Piero Pagliani
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A Geometry of Approximation

Rough Set Theory:
Logic, Algebra and Topology
of Conceptual Patterns

Springer

A Geometry of Approximation

TRENDS IN LOGIC
Studia Logica Library

VOLUME 27

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A Geometry of Approximation

Rough Set Theory: Logic, Algebra and Topology
of Conceptual Patterns

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Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

springer.com

Preface

Rough Set Theory was proposed by Zdzisław Pawlak in 1980. Since inception it has generated an increasing interest in different research fields, namely computer science, mathematics and logic. An impressive amount of research-papers have been devoted to this topic.

A number of good books have appeared in the recent years, in which Rough Set Theory has been investigated under one or more privileged points of view.

We mention Demri & Orłowska [2002] and Polkowski [2002] as two works close to our scientific purpose. Recently in the *Transactions on Rough Sets* a monograph by J. Järvinen has been published, which has close connections with Part II of this book (Järvinen [2007]).

An almost complete bibliography on the subject may be found in the website of the International Rough Set Society (http://roughsets. home.pl/IRSS/).

The present work is not intended to be an introduction about Rough Set Theory, but a sort of journey which starts from a basic idea (that of an elementary "observation system", basically two sets linked by a binary relation) and arrives at a broad "geometrical" construction in which Rough Set Theory is positioned.

The term "observation system" is influenced by the *heuristic* reading (that is, epistemological not ontological) of the phenomenological approach: we become aware of "things" through their observable (perceivable) properties.

On the other hand, the term "geometrical" refers to the fact that to give these "perceptions" a symbolic order, we shall mainly use topological concepts such as "close to", "internal to", and the like.

Moreover, our journey has an explicit purpose: making a reader understand the connections between Rough Set Theory and a wide
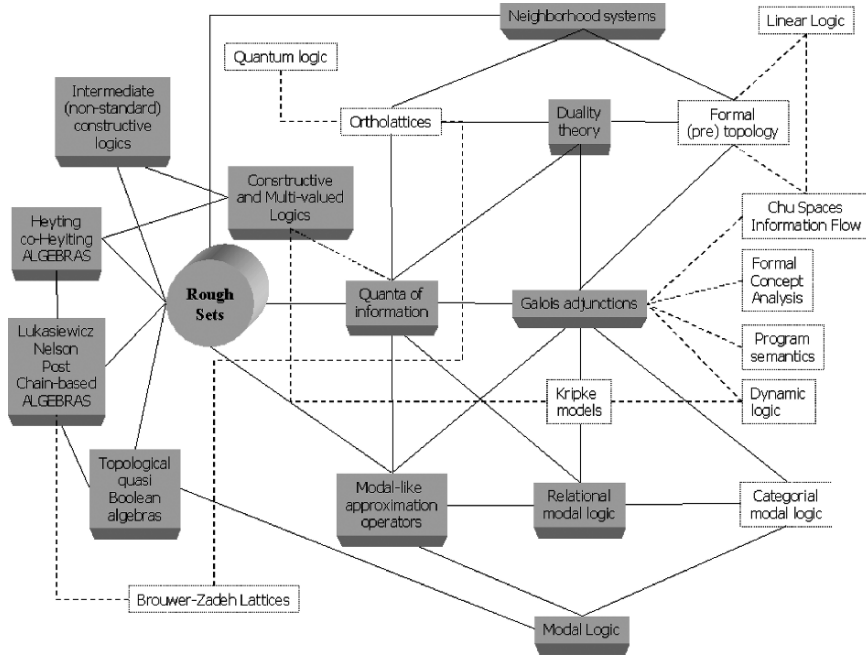
Figure 1: A Rough Set connection

number of mathematical and logical, if not philosophical, fields, topics and concepts, as sketched in Figure 1.

More precisely, in the present book the main bodies of the Chapters shall deal with the topics in the shadowed boxes along the arrows displayed by solid lines. Non shadowed boxes and dotted lines show topics and some interesting links that will be discussed in the *Frame* sections.

The focus of this picture is the concept of an *approximation*, namely Approximation Spaces and Rough Set Theory.

To be sure, some of these connections will sound familiar to mathematicians and logicians (who, in turn, could suggest other interesting links to us).

Probably some will sound new to graduate students, who, we hope, in this book will find suggestions for their future studies and researches. We invite them to explore these connections in order to have a "holistic picture" of a number of logical and mathematical techniques that typically are used and presented in separate fields of interest.

Our logico-mathematical tour may be described, more or less, as follows.

In the Introduction we shall discuss the general connections (and disconnections) between the ideas of "perceiving something", "knowing something", "having a logical understanding of something" and, finally "communicating perception, knowledge or logical understandings to someone". Therefore, we shall discuss some relationships between epistemology, logic and semiotics, in order to understand the reason why (and the extent to which) Rough Set Theory, as an approach to provide "observations" with an intelligible shape, can be given a logical and topological interpretation.

Observation are intelligible when we are able to collect the observed phenomena into classes and compare these classes with different categorisations (induced, for instance, by preceding observations or by theoretical considerations), which are supposed to be able to explain the given observed phenomena.

The comparison between two categorisations leads to the notion of an *approximation*. In fact, some explaining categories may coincide with some categories to be analysed, or they can partially coincide and only approximate the later categories.

In Part I we shall see that the notion of an "approximation" may be mathematically modelled by the notion of a *Galois connection*. Then we show that systems of observations, in this study called *Attribute Systems* or *Property Systems*, induce "perception operators" which fulfill adjunction properties. These operators are generalisation of the topological operators "interior" and "closure". In a particular case they coincide with the notion of an *upper approximation* and a *lower approximation* provided by classical Rough Set Theory.

In fact, the basic philosophical assumption in classical Rough Set Theory is that one fundamental source of our knowledge is the ability to distinguish objects by means of their properties. The more precise these properties are, the finer the grains of our knowledge will be. Thus, given a universe of discourse U and a set of attributes or properties M, the basic gnoseological act is the organization of the elements of U into disjoint categories, in such a way that each category collects the elements of U that are describable by means of the same set of attribute values or properties, that is, the elements that are indiscernible on

the basis of our data or observations. The collection of these disjoint categories define an equivalence relation E on U. Since the properties definable by these attributes or properties are the gnoseological co-ordinates explaining the universe, the *Indiscernibility Space* $\langle U, E \rangle$ provides the "gnoseological geometry" of our world.

However, our "perception operators" are more general and may be described within the theory of point-free (or formal) pre-topology and point-free topology. And in some cases they coincide with the upper and lower approximations that can be defined from a generic relational space $\langle U, R \rangle$, with R any binary relation on U.

In Part II we start with a philosophical consideration about an Indiscernibility Space $\langle U, E \rangle$.

In the best case, that is, when our information about the elements of U is *complete*, E is the identity relation (hence any category is a singleton). In the worst case, that is, when our information is absolutely insufficient, E is the total relation $U \times U$: you can distinguish nothing. But the latter is a pathological situation which, indeed, induces degenerate algebras. Actually, one situation arises when for any element a there are some (but not all) elements a', such that $a \neq a'$ but a and a' are indiscernible. In between these two extreme situations, *viz.* this case and the case of complete information, we have an intermediate possibility: about some elements a we have complete information (so that the indiscernibility class reduces to the singleton $\{a\}$), while other elements cannot be uniquely characterised, so that they belong to more populated classes.

These two situations split the universe of discourse U into two parts, with different logico-algebraic properties.

Let us denote by B the union of the equivalence classes that are singletons, and by P the union of the equivalence classes that have cardinality strictly greater than 1. B and P do not have the same logical role in the construction of a Rough Set System. In fact the elements in B are "exact" in nature and they should enjoy the two principles of Classical Logic reflecting exactness: the law of excluded middle and the law of contradiction.

On the contrary, on P either excluded middle or non-contradiction, or both, may fail because in P we may have *boundary* cases. That is, we have undecided or ambiguous situations.

Thus we may think that Rough Set Systems fulfill two distinct *local* logical behaviours: one is classical and localized on B, whereas the other, localized on P, is three-valued. It is the combination of these local behaviours that characterise the overall logical features of the system. It follows that the construction of a Rough Set System will depend on the parameter B (or P). Moreover, this means that in order to analyse this two-faced situation, one needs some mathematical notion which makes it possible to formalise the concept "*to be locally valid*".

We find that the notion of a *Grothendieck topology* provides us with the required formalism. Moreover, William Lawvere and Myles Tierney gave an abstract logico-algebraic interpretation to Grothendieck topologies, while investigating Topos Theory, giving rise to the techniques of the so-called *Lawvere-Tierney operators*.

So, we shall use this interpretation to describe the logical operators that may be defined in a Rough Set System, namely pseudo and co-pseudo complementation, with the appropriate meaning that which makes it possible to completely characterise the two logical behaviours. It will be shown that these operators induce modalities (with no surprise, since Lawvere et al. showed that "to be locally valid" is a modality, indeed). In turn, these modalities make a Rough Set System into different logico-algebraic structures.

At this point the main result of Part II is ready. Rough Set Systems may vary from *Boolean algebras* (when we have complete information) to *Post algebras of order three* (when we do not have complete information), passing through an intermediate case, *three valued Łukasiewicz algebras* (when we have both complete and incomplete information).

The discriminating point between the two logical behaviours (classical and three-valued) is the existence an intermediate value ("uncertain") in a chain of values. When an intermediate value exists Rough Set Systems may be given the structure of a *Chain-based lattice*, too.

Interestingly enough, it will be seen that the philosophical and formal techniques used in this Part are linked to important problems about maximal intermediate constructive logics and the constructivistic philosophy *tout-court*.

In Part III we revisit this story by analysing these kinds of modalities and operators under three points of view:

- As *external modalities*, that is, modalities defined by means of a double decoding (observations + exogenous interpretation) of primitive data (or primitive events, uninterpreted data, and so on)

- As *Diodorean modalities*, i.e. modalities in which, roughly speaking, an event A is evaluated "possible" only if there is some conceivable state of affairs in which A happens indeed

- As *core/expansion* and *vicinity/contraction* maps induced by *pre-topological spaces* endowed with specific features

Pre-topological spaces play a particular role in our analysis, since they are such generalisations of topological spaces as to make it possible to enlarge the interpretation domains of classical Approximation Spaces and classical Rough Set Theory towards, for instance, dynamic observation systems. On the other hand, positioning classical Approximation Spaces and Rough Set Theory within the framework of pre-topological spaces gives us the possibility for a better understanding of both the strength and weakness of the theory.

Pre-topological operators will be gradually given stronger features so as to arrive at a connection between them and modal operators defined by means of *R-neighboring* operators, for any reflexive binary relation R on the universe of discourse U. Under specific assumptions about R, pre-topological spaces with operators defined by means of the relation R turn into topological spaces.

