N}_b = \{\{a, b\}, U\}$ and $\mathcal{N}_c = \{\{b, c\}, \{a, c\}, U\}$.

Each neighborhood family is an order filter. Thus $\mathcal{N}(U)$ is of type $\mathcal{N}_2$ and it is induced by the basis $\mathfrak{B} = \{\mathcal{B}_a = \{\{a\}\}, \mathcal{B}_b = \{\{a, b\}\}, \mathcal{B}_c = \{\{a, c\}, \{b, c\}\}\}$.

Let us compute $\varkappa(\{a,b\})$:
(a) $\varkappa(\{a,b\}) = \{x : \exists A(A \in \mathcal{B}_x \ \& \ A \subseteq \{a,b\})\}$:
(a.1) a is OK: $\{a\} \in \mathcal{B}_a$ and $\{a\} \subseteq \{a,b\}$.
(a.2) b is OK: $\{a,b\} \in \mathcal{B}_b$ and $\{a,b\} \subseteq \{a,b\}$.
(a.3) c is not OK: none member of $\mathcal{B}_c$ is included in $\{a,b\}$.
Hence, $\varkappa(\{a,b\}) = \{a,b\}$.
Let us compute $\varkappa(\{c\})$: no element of $\mathcal{B}_a$, $\mathcal{B}_b$ or $\mathcal{B}_c$ is included in $\{c\}$; hence $\varkappa(\{c\}) = \emptyset$.

Exercise 12.8.
(a) Give an example of a pre-topological space not of type $\mathfrak{V}_I$ *where* $\{x : A \in \varkappa_x\} \neq \{x : \exists X(X \in \varkappa_x \ \& \ X \subseteq A)\}$.
(b) Exploiting Proposition 12.6.7.(ii), compute $\varepsilon(X)$ *for any* $X \subseteq U$ *in the pre-topological space* $\mathbf{P}_1$ *of Example 12.6.2 above.*
(c) Compute a minimal basis for the pre-topology $\mathbf{P}_1$.
(d) Find a minimal binary relation $R \subseteq U \times X$*, for some set* X*, such that the expansion map* ε *of* $\mathbf{P}_1$ *coincides with the Galois closure operator on* $\wp(U)$*, modulo* R*.*

Although bases are enough, in many examples below we shall also show the entire family of filters or order filters inducing a pre-topology.

Proposition 12.6.9. *If a pre-topological space* $\mathbf{P}$ *is induced by a basis* $\mathfrak{B} = \{\mathcal{B}_x\}_{x \in U}$ *such that for any* $x \in U$, $\mathcal{B}_x$ *is a singleton, then* $\mathbf{P}$ *is of type* $\mathfrak{V}_S$.

Proof. If $\mathcal{B}_x = \{X\}$, then $\Uparrow \mathcal{B}_x = \uparrow X = \{Y \subseteq U : X \subseteq Y)\}$, which is obviously a filter, because $\bigcap \uparrow X = X$ and $X \in \uparrow X$. Therefore, from *Corollary* 12.4.2 we have the result. QED

12.6.2 Excursus. Dynamics 2: The Failure of the Distributivity Laws

In Excursus 12.4.1, we have seen that dynamics and monotonicity may conflict. Here we shall exhibit examples of dynamic data analysis where monotonicity holds. However, distributivity laws fails to hold because of the intrinsic mechanism of these dynamic analyses.

Suppose we are given a universe U and a system of n binary relations on U, $\mathfrak{R} = \{R_i\}_{1 \leq i \leq n}$.

As we have seen, we can think of $R_1, R_2, \ldots, R_n$ as the results of n surveys about the same relation R with respect to n different points of time $t_1, t_2, \ldots, t_n$, respectively, or surveys about n different criteria $C_1, C_2, \ldots, C_n$, respectively.

Definition 12.6.7. *Let U be a set and $\{R_i\}_{i \in I}$ a family of binary relations on U. Then the pair $\langle U, \{R_i\}_{i \in I}\rangle$ is called a* Dynamic Relational System*.*

If each R_i is reflexive, then we can use pre-topology to develop interesting information analyses in which the pre-topological operators are isotonic, although we have still to renounce other nice properties, such as $\varkappa$-cocontinuity and ε-continuity.

Let us list $n \times 2$ basic "use cases" of the above surveys. Given a subset A of U, we have n use cases involving the expansion process, and n use cases involving the contraction process:

1. (*Contraction*): We say that $x \in \varkappa^m(A)$, for $1 \leq m \leq n$, if every y such that $\langle x, y\rangle \in R_i$ belongs to A, at least in m cases. Otherwise stated: $x \in \varkappa^m(A)$ if $R_{1 \leq i \leq n}(x) \subseteq A$ for at least m indices. So, for instance, assume $n = 3$, then $x \in \varkappa^2(A)$ if $R_1(x) \subseteq A$ and $R_2(x) \subseteq A$, or if $R_1(x) \subseteq A$ and $R_3(x) \subseteq A$, or if $R_2(x) \subseteq A$ and $R_3(x) \subseteq A$ (i.e. if $R_1(x) \cup R_2(x) \subseteq A$, or $R_1(x) \cup R_3(x) \subseteq A$, or $R_2(x) \cup R_3(x) \subseteq A$).
2. (*Expansion*): We say that $x \in \varepsilon^m(A)$, for $1 \leq m \leq n$, if A contains at least a y such that $\langle x, y\rangle \in R_i$ in at least $n + 1 - m$ cases. Otherwise stated: $x \in \varepsilon^m(A)$ if $R_{1 \leq i \leq n}(x) \cap A \neq \emptyset$ for at least $n + 1 - m$ indices. So, for instance, assume $n = 3$, then $x \in \varepsilon^3(A)$ if $R_1(x) \cap A \neq \emptyset$, or $R_2(x) \cap A \neq \emptyset$, or $R_3(x) \cap A \neq \emptyset$ (i.e. if $(R_1(x) \cup R_2(x) \cup R_3(x)) \cap A \neq \emptyset$).

According to these use cases, we can compute the families of expansion and contraction operators, $\varepsilon^{1 \leq m \leq n}$ and $\varkappa^{1 \leq m \leq n}$, by transforming the various R_i-neighborhoods into appropriate bases and applying eventually *Proposition* 12.6.7:

Definition 12.6.8. *Let U be a set and let $\mathfrak{R} = \{R_i\}_{1 \leq i \leq n}$ be a system of n binary reflexive relations on U. For $1 \leq m \leq n$, let Γ^m be the family of combinations of m elements out of a set of n elements, γ a combination from Γ^m. Then let us set:*

1. $\varepsilon^m : \wp(U) \longmapsto \wp(U); \varepsilon^m(A) = \{x \in U : \forall F(F \in \mathscr{F}^m_x \Rrightarrow F \cap A \neq \emptyset)\}$.

2. $\varkappa^m : \wp(U) \longmapsto \wp(U); \varkappa^m(A) = \{x \in U : \exists F(F \in \mathscr{F}_x^m \;\&\; F \subseteq A)\}$,

where: $\mathscr{F}_x^m$ *is the (order) filter induced by the basis* $\mathcal{B}_x^m$*, and* $\mathcal{B}_x^m = \{X_\gamma : X_\gamma = \bigcup_{l\in\gamma} R_l(x)\}_{\gamma\in\Gamma^m}$.

Proposition 12.6.10. *Let* $\mathfrak{R}$ *be a system of* n *reflexive binary relations on a set* U*. Then, for each* $m, 1 \leq m \leq n$*,* $\langle U, \varkappa^m, \varepsilon^m\rangle$ *is a pre-topological space of type* $\mathfrak{V}_I$*.*

The proof is immediate. In fact, from *Proposition* 12.6.8, **1** and **N2** hold in $\mathcal{N}^{\varkappa^m}$. Moreover, **Id** and **0** hold because all relations in $\mathfrak{R}$ are reflexive.

Let us apply all the above definitions to a simple example.

Consider the Dynamic Relational System $\langle U, \{R_1, R_2, R_3\}\rangle$, where $U = \{a, b, c\}$ and R_1, R_2 and R_3 are the relations from the example of Excursus 12.4.1.

In view of the above definitions we have:

m	Γ^m	$\mathcal{B}_x^m$
1	$\{\{1\},\{2\},\{3\}\}$	$\{R_1(x), R_2(x), R_3(x)\}$
2	$\{\{1,2\},\{1,3\},\{2,3\}\}$	$\{R_1(x)\cup R_2(x), R_1(x)$ $\cup R_3(x), R_2(x)\cup R_3(x)\}$
3	$\{\{1,2,3\}\}$	$\{R_1(x)\cup R_2(x)\cup R_3(x)\}$

In the following tables we show the basis $\mathfrak{B}^m(U) = \{\mathcal{B}_x^m\}_{x\in U}$, the induced neighborhood system $\mathfrak{F}^m(U) = \{\mathscr{F}_x^m\}_{x\in U}$, and, finally the operators ε^m and $\varkappa^m$:

$\mathcal{B}_x^m$	$\mathcal{B}_a^m$	$\mathcal{B}_b^m$	$\mathcal{B}_c^m$
$\mathcal{B}_x^1$	$\{\{a,b\},\{a,c\}\}$	$\{\{b\},\{a,b\}\}$	$\{\{c\}\}$
$\mathcal{B}_x^2$	$\{\{a,c\},U\}$	$\{\{b\},\{a,b\}\}$	$\{\{c\}\}$
$\mathcal{B}_x^3$	$\{U\}$	$\{\{a,b\}\}$	$\{\{c\}\}$

$\mathscr{F}_x^m$	$\mathscr{F}_a^m$	$\mathscr{F}_b^m$	$\mathscr{F}_c^m$
$\mathscr{F}_x^1$	$\{\{a,b\},\{a,c\},U\}$	$\{\{b\},\{a,b\},\{b,c\},U\}$	$\{\{c\},\{a,c\},\{b,c\},U\}$
$\mathscr{F}_x^2$	$\{\{a,c\},U\}$	$\{\{b\},\{a,b\},\{b,c\},U\}$	$\{\{c\},\{a,c\},\{b,c\},U\}$
$\mathscr{F}_x^3$	$\{U\}$	$\{\{a,b\},U\}$	$\{\{c\},\{a,c\},\{b,c\},U\}$

ε^m	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
ε^1	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	U	U
ε^2	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,c\}$	$\{a,b\}$	$\{a,c\}$	U	U
ε^3	$\emptyset$	$\{a,b\}$	$\{a,b\}$	$\{a,c\}$	$\{a,b\}$	U	U	U

$\varkappa^m$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varkappa^1$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varkappa^2$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varkappa^3$	$\emptyset$	$\emptyset$	$\emptyset$	$\{c\}$	$\{b\}$	$\{c\}$	$\{c\}$	U

Since each R_i is reflexive, $\langle U, \varkappa^1, \varepsilon^1\rangle$ is a pre-topological space. However we can notice, for instance, that $\mathscr{F}_a^1$ is not a filter, because $\{a,b\} \cap \{a,c\} \notin \mathscr{F}_a^1$.

Hence the pre-topological space $\langle U, \varepsilon^1, \varkappa^1\rangle$ is not of type $\mathfrak{V}_D$.

Also, we can directly observe the relationship between ε and $\varkappa$ distributivity and proper filters.

Indeed, since $\mathscr{F}_a^1$ is not a proper filter, there are two minimal distinct elements $A = \{a,b\}$ and $B = \{a,c\}$ of $\mathscr{F}_a^1$ such that $A \cap B \neq \emptyset$ but $A \cap B \notin \mathscr{F}_a^1$. Let us set $Y = B \cap -A = \{b\}$, $Z = A \cap -B = \{c\}$. Therefore, the subset $Y \cup Z = \{b,c\}$ has empty intersection neither with A nor with B; hence $Y \cup Z$ has empty intersections with no members of $\mathscr{F}_a^1$, because A and B are minimal. It follows that a belongs to $\varepsilon^1(Y \cup Z)$. But $a \notin \varepsilon^1(Y)$ and $a \notin \varepsilon^1(Z)$. Henceforth $\varepsilon^1(Y) \cup \varepsilon^1(Z) \subsetneq \varepsilon^1(Y \cup Z)$. Dually for $\varkappa$-codiscontinuity. In fact, $a \in \varkappa^1(A)$ because $A \in \mathscr{F}_a^1$ and $A \subseteq A$. For the same reason $a \in \varkappa^1(B)$. Therefore $a \in \varkappa^1(A) \cap \varkappa^1(B)$. But $A \cap B \subsetneq A$ and $A \cap B \subsetneq B$ (remember that $A \neq B$). Since A and B are minimal in $\mathscr{F}_a^1$, there is not any $F \in \mathscr{F}_a^1$ such that $F \subseteq A \cap B$. Thus $a \notin \varkappa^1(A \cap B)$. Henceforth $\varkappa^1(A \cap B) \subsetneq \varkappa^1(A) \cap \varkappa^1(B)$. As a side consequence, ε^1 is not continuous. In our example:

$$\varkappa^1(\{a,b\} \cap \{a,c\}) = \varkappa^1(\{a\}) = \emptyset \neq \{a\} = \{a,b\} \cap \{a,c\} = \varkappa^1(\{a,b\}) \cap \varkappa^1(\{a,c\}).$$

and

$$\varepsilon^1(\{b\}) \cup \varepsilon^1(\{c\}) = \{b,c\} \subseteq \{a,b,c\} = \varepsilon^1(\{b,c\}) = \varepsilon^1(\{b\} \cup \{c\})$$

[See the Frame section for further details.]