With the addition of some extra features on R, we finally obtain topological spaces of clopen (both open and closed) subsets of the universe. These topological operators may be added to a Boolean algebra in order to obtain *pre-monadic Boolean algebras*, *monadic Boolean algebras*, *topological Boolean algebras* and *monadic topological Boolean algebras*. These structures make it possible to give an algebraic interpretation of *Approximation Spaces* as described by Rough Set Theory.

Particularly, by means of a straightforward manipulation of monadic Boolean algebras it is possible to define *topological quasi-Boolean algebras* and, by means of this notion, algebraic structures that are equal or isomorphic to the Post algebras of order three defined in Part II, called *pre Rough algebras* and *Rough algebras*. This move makes it possible to use Rough Sets Systems as a complete semantics of two logical systems $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

How to read this book

Each Chapter is organised as follows:

- A self-containing exposition of the topics of the Chapter,

- A "Frame section". Each Frame presents additional examples, details and topics related with the main content of a Chapter. Moreover, frames propose additional exercises and provide the reader with subsections dedicated to some history of the concepts used in the Chapter.

- Solutions of the exercises.

For obvious reasons, we could provide neither an all-encompassing history nor all the topics that deserve a due attention. We had to make some choices and some, painful, renounces.

All the preliminary mathematical notions are collected in an Appendix, titled "Mathematical toolkits".

Intended readers

The intended readers of this book are primarily graduate and post graduate students and researchers from mathematics, logic, computer science and philosophy. We hope that insiders of each group will find in the book materials of their specific interest. But for a deeper appreciation of the journey taken up in this writing an interest in all the above disciplines and some general awareness of current trends of thoughts may be helpful.

Acknowledgments

We are deeply indebted to Dr. Mohua Banerjee for our discussions and conversations on the topics of the book and her valuable ideas.

We sincerely thank Professor T. Y. Lin who read the book and made valuable suggestions and remarks.

We want to express our deep gratitude to Professor Ewa Orłowska whose friendship was so important to develop our scientific work for so many years.

We are very grateful to Professor Andrzej Skowron and Professor Lech Polkowski who pioneered and developed many fundamental concepts in Rough Set Theory.

We would like to remember Professor Helena Rasiowa and Professor Zdzsław Pawlak who were so influential to both of us.

Finally, the first author wants to remember Professor Pier Angelo Miglioli and their passionate discussions about logic, mathematics and life.

Roma   *Piero Pagliani*
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February 2007

Contents

List of Figures

Figures drawn with the aid of Paul Taylor's "Commutative Diagrams"
package.

Notation

We enlist more important notations conventions:

Symbol	Meaning
$A, B, ...$	sets (if not otherwise specified)
$a, b, ...$	elements of sets (if not otherwise specified)
$\alpha, \beta, ...$	formulae of a logic language or properties
$[\![\alpha]\!]$	the "extension" of α in a model
$\mathbf{A}, \mathbf{B}, ...$	structures (sets equipped with operations and/or relations)
$\mathcal{A}, \mathcal{B}$	sets of sets (if not otherwise specified)
$\vee$	Linguistic disjunction
$\wedge$	Linguistic conjunction
$\sim$	Strong negation (in Nelson, De Morgan or Kleene algebras)
$\neg$	Pseudo-complementation or complementation (in Heyting and Boolean algebras)
$\lrcorner$	Co-intuitionistic negation
$\dot{\lrcorner}$	Weak negation (in Nelson algebras)
$\overline{\neg}$	Co-weak negation (in Nelson algebras)
$-$	Set-theoretic complement
$\rightarrow$	Material implication
$\rightarrowtail$	Contrapositional implication (in Nelson algebras and related structures)
$\Rightarrow$	Extensional implication (in Rough algebras)
$\Longrightarrow$	Relative pseudo-complementation (in Heyting algebras)
$\Longleftarrow$	Co-relative pseudo-complementation (in co-Heyting algebras)

$\mapsto$	Functional dependence
$\supset$	Pre-intuitionistic implication (in Nelson algebras)
$\subset$	Co-pre-intuitionistic implication (in Nelson algebras)
$\overset{\mathbf{S}}{\Longrightarrow}$	Relative pseudo-supplementation
$\overset{\mathbf{S}}{\Longleftarrow}$	Co-relative pseudo-supplementation
$\Rightarrow$	*Metalinguistic implication*
$\&$	*Metalinguistic conjunction*
$\vee$ (or "*or*")	*Metalinguistic disjunction*
$\neg$ (or "*not)*"	*Metalinguistic negation*

The metalanguage is classical First Order Logic.

Notice that the same operation may be denoted by different symbols, as a structure is transformed.

For instance, in semi-simple Nelson algebras the weak negation ⌐, the co-weak negation ⌐ and the pre-intuitionistic implication $\supset$ coincide, respectively, with the co-intuitionistic negation ⌐, the pseudo-complementation $\neg$ and the relative pseudo-complementation $\Longrightarrow$.

However, in semi-simple Nelson algebras we keep using the symbol $\supset$, while we adopt ⌐ and $\neg$ instead of ⌐ and, respectively, ⌐.

Obviously, the relative pseudo-complementation $\Longrightarrow$ of a Heyting algebra $\mathbf{H}$ turns into a material implication $\to$ if $\mathbf{H}$ is a Boolean algebra. And so on.

If $\mathbf{H} = \langle A, op_1, op_2, ..., op_n \rangle$ is any mathematical structure with carrier A, then we shall denote an element a of the structure by $a \in A$. If there is no risk of confusion we shall also write $a \in \mathbf{H}$.

Abbreviations

JSL=Journal of Symbolic Logic;
SLNAI=Springer Lecture Notes in Artificial Intelligence;
SLNCS=Springer Lecture Notes in Computer Sciences;
SLNM=Springer Lecture Notes in Mathematics.

Introduction

Who looks for life finds the form. Who looks for the form finds death.
— E. De Filippo

1   Perception and Concepts:
   A Phenomenological Approach

The theory of Approximation Spaces, and Rough Sets, the main concern of this book, is essentially a theory of perception. Of course, this is true if we intend "perception" to be taken in a broad sense and not as a mere naturalistic recording of external data by means of our senses.

In order to better explain what is meant here with this notion, we can consider some classical puzzles in the theory of vision. Consider, for instance, the well-known Kanizsa's triangle (Figure 1). It is known that the white triangle in the foreground does not exist and that the strength of its virtual existence is induced by (and is a symptom of) some interrelations between our senses and the elaboration capabili-
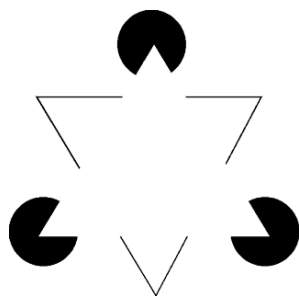
Figure 1: Phenomenal rendering of a white triangle with contours without gradients

ties of our mind. Again, look at Figure 2. Usually people perceive the
convex part as "the object" while the concave part is perceived as the
visible part of the "background" behind the object. However, appar-
ently, we do not have any particular reason for this preference. One of
the most interesting challenges for the theory of vision was and still is
the explanation of these implicit choices: the choice of the convex part
as a *body* against a background, in Figure 2; the choice for an inexis-
tent triangle as a means for getting a sort of equilibrium within the
"possible meanings" of Figure 1.
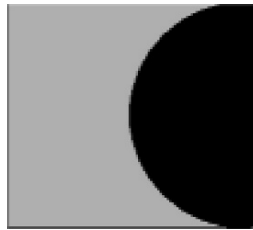
Figure 2: The convex region is perceived as the figure

Whatever the deep nature of these processes could be, nowadays the
schema of Figure 3 is a most reliable hypothesis. According to it, the
source sends data that must be **reconstructed** by the receiver subject.

Figure 3: Sender and receiver concurrently define the meaning of
messages

Indeed, this schema has been assumed by recent semiotics too, in
contrast with the former structuralist interpretations and the pouring
models of the theory of meaning (see Nattiez [1989]).

During the reconstruction process, data are "*selected, analyzed, inte-
grated by means of properties not directly perceivable but induced by
hypotheses, deduced or anticipated, by the intellectual faculties at my
disposal.*" (Kanizsa [1980], page 81. Our translation).

Furthermore, G. Kanizsa points out that "*the mere identification of an object of vision [...] implies an elementary logical operation, its placement in a particular identity category rather than in another. [...] Indeed, it is clear that the simplest of the operations I spoke about, presupposes a "something" which must be identified, that is to say, which must be assigned to an equivalence class, operation that makes it possible to infer the other (probable, but not directly given) features of the object. And this is the premise for any other more complex cognitive activity.*" (*ibidem*, pages 83 and 84).

This long quotation is worthwhile. Indeed we could hardly describe the starting point of Rough Set Theory with different words.

This starting point is, in fact, the partition of a universe of discourse U into disjoint equivalence classes, that is the same as the identification of an equivalence relation over U. This equivalence relation does not come by chance, but strictly depends on the parameters that we have selected in order to interpret the states of affairs about U. Using Kanizsa's words, we can say that in Rough Set Theory, data are perceived by means of *anticipated hypotheses*. In other terms, the set A of observable properties that one decides to use, prepares a *perception grid* to be filled by the data deriving from our observations of U.[1] This means that, since the very beginning, one cannot speak of mere data, but must speak of filtered data: data that one records by means of a well defined and specific perception grid. Data, as such, are Kantian "noumena" that we perceive only within a categorization operation.

In fact, by means of the perception grid we can organize our universe of discourse U into equivalence classes, that constitute our *basic categories*. More precisely, two objects a and b, belonging to U, will be considered equivalent if they enjoy the same observed properties, that is to say, if the set A of observable properties is not able to differentiate them. Thus one can claim that a and b are *indiscernible with respect to A*, since no observables from A is able to *separate a* from b. In this manner, A induces the required equivalence relation, E_A.

Therefore, it follows that any equivalence relation embeds some set of information concerning the elements of a universe U, perceived by

---

[1] We use the term "observable property" and "observable" in a broad sense. For instance, a set of parameters is, in this context, a set of observables.

means of a set of observables A. We shall call the structure $\mathbf{I} = \langle U, A \rangle$ an *Information System*.[2]

This is the first part of the job. But the story is not over.

Our equivalence (or indiscernibility) relation E_A gives only a partial account of U. In a sense, the set of observables A solely induces the particular granulation of a particular "conceptual filter". Of course, we have good reasons for preferring A instead of another set of observables: maybe the observables in A are more significant for the phenomenon we want to analyze; maybe this choice depends on a trade-off between completeness and tractability (more observables, more dimensions, thought more accurate, could lead to infeasible computations). In any case, the reader should notice that A is an *a priori* choice. It depends on some hypotheses.

Definition 1.1. *Let U be a set and $E \subseteq U \times U$ an equivalence relation on U. Then the pair $\langle U, E \rangle$ will be called an* Indiscernibility Space*.*

From what is said before, an *Indiscernibility Space* $\langle U, E_A \rangle$ has no value "*an sich*". It is inert. We must contrast it against another space in order to have a useful information. In other words, we must compare at least two different categorisations $\mathbf{E} = \langle U, E_A \rangle$ and $\mathbf{E}' = \langle U, E_{A'} \rangle$ of the same universe U, that is, by means of two Information Systems $\mathbf{I}$ and $\mathbf{I}$'. As always happens, we need two different states to initiate the dynamic process of information collection: $0/1$, background/foreground, black/white, ...

What is the nature of $\mathbf{E}$ and $\mathbf{E}'$?

1.1   Monological Approach and Dialogical Approach

If we assume that one of the two categorisations, say $\mathbf{E}$, must provide the conceptual explanation of the other, $\mathbf{E}'$, then it should be assumed that $\mathbf{E}$ is charged with some founding role in our gnoseological dynamics. Although the choice of $\mathbf{E}$ is an "a priori", depending for instance on the meaning of the parameters in A, once we have decided that $\mathbf{E}$ has to provide the basic categorisation, $\mathbf{E}$ is given a privileged role. Such a role imposes a particular direction to the gnoseological dynamics so that $\mathbf{E}$ cannot be influenced by $\mathbf{E}'$.

---

[2]This notion of an "Information System" is a generalisation of that adopted in Rough Set Theory (see below).

For instance, A' might be a set of decisions about members of U (for instance, decisions for investments in a group of regions) while A might be a set of evaluations related to those decisions (for example, socio-economic evaluations about those regions). In this case we should like, for instance, to check the consistency of our decisions, with respect to A in order to compute the best consistent decisions, or we would like to know the reliability of possible extensions of a decision to other members of U. This is the usual approach in Rough Set Analysis and we call it *monological approach*, because of the directedness from a privileged categorisation towards the other.

But one can also think of $\mathbf{E}$ and $\mathbf{E}'$ as categorisations induced by the perception of two distinct gnoseological subjects, $\mathbf{S}$ and $\mathbf{S}'$. In this case we cannot privilege one of them. On the contrary we should be interested in questions like: how do $\mathbf{S}$ and $\mathbf{S}'$ interact? What are the conditions for such an interaction being possible? How can $\mathbf{E}$ and $\mathbf{E}'$ be re-formed so that the subjects $\mathbf{S}$ and $\mathbf{S}'$ may arrive at a consensus on a claim like "x is X", where X is not a sharply defined property (or, equivalently, X is not the extension of a property)? How can $\mathbf{E}$ and $\mathbf{E}'$ merge together? In other words, now we do not have a privileged direction forcing one categorisation to be subordinated by the other. Therefore this reading is called *dialogical approach*.

This approach is a novelty and at in embryo stage within the specialized literature but we mention it in order to underline the philosophical and mathematical differences from the monological approach (some details about the so-called *Dynamic spaces* are shown in Part III).

The monological variant is clearly an instance of the dialogical approach. In accordance with it, one of the two Indiscernibility Spaces, say $\mathbf{E}$, will act as background while $\mathbf{E}'$ becomes the set of perceptions that must be focused in contrast to this background. To some extent the monological approach has a metaphysical flavour since in this approach $\mathbf{E}'$ plays the role of "object" while $\mathbf{E}$ plays the role of "subject". In this framework, we call $\mathbf{E}$ an "internal categorisation" and $\mathbf{E}'$ an "external categorisation".

Nevertheless we shall be primarily concerned with the monological approach since researches in this direction has almost reached a status of well-established scientific "corpus". Moreover, the monological approach is able to supply the basis for understanding the problems behind the dialogical approach.

2 Monological Approach to Perception and Concepts

So, let us assume that one of the two categorisations has to be considered as the internal categorisation that must be referred to by the external one in order to obtain a conceptual reading of the external categories.

Hence we are back to the initial puzzles. Indeed these puzzles and the process just described, share some basic mechanisms.

Researchers achieved an almost general agreement in focusing on a number of most promising notions for the explanation of these kinds of puzzles. These notions, generally, describe some "topological" relation: *vicinity, similarity, direction, continuity, closure.* Others belong to higher order factors, such as the principle of "good Gestalt" (or "Prägnanz"). Finally, others pertain to the history of the subject, like her/his past experience.[3]

Among the topology related factors, the notion of a *completion* covers an outstanding position. Indeed, in Figure 1 one virtually forms, by means of a *completion* process, three black discs and a white triangle with black contours, that "justify" the presence, on an upper stratum, of the white triangle with contours without gradients (or with "cognitive contours").

This completion process is called "amodal completion" since the resulting figures do not enjoy the characteristics of the visual modality.

What we can here appreciate is that probably we are naturally inclined, first of all, to *close* a figure (in a sense, we suffer a "*horror vacui*") and, second, to take the *maximal coherent* part of this closure (in order to be able both to separate a figure from another figure and a figure from the background). This double move is what is called here a "completion".

---

[3]From this point on, we are trying to give a reliable, albeit simple, mathematical interpretation of these cognitive features, with the final goal of providing a particular mathematical approach to data analysis with an intuitive background. Otherwise stated, we do not claim that the aforementioned cognitive psychology analysis are given a mathematical support by our model. On the contrary, we just hope that our mathematical approach could be better understood by means of the psychological and philosophical concepts here discussed. Anyway we think that the interpretation developed here has some deep cognitive motivation (to be better explored in the future).
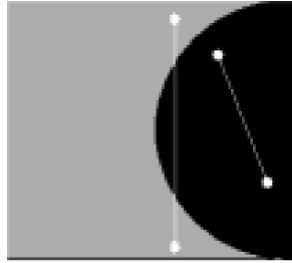
Figure 4: A straight segment connecting two points inside a convex region lies in that region
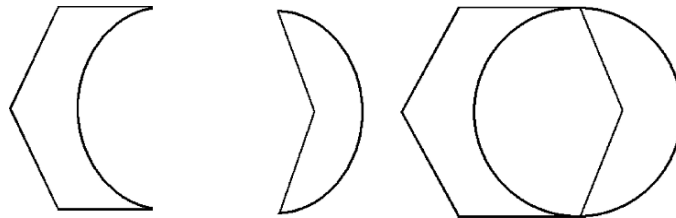
Figure 5: Unification of two figures with incoherent contours

Also in Figure 2, the notion of a completion is decisive. In fact, a convex area has the property that any two distinct points lying in it can be linked by a segment completely inside the area. This is a completion feature too and, in our opinion, it is the best explanation of our propensity to perceive convex areas as "bodies".[4] Indeed, "bodies" are *solid*, cannot be *interpenetrated* and enjoy an *internal coherence*. All this is assured by their completion features (see Figure 4).

We do not go far from the notion of a "completion", even when the "good Gestalt" principle is considered. This principle is a mix of "topological" factors, like "simplicity", "order", "symmetry" and "regularity". According to this principle, phenomena are perceived so as to get a *maximum of "structural coherence"*. The completion process is accomplished in view of this result. For instance, if one unifies the two shapes on the left in Figure 5, one perceives two new figures (a circle and a hexagon) according to the principle of maximal structural coherence. The term "maximum" is somewhat misleading. We are not speaking of the maximality of the area enjoying structural coherence,

---

[4]This elementary property defines convex figures; curiously enough, as far as we know, it does not appear among the explanations of this phenomenon.

which would be an extensional maximality; on the contrary, we are speaking of a sort of intensional maximality or, better, we are speaking of *fixpoints* in the search process for structural coherence. In the same way, by a "completion" we do not intend uniformly a coherent extension of a figure. A figure could loose part of its area if this sacrifice allows us to accomplish a coherent completion (actually, a completion is not referred only to the figure at hand, but to the geometry of the entire perception field). We give an example of this fact.

Example 2.1.
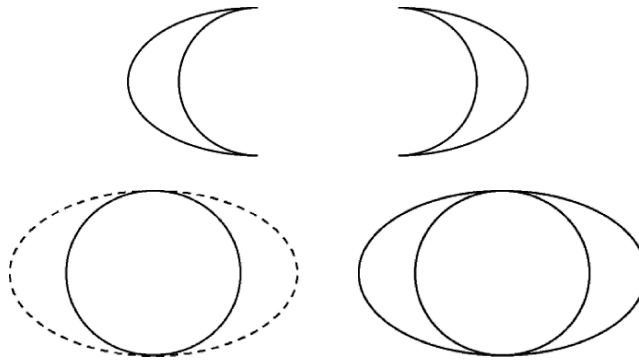
Figure 6: Two admissible unifications

The two crescents in Figure 6 may be unified either as a circle inscribed in an oval, or as two intersected eggs. This is a case of different *fixpoints* of a completion processes.

As for the "past experience", one has to consider that we distinguish "figures" (or "bodies", or "objects") within a context (or universe of discourse) in such a way as to make them behave as usually experienced figures, (bodies or objects).[5] For instance we want them to fulfill some "invariance property" and some "persistence property". Otherwise stated, we require a sort of *structural stability*. It is

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[5]In this context, the terms "body", "object" and "figure" are interchangeable. The reason should be already clear. Anyway it will be explained later on in Subsection 5.1.

this structural stability which makes us perceive things as a "*pertinent world*", provided that "*what is regarded as a pertinent world is inseparable from the structure of the perceiver.*" (Varela [1994].)

These remarks oblige us to be a little bit more precise. In fact, what has been described was called "the completion" of an object, together with its relation to the notion of a *closure* of a region, with some license, while we should rather speak of a *connected space.* Indeed, this seems to be the property characterizing the *domain of existence of an object* (cf. Thom [1977]). Nevertheless, in view of the mathematical nature of the models that we shall use, the notion of a *closure* as noticed, must be supplemented by some invariance condition. Intuitively, an invariance condition assures that a body is determined and individualized within its relationships with other objects. A minimal requirement to be satisfied by a "body", therefore, is its separability from the other "bodies". When we are dealing with usual sets, we have such features: the closure property is given by the comprehension schema which defines a set by enclosing all the elements satisfying a given property, while the required invariance condition is guaranteed by the fact that any set, as such, is *separable* from all the others by means of the set-theoretic complementation, which is involutive: $--X = X$. That is what induces the Boolean structure of the powerset of any set. Thus closure and separability coincide in this case.

However, in more complex and refined situations, the closure property is insufficient and it cannot guarantee separability. In this case we have to require a "body" X to be characterized by the inductive property stating that given any point x of X, X comprehends all the points which are close to x but that do not belong to the border region of X.

It is this double condition which gives sets their "rough" nature.

Indeed, if we do not include, for any sub-region of a body, the points which are close to it, we would lack the principle of coherence. On the other side, if we do not exclude the border region, we would admit external interferences so that the principle of stability would be lacking.

Therefore, we require a "body" to be a *regular set*, topologically speaking, *viz.* a set which coincides with the *interior of its closure* (for these and other basic topological notions see Mathematical toolkit 16.4 and the text).