Example 12.6.7. A pre-topological space of type $\mathfrak{V}_D$ which is not topological: $\varkappa$-idempotence is independent of $\varkappa$−distributivity and isotonicity

Consider the pre-topological space $\mathbf{P}_2 = \langle U, \varepsilon, \varkappa \rangle$ such that $U = \{a, b, c\}$ and $\varkappa$ and ε are given by the following table:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varepsilon(x)$	$\emptyset$	$\{a,b\}$	U	$\{c\}$	U	U	U	U
$\varkappa(x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\emptyset$	$\{c\}$	U

By easy inspection we can verify that $\varkappa$ distributes over meets and ε distributes over unions.

However, the two operators are not idempotent: $\varepsilon(\{a\}) = \{a, b\} \neq \{a, b, c\} = \varepsilon(\{a, b\})$; $\varkappa(\{b, c\}) = \{c\} \neq \emptyset = \varkappa(\{c\})$. So this is a case of distributive operators that are not idempotent. Since distributivity implies isotonicity, we have the required example. Therefore property (τ) is not valid.

But (τ) is a typical property of neighborhoods in topological spaces – see further in the text.

The same happens for the structure $\langle U, \varepsilon^3, \varkappa^3 \rangle$ in Excursus Dynamics 2, § 12.6.2. Indeed, consider the neighborhood systems $\mathscr{F}_a^3 = \{\{a, b, c\}\}$, $\mathscr{F}_b^3 = \{\{a, b\}, \{a, b, c\}\}$, $\mathscr{F}_c^3 = \{\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Given the element $\{a, b\}$ of $\mathscr{F}_b^3$, there is not any $X \in \mathscr{F}_b^3$ such that $\{a, b\} \in \mathscr{F}_x^3$ for any $x \in X$. In fact, clearly $\{a, b, c\}$ is not such an X. As for the remaining element of $\mathscr{F}_b^3$, $\{a, b\}$ itself, it does not belong to $\mathscr{F}_a^3$. This is the reason for $\varkappa^3(\varkappa^3(\{a, b\})) = \emptyset \neq \{b\} = \varkappa^3(\{a, b\})$. Dually, this is the reason for $\varepsilon^3(\varepsilon^3(\{c\})) = \{a, b, c\} \neq \{a, b\} = \varepsilon^3(\{c\})$.

Exercise 12.9.
(a) Compute the family $\mathscr{F} = \{\varkappa_a, \varkappa_b, \varkappa_c\}$ from $\mathbf{P}_2$.
(b) Check that every member of $\mathscr{F}$ is a filter and not only an order filter.
(c) Compute a minimal basis for the pre-topology $\mathbf{P}_2$.
(d) Verify that property (τ) fails to hold in $\mathbf{P}_2$.

12.7 Pre-Topological Spaces and Binary Relations

Now we are in a good position for understanding how relations and relation neighborhoods are connected with pre-topological spaces.

First of all, let us underline that not every pre-topological space is connected with a binary relation and not every binary relation induces a pre-topology. We know that pre-topologies can be associated with relations that are *at least reflexive*. Taking into account this proviso, let us formalise in a definition the construction discussed in the above Excursus 12.6.2:

Definition 12.7.1. *Let U be a set and let $\mathfrak{R} = \{R_i\}_{i \in I}$ be a system of reflexive binary relations on U.*

(a) *If $\mathcal{N}(U) = \{R_i(x)\}_{i \in \mathfrak{R}}$ then we say that $\mathcal{N}(U)$ and the pre-topological space $\mathbf{P}(\mathfrak{R}) = \langle U, \varepsilon, \varkappa \rangle$ are* connected *with $\mathfrak{R}$, where for any $x \in U$, $\varkappa(x) = G(x)$. In this case we shall also write $\mathcal{N}(\mathfrak{R})$.*

(b) *The pre-topological space induced by the basis $\mathfrak{B}^m(U) = \{\mathcal{B}_x^m\}_{x \in U}$ is said to be $m-$*associated *with the system $\mathfrak{R}$ and denoted by $\mathbf{P}^m(\mathfrak{R}) = \langle U, \varepsilon^m, \varkappa^m \rangle$.*

(c) *In particular, if $\mathfrak{R} = \{R\}$, then the pre-topological space induced by the basis $\{R(x)\}_{x \in U}$ is said to be* associated *with the relation R and denoted by $\mathbf{P}(R) = \left\langle U, \varepsilon^R, \varkappa^R \right\rangle$.*

One should not confuse $\mathbf{P}(\mathfrak{R})$ with $\mathbf{P}(R)$.

Example 12.7.1. Difference between pre-topological spaces using $\{R_i(x)\}_{i \in I, x \in U}$ as a neighborhood system or as a basis for a neighborhood system

Here we show the difference between pre-topological spaces *connected* with a system $\mathfrak{R}$ of reflexive binary relations, and pre-topological spaces *induced* by $\mathfrak{R}$.

Consider the following system of relations $\mathfrak{R} = \{R_1, R_2\}$:

R_1	a	b	c
a	1	1	0
b	0	1	1
c	1	0	1

R_2	a	b	c
a	1	0	1
b	1	1	0
c	0	1	1

If we intend $\{R_1(x), R_2(x)\}_{x \in U}$ as a neighborhood system for a pre-topological space $\mathbf{P}(\mathfrak{R}) = \langle U, \varepsilon, \varkappa \rangle$, then $\varkappa(\{a, b, c\}) = \emptyset$. Actually, $\mathbf{P}(\mathfrak{R})$ is not of type $\mathfrak{V}_I$ because neither $\{R_1(a), R_2(a)\}$ nor $\{R_1(b), R_2(b)\}$ are filters. Therefore we must use the definition $\varkappa(X) = G(X) = \{x : X \in \mathcal{N}_x\}$. But for all $x \in U$, $\{a, b, c\} \notin R_1(x)$ or $R_2(x)$. On the contrary, if we intend $\mathfrak{R}$ as a basis then we obtain $\mathbf{P}^1(\mathfrak{R}) = \langle U, \varepsilon^1, \varkappa^1 \rangle$. In this case $\varkappa^1(\{a, b, c\}) = \{a, b, c\}$. One can observe that in $\mathbf{P}(\mathfrak{R})$, $\varkappa(X) \neq \{x : \exists R_i(R_i(x) \subseteq X)\}$ (indeed for any $i \in \{1, 2\}$, and for any $x \in U$, $R_i(x) \subseteq \{a, b, c\}$). Indeed, $\mathbf{P}^1(\mathfrak{R})$ has type $\mathfrak{V}_{Cl}$ while $\mathbf{P}(\mathfrak{R})$ has type $\mathfrak{V}_{Id}$.

Therefore, in $\mathbf{P}^1(\mathfrak{R})$ we can apply *Proposition* 12.5.4 and *Proposition* 12.6.7, while in $\mathbf{P}(\mathfrak{R})$ we can just set $\varkappa = G$.

However, since for any $x \in U$, $\{a, b, c\} \in \Uparrow (\mathcal{B}_x^1)$, we have that the relation $\mathbf{1} = U \times U$ belongs to the pseudo-uniformity $U(\mathfrak{R})$ see below, because $\mathbf{1} \supseteq R_i$, any $R_i \subseteq U \times U$. It follows that $\mathbf{P}(U(\mathfrak{R})) = \mathbf{P}(\mathfrak{R})$.

Proposition 12.7.1. *Let* $\mathbf{P}^1(\mathfrak{R}) = \langle U, \varepsilon^1, \varkappa^1 \rangle$ *be a pre-topological space 1-associated with a system of reflexive binary relations* $\mathfrak{R} = \{R_i\}_{i\in I}$*. Then;*

1. $\mathbf{P}^1(\mathfrak{R})$ *is of type* $\mathfrak{V}_I$.
2. $\mathfrak{B}^1(U) = \{\mathcal{B}^1_x\}_{x\in U} = \{R_i(x)\}_{i\in I, x\in U}$.
3. $\varkappa^1(X) = \{x : \exists R_i(R_i \in \mathfrak{R} \;\&\; R_i(x) \subseteq X)\}$.
4. $\varepsilon^1(X) = \bigcap_{i\in I} R_i^{\smile}(X)$.

Proof. (**1**) Obvious. (**2**) Obvious. (**3**) In view of (1), $\mathbf{P}^1(\mathfrak{R})$ fulfills **N2**, hence we can apply *Proposition* 12.6.7. Therefore, $\varepsilon^1(X) = \{x : \forall R_i(R_i \in \mathfrak{R} \Rrightarrow R_i(x) \cap X \neq \emptyset)\}$. But $R_i(x) \cap X \neq \emptyset$ if and only if $\exists x'(x' \in X \;\&\; \langle x, x'\rangle \in R_i)$. Thus, $\varepsilon^1(X) = \{x : \forall R_i(R_i \in \mathfrak{R} \Rrightarrow x \in R^{\smile}(X))\}$. QED

Exercise 12.10.
(a) Consider the system $\mathfrak{R}$ *collecting the following two equivalence relations on* $U_4 = \{a, b, c, d\}$*:*

E_1	a	b	c	d
a	1	1	1	0
b	1	1	1	0
c	1	1	1	0
d	0	0	0	1

E_2	a	b	c	d
a	1	1	0	0
b	1	1	0	0
c	0	0	1	1
d	0	0	1	1

(a.1) Compute the operators ε^1, ε^2, $\varkappa^1$ *and* $\varkappa^2$ *starting from the two bases* $\{\mathcal{B}^1_x\}_{x\in U_4}$ *and* $\{\mathcal{B}^2_x\}_{x\in U_4}$.
(a.2) Consider the pre-topologies $\mathbf{P}^1(\mathfrak{R}) = \langle U_4, \varepsilon^1, \varkappa^1\rangle$ *and* $\mathbf{P}^2(\mathfrak{R}) = \langle U_4, \varepsilon^2, \varkappa^2\rangle$*. Do* $\mathbf{P}^1(\mathfrak{R})$ *or* $\mathbf{P}^2(\mathfrak{R})$ *coincide with the Approximation Space induced by* $\langle U_4, E_1 \cap E_2\rangle$*?*
(b) Consider the following relations on $U_3 = \{a, b, c, \}$*:*

R_1	a	b	c
a	1	0	0
b	0	1	0
c	0	1	0

R_2	a	b	c
a	1	1	0
b	1	1	0
c	0	0	1

(b.1) Compute ε^1, ε^2, $\varkappa^1$ *and* $\varkappa^2$ *by starting with the two bases* $\{\mathcal{B}^1_x\}_{x\in U_3}$ *and* $\{\mathcal{B}^2_x\}_{x\in U_3}$.
(b.2) Are $\langle U_3, \varepsilon^1, \varkappa^1\rangle$ *and* $\langle U_3, \varepsilon^2, \varkappa^2\rangle$ *pre-topological structures?*

We must distinguish pre-topological spaces that are *connected* with a system of relations $\mathfrak{R}$ and pre-topological spaces that are *induced* by $\mathfrak{R}$. However, if $\mathbf{P}$ is induced by $\mathfrak{R}$ we can find a system of relations $\mathfrak{R}'$ such that $\mathbf{P}$ is connected with it, in a straightforward way.

Indeed, so far we have discussed spaces generated by arbitrary families of binary reflexive relations. However we can prove that particular types of spaces are generated by families of binary reflexive relations organised in a specific manner.

If $\mathfrak{R} = \{R_i\}_{i \in I}$ is a system of binary reflexive relations and $\mathbf{P}^1(\mathfrak{R}) = \langle U, \varepsilon^1, \varkappa^1 \rangle$ is the pre-topological space 1-associated with $\mathfrak{R}$, then we can regard any R_i as a vicinity (nearness) relation on U. Clearly, if $R'_i \supseteq R_i$, then $R'_i(x) \in \varkappa^1_x$, because by definition $R_i(x) \in \varkappa^1_x$. Therefore we can think of I principal filters of relations ordered by $\subseteq$, generated by $\mathfrak{R} = \{R_i\}_{i \in I}$. The collection of these filters is called a *pseudo-uniformity* generated by $\mathfrak{R}$, and denoted by $U(\mathfrak{R})$ (see Figure 12.7).

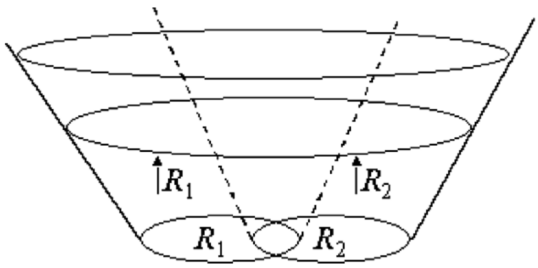


Figure 12.7: A pseudo-uniformity

Definition 12.7.2. *Let $\mathfrak{R} = \{R_i\}_{i \in I}$ be a system of relations, then the family $U(\mathfrak{R}) = \Uparrow \{R_i\}_{i \in I}$ is called a* pseudo-uniformity.

REMARKS. Notice that a pseudo-uniformity is a system of relations, not a system of relation neighborhoods.