From now on, by $\mathbb{I}(X)$, $\mathbb{C}(X)$, $\mathbb{B}(X)$ and $-X$, we shall denote the *interior*, the *closure*, the *boundary* and the set-theoretic *complement* of X, respectively (where $\mathbb{B}(X) = \mathbb{C}(X) \cap -\mathbb{I}(X)$).[6]

It is not difficult to notice that the notion of a regular element is well suited for our purposes. Of course, it is appropriate when we deal with *unstructured* sets and simple categorization operations induce a Boolean algebra of both *open and closed* (*clopen*) subsets of U (as a topology, this algebra is said to be 0-*dimensional*; this concept renders the fact that we have not to take into account any deeper categorisation of U, any previous structure).

But it is appropriate also when we deal with a universe structured by means of more general topological relations. Actually, the simple set-theoretic complement together with the notion of an open set could not be sufficient, since it would only separate an open set from the background, but not a body from another body. In fact if we accept that the "coherence" requirement implies that a "figure" a has to be an open set, then its complement $-a$ is surely a closed one, thus generally it is not a "figure". Therefore we must take the interior of $-a$, $\mathbb{I}(-a)$, in order to obtain a figure. Call it the *complementary figure* (the *complementary body*) of a, denoted by $\neg a$. However, usually it happens that there is not a 1-1 correspondence between open sets and complementary figures, because the complementary figure of $\neg a$ could be different from a: $-(\neg a)$ is a closed set and when we take its interior we obtain $\neg\neg a = \mathbb{I} - (\neg a) = \mathbb{I} - \mathbb{I}(-a) = \mathbb{IC}(a)$ which, generally, is larger than a. Indeed we can even have incomparable a and a' such that $\neg a = \neg a'$. Hence we observe that an open set could be structurally unstable, because the separation capability of the operation $\neg$ is just one way and we could be unable to recover a from its complementary figure. In other words, when we go out entering the exterior of a and then come back, we are not sure to find the same set we have left. In this case a is not clearly separable from the rest of its ambient universe. An open set a will be structurally stable only if $a = \neg\neg a$, and this means, precisely, that a is regular, from a topological point of view.

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[6]By now the reader can think of a point x which belongs to the interior of a set X as a point which has links just with points belonging to X, while x is close to X if it has at least a link with some elements of X. Finally, x is on the border of X if it is close to X but has at least a link with a point not belonging to X.

We give a simple but general enough example.

Example 2.2.

$$c$$
$$|$$
$$b \qquad\qquad d$$
$$\diagdown \quad \diagup$$
$$a$$
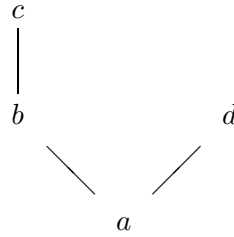
A simple structured space

We say that a set $X \subseteq \{a, b, c, d\}$ is an *upset* or *order filter* (respectively, a *downset* or *order ideal*) if whenever $x \in X$ and x' is above (respectively, below) x, then x' is in X too (see Mathematical toolkit 16.1). Upsets (respectively, downsets) fulfill the open set (respectively, closed set) axioms. It follows that a point x is close to a set Z if there is a $z \in Z$ such that x is below z. Hence, assuming this *upward topology* (a topology where the open sets are all and only the upsets) over $\{a, b, c, d\}$, we can easily notice that, for instance, the set $\{c\}$ is open, its closure is $\{c, b, a\}$ and its boundary is $\{c, b, a\} \cap -\{c\} = \{b, a\}$. The set-theoretic complement of $\{c\}$ is $\{a, b, d\}$ which is a closed set. Thus the set-theoretic complement $-\{c\}$ cannot be the complementary figure of $\{c\}$. To obtain the complementary figure of $\{c\}$ we have to take the interior of the complement, $\mathbb{I}(-\{c\}) = \{d\}$, which is an up-set. But the complementary figure of $\{d\}$ is $\mathbb{I}\mathbb{C}(\{c\}) = \{b, c\} \supsetneq \{c\}$. It follows that $\{c\}$ is not a regular element. On the contrary, $\{d\}, \{b, c\}$, the whole universe $\{a, b, c, d\}$ and the empty set $\emptyset$ are regular.

Throughout this study the notion of a regular element will be fundamental, at any level. By means of this notion Boolean elements hidden inside different logico-algebraic structures shall be identified. And this is important because these Boolean elements, will represent the structurally stable "islands", surrounded by more complex and unstable seas.

As a matter of fact, at the very beginning given a universe U we take into account a particular Boolean algebra of subsets of U.

In fact, concerning an Indiscernibility Spaces $\mathbf{E} = \langle U, E_A \rangle$, it is noticed that the family of equivalence classes $U/E_A = \{[x]_{E_A}\}$, for x ranging over U, can be regarded as the family of atoms of a Boolean algebra that will be denoted by $\mathbf{AS}(U/E_A)$. Therefore, any element of $\mathbf{AS}(U/E_A)$ is a union of equivalence classes.

Any such Boolean algebra is induced by an Indiscernibility Space and can be considered as the fundamental (albeit elementary) phenomenological space of our construction. From this point of view, it receives the name of "*Approximation Space*" (of U, with respect to the indiscernibility relation E_A). Because of our definition, an element $X \in \mathbf{AS}(U/E_A)$ is a set definable by means of a basic property, if it is an atom, or by means of a set of basic properties, otherwise.

A simple Information System and its Approximation Space are shown below in Figure 7.

Example 2.3.

	P_1	P_2	P_3
a	1	2	f
b	1	3	h
c	1	2	f
d	0	3	f
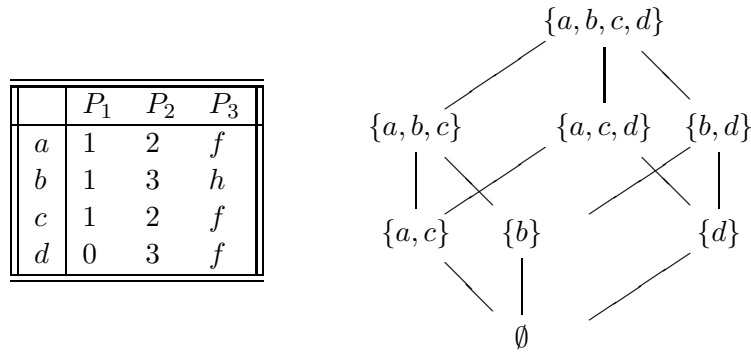
Figure 7: A simple Information System and the Hasse diagram of its Approximation Space

$A = \{P_1, P_2, P_3\}, U = \{a, b, c, d\}, E_A = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle, \langle a, c \rangle, \langle c, a \rangle\}$, so the basic categories are $\{a, c\}, \{b\}$ and $\{d\}$, corresponding to the basic properties "$P_1 = 1 \wedge P_2 = 2 \wedge P_3 = f$", "$P_1 = 1 \wedge P_2 = 3 \wedge P_3 = h$", and "$P_1 = 1 \wedge P_2 = 3 \wedge P_3 = f$", respectively. The element $\{a, b, c\}$ is given by $\{a, c\} \cup \{b\}$; hence it is describable by the property ("$P_1 = 1$" $\wedge$ "$P_2 = 2$" $\wedge$ "$P_3 = f$") $\vee$ ("$P_1 = 1$" $\wedge$ "$P_2 = 3$" $\wedge$ "$P_3 = h$").

In an Approximation Space any equivalence class X is a structurally stable object, because it is clopen. Indeed, $X = \mathbb{C}(X) = \mathbb{I}(X) = \mathbb{I}\mathbb{C}(X)$.[7] This property is inherited by joins.

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[7]This is not new. For instance, "[If E is a topological space] *in order for a G-equivalence class F to be structurally stable, it needs and suffices that the totality of the points in E of this equivalence class constitutes an open set of the space.*" (Thom [1980], Ch. 2, § 2.1, C).

REMARKS. Concerning the last point, it must be mentioned that, strictly speaking, only the atoms of $\mathbf{AS}(U/E_A)$ are objects because they are connected by the indiscernibility relation, while non-atomic elements (formed by joins) are separable but not connected. Indeed non atomic objects are compound objects, in that they are given by an assemblage of atomic (hence connected) objects. The compound but not connected nature of non-atomic elements will be clearly illustrated by the construction of the dual space of an Approximation Space. Moreover, we shall see that there is an important difference between compound objects and complex objects which are given, for instance, by Galois adjunctions, because the latter enjoy a sort of organic unity, as we shall see in Part I. In a complex object, elements are merged together and, in a sense, their properties are multiplied. In a compound objects, elements are juxtaposed and their properties are added.

Being aware of the above proviso, we can work with $\mathbf{AS}(U/E_A)$ and consider it as the lattice of clopen subsets of a 0-dimensional topological space $\langle U, \mathbf{AS}(U/E_A) \rangle$. In a 0-dimensional space if $X \in \mathbf{AS}(U/E_A)$, then $-X$ is in $\mathbf{AS}(U/E_A)$ too. Moreover, $--X = X$. It is worthwhile remarking this property, since it has some relevance not only from a mathematical, but also from a philosophical point of view. In fact, in the history of Philosophy, it is often found that "individual objects" are defined as "entities" fulfilling properties that can be interpreted by means of mathematical notions such as "closure", "interior" and "separability".

For instance, in Modern Age, Leibniz observed that "*the nature of an individual substance, or complete being, is to have such a perfect notion so that it comprehends, and makes it possible to deduce, all the predicates of the subject to which this notion is ascribed.*" If this is not the case, then Leibniz speaks of "*accidental* beings", that is, *not necessary* beings. Hence a complete being is both (deductively) *closed* and *necessary* (Leibniz, "*Discourse on Metaphysics*").

These requirements are usually in direct contrast in mathematical logic, since "closure" is strictly connected with the notion of "possibility", while the notion of "necessity" is reflected by that of "interior". However, when we deal with a universe of both closed and open sets, any possible event is also necessary and the contradiction does not remain further.

In fact, we shall investigate Approximation Spaces as topo-algebraic models of $\mathbf{S}5$ modal system which is characterized by this property.

To conclude, Approximation Spaces provide us with a good framework for a conceptual organization.

But concept forming really occurs when at least two phenomenological spaces arising from the same universe of discourse U are compared. So, we must contrast an Approximation Space with another which is derived by a different basic categorization.

First it has to be decided which Approximation Space will act as background. Here we are speaking of a "conceptual background", in the sense that the parameters that generate the background space are supposed to be able to explain phenomena recorded in the structure of the other Approximation Space.

Once we have decided which is the *background Approximation Space*, say $\mathbf{AS}(U/E_A)$, we have to fit in it the objects from the second space, say $\mathbf{AS}(U/E_{A'})$, that therefore shall be called a *foreground Approximation Space*. An element $X \in \mathbf{AS}(U/E_{A'})$ (an "object" or a "figure") might fail to be an element (object, figure) of $\mathbf{AS}(U/E_A)$. This means that a set exactly described by properties of the foreground space A' (*foreground properties*) might not be exactly described by properties of the background space A (*background properties*). Or, otherwise stated, a set which is structurally stable with respect to the topological space $\langle U, \mathbf{AS}(U/E_{A'}) \rangle$, might be structurally unstable in $\langle U, \mathbf{AS}(U/E_A) \rangle$. We can then say that a figure of the foreground space is a *pre-figure* in the background space, a sort of fuzzy shadow which must be focalised (remember that this is a "one way" operation, in the monological approach).

Let us consider an arbitrary object X of the foreground space $\mathbf{AS}(U/E_{A'})$. Two possibilities are given: either $X \in \mathbf{AS}(U/E_A)$, too, or not. In the first case one can exactly describe X by means of a property, which in view of the preceding discussion can be named a *background property*. In the second case background properties cannot be used for a direct description of X, but one can approximate it by means of properties from $\mathbf{AS}(U/E_A)$ (see Figure 8).

More precisely, there is an approximation from above, called an *upper approximation*, $(uE_A)(X)$, and an approximation from below, called a *lower approximation*, $(lE_A)(X)$.

REMARKS. If the set A of observables is understood or immaterial, we shall use the symbols $(uE)(X)$ and $(lE)(X)$ instead of $(uE_A)(X)$ and $(lE_A)(X)$, respectively.

We shall see that if we consider an Approximation Space $\langle U, \mathbf{AS}(U/E)\rangle$ as a topological space, then the upper approximation of X coincides with the topological closure $\mathbb{C}(X)$ and the lower approximation of X with the interior $\mathbb{I}(X)$.

It follows that the *lower approximation* of X, $(lE)(X)$, is the *largest body contained* in it, while the *upper approximation* of X, $(uE)(X)$, is the *smallest body that contains* it.

However, in general between $(lE)(X)$ and $(uE)(X)$ we have the topological boundary of X: $\mathbb{B}(X) = \mathbb{C}(X) \cap -\mathbb{I}(X) = (uE)(X) \cap -(lE)(X)$. Notice that the boundary of X is the set of the points that are neither in its lower approximation, nor in the complement of its upper approximation: $\mathbb{B}(X) = \mathbb{C}(X) \cap -\mathbb{I}(X) = -(-\mathbb{C}(X) \cup \mathbb{I}(X)) = -(-(uE)(X) \cup (lE)(X))$.

Figure 9 displays a picture of the approximation operations. This is the basic intuition behind Rough Set Theory. Otherwise stated, the philosophical assumption in Rough Set Theory (see Pawlak [1991]) is that the ability to distinguish objects by means of their properties is a basic source of knowledge. Such properties may be induced by direct observations or by theoretical considerations: in any case the more precise these properties are, the finer the grains of our knowledge will be. Thus, given a universe of discourse U and a set of observables A, the basic gnoseological act is the organization of U into disjoint categories, in such a way that each category collects the elements of U that are describable by means of the same observed properties or,
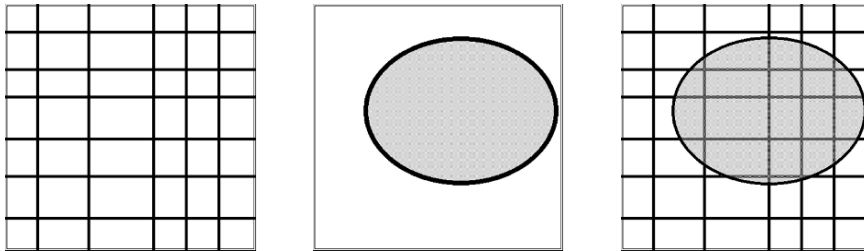
Figure 8: Background space – Foreground space – Contrast
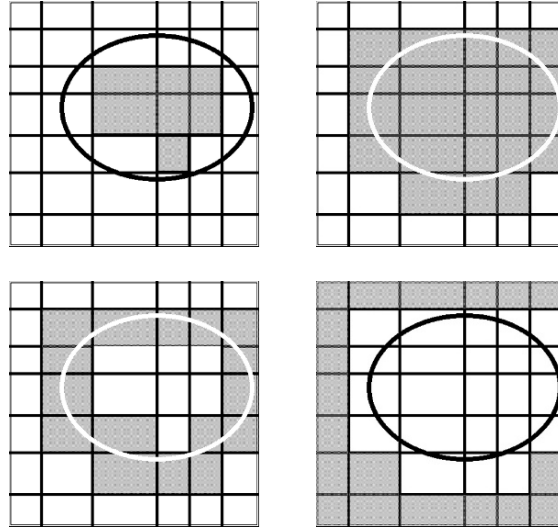
Figure 9: Lower approximation – Upper approximation – Boundary – External part

in other terms, that are *indiscernible* by means of the observables considered.

Therefore, the atoms of $\mathbf{AS}(U/E_A)$ and $\mathbf{AS}(U/E_{A'})$ may be intended as extensions of basic properties which are definable from two information systems, **I** and **I'**. Clearly, any element of the background and foreground Approximation Spaces is the extension of a disjunction of basic properties.

Example 2.4. Information Systems

In the Information System depicted in Table 1 below, the set $A' = \{Comfort\}$ has to be compared with the set $A = \{Temperature,\ Hemoglobin,\ Blood\ Pressure,\ Oxygen\ Saturation\}$ in order to find some regularity connecting these two sets of observations. The parameter in A' is supposed to depend on parameters in A. Therefore the latter are considered *explicantes*. They will induce the background Approximation Space $\mathbf{AS}(U/E_A)$, and the *explicandum* will give the foreground Approximation Space $\mathbf{AS}(U/E_{A'})$.

In this Information System, the set $\{d, e, f, g\}$ is an object of $\mathbf{AS}(U/E_{A'})$ described by the basic property "*Comfort* = *medium*". This fact will be denoted by $\{d, e, f, g\} = [\![\text{"}Comfort = medium\text{"}]\!]$, to be read: "the set $\{d, e, f, g\}$ is the *extension* of the basic property "*Comfort* = *medium*". But $\{d, e, f, g\}$ is not an object of $\mathbf{AS}(U/E_A)$ because we cannot obtain it by means of any union of its atoms (i.e. equivalence classes from $\langle U, E_A \rangle$ which are: $\{\{a\}, \{b\}, \{c, d\}, \{e, f\}, \{g\}, \{h\}, \{i\}\}$). So we have to approximate it, as illustrated in the next example.

Table 1: A medical information system (Slight modification of a database presented in Grzymala-Busse [1992])

	Temperature	*Hemoglobin*	*Blood pressure*	*Oxygen saturation*	*Comfort*
a	low	fair	low	fair	low
b	low	fair	normal	poor	low
c	normal	good	low	good	low
d	normal	good	low	good	medium
e	low	good	normal	fair	medium
f	low	good	normal	fair	medium
g	normal	fair	normal	good	medium
h	normal	poor	high	good	very low
i	high	good	high	fair	very low

Sets which have the same upper approximation and the same lower approximation induced by an indiscernibility space $\langle U, E \rangle$, cannot be distinguished by means of the information in **I**. Under this respect they are equivalent and their equivalence class is called a *rough set*. The family of all rough sets is called a *Rough Set System*, denoted by $RS(U/E)$.

Example 2.5. Approximation operations and rough sets
(a) Rough sets

In Example 2.3, $\{\{c\}, \{a\}\}$ is a rough set belonging to the Rough Set System $RS(U/E_A)$, because $(lE_A)(\{c\}) = (lE_A)(\{a\}) = \emptyset$ and $(uE_A)(\{c\}) = (uE_A)(\{a\}) = \{a, c\}$ and no other set has these approximations. The family $RS(U/E_A)$ of all rough sets is: $\{\{\{a\}, \{c\}\}, \{\{b, a\}, \{b, c\}\}, \{\{d, a\}, \{d, c\}\}, \{\{b, d, a\}, \{b, d, c\}\}\}$, plus the set $\{X\}$ for any $X \in \mathbf{AS}(U/E_A)$.

(b) Boundaries

Consider the Information System of Example 2.4. If $X = \{d, e, f, g\}$, then $\mathbb{B}(X) = (uE_A)(X) \cap -(lE_A)(X) = \{c, d, e, f, g\} \cap -\{e, f, g\} = \{c, d\}$. But if X is an element of the foreground Approximation Space $\mathbf{AS}(U/E_{A'})$, then its boundary is empty. Thus X must be describable by some property of $\mathbf{AS}(U/E_{A'})$. Indeed, it is the extension of the (basic) property "*Comfort = medium*". On the contrary, the set $\{h, i\} = [\![$ "*Comfort = verylow*" $]\!]$ is given by $\{h\} \cup \{i\}$, which is given by a disjunction of two atoms of $\mathbf{AS}(U/E_A)$. Actually, $\mathbb{B}(\{h, i\}) = (uE_A)(\{h, i\}) \cap -(lE_A)(\{h, i\}) = \{h, i\} \cap -\{h, i\} = \emptyset$ and $\{h, i\}$ is the extension of the (complex) property "("*Temperature = normal*" $\wedge$ "*Hemoglobin = poor*" $\wedge$ "*Blood Pressure = high*" $\wedge$ "*Oxygen Saturation = fair*") $\vee$ ("*Temperature = high*" $\wedge$ "*Hemoglobin = good*" $\wedge$ "*Blood Pressure = high*" $\wedge$ "*Oxygen Saturation = good*"".)

We conclude that observing is the act of comparing properties.

So a number of questions arises: How can we perform our comparison? What geometry emerges from this contrast? What figures, what bodies, can we recognize?

Rough Set Theory answers these questions in a *logical* way, well-founded on the notion of a "completion".

We shall develop the logical analysis of Rough Set Theory in Part II and Part III. Now we have to explain what is intended of the term "logical".

3   Phenomenology and Logic

3.1   Semantics *vs* Syntax

The main topic of this book is the logico-algebraic interpretation of Approximation Spaces and Rough Set Systems.

Thus three principal questions arise: (a) Is there any perspicuous relationship between phenomenology and logics? (b) What have we to intend with the word "interpretation"? (c) What is the status of logico-algebraic models (or, more generally, of semantics) within formal logics?

These questions have to be answered because, in order to understand what we are going to do, we must correctly situate the logical analysis of Rough Set Theory. For instance, we can legitimately ask whether there is any use in developing syntactic calculi for which our algebraic models (derived from *concrete* Approximation Spaces and *concrete* Rough Set Systems) are complete. Or, on the opposite side, one may wonder whether semantics has a meaning at all for Mathematical Logic. More simply, we can even doubt if a logical interpretation of Approximation Spaces and Rough Set Systems is of any interest for their development.

Not only the three questions above are unavoidable, but one cannot answer them separately. Therefore we hope to be excused by the reader, because we are going to provide an answer within a somewhat unsystematic and extremely synthetic survey of different topics.