Proposition 12.7.2. *Let $U(\mathfrak{R})$ be a pseudo-uniformity such that each $R \in \mathfrak{R}$ is reflexive. Then,*

1. *The pre-topological space $\mathbf{P}(U(\mathfrak{R}))$ connected with $U(\mathfrak{R})$ coincides with the pre-topological space $\mathbf{P}^1(\mathfrak{R})$ 1-associated with $\mathfrak{R}$.*
2. *$\mathbf{P}(U(\mathfrak{R}))$ is a pre-topological space of type $\mathfrak{V}_I$.*

Proof. Directly from *Definitions* 12.7.2 and *Proposition* 12.7.1. QED

So, pseudo-uniformities provide us with the intuitive concept of a family of vicinity relations, or, under a slightly different point of view, they provide us with a qualitative (non numerical) notion of nearness. In this intuitive context, the requirement that in any pseudo-uniformity $U(\mathfrak{R})$, if $R \in U(\mathfrak{R})$ and $R \subseteq R'$, then $R' \in U(\mathfrak{R})$, rests on the intuition that if two points x and y are estimated to be near with respect to a given point of view (or resolution) then they are near also with respect to a less refined point of view (i.e. with respect to a coarser resolution). The opposite, of course, does not hold, because a better resolution can separate x and y.

Moreover a different scenario is given by the requirement that if x and y are estimated to be near with respect to both the relations R and R', then they must be estimated near also with respect to the relation $R \cap R'$. This is the behaviour of pre-topological spaces of type $\mathfrak{V}_D$, so that the situation in which $U(\mathfrak{R})$ is closed under intersections needs a new name:

Definition 12.7.3. *If* $\mathfrak{R} = \{R_i\}_{i \in I}$ *is a system of relations, and* $U(\mathfrak{R})$ *a pseudo-uniformity such that* $R, R' \in \mathfrak{R}$ *implies* $R \cap R' \in \mathfrak{R}$*, then* $U(\mathfrak{R})$ *is called a* pre-uniformity*.*

Notice that since both R and R' are required to be reflexive, the so called diagonal $\Delta(U) = \{\langle x, x\rangle : x \in U\}$ is always included in $R \cap R'$, so that the intersection of elements of $\mathfrak{R}$ is never empty (see Figure 12.8).

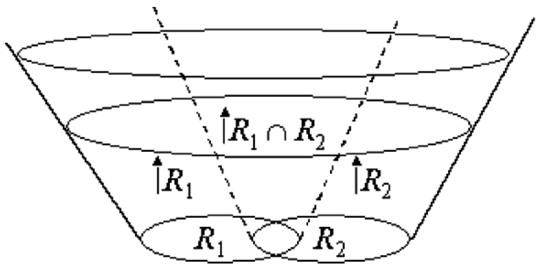


Figure 12.8: A pre-uniformity

Corollary 12.7.1. *Let* $U(\mathfrak{R})$ *be a pre-uniformity such that each* $R \in \mathfrak{R}$ *is reflexive and* $\mathfrak{R}' = \{R\}$*, for* R *reflexive. Then,*

1. $\mathbf{P}(U(\mathfrak{R}))$ *is a pre-topological space of type* $\mathfrak{V}_D$*.*
2. $U(\mathfrak{R}') = \uparrow R$ *is a pre-uniformity.*

Proof. From *Definitions* 12.7.1, 12.7.3 and *Proposition* 12.6.2, because if $U(\mathfrak{R})$ is a pre-uniformity then $\mathcal{N}(U(\mathfrak{R}))$ is a filter. QED

The catalogue of the interesting pre-topologies does not reduces to the above cases. Indeed, it is not complete if we miss the following important case:

Definition 12.7.4. *A pre-topological space* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *is said to be of type* $\mathfrak{V}_{Cl}$ *if the operator* ε *is a* closure operator*, that is, inflationary, isotonic and idempotent and* $\varkappa$ *is an* interior operator*, that is, deflationary, isotonic and idempotent.*

The pre-topological space $\mathbf{P}_1$ of Example 12.6.2 is of type $\mathfrak{V}_{Cl}$ (notice that, however, it is not of type $\mathfrak{V}_D$. Therefore, distributivity might not hold).

Proposition 12.7.3. $\mathcal{N}(U)$ *is a neighborhood system of type* $\mathcal{N}_{2Id}$ *if and only if its induced pre-topological space is of type* $\mathfrak{V}_{Cl}$.

Proof. From *Proposition* 12.4.10 and *Lemma* 12.4.1.(N1) and (N2), G is inflationary and isotonic if and only if **N1** and **N2** hold in $\mathcal{N}(U)$. QED

Moreover,

Proposition 12.7.4. *Let* $\mathfrak{R} = \{R_i\}_{i \in I}$ *be a system of preorder relations on* U*. Then,*

1. *In the pre-topological space* $\mathbf{P}^1(\mathfrak{R}) = \langle U, \varepsilon^1, \varkappa^1 \rangle$ *the operators* ε^1 *and* $\varkappa^1$ *are isotonic and idempotent.*
2. *The family* $\{\Uparrow \mathcal{B}^1_x\}_{x \in U}$ *is a neighborhood system of type* $\mathcal{N}_{2Id}$.

Proof. We prove only (**1**) because $\{\varkappa^1_x\}_{x \in U} = \{\Uparrow \mathcal{B}^1_x\}_{x \in U}$ and (1) implies that $\{\varkappa^1_x\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$.

Isotonicity derives from the construction of $\mathbf{P}^1(\mathfrak{R})$ via $\{\Uparrow \mathcal{B}^1_x\}_{x \in U}$.

Let us prove the assertion about the idempotence of $\varkappa^1$ through its contraposition. If $\varkappa^1$ is not idempotent then there is an $A \subseteq U$ such that $\varkappa^1(A) \nsubseteq \varkappa^1(\varkappa^1(A))$ (indeed, $\varkappa^1(\varkappa^1(A)) \subseteq \varkappa^1(A)$ always holds). In this case there is a $y \in \varkappa^1(A)$ such that $y \notin \varkappa^1(\varkappa^1(A))$. Therefore there is a set $B \in \mathcal{B}^1_y$ such that $B \subseteq A$ (so that $y \in \varkappa^1(A)$), but for every $B' \in \mathcal{B}^1_y$, $B' \nsubseteq \varkappa^1(A)$ (so that $y \notin \varkappa^1(\varkappa^1(A))$). In particular

$B \nsubseteq \varkappa^1(A)$. It follows immediately that there is an element $b \in B$ such that $b \notin \varkappa^1(A)$. This means that for all $B'' \in \mathcal{B}^1_b$, $B'' \nsubseteq A$. But B has the form $R_i(y)$, for some index i, and B'' has the form $R_j(b)$, for every index j. Therefore we can put $i = j$. To sum up, there is a $y \in U$ such that for all $b \in R_i(y), b \in A$ but for some $b \in R_i(y)$ there is a $b' \in R_i(b)$ such that $b' \notin A$. It follows that $b' \notin R_i(y)$. Hence R_i is not transitive, so that it is not true that all the members of $\mathfrak{R}$ are preorders. QED

The converse of the above *Proposition* does not hold because we can have systems of relations $\mathfrak{R}$ such that none of their components is a preorder but, nonetheless, in $\mathbf{P}^1(\mathfrak{R})$ the operator $\varkappa^1$ (resp. ε^1) is idempotent.

Example 12.7.2. A system $\mathfrak{R}$ of non-preorder relations, which induces a pre-topological space of type $\mathfrak{V}_{Cl}$

Notice: Under the assumptions of *Proposition* 12.6.7, in what follows we shall work on pre-topological bases, instead of induced filters.

Let $\mathfrak{R}$ be the collection of relations of Example 12.7.1.

It is easy to check that neither relation is transitive ($\langle a, c\rangle \in R_2, \langle b, a\rangle \in R_2$ but $\langle b, c\rangle \notin R_2$; $\langle a, b\rangle \in R_1, \langle b, c\rangle \in R_1$ but $\langle a, c\rangle \notin R_1$).

The basis $\mathfrak{B}^1$ is given by

$\mathcal{B}^1_x$	$\mathcal{B}^1_a$	$\mathcal{B}^1_b$	$\mathcal{B}^1_c$
$\mathcal{B}^1_x$	$\{\{a, b\}, \{a, c\}\}$	$\{\{b, c\}, \{a, b\}\}$	$\{\{a, c\}, \{b, c\}\}$

Therefore the operator $\varkappa^1$ is given by:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U
$\varkappa^1(x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U

Hence $\varkappa^1$ is idempotent. Indeed, the family $\{\varkappa_x\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$.

So it is observed that the important mathematical notion of a closure (interior) operator is connected, in particular contexts, with pre-topological spaces induced by systems of relations featuring specific properties, namely preorders.

We sum-up the above results in Table 12.4.

The last row will be the target of what follows.

We recall that from *Proposition* 12.5.4, if $\langle U, \varepsilon, \varkappa\rangle$ is of type $\mathfrak{V}_I$, then for any $X \subseteq U$, $\varkappa(X) = \{x \in U : \exists N(N \in \mathcal{N}_x \ \& \ N \subseteq X)\}$ and

Table 12.4: Correspondence between pre-topology types and neighborhood system types

$\langle U, \varepsilon, \varkappa \rangle$ is said of type:	Characteristic properties	if $\mathcal{N}^{\varkappa}(U)$ is of type:
$\mathfrak{V}_{Id}$	$\varkappa(\varkappa(X)) = \varkappa(X)$ $\varepsilon(\varepsilon(X)) = \varepsilon(X)$	$\mathcal{N}_{1Id}$
$\mathfrak{V}_{I}$	$X \subseteq Y \Rightarrow \varkappa(X) \subseteq \varkappa(Y)$ [resp. $\varepsilon(X) \subseteq \varepsilon(Y)$]	$\mathcal{N}_2$
$\mathfrak{V}_{D}$	$\varepsilon(X \cup Y) = \varepsilon(X) \cup \varepsilon(Y)$ [resp. $\varkappa(X \cap Y) = \varkappa(X) \cap \varkappa(Y)$]	$\mathcal{N}_3$
$\mathfrak{V}_{Cl}$	ε [resp. $\varkappa$] is a closure [resp. interior] operator	$\mathcal{N}_{2Id}$
$\mathfrak{V}_{S}$	$\varepsilon(X) = \bigcup_{x \in X} \varepsilon(\{x\})$	$\mathcal{N}_4$
topological	ε [resp. $\varkappa$] is a topological closure [resp. interior] operator	$\mathcal{N}_{3Id}$, $\mathcal{N}_{4Id}$

$\varepsilon(X) = \{x \in U : \forall N(N \in \mathcal{N}_x \Rrightarrow N \cap X \neq \emptyset)\}$, that is, in $\mathfrak{V}_I$ spaces the contraction operator (the expansion operator) has the same definition as the interior (closure) operator in usual topological spaces. Moreover, notice that if U is finite, then the notions of $\mathfrak{V}_S$ and $\mathfrak{V}_D$ pre-topological spaces coincide.

REMARKS. If we think of a neighborhood system as the image of a relation $R \subseteq U \times \wp(U)$, then we can ask what are the relationships between $\varkappa$, G and the perception operator *int* introduced in Chapter 2, and the role played by **Id**, **N1**, **N2** and so on in these relationships. This point will be developed il *Lemma* 15.14.4 of Frame 15.14.

In what follows we abandon systems of relations and from now on we shall focus on single relation based pre-topologies. About them we have a first set of results:

Corollary 12.7.2. *Let* $\mathbf{P}(R) = \langle U, \varepsilon^R, \varkappa^R \rangle$ *be a pre-topological space associated with a reflexive binary relation* R*. Then for any* $X \subseteq U$*:*

1. $\mathbf{P}(R)$ *is of type* $\mathfrak{V}_S$*.*
2. $\varkappa^R(X) = \{x : R(x) \subseteq X\}$*.*
3. $\varkappa^R(X) = \bigcup\{Y : R(Y) \subseteq X\}$*.*

4. $\varepsilon^R(X) = R^{\smile}(X)$.

5. $\varkappa^R_x = \Uparrow \{R(x)\}$, *for all* $x \in U$.

Proof. (**1**) Trivially, from *Proposition* 12.6.9 since $R(x)$ is a single subset of U. (**2**) From *Proposition* 12.7.1. (**3**) From the additivity of R-neighborhoods. (**4**) From *Proposition* 12.7.1. (**5**) From *Proposition* 12.7.1 and the definition of a basis of a filter. QED

Notice that we cannot prove $\varkappa^R(X) = R(X)$, in contrast with $\varepsilon^R(X) = R^{\smile}(X)$; conversely, we cannot prove $\varepsilon^R(X) = \bigcap\{Y : R^{\smile}(Y) \supseteq X\}$, in contrast to $\varkappa^R(X) = \bigcup\{Y : R(Y) \subseteq X\}$.

Example 12.7.3. If R is not a preorder, then $R^{\smile}(X) \neq \bigcap\{R^{\smile}(Z) : X \subseteq R^{\smile}(Z)\}$

We show that reflexivity is not enough in order to turn the above inequality into equality. Consider the following reflexive but not transitive relation:

R	a	b	c
a	1	0	1
b	1	1	1
c	0	1	1

$\{R^{\smile}(Z) : \{b\} \subseteq R^{\smile}(Z)\} = \{\{a,b\},\{b,c\},\{a,b,c\}\}$. Thus $\bigcap\{R^{\smile}(Z) : \{b\} \subseteq R^{\smile}(Z)\} = \{b\} \neq R^{\smile}(\{b\}) = \{b,c\}$. Indeed, the problem is that $b \in R(c)$, $a \in R(b)$ but $a \notin R(c)$. Thus $c \notin R^{\smile}(\{a\})$ so that $c \notin \bigcap\{R^{\smile}(Z) : \{b\} \subseteq R^{\smile}(Z)\}$, although $R^{\smile}(\{a,b\}) \in \{R^{\smile}(Z) : \{b\} \subseteq R^{\smile}(Z)\}$.