The relationship between phenomenology and logic is a "*vexata quaestio*" (we only recall Husserl's work and address the reader to the "Concluding Remarks"). Here we shall narrow down the scope by delimiting the first term, on the one hand, and by discussing some issues about the second term, on the other.

The term "phenomenon", within the limits of the present work, is strictly framed in the process described above: a state of affairs is a phenomenon whenever it can be categorized by means of some parameter grid.

Once this term is fixed, in order to answer the above questions, it should be specified what we intend with "Logic".

This term, as a matter of fact, has some ambiguity in that it can be intended either:

- As the art that makes it possible to distinguish a correct argumentation from an incorrect one

  or

- As the art that makes it possible to distinguish truth from falsity

In other words, it is not fairly clear if the central notion in Logics is "proof" or "truth", is "syntax" or "semantics". As is well-known, completeness theorems are bridges between the two horns of the dilemma, but this is not decisive for answering the above questions, since we can construct a *syntax* from a given mental *model* of what we are interested in, or can construct a *semantics* from a given *abstract language* that is supposed appropriate for speaking about what we are concerned with. And it is very difficult to privilege one starting point instead of the other. It is also incorrect to say that in view of completeness theorems there is no difference, since this is a mere tautological comment "ex post", when it applies; and sometimes it applies not even.[8] As a matter of fact, what is discussed here is a pre-logical, pre-categorical, representation of our own mental processes and therefore it is not a scandal if we find unresolved options. Exactly as there is no scandal, for instance, in unresolved dichotomies like "grammar/meaning", in semiotics.

Roughly speaking, it is difficult to decide if we have to accept the rules

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta} \ (\wedge - introduction); \qquad \frac{\alpha \wedge \beta}{\alpha} \quad \frac{\alpha \wedge \beta}{\beta} \ (\wedge - elimination);$$

because we accept the truth-table

$\wedge$	0	1
0	0	0
1	0	1

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[8]For instance, we know the syntax of Kleene's Recursive Realizability, but we do not know its models. In contrast, we know the models of Medvedev's Logic, but not its syntax, for the time being.

or we accept this table because the conjunction "and" seems to be currently used as described by the above introduction and elimination rules.

Indeed, this ambiguity runs through the history of modern logics since their origins: although Boole claimed that symbols represent nothing but mental processes and are dispensable in principle, for Frege the linguistic apparatus plays a pre-eminent role if we want (as we should, according to him) ban from Logic any contamination of perceptive and psychological categories.

The Fregean setting, that we call *"formal"*, widely influenced researchers in the XX century, in any field. So, the Boolean point of view, that we shall call *"conceptual"*, was surprisingly denied, for instance, also by scholars who probably would have got a significant gain from the conceptual approach.

Consider the case of Piaget: he insisted on separating Logic from the operational-conceptual structures elaborated during his researches, justifying this position by a methodological assumption: the separation of a *"normative epistemology"* from a *"genetic epistemology"*. By definition, a "normative epistemology" does not deal with the activities of the cognitive subject but tries to determine the most general norms that truth-conditions rely on, for a given knowledge domain; dually, a "genetic epistemology" does not deal with any condition normalizing the notion of "truth", but tries to determine the activities which eventually allow the subject to construct truth norms (see Piaget [1957], page 24).

This methodology was clearly influenced by Frege's point concerning the separation of psychology and logic and it is justified only within the framework of the formal approach and its claim that logic is Classical Logic *tout court*. Thus, Piaget's explicit target was to explain how normative and genetic epistemology converge. Nevertheless his ideological apparatus prevented him from trying to logically (although partially and not classically) interpret the operational-conceptual structures that he was using in order to formalise the intermediate steps of the mental evolution of children.[9]

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[9]For instance, Piaget's so-called First Stage Logic is characterised by the fact that 5–6 years old children do not control inverse operations, yet. For example, they are able to split a set B of black or white pearls into the set A of black and the set A' of white pearls and compare them. But they are not able to compare A or A' with B, since B has been irreversibly "destroyed" by this splitting and cannot

For Piaget, as for Frege, logic is Classical Logic and this logic is the natural accomplishment of the (genetic) mental development of a child.[10] This target is achieved by means of subsequent filtrations of those less mature operational-spatial structures that are acquired by a child during her/his mental development and that do not as yet fit the perfect architecture of Classical Logic. In order to achieve this goal,

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be recovered from the join of A and A' (see Piaget [1950], Part I, Section 1, § 3). It seems that although A is the complement of A', nonetheless the splitting move makes children consider A and A' separately, one at a time. Thus, in a sense, this separated perception has the same effect as adding a non void topological border between A and A'; therefore, recalling that $-A \cap -\mathbb{B}(A) = \mathbb{I}(-A) = \neg A$, we can model this situation by means of the equations $A = \neg A'$, $A' = \neg A$ (hence $A' = \neg\neg A'$ and $A = \neg\neg A$).

Henceforth, in our terminology, although they recognize that A and A' are complementary figures in the universe B, nevertheless they are not able to apply the operation $\neg\neg$ to the join $A \cup A'$ (that is, $A \cup \neg A$) in order to obtain B. Indeed Piaget registers that at this age $\alpha \vee \neg\alpha \neq 1$, but he does not try to connect this particular failure of the excluded middle to the topological modeling of intuitionistic logic, to its related notion of a creative subject and to Gödel-Glivenko results such as $\neg\neg(\alpha \vee \neg\alpha) = 1$. According to this non-classical reading, one could argue that the unity body B is unrecoverable from A and $\neg A$, because although both A and A' are regular, nonetheless $A \cup A' = (\neg\neg A \cup \neg\neg A') \subseteq \neg\neg(A \cup A') = B$. Hence, this typical Intuitionistic inequality might interpret the difficulty, for 5–6 years old children, to apply a completion operation to complex objects.

Other operational characteristics of this age could be analysed in terms of structural rules. This is the case of the so-called "join by contiguity and subtraction": if the set C is split into the subsets B and B', and in turn B is split into A and A', then the join of A and B', $A \cup B'$, is given only by means of the operation $C \cap -A'$. That is, "C *but not* A'", since classes are formed by children only thanks to the presence or absence of a "quality" (*ibidem*). Now, we can notice that a general form for $C \cap -A'$, is an operation $a \otimes b$ defined by $min(b, 1-a)$ (with "*min*" given by some order) that is formally neither associative nor commutative. Thus it would be interesting to understand how children, in their development, transform $\otimes$ into an associative and commutative operation similar to $min(a, b)$, using only $+$ and $-$. This could be interpreted as the adoption of a sort of exchange rule (exchanging the position of the operation $-$), which is a structural rule, not a logical rule.

[10]Piaget was aware of the Intuitionistic approach, but he considered its constructive point of view only within the analysis of the genesis of the concept of a "number" (and he rejected it). Moreover, we can notice that Piaget's conception of Classical Logic as a *normative* logic, does not contrast Frege's claim that Classical Logic is a *descriptive* logic: the "third" (Platonistic) realm, between the perceptive and the physical ones, which, according to Frege, is described by Classical Logic, is substituted in Piaget's work with the *normal* "fine tuning" of human conceptual capabilities, the outcome of a child's *normal* mental evolution. Of course, Piaget cannot accept an "a priori" norm, but only a phylogenetic one. Nonetheless, Piaget's "ex-post" norm is, roughly speaking, nothing else but Frege's "ex ante" mental world.

mankind uses powerful instruments: the abstraction from contingent details and the idealization of the concrete elements of the fundamental structures that model human mental capabilities progressively.[11]

Although in the Formal approach models are claimed to be independent, nonetheless technically they do not have any independent existence; on the contrary they are generated by means of manipulations of the syntax. In a sense, models are just metaphors of syntax.[12] This point of view made the development of the so-called "quantitative proof theory" possible. The first season of modern logic is characterized by this approach, according to which *what* is provable and *to what extent* are the important things to understand. That is, logicians had to investigate two parallel bi-partitions: "theorems/non theorems", "valid formulas/non valid formulas", since the target, at the beginning of modern Formal Logic, was to know the *extension* of the power of formalism. And we had the first important results: the completeness and the (non) categoricity theorems for First Order Logic.

But when theories were taken into account, instead of pure logical systems, when the notion of an "intended model" was considered (for instance, naive Set Theory or elementary Arithmetic), the two notions of "proof" and "truth" suddenly diverged: the problem of the Continuum Hypothesis and Gödel's incompleteness theorems opened a drastic reflection about the possible ineffectiveness of pure formalism.

The Formal approach can be criticized in different ways: (A) *Radical constructivism.* As is well-known, the Dutch mathematician L. E. J. Brouwer denied any legitimacy to the mathematical linguistic apparatus, since, according to him, it does not have any actual role in mathematical creation. On the contrary, sometimes it may even be an obstacle, since it may breed antinomies. Henceforth, Classical Logic and Formalism do not have either a descriptive or a normative dignity. (B) *Neo-conceptualism.* The linguistic apparatus is accepted, but it must be intrinsically connected with the mathematical structures we are interested in: "*The essential role of a theory is to describe its models*" (W. Lawvere).

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[11]This activity resembles the usual filtration of a Heyting algebra by means of the filter of its dense elements. As is well-known, also in this case we obtain a Boolean algebra isomorphic to the Boolean algebra of the *regular elements* of the given Heyting algebra. And this structure is obtained by *simplifying* the topological relations of the dual space of the original Heyting algebra (for these notions see Part II).

[12]Of course, metaphors are very useful in knowledge creation.

In order to develop Brouwer's ideas, A. Heyting introduced a particular semantics: the meaning of a formula is any deduction ending with that formula.[13] Thus, the meaning of a formula is, strictly speaking, a syntactic object. So in order to avoid the short-circuit "syntax-syntax" we need a very refined notion of calculus, in which one can develop operational concepts like that of "extraction of information from a proof". Following this issue, one finds the central role played by Gentzen's normal proofs, that is, *continuous* proofs which can be read backwards since there are not jumps (cuts) from the assumptions to the conclusion. Linear Logic is a recent development of this program, that can be named "qualitative proof theory": now the target is to understand *how* one deduces (see Girard [1982].) In a sense, the manifesto of this program sounds like "The real meaning of syntax is inside the internal harmony of syntax itself". In principle semantics is avoidable, because it does not add a real value (and, even worst, sometimes it is conceptually misleading).

On the other hand, Neo-conceptualism developed an impressive number of model-oriented frameworks, linked by fundamental concepts such as that of an "adjoint functor", each one based on the idea that "models" are first off all, and before being models, rich mathematical structures possibly with an intrinsic particular logic.[14] In a sense, the syntactic apparatus follows its model. We illustrate this point by means of the following example of a process: (a) first we assume that presheaves (of functions over a topological space) are the mathematical objects we are interested in; (b) then we decide that presheaves have to be logically interpreted; (c) finally, by means of our analysis we discover that presheaves are models of Intuitionistic Logic. We may infer, dually, that presheaves are describable by means of Intuitionistic Logic so that we can call this process the "elicitation of the logic of presheaves".

However, saying that presheaves intrinsically have an intuitionistic logic is a sentence that sounds, to us, too much unbalanced towards

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[13]So, for instance, the meaning of $\alpha \wedge \beta$ is any proof ending with $\alpha \wedge \beta$, e.g. $\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$.

[14]This position reminds us Karl Marx's remark that "*any specific object has its own specific logic*". Actually, this principle guided some researchers towards deep results, according to the Neo-conceptualist approach (cf. Lawvere's work).

the ontological side. We prefer to interpret this process by saying that presheaves fulfill structural features that are shared by Intuitionistic Logic. Similarly, Boolean algebras have the same structure as (the Lindenbaum algebras of) Classical Logic and we shall see that in the framework of Rough Set Theory, Boolean algebras are used to model complete information systems, in contrast with three-valued algebras which in the same framework model incomplete information systems. Nonetheless, it can not be claimed that Classical Logic *is* the logic of complete information. In a framework different from Rough Sets Theory, this claim may be falsified from an algebraic point of view (see, for instance, how Boolean algebras may model not completely defined objects in Scott-Solovay proof of the independence of the Continuum Hypothesis). Moreover, the above claim is also incorrect from a strictly logical point of view.[15]

Therefore, although within the scope of this book "Boolean algebra" is synonym of "exactness", nevertheless we have to take great care if we are to generalise this situation.

3.2   Information and Interpretation: Correspondence Theory of Truth *vs* Pragmatism

Although the Fregean conception is justified by a picture of Logic as a descriptive science, on the opposite side it led to a paradoxical result: the actual non-distinction between derivability and validity, syntax and semantics, inference and truth. This is because, in this setting, semantics as completely subordinated to syntax is a mere tool for mining results in logical theories. As a matter of fact, if syntax describes something, it describes a linguistic variant of itself. The most

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[15]Assume there are three aligned boxes, A, B and C. We know that A is green and C is blue. We do not know the color of B. We want to know if there is a green box near a non-green box. Using Classical Logic we can solve the problem: if B is green, then B is a green box near a non-green box, namely C. If B is not green, then A is a green box near a non-green box, namely B. Thus, the answer is "Yes".

According to Moore, here we use the following features: (a) the ability to prove that an existential predicate $A(x)$ is true without knowing which term ("object") t makes $A(t)$ true; (b) the possibility to say that for any sentence A either A is true or $\neg A$ is true; (c) the ability to reason by cases.

The first two features are strictly related to Classical Logic (cf. Moore [1982]).

visible result of this confusion, namely the identification of *meaning* and *truth-conditions*, opened up a measureless distance between logic and semiotics.

Indeed, semioticians are wary of Formal Logic, since they do not perceive any essential relation between the concept of "meaning" and that of "truth". They claim that as soon as there is a meaning, one is in position to lie, and vice-versa. Otherwise stated, I lie when I utter a sentence that you understand but that it is not true. So that the meaning of a sentence has an existence independent of its truth conditions.

However, after Gödel incompleteness results, besides the so-called "intentional proof theory" (originating from Herbrand and Gentzen), more refined semantics began to be studied, that aimed at tackling those problems where verification criteria (hence referential issues) have a real priority. Therefore, while according to the correspondence theory of truth, the meaning of a sentence is equated to the result of the verification process ("true" or "false"), on the contrary, the alternative semantic criteria stress the importance of the *verification process* itself.

Typical problems that are faced by these semantics are the so-called "referentially opaque" contexts, that is to say contexts in which a sentence is within the scope of some modalised expression, like "to know", "to believe", "it is necessary", "it is possible", "it is provable" and so on ("*The subject X beliefs that Y*", "*There is an X that necessarily enjoys the property P*", etc.).

With a parallel action, constructivistic criticism urgently stated the necessity of a revision of the pair ⟨*syntax, correspondence theory of truth*⟩ in favour of the analysis of the so-called "creative subject" (see Kreisel [1965]).

Through these efforts, Logic stealthily began recovering with an addition of a hitherto neglected dimension that was already present in semiotics: *pragmatics*.

3.2.1 Meaning-conditions *vs* Truth-Conditions

Semiotics distinguishes between *meaning-conditions* and *truth-conditions*. To enter the topic with a paradigmatic case, let us compare Peirce's and Frege's interpretations of the semiotic triangle:

Sense (A)

Sign (B) $\longrightarrow$ Denotation (C)

Semiotic triangle

Interpretant (A) Sinn (A)

Representamen (B) $\rightarrow$ Reference (C) Zeichen (B) $\longrightarrow$ Bedeutung (C)

Peirce's triangle   Frege's triangle
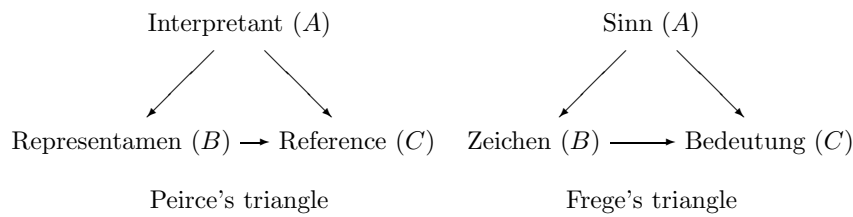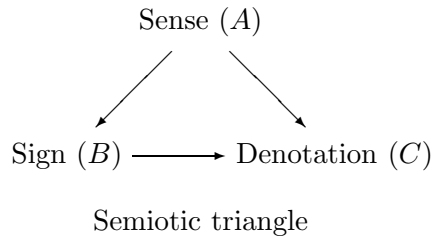
Semiotics, as the theory of the functions of sign, is concerned with the relation between A and B. However we can notice that, according to Peirce, "this three-relative inference is not solvable by an action between pairs, at all". So we need C. But one cannot accept C if it is intended as an extensional term instead of an intensional one. On the contrary, the extensional reading of C is the basis of Frege's truth-functional interpretation of sentences. More precisely, in Frege's version C is a set and our interpretation reflects our ability to evaluate the characteristic function of C *via* A. Otherwise stated, we have the following interpretation function: $A : B \longmapsto C$. On the contrary, the intensional reading of C requires a somewhat less comfortable analysis of the semiotic triangle, as is suggested by Peirce's "theory of the interpretant". In accordance with this point of view, U. Eco explains that a Fregean *Bedeutung* is captured *via* the series of its proper *Sinnen.* Hence, a semiotic triangle is not just a simple commutative diagram made up of a Sign, a Sense (or Connotation) and a Reference (a Denotation). Since a Bedeutung itself must be interpreted, thus it requires a new triangular process. We can give a mathematical interpretation of Eco's remarks. In a sense, we can say that the interpretation function is parameterized by the "interpretant", so that we obtain $i_A : B \longmapsto C$, which is tantamount to a function $i : A \times B \longmapsto C$. However, i is not a term of the triangle, but the unlimited interpretation process itself, the process that links all the three terms. Since this process behaves as
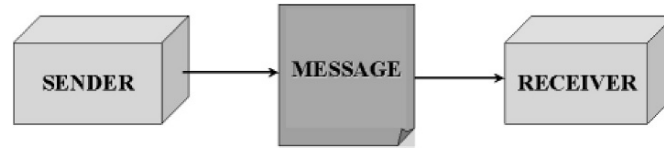
Figure 10: Structuralist pouring model

a cultural reference for the interpreter, then C can be intended to be a "denotation" only naively, since as a matter of fact it is a "cultural unity" (something that culture defines as a unity distinguished from other unities). According to this conception, a meaning is not unacceptable because we cannot understand it (for example "if snow is made of chocolate, then dogs are mammals"), "*but it is unacceptable because if it were, we should re-organize our rules of comprehension.*" (See Eco [1975], § 2.5).

If this is true, we face an *unlimited semiosis* that circumscribes cultural unities, by means of a sequence of approximations. The "final interpretant" (which justifies the Fregean *Bedeutung*, actually) is a passage to the limit.

Although it is not that evident what morphisms between *Bedeuntung* can be defined, surely they are transformations of a "state of affairs" into another. If $x \in C$ is a fixpoint under such a transformation, then it fulfills some kind of *invariance property* that makes it look very closely like an "object". For the time being we do not have sufficient elements for carrying on this modeling of the "unlimited semiosis" further, but we think that it is evocative enough for suggesting that the "unlimited semiosis" cannot end without somehow giving the "reference" (C) the status of an "object". This asymptotic interpretation process is what makes it possible the comunicative use of signs for referring to things: it does not require to be resolved into a physical or Platonistic entity (the "third realm" envisioned by Frege).

This also means that we must revise the usual pouring model in favour of the *poiesis-aisthesis* concurrent activity described in Figure 10 above.

Indeed, the "final sign" is not really a sign, but it is the "*entire semantic field as a structure which connects signs to each other*" (U. Eco, *ibidem*, § 2.7).[16]

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[16]If this asymptotic limit were regarded as an actual entity, then it should look like a "peradam", the mythical stone of the "*Mont Analogue*". In this novel, René

In Mathematical Logic, after Heyting's work we started seeing the possibility to understand, to some extent, semantics as a "theory of meaning"; that is to say, as a notion released from the concept of "truth", as we shall see in Part II.

Gentzen's approach is not different: he aimed at releasing the notion of "coherence" from that of "truth". So the point of view of semiotic was assumed, in some form, by the so-called intensional proof theory. Of course, within the limits of the expressive power of First Order Logic.

It is the harmony of the overall architecture of the so-called Natural Calculus that guarantees its own coherence and that makes it possible to speak of proofs as meaningful-in-themselves entities (proof semantics). And it is this harmony together with the constructive features of the system that makes it possible to communicate, through the dialectic "*poiesis* (connective introduction, or right part of a sequent read top-down)/*aisthesis* (connective elimination, or left part of a sequent

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Daumal tells about expeditions that have to conquer the top of a mountain, called the Mont Analogue. This mountain is higher than any mountain, hence higher than itself. But when, after a "non-Euclidean navigation", a team reaches the island where Mont Analogue rises, it suddenly faces the following problem: it has to buy the equipment for the expedition, exchanging them with a particular form of money: the "peradam". The "peradam" is an almost invisible stone, that can be found almost exclusively in high lands. Alternatively the team members can work for getting a more conventional form of money: counters. Thus a number of teams instead of starting climbing, try to gain counters to prepare the expedition that they will never begin, because they have to work.