In view of *Definition* 12.1.2 we obtain immediately the following

Corollary 12.7.3. *Let* $\mathbf{P}(R) = \langle U, \varepsilon^R, \varkappa^R\rangle$ *be a pre-topological space associated with a reflexive binary relation* R. *Then for any* $X \subseteq U$,

1. $\varkappa^R(X) = L_R(X)$.

2. $\varepsilon^R(X) = M_R(X)$.

12.7.1 Excursus: Pre-topological Spaces and Modal Algebras

Let us set $\Omega_{\varkappa^R}(U) = \{X \subseteq U : \varkappa^R(X) = X\}$. We should now ask if the system $\langle \mathbf{B}(U), \Omega_{\varkappa^R}(U)\rangle$ is a modal system. From the point of view of *Definition* 11.5.4 the answer is negative because in general we cannot define $\varkappa^R(X)$ (i.e. $L_R(X)$) as $U \overset{\Omega_{\varkappa^R}}{\Longrightarrow} (X)$, that is, $\varkappa^R(X)$ does not coincide with the greatest element of $\Omega_{\varkappa^R}(U)$ below X, because of the trivial reason that this element might not exist. This limitation is due to the fact that without further constraints, generally $\Omega_{\varkappa^R}(U)$ is not a sup-subsemilattice of $\wp(U)$. Moreover notice that this fact is independent of the distributive properties of $\varkappa^R$ (see Example 12.8.1).

To analyse this topic, we shall use the following Lemma:

Lemma 12.7.1. *Let $\langle \mathbf{B}(U), k(\mathbf{B}(U))\rangle$ be a k-modal system such that the knowledge map k is connected with a relation $R \subseteq U \times U$. Let, for any $X \subseteq U$, $L^*_R(X) = \bigcup\{R(Z) : R(Z) \subseteq X\}$ and $!_R(X) = U \overset{k(\mathbf{B}(U))}{\Longrightarrow} X = \max\{Z \in k(\mathbf{B}(U)) : Z \subseteq X\}$. Then for any $X \subseteq U$,*

1. *$L^*_R(X) = !_R(X)$.*
2. *If R is reflexive, then $L_R(X) \subseteq L^*_R(X)$.*
3. *If R is a preorder, then $L_R(X) = L^*_R(X)$.*

Proof. (**1**) $!_R(X) = \max\{Y \in k(\mathbf{B}(U)) : Y \subseteq X\} = \max\{Y \in \{R(Z)\}_{Z \subseteq U} : Y \subseteq X\} = \max\{R(Z) : R(Z) \subseteq X\}$. But from the additivity of R-neighborhoods, $\max\{R(Z) : R(Z) \subseteq X\} = \bigcup\{R(Z) : R(Z) \subseteq X\} = L^*_R(X)$. (**2**) Suppose $a \in L_R(X)$. Then $R(a) \subseteq X$, so that $R(a) \subseteq \bigcup\{R(Z) : R(Z) \subseteq X\}$. If R is reflexive, $a \in R(a)$ and, therefore, $a \in \bigcup\{R(Z) : R(Z) \subseteq X\} = L^*_R(X)$ (the reverse inclusion does not hold – cf. Example 12.8.1). (**3**) In view of (2) we have only to prove the reverse inclusion. Suppose R is a preorder and $a \in \bigcup\{R(Z) : R(Z) \subseteq X\}$. Then there is a b such that $a \in R(b)$ and $R(b) \subseteq X$. Now, for any c such that $c \in R(a)$, $c \in R(b)$ by transitivity of R. Therefore $c \in X$. It follows that $R(a) \subseteq X$ and we can conclude that $a \in \{x : R(x) \subseteq X\}$. QED

We show some instances of this point in Example 12.8.1 below.

So, we have partially solved the problem issued at the end of Section 12.1: $L_R(X) = L^*_R(X)$ and $M_R(X) = M^*_R(X)$ if and only if R is a preorder. Otherwise stated:

Corollary 12.7.4. *If $\langle \mathbf{B}(U), k(\mathbf{B}(U))\rangle$ is a k-modal system such that the knowledge map k is connected with a relation $R \subseteq U \times U$, then $\langle \mathbf{B}(U), k(\mathbf{B}(U))\rangle$ is a modal system if and only if for any $X \subseteq U$, $X \subseteq k(X)$ and if $X' \subseteq k(X)$, then $k(X') \subseteq k(X)$.*

However, pre-topological spaces are naturally connected with a more general class of modal structures called *modal algebras*:

Definition 12.7.5. *A* modal algebra *is a pair* $\langle \mathbf{B}, \Box \rangle$*, where* $\mathbf{B}$ *is a non degenerate Boolean algebra closed under a unary operation* $\Box$*.*

We have trivially:

Proposition 12.7.5. *Let* U *be a set and* $\mathcal{N}(U)$ *a neighborhood system over* U*. Then* $\langle \mathbf{B}(U), G \rangle$ *is a modal algebra.*

Conversely,

Proposition 12.7.6. *Let* $\langle \mathbf{B}(U), \Box \rangle$ *be a modal algebra of the subsets of a set* U*. Let us set for all* $X \subseteq U$*,* $X \in \mathcal{N}_x^{\Box}$ *if and only if* $x \in \Box(X)$*. Then* $\mathcal{N}^{\Box}(U) = \{\mathcal{N}_x^{\Box}\}_{x \in U}$ *is a neighborhood system.*

A more general form of duality between modal algebras and neighborhood system is discussed in Frame 15.15.

Now we continue our analysis of pre-topological spaces.

In order to compare two pre-topological spaces associated with two binary relations it is sufficient to compare the relations themselves:

Corollary 12.7.5. *Let* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *be two pre-topological spaces on the same universe* U*, associated with two reflexive binary relations on* U*,* R_1 *and* R_2*, respectively. Then* $\mathbf{P_2} \trianglelefteq \mathbf{P_1}$ *if and only if for any* $x \in U, R_1(x) \subseteq R_2(x)$*, if and only if* $R_1 \subseteq R_2$*.*

Dually, given a pre-topological space, we can define a reflexive binary relation associated with it:

Proposition 12.7.7. *(T-association) Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space. Let us set:*

$$\langle x, y \rangle \in R^T(\mathbf{P}) \text{ iff } y \in \bigcap\{X : X \in \varkappa_x\} \qquad (T)$$

Then $R^T(\mathbf{P})$ *is a reflexive binary relation on* U*.*

Proof. Trivial: since by definition $x \in \bigcap\{X : X \in \varkappa_x\}$, then $\langle x, x \rangle \in R^T(\mathbf{P})$. QED

We shall say that $R^T(\mathbf{P})$ is *T-associated* with the pre-topology $\mathbf{P}$ and denote this relation with R^T whenever the pre-topological space $\mathbf{P}$ is understood.

In general, given a pre-topological space $\mathbf{P}$, it is possible to link it to a pre-topological space that is associated with a reflexive binary relation, in a unique way.

Proposition 12.7.8. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space. Then,*

1. $\mathbf{P} \trianglelefteq \mathbf{P}(R^T(\mathbf{P}))$.
2. *If* $\mathbf{P}$ *is of type* $\mathfrak{V}_S$*, then* $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}))$.
3. *If* $\mathbf{P}$ *is associated with a relation* R*, then* R *is T-associated with the pre-topological space* $\mathbf{P}$*, that is,* $R = R^T(\mathbf{P}(R))$.

Proof. (**1**) $\mathbf{P}(R^T(\mathbf{P}))$ $\varkappa_x^{R^T}$ is induced by the family $R^T(x)$, and for any $x \in U$, $R^T(x) = \bigcap\{X : X \in \varkappa_x\}$, so that $R^T(x) \subseteq X$, for any $X \in \varkappa_x$. Therefore, $\varkappa_x \subseteq \varkappa_x^{R^T}$. (**2**) Suppose $\mathbf{P}$ is of type $\mathfrak{V}_S$. Then for any $x \in U$ there is a subset X of U such that $\varkappa_x =\uparrow X$. Since $y \in R^T(x)$ iff $y \in \bigcap\{X : X \in \varkappa_x\} = X$, we obtain $\varkappa_x = \varkappa_x^{R^T}$. (**3**) From *Definition* 12.7.1 if $\mathbf{P}$ is associated with a relation R, then it is induced by the basis $\{R(x)\}_{x \in U}$. Therefore, $\varkappa_x =\Uparrow \{R(x)\} = \{\uparrow R(x)\}$. Hence, from *Proposition* 12.6.4, $\mathbf{P}$ is of type $\mathfrak{V}_S$, so that from point (2) we obtain the result. QED

REMARKS. The above *Proposition* 12.7.8 guarantees that given a pre-topological space $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ of type $\mathfrak{V}_S$, we can derive the properties of $\mathbf{P}$ from those of $\mathbf{P}(R^T(\mathbf{P}))$.

As to the inequality (1) of *Proposition* 12.7.8, it is possible to show that if $\mathbf{P}$ is of type $\mathfrak{V}_I$, then $\mathbf{P}(R^T(\mathbf{P}))$ is the coarsest pre-topology among those of type $\mathfrak{V}_S$ that are finer than $\mathbf{P}$ (see Frame 15.3 for a proof).

We can also associate a pre-topology to a tolerance (i.e. reflexive and symmetric) relation:

Proposition 12.7.9. *(B-association) Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space. Let us set, for all* $x, y \in U$*:*

$$\langle x, y \rangle \in R^B(\mathbf{P}) \text{ iff } y \in \bigcap\{X : X \in \varkappa_x\} \Rrightarrow x \in \bigcap\{Y : Y \in \varkappa_y\} \quad (B)$$

Then R *is a reflexive and symmetric relation.*

Proof. Trivial. QED

We shall say, that $R^B(\mathbf{P})$ is *B-associated* with the pre-topology $\mathbf{P}$.

12.7.2 Excursus. Pre-Topological Spaces and Approximation Spaces

From *Proposition* 12.7.8 we know that $\mathbf{P} \trianglelefteq \mathbf{P}(R^T(\mathbf{P}))$ and $R^T(\mathbf{P}(R)) = R$. So we wonder what information $\mathbf{P}(R^B(\mathbf{P}))$ and $R^B(\mathbf{P}(R))$ carry. We shall give the answer as a corollary of the following more general statement about families of binary relations, as treated in Excursus 12.4.1.

Proposition 12.7.10. *Let U be a set and $\{R_j\}_{1\leq j\leq n}$ a family of n reflexive binary relations on U. Let $1 \leq m \leq n$ and let $\mathbf{P}^m = \langle U, \varepsilon^m, \varkappa^m\rangle$, with the operators ε^m and $\varkappa^m$, as defined by* Definition *12.6.8. Moreover let us set $R^* = \bigcup_{1\leq j\leq n} R_j$ and $R_* = \bigcap_{1\leq j\leq n} R_j$. Then,*

1. *$R^T(\mathbf{P}^n) = R^*$.*
2. *$R^T(\mathbf{P}^1) = R_*$.*
3. *$R^B(\mathbf{P}^n)$ is the largest tolerance relation included in R^*.*
4. *$R^B(\mathbf{P}^1)$ is the largest tolerance relation included in R_*.*

The proof is given in Frame 15.3

Corollary 12.7.6. *For any family of n reflexive binary relations, $\mathbf{P}(R^T(\mathbf{P}^n)) \trianglelefteq \mathbf{P}(R^B(\mathbf{P}^n))$.*

Therefore, trivially, if we are given just one reflexive binary relation R, then $R^B(\mathbf{P}(R)) \subseteq R$, because $R^B(\mathbf{P}(R))$ is the largest tolerance relation included in R, while if we are given a pre-topological space $\mathbf{P}$, then $\mathbf{P}(R^B(\mathbf{P}))$ is the pre-topological space associated with the largest tolerance relation included in $R^T(\mathbf{P})$. It follows that $\mathbf{P} \trianglelefteq \mathbf{P}(R^B(\mathbf{P}))$ (the equality is not uniformly valid even if $\mathbf{P}$ is of type $\mathfrak{V}_S$; in fact in this case we have, generally, $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}) \trianglelefteq \mathbf{P}(R^B(\mathbf{P})))$.

Corollary 12.7.7. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa\rangle$ be a pre-topological space. Then $\mathbf{P}(R^B(\mathbf{P}))$ is the coarsest pre-topology among the pre-topological spaces finer than $\mathbf{P}$ and associated with a tolerance relation.*

A direct proof is in Frame 15.4.

Corollary 12.7.8. *Let U be a set and $\{R_j\}_{1\leq j\leq n}$ a system of n reflexive binary relations on U, such that $R^* = \bigcup_{1\leq j\leq n} R_j$ (such that $R_* =$*

$\bigcap_{1\leq j\leq n} R_j$*) is transitive. Then,* $\mathbf{P}(R^B(\mathbf{P}^n))$ *(respectively* $\mathbf{P}(R^B(\mathbf{P}^1))$*) is the Approximation Space induced by the largest tolerance relation included in* R^* *(respectively included in* R_**).*

Particularly, if $R^T(\mathbf{P})$ is a binary transitive relation on U, then $\mathbf{P}(R^B(\mathbf{P}))$ is the Approximation Space induced by the largest tolerance relation included in $R^T(\mathbf{P})$, so that we can say that $\mathbf{P}(R^B(\mathbf{P}))$ is the coarsest Approximation Space finer than the pre-topological space $\mathbf{P}$.

Definition 12.7.6. *Let* $\mathbf{P}$ *be a pre-topological space, then:*

1. *If* $R^T(\mathbf{P})$ *is a tolerance relation, then* $\mathbf{P}$ *is said to be* weakly symmetric*.*
2. *If* $R^T(\mathbf{P})$ *is an equivalence relation, then* $\mathbf{P}$ *is said to be* strongly symmetric*.*

Therefore, any pre-topological space of the form $\mathbf{P}(R^B(\mathbf{P}))$, is weakly symmetric and any pre-topological space of the form $\mathbf{P}(R^B(\mathbf{P}))$ such that $R^T(\mathbf{P})$ is a transitive and reflexive, is strongly symmetric.

Corollary 12.7.9. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa\rangle$ *be a pre-topological space, then:*

1. $\mathbf{P}$ *is weakly symmetric if given* $x, y \in U, x \in \varepsilon(\{y\})$ *implies* $y \in \varepsilon(\{x\})$.
2. $\mathbf{P}$ *is strongly symmetric if given* $x, y \in U, x \in \bigcap\{X : X \in \varkappa_y\}$ *implies* $\varkappa_x = \varkappa_y$.
3. $\mathbf{P}$ *is strongly symmetric if* $\{\bigcap \varkappa_x : x \in U\}$ *forms a partition of* U.

Weakly symmetric pre-topological spaces are connected with a particular kind of pre-uniformity structures:

Definition 12.7.7. *Let* $U(\mathfrak{R})$ *be a pre-uniformity of reflexive relations over a set* U*, such that for all* $R \subseteq U \times U$, $R \in U(\mathfrak{R})$ *implies* $R^{\smile} \in U(\mathfrak{R})$*. Then* $U(\mathfrak{R})$ *is called a* semi-uniformity*.*

Proposition 12.7.11. *If* $U(\mathfrak{R})$ *is a semi-uniformity then its connected pre-topological space* $\mathbf{P}(U(\mathfrak{R}))$ *is weakly symmetric.*

Example 12.7.4. Associating pre-topologies with reflexive binary relations
Some examples.

T-association: $R^T(\mathbf{P}^1)$, $R^T(\mathbf{P}^n)$

Consider the system of relations $\mathfrak{R}$ worked in Excursus Dynamics 2.

In view of the basis computed in Section 12.6.2, we derive the following table:

$\bigcap \mathcal{B}_x^m$	$\bigcap \mathcal{B}_a^m$	$\bigcap \mathcal{B}_b^m$	$\bigcap \mathcal{B}_c^m$
$\bigcap \mathcal{B}_x^1$	$\{a\}$	$\{b\}$	$\{c\}$
$\bigcap \mathcal{B}_x^2$	$\{a, c\}$	$\{b\}$	$\{c\}$
$\bigcap \mathcal{B}_x^3$	U	$\{a, b\}$	$\{c\}$

Consider $\mathbf{P}^1(\mathfrak{R})$ and call it $\mathbf{P}^1$, for short. Let us compute $R^T(\mathbf{P}^1)$. Since for all $x \in U$, $\varkappa_x^m = \Uparrow \bigcap \mathcal{B}_x^m$, we can work on the generators of the basis.

(i) $a \in \bigcap \mathcal{B}_x^1$, for $x = a$; (ii) $b \in \bigcap \mathcal{B}_x^1$, for $x = b$; (iii) $c \in \bigcap \mathcal{B}_x^1$, for $x = c$.

Therefore we obtain:

$R^T(\mathbf{P}^1)$	a	b	c
a	1	0	0
b	0	1	0
c	0	0	1

We immediately see that $R^T(\mathbf{P}^1)$ and R_* coincide. Notice, anyway, that $R^T(\mathbf{P}^1)$ is a transitive relation by chance.

Incidentally, here we can verify that $\mathbf{P}^1 \trianglelefteq \mathbf{P}(R^T(\mathbf{P}^1))$. Indeed, $\mathbf{P}(R^T(\mathbf{P}^1))$ has the following family of basis: $\mathcal{B}_a = \{\{a\}\}, \mathcal{B}_b = \{\{b\}\}, \mathcal{B}_c = \{\{c\}\}$. Clearly $\mathcal{B}_a$ induces a filter $\mathscr{F}_a = \uparrow \{a\}$ which is finer than $\mathscr{F}_a^1$ (i.e. $\{\{a, b\}, \{a, c\}, \{a, b, c\}\}$ – cf. Section 12.6.2). Indeed, we can recognize that $\mathbf{P}^1$ is not of type $\mathfrak{V}_S$, because $\bigcap \mathcal{B}_a^1 \notin \mathcal{B}_a^1$.

Now let us compute $R^T(\mathbf{P}^3)$:

(i) $a \in \bigcap \mathcal{B}_x^3$, for $x \in \{a, b\}$; (ii) $b \in \bigcap \mathcal{B}_x^2$, for $x \in \{a, b\}$; (iii) $c \in \bigcap \mathcal{B}_x^3$, for $x \in \{a, c\}$.

So, for instance, since $c \in \bigcap \mathcal{B}_a^2$, $\langle a, c\rangle \in R^T(\mathbf{P}^3)$.

Summing up, we obtain:

$R^T(\mathbf{P}^3)$	a	b	c
a	1	1	1
b	1	1	0
c	0	0	1

We immediately verify $R^T(\mathbf{P}^3) = R^*$.

B-association: $R^B(\mathbf{P}^n), R^B(\mathbf{P}^1)$**.**

Let $U = \{a, b, c, d, e\}$ and R_1, R_2, R^*, R_* be given by

R_1	a	b	c	d	e
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
d	0	0	0	1	1
e	0	0	0	1	1

R_2	a	b	c	d	e
a	1	1	1	1	1
b	1	1	1	1	1
c	0	0	1	0	0
d	0	0	0	1	1
e	0	0	0	1	1

R_*	a	b	c	d	e
a	1	1	1	1	1
b	1	1	1	1	1
c	0	0	1	0	0
d	0	0	0	1	1
e	0	0	0	1	1

R^*	a	b	c	d	e
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
d	0	0	0	1	1
e	0	0	0	1	1

By easy computation, applying the two formulas $\mathcal{B}_x^1 = \{R_1(x), R_2(x)\}$ and $\mathcal{B}_x^2 = \{R_1(x) \cup R_2(x)\}$ we obtain:

$\mathcal{B}_x^m$	$\mathcal{B}_a^m$	$\mathcal{B}_b^m$	$\mathcal{B}_c^m$	$\mathcal{B}_d^m$	$\mathcal{B}_e^m$
$\mathcal{B}_x^1$	$\{U\}$	$\{U\}$	$\{\{c\}, U\}$	$\{\{d, e\}\}$	$\{\{d, e\}\}$
$\mathcal{B}_x^2$	$\{U\}$	$\{U\}$	$\{U\}$	$\{\{d, e\}\}$	$\{\{d, e\}\}$

From the above table we derive the following one:

$\bigcap \mathcal{B}_x^m$	$\bigcap \mathcal{B}_a^m$	$\bigcap \mathcal{B}_b^m$	$\bigcap \mathcal{B}_c^m$	$\bigcap \mathcal{B}_d^m$	$\bigcap \mathcal{B}_e^m$
$\bigcap \mathcal{B}_x^1$	U	U	$\{c\}$	$\{d, e\}$	$\{d, e\}$
$\bigcap \mathcal{B}_x^2$	U	U	U	$\{d, e\}$	$\{d, e\}$

Let us compute $R^B(\mathbf{P}^1)$:

(i) $a \in \bigcap \mathcal{B}_x^1$, for $x \in \{a, b\}$; (ii) $b \in \bigcap \mathcal{B}_x^1$, for $x \in \{a, b\}$; (iii) $c \in \bigcap \mathcal{B}_x^1$, for $x \in \{a, b, c\}$; (iv) $d \in \bigcap \mathcal{B}_x^1$, for $x \in \{a, b, d, e\}$; (v) $e \in \bigcap \mathcal{B}_x^1$, for $x \in \{a, b, d, e\}$.

Therefore, for instance, $\langle a, b\rangle$ and $\langle b, a\rangle \in R^B(\mathbf{P}^1)$, while $\langle a, e\rangle$ and $\langle e, a\rangle \notin R^B(\mathbf{P}^1)$ because although $e \in U = \bigcap \mathcal{B}_a^1$, $a \notin \{d, e\} = \bigcap \mathcal{B}_e^1$ (in order to understand if $\langle x, y\rangle \in R^B(\mathbf{P}^1)$, it is sufficient to compare the ranges of validity of the membership relation for x and y: $\{x, y\}$ is included in both of them, hence $\langle x, y\rangle \in R^B(\mathbf{P}^1)$).

Summing up, we obtain:

$R^B(\mathbf{P}^1)$	a	b	c	d	e
a	1	1	0	0	0
b	1	1	0	0	0
c	0	0	1	0	0
d	0	0	0	1	1
e	0	0	0	1	1

Comparing this relation with R_*, we immediately see that it is the largest tolerance relation included in R_*. Incidentally, since R_* is transitive, $R^B(\mathbf{P}^1)$ is also an equivalence relation.

Now let us compute $R^B(\mathbf{P}^2)$:

(i) $a \in \bigcap \mathcal{B}_x^2$, for $x \in \{a,b,c\}$; (ii) $b \in \bigcap \mathcal{B}_x^2$, for $x \in \{a,b,c\}$; (iii) $c \in \bigcap \mathcal{B}_x^2$, for $x \in \{a,b,c\}$; (iv) $d \in \bigcap \mathcal{B}_x^2$, for $x \in \{a,b,d,e\}$; (v) $e \in \bigcap \mathcal{B}_x^1$, for $x \in \{a,b,d,e\}$.

Therefore, for instance, now $\langle a,c\rangle$, $\langle b,c\rangle$, $\langle c,a\rangle$ and $\langle c,b\rangle \in R^B(\mathbf{P}^2)$.

We obtain:

$R^B(\mathbf{P}^1)$	a	b	c	d	e
a	1	1	1	0	0
b	1	1	1	0	0
c	1	1	1	0	0
d	0	0	0	1	1
e	0	0	0	1	1

Comparing this relation with R^*, we immediately see that it is the largest tolerance relation included in R^*. Also in this case, since R^* is transitive, $R^B(\mathbf{P}^2)$ is an equivalence relation.

We conclude noticing that if the pre-topological space $\mathbf{P}$ is of type $\mathfrak{V}_S$, then it can be associated with a pre-topology which is the finest among the pre-topology T-associated with a tolerance relation and that are coarser than $\mathbf{P}$. Suffices it to consider for any $x \in U$ the basis $\mathcal{B}_x = \bigcap \varkappa_x \cup \{y : x \in \varkappa_y\}$.

Exercise 12.11.
Draw directed graphs representing $R_1, R_2, R^B(\mathbf{P}^1)$ *and* $R^B(\mathbf{P}^2)$.

A relation R such that $R^B(\mathbf{P}(R))$ is a tolerance but not an equivalence relation

Consider the following relation R:

R	a	b	c	d
a	1	1	0	0
b	1	1	1	0
c	0	1	1	1
d	0	0	0	1

By easy inspection we can observe that R is not transitive. For instance $\langle a,b\rangle \in R$, $\langle b,c\rangle \in R$, but $\langle a,c\rangle \notin R$.

If we transform it into the relation $R^B(\mathbf{P}(R))$, then we obtain a tolerance relation and not an equivalence (because of the lack of transitivity). The basis for $\mathbf{P}(R)$ is:

x	a	b	c	d
$\mathcal{B}_x$	$\{\{a,b\}\}$	$\{\{a,b,c\}\}$	$\{\{b,c,d\}\}$	$\{\{d\}\}$

(i) $a \in \bigcap \mathcal{B}_x$, for $x \in \{a,b\}$; (ii) $b \in \bigcap \mathcal{B}_x$, for $x \in \{a,b,c\}$; (iii) $c \in \bigcap \mathcal{B}_x$, for $x \in \{b,c\}$; (iv) $d \in \bigcap \mathcal{B}_x$, for $x \in \{c,d\}$. Therefore, for instance, $a \in \bigcap \mathcal{B}_b$ and $b \in \bigcap \mathcal{B}_a$, so that $\langle a,b\rangle \in R^B(\mathbf{P}(R))$; $d \in \bigcap \mathcal{B}_c$ but $c \notin \bigcap \mathcal{B}_d$, so that $\langle c,d\rangle \notin R^B(\mathbf{P}(R))$.

Summing up:

$R^B(\mathbf{P}(R))$	a	b	c	d
a	1	1	0	0
b	1	1	1	0
c	0	1	1	0
d	0	0	0	1

and we can easily notice that $R^B(\mathbf{P}(R)) \subseteq R$ (in fact, $\langle c, d \rangle \in R$, while $\langle c, d \rangle \notin R^B(\mathbf{P}(R))$).

Exercise 12.12.
Draw two directed graphs representing R and $R^B(\mathbf{P}(R))$.

12.8 Topological Spaces and Binary Relations

We are eventually at one step from our main goal in the Subsection: linking topologies and relations.

Until now, we have considered pre-topological spaces and specifically, pre-topological spaces associated with reflexive or tolerance relations.

Now we have to understand what happens in case of a topological space.

First of all, let us define topological spaces and understand the basic differences between them and their closest relatives: pre-topological spaces of type $\mathfrak{V}_{Id}$, $\mathfrak{V}_{Cl}$ and $\mathfrak{V}_S$.

Definition 12.8.1. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ of type $\mathfrak{V}_D$ is a topological space if for any x, $\varkappa_x$ satisfies property (τ).*

For the reader's convenience, we recall here this property: for any $x \in U, X \subseteq U$,

$$if\ X \in \varkappa_x,\ then\ there\ is\ a\ Y \in \varkappa_x\ such\ that\ for\ any\ y \in Y, X \in \varkappa_y$$

Usually, pre-topological spaces do not fulfill this property.

A pre-uniformity connected with a pre-topological space fulfilling (τ), satisfies the property described in the following definition:

Definition 12.8.2. *Let $U(\mathfrak{R})$ be a pre-uniformity such that for each $R \in U(\mathfrak{R})$ there is an $R' \in U(\mathfrak{R})$ such that $R' \otimes R' \subseteq R$. Then $U(\mathfrak{R})$ is called a* quasi-uniformity*.*

We remind that given two relations R and R' on a set U, $R \otimes R'$ is the concatenation $\{\langle x, y \rangle : \exists z(\ \langle x, z \rangle \in R\ \&\ \langle z, y \rangle \in R')\}$.

Intuitively, quasi-uniformities provide us with a notion of "non-discontinuity": if $\langle a, c \rangle \in R$, then there is a b in between a and c, that is, a b such that $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$. Conversely if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$, too.

Proposition 12.8.1. *Let $U(\mathfrak{R})$ be a pre-uniformity of reflexive relations. Then, if $U(\mathfrak{R})$ is a quasi-uniformity, its connected pre-topological space $\mathbf{P}(U(\mathfrak{R}))$ fulfills (τ).*

We have seen at the very beginning of this story that isotonicity plus property τ give idempotence. Therefore, if a neighborhood system fulfills **0**, **1**, **N1**, **N2** and (τ), then the operators $\varkappa$ and ε are idempotent in the induced pre-topological space.

In this Section we want to analyse the connections between topological properties and binary relations. More precisely, let $\langle U, \varepsilon^R, \varkappa^R\rangle$ be a pre-topological space associated with a binary relation R. We wonder whether R has to enjoy specific properties whenever $\langle U, \varepsilon^R, \varkappa^R\rangle$ is a topological space. The answer is positive: there is a strict connection between topological spaces and preorders, i.e. binary reflexive and transitive relations.

Proposition 12.8.2. *Let $\mathbf{P} = \langle U, \varepsilon^R, \varkappa^R\rangle$ be a pre-topological space associated with a reflexive binary relation $R \subseteq U \times U$. Then $\mathbf{P}$ is a topological space if and only if R is transitive.*

Proof. (A) $\Rrightarrow$: Assume that (τ) holds. In case of pre-topological spaces induced by a reflexive binary relation R, property (τ) reads:

$$(*) \quad \forall x \in U, \forall X \subseteq U(X \in \varkappa_x^R \Rrightarrow \exists Y(Y \in \varkappa_x^R \& \forall y \in Y(X \in \varkappa_y^R))).$$

So, take $X = R(x)$. Assume (*) holds for some Y. Then, (i) X is the least element of $\varkappa_x^R$. (ii) $\forall y \in Y, X \supseteq R(y)$ (from the assumption and *Corollary* 12.7.2.(5)). (iii) $Y \subseteq X$, because R is reflexive. Hence, (iv) $X = Y$. Therefore, (v) for all $x' \in R(x)$, $R(x') \subseteq R(x)$, that is, $R(x) \supseteq R(x')$ for all $x' \in X$. Hence, (vi) $R(x) \supseteq R(y)$. But $X = Y$. Therefore $R(x) \supseteq R(R(x))$, which is the axiom for transitivity.

(B) $\Lleftarrow$: Suppose (*) does not hold. Therefore we have:

$$(**) \quad \exists x, \exists X(X \in \varkappa_x^R \& \forall Y(Y \in \varkappa_x^R \Rrightarrow \exists y(y \in Y \& X \notin \varkappa_y^R))).$$

So, choose $Y = R(x)$. We have elements x and y such that: (i) $\langle x, y\rangle \in R$ and $X \notin \varkappa_y^R$; hence (ii) $\langle x, y\rangle \in R$ and $X \notin \uparrow R(y)$; (iii) $\langle x, y\rangle \in R$ and $R(y) \nsubseteq X$; (iv) $\langle x, y\rangle \in R$ and there exists a z such that $\langle y, z\rangle \in R$ and $z \notin X$. But $X \in \varkappa_x^R$, so that $R(x) \subseteq X$. We obtain: (v) $\langle x, x'\rangle \in R$ implies $x' \in X$. From (iv) and (v) we conclude that $\langle x, z\rangle \notin R$. Hence R is not transitive. QED

Proposition 12.8.3. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a topological space induced by a basis* $\mathfrak{B} = \{\mathcal{B}_x\}_{x \in U}$*. Then,*

1. *For any* $x \in U$*:* (a) $\bigcap \varkappa_x$ *is open;* (b) $\bigcap \mathcal{B}_x$ *is open:* (c) $\bigcap \varkappa_x$ *is the least open set containing* x*.*
2. *For any* $X \subseteq U$*,* $\varkappa(X) = \bigcup\{\varkappa(Z) : \varkappa(Z) \subseteq X\}$*.*

Proof. (**1**) (a) For any $x \in U$, $\varkappa_x = \Uparrow \mathcal{B}_x$ and from *Proposition* 12.6.1 we have $\bigcap \Uparrow \mathcal{B}_x \in \mathcal{B}_x$. Thus we obtain $\bigcap \varkappa_x \in \varkappa_x$. So, from the topological property (τ), there is a $Y \in \varkappa_x$ such that $\bigcap \varkappa_x \in \varkappa_y$ for any $y \in Y$. But this means that any y belonging to Y belongs to $\bigcap \varkappa_x$, too (because, by definition, if $A \in \varkappa_y$, then $y \in A$). Henceforth, $Y \subseteq \bigcap \varkappa_x$. But $\bigcap \varkappa_x$ is the least element of $\varkappa_x$ and Y belongs to $\varkappa_x$. It follows that $Y = \bigcap \varkappa_x$ and we can conclude that $\bigcap \varkappa_x$ is a neighborhood of all its own elements. Hence it is open. (b) is straightforward from the equality $\varkappa_x = \Uparrow \mathcal{B}_x$. (c) is obvious, because if $x \in \varkappa(A) = A$, then $A \in \varkappa_x$, so that $A \supseteq \bigcap \varkappa_x$. (**2**) If $\mathbf{P}$ is a topological space, then $\varkappa$ is isotonic and idempotent. Therefore, if $\varkappa(Z) \subseteq X$, then, for isotonicity, $\varkappa(\varkappa(Z)) \subseteq \varkappa(X)$. That is to say, for idempotence, $\varkappa(Z) \subseteq \varkappa(X)$. Conversely, if $\varkappa(Z) \subseteq \varkappa(X)$, since $\varkappa$ is deflationary $\varkappa(Z) \subseteq \varkappa(X) \subseteq X$. It follows, immediately, $\varkappa(X) = \bigcup\{\varkappa(Z) : \varkappa(Z) \subseteq X\}$. QED

Notice that *Proposition* 12.8.3.(2) is not that trivial when we frame it in topological spaces connected with binary relations, as we are going to see in the next corollary. Indeed, if isotonicity or idempotence fails, then this result does not hold any longer.

Example 12.8.1. Idempotence, preorders and contractions

The pre-topology $\mathbf{P}_2$ of Example 12.6.7 provides us with an example of the fact that the lack of idempotence makes the equivalence $\varkappa(X) = \bigcup\{\varkappa(Y) : \varkappa(Y) \subseteq X\}$ fail. In fact, $\varkappa(\{c\}) = \emptyset$, while $\bigcup\{\varkappa(Y) : \varkappa(Y) \subseteq X\} = \{c\}$.

It must be noticed that, a fortiori, $\varkappa(X) \neq \bigcup\{Y : \varkappa(Y) \subseteq X\}$. So do not confuse the set $\{x : \varkappa^R(x) \subseteq X\}$ (which has no meaning) with the set $\{x : R(x) \subseteq X\}$ (which gives $\varkappa^R(X)$ in case of pre-topological spaces of type $\mathfrak{V}_S$) and the set $\{R(Y) : R(Y) \subseteq X\}$ (whose union gives $\varkappa^R(X)$ in case of a preorder R).

For another example consider the following non-transitive relation:

R_3	a	b	c
a	1	1	0
b	0	1	1
c	0	0	1

It gives the basis $\mathfrak{B}^R$:

x	a	b	c
$\mathcal{B}_x^{R_3}$	$\{\{a,b\}\}$	$\{\{b,c\}\}$	$\{\{c\}\}$

By applying in $\mathbf{P}(R_3)$ the formula $L_{R_3}(X) = \varkappa^{R_3}(X) = \{x : \exists B(B \in \mathcal{B}_x^{R_3}\ \&\ B \subseteq X)\}$ we obtain:

X	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\emptyset$	U
$L_{R_3}(X)$	$\emptyset$	$\emptyset$	$\{c\}$	$\{a\}$	$\{c\}$	$\{b,c\}$	$\emptyset$	U

Thus, $L_{R_3}(\{a,b\}) = \{a\} \neq \bigcup\{R_3(x) : R_3(x) \subseteq \{a,b\}\} = \bigcup\{R_3(\{a\})\} = R_3(\{a\}) = \{a,b\}$. This is due to the fact that although $b \in R_3(\{a\})$ and $R_3(\{a\}) \subseteq \{a,b\}$, nonetheless $R_3(\{b\}) = \{b,c\} \not\subseteq \{a,b\} = R_3(\{a\})$, because of the failure of transitivity. Hence, $b \notin \{x : R_3(x) \subseteq \{a,b\}\}$. Therefore $\bigcup\{R_3(Z) : R_3(Z) \subseteq \{a,b\}\}$ contains elements (*viz.* b) that do not fulfill the universal proviso of the necessity operator (i.e. "$L_{R_3}(X) = \{x : \forall y, \langle x,y\rangle \in R_3 \Rrightarrow y \in X\}$"). It follows that $\bigcup\{R_3(Z) : R_3(Z) \subseteq X\}$ is not a suitable formula for $L(X)$, in this case.

Moreover notice that in case of lack of transitivity, the set of R-neighborhoods, $\{\emptyset, \{a,b\}, \{b,c\}, \{c\}, U\}$, and the set of necessitations $\{\emptyset, \{a\}, \{c\}, \{b,c\}, U\}$ do not coincide. Henceforth we can observe that in this case it is not true that for any $A \subseteq U$, if $X = L_{R_3}(A)$, then $X = R_3(Y)$ for some $Y \subseteq U$.

Also, observe that $\Omega_{R_3}(U) = \{L_{R_3}(X) : X \subseteq U\}$ is not a distributive sublattice of $\mathbf{B}(U)$. Due to this fact we can see that the element $\max\{X \in \Omega_{R_3}(U) : X \subseteq Z\}$ might not exist for some subset Z of U. Indeed this is the case of $\{a,c\}$. Actually, $\{X \in \Omega_{R_3}(U) : X \subseteq \{a,c\}\} = \{\{a\},\{c\}\}$ which does not have a greatest element. It can be noticed that $\bigvee\{X \in \Omega_{R_3}(U) : X \subseteq \{a,b\}\} = U$, but $U \not\subseteq \{a,c\}$.

To sum up, if a relation R is not a preorder, then $\mathfrak{B}^R$ is not a basis for $\Omega_R(U)$ in the topological sense. That is, there can be $X, Y \in \mathfrak{B}^R$ such that $X \cup Y \neq L_R(Z)$ for all $Z \subseteq U$.

Finally, it is worth noticing that if R lacks reflexivity, then $\{x : R(x) \subseteq X\}$ is not included in $\bigcup\{R(Z) : R(Z) \subseteq X\}$. For instance, consider the (non-reflexive) relation R of Example 12.1.1.

Then, $\{x : R(x) \subseteq \{b,c\}\} = \{a,b,c\}$, while $\bigcup\{R(Z) : R(Z) \subseteq \{b,c\}\} = \{b,c\}$.

Indeed, the inclusion here failed was proved in *Lemma* 12.7.1 by exploiting reflexivity. On the contrary, the reverse inclusion $\bigcup\{R(Z) : R(Z) \subseteq X\} \subseteq \{x : R(x) \subseteq X\}$ requires just transitivity. For instance, to prove that if $b \in \bigcup\{R(Z) : R(Z) \subseteq \{b,c\}\}$ then $b \in \{x : R(x) \subseteq \{b,c\}\}$, first we need to notice that the antecedent is valid because $b \in R(\{a\})$ and $R(\{a\}) \subseteq \{b,c\}$; second, we apply the transitivity of R to show that $R(\{b\}) \subseteq R(\{a\})$. So we conclude $R(\{b\}) \subseteq \{b,c\}$ and can derive $b \in \{x : R(x) \subseteq \{b,c\}\}$.

We can restate the property (τ) of *Definition* 12.8.1, in terms of idempotence of ε^R and $\varkappa^R$:

Corollary 12.8.1. *Let* $\mathbf{P} = \langle U, \varepsilon^R, \varkappa^R\rangle$ *be a pre-topological space associated with a reflexive binary relation* $R \subseteq U \times U$. *Then* ε^R *and* $\varkappa^R$ *are idempotent if and only if* R *is transitive.*

Proof. (A) $\Rrightarrow$: Suppose $y \in R(x)$ and $z \in R(y)$. It follows, from *Proposition* 12.7.2, that $y \in \varepsilon^R(\{z\})$ and $x \in \varepsilon^R(\{y\})$. Since $\mathbf{P}$ is of type $\mathfrak{V}_S$ (again from *Proposition* 12.7.2), the operator ε^R is isotonic. Hence $x \in \varepsilon^R(\varepsilon^R(\{z\}))$. But if ε^R is idempotent, $x \in \varepsilon^R(\{z\})$ and, as a consequence, $z \in R(x)$, which proves that R is transitive.

(B) $\Lleftarrow$: Suppose R is transitive. Then, since by default R is also reflexive, we have $R(R(x)) = R(x)$. Hence $R^{\smile}(R^{\smile}(x)) = R^{\smile}(x)$. So, from *Proposition* 12.7.2 $\varepsilon^R(\varepsilon^R(x)) = \varepsilon^R(x)$. For $\varkappa^R$ the proof is by duality. QED

Therefore, reflexive and transitive relations, i.e. preorders, are tightly linked with topological spaces.

Example 12.8.2. A pre-topological space associated with a reflexive and transitive relation R

Let $U = \{a, b, c, d, e\}$ and let R be the following preorder:

R	a	b	c	d	e
a	1	1	1	1	1
b	0	1	0	1	1
c	0	0	1	0	0
d	0	0	0	1	1
e	0	0	0	1	1

Consider the family $\mathfrak{B}^R = \{R(x)\}_{x \in U} = \{\{c\}, \{d, e\}, \{b, d, e\}, U\}$. Let us compute the family $\{\mathcal{B}^R_x\}_{x \in U}$, where for any $x \in U$, $\mathcal{B}^R_x = \{X : X \in \mathfrak{B}^R \ \& \ x \in X\}$:

x	a	b	c	d	e
$\mathcal{B}^R_x$	$\{U\}$	$\{\{b, d, e\}, U\}$	$\{\{c\}, U\}$	$\{\{d, e\}, \{b, d, e\}, U\}$	$\{\{d, e\}, \{b, d, e\}, U\}$

We can observe what follows

$$\begin{aligned} &\textit{for any } x \in U, \textit{ if } X \in \mathcal{B}^R_x \textit{ then there is a } Y \in \mathcal{B}^R_x \\ &\textit{such that for any } y \in Y, X \in \mathcal{B}^R_y \end{aligned} \tag{12.8.1}$$

Indeed we can chose $Y = X$. Moreover for any x, $\mathscr{F}_x = \Uparrow \{R(x)\} = \Uparrow \mathcal{B}^R_x$ (this is proved in the following way: Let $a \in R(b)$ for $b \neq a$. Suppose $R(a) \nsubseteq R(b)$. Then there is an x such that $x \in R(a)$ and $x \notin R(b)$. But $a \in R(b)$; it follows that R is not transitive). Obviously, property (τ) is inherited by $\mathscr{F}_x$ from $\mathcal{B}^R_x$ and this makes the topological property (τ) hold. In fact take any $x \in U$ and any $F \in \mathscr{F}_x$. Let us look for an $X \in \mathscr{F}_x$ such that for any $y \in X$, $F \in \mathscr{F}_y$. It is sufficient to take any member Y of $\mathcal{B}^R$, such that $Y \subseteq F$ (and it exists, because $\mathscr{F}_x = \Uparrow \{R(x)\}$ and $R(x) \in \mathfrak{B}^R$). In fact, since Y is open, it is a neighborhood of all its points. But since $Y \subseteq F$, then F is a neighborhood of all the points of Y, too.

For instance, let $x = d$ and take $F = \{b, c, d, e\}$ which is a member of $\mathscr{F}_d$. Take $Y = \{b, d, e\}$. Y belongs to $\Uparrow \{R(b)\}, \Uparrow \{R(d)\}$ and $\Uparrow \{R(e)\}$. But since $Y \subseteq F$, we obtain $R(b) = \{b, d, e\} = Y \subseteq \{b, c, d, e\} = F, R(d) = \{d, e\} \subseteq Y \subseteq F$

and $R(e) = \{d, e\} \subseteq Y \subseteq F$. That is to say, $F \in\Uparrow \{R(b)\}, F \in\Uparrow \{R(d)\}$ and $F \in\Uparrow \{R(e)\}$. Which is the same thing as saying that F is a neighborhood of b, d and e.

Therefore $\mathscr{F} = \{\mathscr{F}_x\}_{x\in U}$ is a neighborhood system for a topology $\Omega(U)$ with subbasis $\mathfrak{B}^R$. But $\mathscr{F}$ is a neighborhood system for $\mathbf{P}(R)$, too. So we conclude that $\langle U, \Omega(U)\rangle = \mathbf{P}(R)$. Hence $\mathbf{P}(R)$ is a topological space and the interior operator $\mathbb{I}_R$ coincides with the contraction operator $\varkappa^R$.

Let us compute, for instance, $\varkappa^R(\{a, c, d, e\})$ and $\mathbb{I}_R(\{a, c, d, e\})$:

$\varkappa^R(\{a, c, d, e\}) = \{x : \exists X(X \in\Uparrow \{R(x)\} \ \&\ X \subseteq \{a, c, d, e\})\} = \{x : R(x) \subseteq \{a, c, d, e\}\} = \{c, d, e\}$;

$\mathbb{I}_R(\{a, c, d, e\}) = \bigcup\{X : X \in \mathfrak{B}^R \ \&\ X \subseteq \{a, c, d, e\}\} = \bigcup\{\{c\}, \{d, e\}\} = \{c, d, e\}$.

It is immediate that $\Omega_{\varkappa^R}(U) = \{\emptyset, \{c\}, \{d, e\}, \{c, d, e\}, \{b, d, e\}, \{b, c, d, e\}, U\}$.

Conversely, suppose we are given a topological space $\langle U, \Omega(U)\rangle$, such that $\Omega(U) = \{\emptyset, \{c\}, \{d, e\}, \{c, d, e\}, \{b, d, e\}, \{b, c, d, e\}, U\}$.

Let us set $\mathcal{O}_x = \{O : O \in \Omega(U) \ \&\ x \in O\}$. Then we obtain a reflexive and transitive binary relation S in the following way:

$$\langle x, y\rangle \in S \textit{ iff } y \in \bigcap \mathcal{O}_x \qquad (12.8.2)$$

x	a	b	c	d	e
$\bigcap \mathcal{O}_x$	U	$\{b, d, e\}$	$\{c\}$	$\{d, e\}$	$\{d, e\}$

Clearly, $\bigcap \mathcal{O}_x = \bigcap \mathcal{B}^R_x$. Summing up, we found $S = R$.

Now we have a list of results connected with the fact that preorder is the relational counterpart of the topological property "A set X is open if and only if it is a neighborhood of all its own points". A sentence which, in turn, reflects, as we know, the intuitive reading "If a set X is close to a point x, then it is close to all the points that are sufficiently close to x".

Proposition 12.8.4. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa\rangle$ *be a pre-topological space of type* $\mathfrak{V}_S$. *Then* $R^T(\mathbf{P})$ *is a preorder if and only if* $\mathbf{P}$ *is a topological space.*

Proof. In any pre-topology $\mathbf{P}$ of type $\mathfrak{V}_S$, $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}))$ (*Proposition* 12.7.8). Hence from *Proposition* 12.8.2 we obtain the result. QED

Example 12.8.3. A pre-topological space $\mathbf{P}$ which is not a topological space, even if $R^T(\mathbf{P})$ is a preorder

This example seems to question *Proposition* 12.8.4. But there is a trick. The topological space $\mathbf{P}^1$ of Example 12.7.4 is such an example. Indeed, it is not difficult to see that $\mathbf{P}^1$ is not a topological space (we have seen that $\varkappa^1(\{a,b\}) \cap \varkappa^1(\{a,c\}) = \{a\} \neq \emptyset = \varkappa(\{a,b\} \cap \{a,c\})$; however, in Example 12.7.4 we have shown that $R^T(\mathbf{P}^1)$ is the identity relation, hence reflexive and transitive).

Notice, anyway, that $\mathbf{P}^1$ is not of type $\mathfrak{V}_S$. Indeed, there are not pre-topological spaces $\mathbf{P}$ of type $\mathfrak{V}_S$ such that $R^T(\mathbf{P})$ is a preorder but $\mathbf{P}$ is not a topological space [for a related example, see Frame 15.7].

Exercise 12.13. *Give a direct proof of Proposition 12.8.4.*

Moreover, it is important to point out that there are pre-topologies $\mathbf{P}$ such that $\Omega_{\varkappa}(U)$ is a distributive lattice, so that $\langle U, \Omega_{\varkappa}(U)\rangle$ is a topological space, but $\mathbf{P}$ is not topological (see a counter example below). This may happen because the interior operator induced by $\Omega_{\varkappa}(U)$ as a topology on U and $\varkappa$ may fail to coincide. However, this cannot happen if $\mathbf{P}$ is topological.

Example 12.8.4. A pre-topological space $\mathbf{P}$ which is not topological such that $\Omega_{\varkappa}(U)$ is a lattice of sets

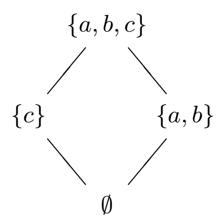
Consider the pre-topological space $\mathbf{P}_2$ of Example 12.6.7.

Let us compute the family $\{\varkappa_x\}_{x \in U}$:

x	a	b	c
$\varkappa_x$	$\{\{a,b\},U\}$	$\{\{a,b\},U\}$	$\{\{b,c\},U\}$

$\mathbf{P}$ is of type $\mathfrak{V}_s$ because each $\varkappa_x$ is a principal filter. However $\mathbf{P}$ is not topological because $\varkappa_c$ does not satisfy τ. Indeed, given the member $\{b,c\}$ of $\varkappa_c$ there is no $X \in \varkappa_c$ such that $\{b,c\}$ belongs to $\varkappa_x$ for every $x \in X$. In fact if we hope to have some chance we must consider the least element $\{b,c\}$ of $\varkappa_c$. But $\{b,c\} \notin \varkappa_b$ (indeed, $\varkappa(\varkappa(\{b,c\})) = \emptyset \neq \{c\} = \varkappa(\{b,c\})$), so that idempotence fails.

However, $\Omega_{\varkappa}(U)$ is clearly a lattice of sets, hence a distributive lattice:



Thus, $\langle U, \Omega_{\varkappa}(U)\rangle$ is a topological space. Let $\mathbb{I}_{\Omega_{\varkappa}}$ and $\preceq_{\Omega_{\varkappa}}$ denote the interior operator and, respectively, the specialization preorder induced by $\Omega_{\varkappa}(U)$. Since $\mathbf{P}$

is not topological, $\varkappa$ and $\mathbb{I}_{\Omega_\varkappa}$ cannot coincide. Indeed here is a counterexample: $\mathbb{I}_{\Omega_\varkappa}(\{c\}) = \{c\} \neq \emptyset = \varkappa(\{c\})$.

Finally, neither $R^T(\mathbf{P})$ and $\preceq_{\Omega_\varkappa}$ coincide. Indeed $\langle b, c\rangle \in R^T(\mathbf{P})$ because $b \in \bigcap \varkappa_c = \{b, c\}$, while $\langle b, c\rangle \notin \preceq_{\Omega_\varkappa}$ because, $c \in \{c\}$ but $b \notin \{c\}$.

The proof of this fact makes it possible to have a brief tour through some of the results so far achieved.

Proposition 12.8.5. *Let* $\mathbf{P} = \langle U, \varkappa, \varepsilon\rangle$ *be a pre-topology. If* $\mathbf{P}$ *is topological then* $\varkappa$ *and the interior operator induced by* $\Omega_\varkappa(U)$ *as a topology on* U *coincide.*

Proof.

1. $R^T(\mathbf{P}) = \preceq_{\Omega_\varkappa}$, where $\preceq_{\Omega_\varkappa}$ is the specialization preorder induced by $\Omega_\varkappa(U)$ *qua* topology on U. In fact, by definition $\langle x, y\rangle \in R^T(\mathbf{P})$ if and only if $y \in \bigcap \varkappa_x$. But from *Proposition* 12.8.3.(1) $\bigcap \varkappa_x$ is the least open set containing x. It follows that $\langle x, y\rangle \in R^T(\mathbf{P})$ if and only if $x \in \bigcap \varkappa_x \Rrightarrow y \in \bigcap \varkappa_x$ if and only if $x \preceq_{\Omega_\varkappa} y$ (remember that $x \in \bigcap \varkappa_x$ always holds). This means that the specialization preorder $\preceq$ and $R^T(\mathbf{P})$ coincide.
2. Hence, $R^T(\mathbf{P})$ is a preorder. Recalling that $\mathbf{P}$ is finite and topological, thus of type $\mathfrak{V}_S$, it follows that:
   (a) $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}))$ (from *Proposition* 12.7.8.(2)), so that $\varkappa = \varkappa^{R^T(\mathbf{P})}$.
   (b) $\varkappa^{R^T(\mathbf{P})} = L_{R^T}(\mathbf{P})$ (see *Corollary* 12.7.3).
   (c) $\mathbf{F}(\langle U, \preceq\rangle) = \Omega_\varkappa(U)$.
   (d) $L_{R^T}(\mathbf{P}) = !_\preceq$ (suffice it to substitute $\preceq$ for R in *Lemma* 12.7.1). But $!_\preceq$ is indeed the interior operator induced by $\mathbf{F}(\langle U, \preceq\rangle)$.

   Hence
3. $!_\varkappa$ is the interior operator induced by $\Omega_\varkappa(U)$.

From 1 and 3 we obtain the result. QED

REMARKS. Moreover, notice that we can have neighborhood systems $\mathcal{N}(U)$ with a related core map G such that $\{G(X)\}_{X\subseteq U}$ is a lattice of sets but such that G is not a contraction operator. Obviously, in this case G does not coincide with the interior operator induced by $\{G(X)\}_{X\subseteq U}$

qua frame of the open subsets of a topological space. Obviously, in view of *Proposition* 12.4.11, $\mathcal{N}(U)$ cannot be of type $\mathcal{N}_1$. The reader is referred to Example 12.8.5.

Example 12.8.5. A neighborhood system whose core map G induces a topological space but such that G is neither a contraction operator, nor coincides with the interior operator

In Example 12.6.7 we have shown a pre-topological space which is not topological but such that $\Omega_{\varkappa}(U)$ is a topology. Now we exhibit a neighborhood system such that G is not an interior operator but such that $\Omega_G(U)$ is a topology.

Let $U = \{a, b, c\}$ and let the neighborhood system $\mathcal{N}(U)$ be given by

x	a	b	c
$\mathcal{N}_x$	$\{\{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, U\}$	$\{\{b\}, \{a, b\}, \{b, c\}, U\}$	$\{U\}$

Therefore the core map G is:

X	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\emptyset$	U
$G(X)$	$\{a\}$	$\{b\}$	$\{a\}$	$\{a, b\}$	$\{a\}$	$\{a, b\}$	$\emptyset$	U

It is easy to check that $\mathcal{N}(U)$ fulfills **N2**, **N3** and **Id**, so that G is idempotent. Moreover, $\{G(X)\}_{X \subseteq U}$ is a distributive lattice. It follows that $\mathbf{T} = \langle U, \{G(X)\}_{X \subseteq U} \rangle$ is a topological space. However G does not coincide with the interior operator of $\mathbf{T}$. In fact, $G(\{b, c\}) = \{a, b\}$ while $\mathbb{I}_{\mathbf{T}}(\{b, c\}) = \bigvee\{A \in \{G(X)\}_{X \subseteq U} : A \subseteq \{b, c\}\} = \{b\}$. Indeed, $\mathcal{N}(U)$ does not fulfill **N1**, so that G cannot be a contraction operator (cf. $G(\{b, c\})$).

Therefore, the lack of **N1** is the reason why $\mathcal{N}(U)$ does not induce a topological interior operator.

Exercise 12.14.
(a) Draw a graph for $\Omega_{\varkappa}(U)$ *where* $\mathbf{P}$ *is the pre-topological space of Example 12.4.8.*
(b) Is $\Omega_{\varkappa}(U)$ *a lattice of sets?*
(c) Is $\Omega_{\varkappa}(U)$ *distributive?*
(d) What known properties does $\Omega_{\varkappa}(U)$ *fulfill?*

Corollary 12.8.2. *Let* $\mathbf{P}(R) = \langle U, \varepsilon^R, \varkappa^R \rangle$ *be a topological space associated with a preorder* $R \subseteq U \times U$. *Then:*

1. *For any* $x \in U$, $R(x)$ *is the least open set containing* x.
2. *For any open set* O, $O = \bigcup_{x \in O} R(x)$ *(any open set is a union of minimal* R*-neighborhoods). We record this fact by saying that*

$\{R(x)\}_{x \in U}$ is a basis of open subsets *for the topological space $\mathbf{P}(R)$.*

3. *For any $X \subseteq U$, $R(X)$ is open.*
4. *X is open if and only if $X = R(X)$.*
5. *For any $x \in U$, $R^{\smile}(x)$ is the least closed set containing x.*
6. *For any $X \subseteq U$, $R^{\smile}(X)$ is closed.*
7. *X is closed if and only if $X = R^{\smile}(X)$.*
8. *$R^T(\mathbf{P}(R)) = \preceq$ (where $\preceq$ is the specialization preorder induced by $\mathbf{P}(R)$).*

Proof. (**1**) From *Proposition* 12.8.3, for any $x \in U$, $\bigcap \varkappa_x^R$ is open. But $\bigcap \varkappa_x^R = R(x)$ (alternative proofs are reported in Frame 15.5).

(**2**) Since O is open, from *Proposition* 12.7.2. (2) we have $O = \varkappa^R(O) = \{x : R(x) \subseteq O)\}$. This means that for any $x \in O$, $R(x)$ is included in O. Thus $\bigcup_{x \in O} R(x)$ is included in O. Moreover, if $x \in O$, then $\langle x, x \rangle \in R(x)$, for reflexivity. Hence $x \in \bigcup_{x \in O} R(x)$, so that O, in turn, is included in $\bigcup_{x \in O} R(x)$. (**3**) For any $x \in X$, $R(x)$ is open; therefore, $R(X) = \bigcup_{x \in X} R(x)$ is open (because unions of open sets are open and R-neighboring is additive). (**4**) If $X = R(X)$, then from (3) X is open. Conversely, suppose X is open. Then, from point (2), $X = \bigcup_{x \in X} R(x) = R(X)$. (**5**) From *Proposition* 12.7.2.(4), $\varepsilon^R(\{x\}) = R^{\smile}(x)$. But $\varepsilon^R(\{x\})$ is closed and contains x. (**6**) From (5), using R-neighborhood additivity. (**7**) X is closed if and only if $X = \varepsilon^R(X)$ if and only if $X = R^{\smile}(X)$. (**8**) from Proposition 12.7.8.(3), $R^T(\mathbf{P}(R)) = R$. Obviously, $\mathbf{P}(R)$ is induced by the basis $\{R(x)\}_{x \in U}$. Thus we have to prove: $a \preceq b$ if and only if $\langle a, b \rangle \in R$. But $a \preceq b$ if and only if $a \in R(x) \Rrightarrow b \in R(x)$, for all x. Therefore if $a \preceq b$ then $a \in R(a) \Rrightarrow b \in R(a)$. But $a \in R(a)$, since R is reflexive. Thus $b \in R(a)$, that is, $\langle a, b \rangle \in R$. Conversely suppose $b \in R(a)$. Therefore if $a \in R(x)$ then, by transitivity, $b \in R(x)$. Hence $a \preceq b$. QED

From *Corollary* 12.8.2.(1) and (2) we obtain a well-known fact about topological spaces, namely that any topological space is induced by a basis of open sets by means of union formation.

Corollary 12.8.3. *Let R be a preorder. Then $\mathfrak{B}^R = \{R(x)\}_{x \in U}$ is a topological basis of a topological space $\langle U, \Omega(U)\rangle$ such that for any $X \subseteq U$, $\mathbb{I}(X) = \bigcup\{Y \in \mathfrak{B}^R : Y \subseteq X\} = \varkappa^R(X)$.*

This is specific of preorder relations. For counterexamples see Example 12.7.4.

Terminology and Notation. From now on, given a topological space $\langle U, \varepsilon^R, \varkappa^R\rangle$ associated with a preorder R, the frame of open subsets of U (i.e. the family of open subsets of U equipped with the operations $\cap, \cup, -, U, \emptyset$) will be denoted by $\Omega_R(U)$ and, consequently, the topological space will be also denoted by $\langle U, \Omega_R(U)\rangle$. By $\Gamma_R(U)$ we shall mean the family of closed subsets of U. The interior and the closure operators induced by $\Omega_R(U)$, will be denoted by $\mathbb{I}_R$ and, respectively, $\mathbb{C}_R$ (we recall that $\mathbb{I}_R(X) = \bigcup\{Y \in \Omega_R(U) : Y \subseteq X\}$; $\mathbb{C}_R$ is defined dually). Moreover, remember that $\Omega_{\varkappa^R}(U)$ denotes the set $\{\varkappa^R(X) : X \subseteq U\}$.

Corollary 12.8.4. *Let $\langle U, \varepsilon^R, \varkappa^R\rangle$ be a topological space associated with a preorder $R \subseteq U \times U$. Then for any $X \subseteq U$:*

1. $\Omega_R(U) = \Omega_{\varkappa^R}(U)$; $\Gamma_R(U) = \{\varepsilon^R(X) : X \subseteq U\}$.
2. $\mathbb{I}_{\varkappa^R}(X) = \varkappa^R(X) = \mathbb{I}_R(X)$; $\mathbb{C}_{\varepsilon^R}(X) = \varepsilon^R(X) = \mathbb{C}_R(X)$.
3. *For any* $X \subseteq U$, $\mathbb{I}_R(X) = \bigcup\{R(Z) : R(Z) \subseteq X\}$.
4. *For any* $X \subseteq U$, $\mathbb{C}_R(X) = \bigcap\{R^{\smile}(Z) : X \subseteq R^{\smile}(Z)\}$.
5. $\mathbb{I}_R(X) = \bigcup\{Z : Z \in \Omega_R(U) \;\&\; Z \subseteq X\} = U \overset{\Omega_R}{\Longrightarrow} X$.
6. $\langle \mathbf{B}(U), \Omega_R(U)\rangle$ *is a modal system.*

Proof. (**1**) $\Omega_R(U) = \{X : X = \varkappa^R(X)\}$. So, since $\varkappa^R(X) = \varkappa^R(\varkappa^R(X))$ the result is obvious. (**2**) Immediately from (1) and *Definition* 12.4.6. (**3**) Directly from *Lemma* 12.7.1. (**4**) Let $\mathcal{I} = \{R^{\smile}(Z) : X \subseteq R^{\smile}(Z)\}$. Clearly, since R is reflexive, so is $R^{\smile}$. Thus $X \subseteq R^{\smile}(X)$, so that $R^{\smile}(X) \in \mathcal{I}$. Moreover, because R is transitive, so is $R^{\smile}$. Thus if $X \subseteq R^{\smile}(Y)$, for some $Y \subseteq U$, then $R^{\smile}(X) \subseteq R^{\smile}(Y)$. It follows that $R^{\smile}(X)$ is the least element of $\mathcal{I}$. We conclude that $\bigcap \mathcal{I} = R^{\smile}(X)$. (**5**) Since $\Omega_R(U) = \{R(X) : X \subseteq U\}$, the thesis is just a translation of (3). (**6**) Directly from (5) QED.

Corollary 12.8.4.(3) tells us that an open set O is a fixpoint of the process of formation of R-neighborhoods limited by some set X. We can restate this image, saying that an open set O is the result of a process π of approximation of a phenomenon X by means of some basic pieces of information: $O = \pi(X)$. As such it is stable: $\pi(\pi(X)) = \pi(X) = \pi(O) = O$. Otherwise stated, it is the core of a "phenomenon", modulo a perception process π. This stability is precisely the nice property we can derive from the topological property (τ) discussed above, which, in turn is strictly connected with transitivity. Indeed, transitivity makes it possible to drill down until the limit, or to collect everything that immediately or mediately is connected with a given perception point x.

We have seen that in order to obtain this nice property we have to renounce some dynamic features. Classical Rough Set Theory is within this choice. And in this framework we can review the story of the modal operators we have suspended at the end of the last paragraph.

Let us continue it.

In view of *Proposition* 12.8.2.(2) we have that if O is open, $x \in O$ and $\langle x, y \rangle \in R$, then $y \in O$. From this, one can easily understand why open sets are images of the necessity operator L.

In fact, compare the last property with the definitions of L as shown in the table at the end of Section 12.1. In view of those definitions, in set-theoretical terms we have: $x \in [\![L(\alpha)]\!]$ iff $\forall y, \langle x, y \rangle \in R \Rrightarrow y \in [\![\alpha]\!]$. That is to say, $x \in [\![L(\alpha)]\!]$ iff $R(x) \subseteq [\![A]\!]$. Therefore if we are given a topological space $\langle U, \varepsilon^R, \varkappa^R \rangle$ and for any formula α, $[\![\alpha]\!]$ is a subset of U, then $[\![L(\alpha)]\!]$ is to be interpreted as the largest open subset included in $[\![\alpha]\!]$.

Indeed, we have:

Corollary 12.8.5. *Let $\langle \mathbf{B}(U), L_R \rangle$ be a pre-monadic Boolean algebra of the powersets of a set U. Let R be a preorder and $X \subseteq U$. Then,*

1. $L_R(X) = \bigcup\{R(Z) : R(Z) \subseteq X\}$.
2. $M_R(X) = \bigcap\{R^{\smile}(Z) : R^{\smile}(Z) \supseteq X\}$.

Proof. Straightforwardly, from *Corollaries* 12.7.3, 12.7.1 and 12.8.4. QED

In Frame 15.6 we give a direct proof of the second equation. Therefore, we have accomplished almost all the moves listed in the last table of Section 12.1.