   As is well-known, the monetary exchange is the typical symbolic exchange, so that this loop seems to point out that any symbol negatively and unsolvably hides an essence. To use Hegel's, and Marx' concepts, it continuously turns from the condition of being an "*Erscheinung*" (a phenomenon of something, a manifestation of an essence) into that of being a "*Schein*" (mere appearance, illusion). As Daumal points out in the working notes for his novel, that he was not able to finish since he died before his time, only who starts climbing anyway, has a possibility to find the "peradam", a "curved crystal" that enjoys the "same refraction as the air", hence a sort of non-Euclidean object, a sort of "object non-object". Thereafter, we say, the "peradam" is a symbol non-symbol, or, rather, it is the symbol of all symbols, the means required to climb the top of the Mont Analogue which, in turn, is the denotation of all denotations (hence the risk of entering a loop when trying to fix this denotation as an actual entity by means of static symbols – the counters). In a sense, who enters the loop is an interpreter *á la* Frege, whereas who starts climbing anyway is an interpreter *á la* Peirce.

read bottom-up)", "*construction/re-construction*".[17] The pillar of this architecture is the Hauptsatz, the cut elimination theorem. It is this result that allows us to go forth and back through a proof, to extract information from it and, therefore, to amend Thom's perspicuous and fatal issue concerning deductions: "*The engine of any logical deduction is the loss of informational content: 'Socrates is mortal' is less informative than 'Socrates is a man'.*" (Thom [1980], Ch. 10, § 10.2, footnote).

The target of intensional proof theory is, in a sense, the famous "final interpretant":

- *Synchronously*: if $\Pi_1, \Pi_2, ..., \Pi_n$ are different proofs of a formula α, then we can intend them as different "readings"of the same inference,[18] but also as different "Sinnen" of the same "Bedeutung". The unique *normal proof* of α is then *the* proof of α, representing all the other proofs. It is the *prototype* of this inference, not just a representative of an equivalence class of proofs (modulo the root formula). It is a *type* whereas $\Pi_1, \Pi_2, ..., \Pi_n$ are *tokens*.

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[17]If the $\wedge$-introduction rule reads "From the contemporary evidence of both α and β we can assert $\alpha \wedge \beta$", then the $\wedge$-elimination rule reads "From the assertion $\alpha \wedge \beta$ we can deduce both the evidence for α and the evidence for β."

"To understand" is then "to deduce", because I understand you if I can recover the conceptual roles of the components of your phrase; and I can infer their conceptual roles if I recognize their positions with respect to the fundamental constructors of the sentence. In its turn, this means that I must be able to eliminate any connective that you introduce. Indeed, this elimination is the evidence that I am able to recover the meaning of the connective (if you say "John *and* are coming back" and I am satisfied, then surely I am not understanding, or listening to you: in fact I have not understood the meaning of the conjunction "and". But if I really understand the meaning of "and", I stop you and ask "John and who?", because I know that the elimination rule for "and" is double and makes it possible to infer "John is coming back" and "(I don't know who, because "and" was ill-used) is coming back, too". Thus, to understand is to accomplish a deduction. For better fitting natural languages (or at least some families of natural languages) the calculus has been refined in order to distinguish left conceptual positions from right conceptual positions and to take into account also the multiplicity of terms. Along this line we have Lambek calculi and Non-commutative Linear Logic – see for instance Abrusci [1991].

Besides the specific connections between Chomski's grammars and logics (see Buszkowski [1988]), the first interpretation of Intuitionistic Natural Deduction as a system describing language competence, appeared in Prawitz [1980].

[18]Cf. Van Benthem [1991], Ch. IV.

Obviously Natural Calculus accounts for a theory of meaning only partially. Indeed, it accounts for a purely structural and relational understanding of "meaning". As such, it is a genuine Sausurrean framework. However, the notion of a normal proof is a fair step towards a system linking all the possible "readings" of a sentence.

- *Diachronously*: the ultimate system cannot be a mono-logical system (classical, or intuitionistic or whatever you like), but a *pluralistic* system, a system in which several codes can communicate.

At this point we have to underline that the term "pluralistic system" has essentially two meanings in contemporary logical researches:

- A monadic system with a plurality of logical contexts inside (this is the "Unity of Logic" program, as derived from Linear Logic – see Girard [1993])

- A plurality of monadic systems linked by a metacontext (this is, roughly speaking, the program that can be founded upon Labelled Deductive Systems and Fibred Semantics – see [Gabbay 1997 and 1996])

In accordance with the latter program we have different systems linked by a logical middleware. In accordance with the former, the target system itself must be a metasystem, at the same time.[19]

3.2.2   Logic, Meaning and Rough Set Theory

Rough Set Theory, *as a theory of perception*, aims at interpreting observations by means of concepts. In this framework, concepts are disjunctions of linguistically describable properties, represented by unions of basic categories (as sketched above).

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[19]It is worthwhile to quote U. Eco again: "*When one speaks of a 'language' as a 'code', one has to think of a wide series of small semantic systems (or fields), which couple with the unities of the meaning system, in different ways. Concerning this point the code starts appearing as [...] the system of the semantic systems and of the rules of semantic combination of the different unities [...]*" (Eco [1968], page 110. Translation by the author). Hence, Eco thinks of the concept of a code as a metacode.

Nevertheless, we must recognize that in general Rough Set Theory is considered, developed and exploited, as a theory of partial information, data mining or knowledge discovery in databases. Generally the notion of "meaning" is not explicitly used by Rough Set theoreticians. But this is just a historical incident, since, generally speaking, this notion cannot be ignored by qualitative data analysis and knowledge discovery. On the other hand, from a technical point of view Rough Set Theory can be developed also in the direction of a theory of meaning and formal ontology (for a general introduction to this topic, see Subsection 5.1).

In any case, both as a theory of *incomplete information* and as a theory of *incompletely defined objects*, Rough Set Theory induces logico-algebraic models that are polymorphic in nature: in the same model we have intuitionistic, co-intuitionistic, classical, modal and three-valued logical environments.

This list is definitely the result of an application of the *Neo-conceptualistic approach*: we have a mathematical object and we elicit its logic (if any). In our case we obtain an unexpected number of different logical behaviours, because we deal with systems that may represent, at the same time both inexact and exact information mixed together (as it actually happens: it is difficult to find completely fuzzy or completely sharp information systems).

Indeed, neo-conceptualism is the declared framework of this book. However, we must underline that neo-conceptualism is adopted as a methodological framework in this book. It is not a philosophical commitment.

So our approach should look far from intentional proof theory. But it is not completely true. Indeed a "mixed radix" system of calculus, as conceivable within the pluralistic approach, could be of great use for such multifarious logico-algebraic systems. This is true, of course, if we think that a logical global calculus is of some interest for Rough Set Theory. But one can also suppose that what one really needs is only an algebraic system and, eventually, a local logical calculus, that is a logical calculus which enables us to infer formulas that are valid in a *particular* model of a given class.

We shall resume the problem of the relationships between logic and language at the end of this Introduction, specifically embedded into the rough set framework. Therefore it has to be explained why we think that Rough Set Theory has a genuine logical flavour.

4   The Logico-Algebraic Interpretation of Rough Set Systems

In logic-literature we find several systems that have been studied in order to generalize Classical Logic because bivalence was considered an unrealistic limitation and/or because principles like the law of *excluded middle* or the *law of contradiction* were considered as being founded on metaphysical assumptions or as being sources of paradoxes.

Although the motivating ideas and purposes were, evidently, different in nature, during the development of these systems the notions of "information" and "knowledge" have been often used as a means to justify their logical architecture.

We can say that at present these notions have gained a citizenship in contemporary Logic thanks to such efforts, although it has been rediscovered as a consequence of the new problems posed by Information Technology and Computer Science.

As we shall see, Rough Set Systems can be represented by more than one non-classical system and, surprisingly enough, by systems with different and sometimes contrasting properties. For instance, some of them present intermediate values while some others do not. Some present a chain of values while some others do not.

One can legitimately ask whether this contradictory picture is a symptom of the fact that these logical interpretations of Rough Set Systems is just formal but not substantial in character, or we can try and see whether, in view of the heuristic intuition recalled above, these relationships can be explained in terms of the notion of "information". Indeed, we shall see that an intermediate value has a *non-metaphorical* information content, where *information* is intended to be as a *concrete organization* of *data*.

In Part II we shall investigate the deep connections between these informational features and the algebraic properties of Rough Set Systems.

In the first part we shall analyse the *local* logical behaviours suggested by the inherent philosophy of Rough Set Theory as related to the notion of information "granularity" or "quality/precision", as well as the logico-algebraic systems induced by such local behaviours. With the term "local behaviours" we intend to mean that one can observe

certain logical relationships to hold in some parts of a domain, while in other parts quite different relationships hold.

Indeed, this "mixed behaviour" depends on the information embedded in a given Rough Set System and results in the *polymorphism* of Rough Set Systems as logical models.

Therefore, we are not seeking a "Rough Set interpretation" of some given logic, but, on the contrary, according to the conceptual point of view, we are to explore in the opposite direction, that is, finding the logic which is inherent to Rough Set Systems.

Given a universe of discourse U, according to Rough Set Theory the basic gnoseological act is the organization of U into disjoint categories. These categories are induced by our observations and the properties that correspond to them are the gnoseological co-ordinates explaining the universe U, that is, they are the starting points in order to organize U from a cognitive point of view.

It is this "gnoseological geometry" which induces the aforementioned polymorphism of Rough Set Systems.

This polymorphism is linked with important problems related to Logic in a broad and general sense which induces the necessity to make classical and constructive logics coexist.

Indeed, a sentence may be regarded as either bearing information which deserves to be constructively analysed or as bearing information about facts which need not be analysed.

It will be proved that from both a philosophical and a mathematical point of view the "mixed behaviour" of Rough Set Systems is a special case of a pluralistic approach in Logic.

We know that given an Indiscernibility Space $\langle U, E \rangle$, an Approximation Space $\mathbf{AS}(U/E)$ is a subalgebra of the Boolean algebra of sets $\wp(U)$. With a slight abuse of terminology from the "concrete" level, we call any element of $\mathbf{AS}(U/E)$ an *exactly describable subset* or an *exact subset*, and its atoms *basic subsets*. Basic subsets are extensions of basic properties, so that exact sets are disjunctions of basic properties.

Moreover, we have seen that given an arbitrary subset $X \subseteq U$, we have two possibilities: either X is exact or not. In the first case X is either a basic subset or a disjunctions of basic subsets.[20] In the

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[20]Strictly speaking, in both cases X is the disjunction of basic subsets, evidently, because $X = \bigcup X$.

second case we cannot use basic subsets for a direct description of X, but we can approximate it by means of an *upper approximation*, $(uE)(X)$, and a *lower approximation* $(lE)(X)$. We know that the former coincides with the topological closure $\mathbb{C}(X)$ and the latter with the topological interior $\mathbb{I}(X)$ of the space $\langle U, \mathbf{AS}(U/E) \rangle$, intended as a topological space.

Finally, we have seen that two sets which have the same upper and lower approximations are to be considered equivalent: they cannot be distinguished using our description capabilities. Any such equivalence class is called a *rough set*.

We can give a logical interpretation to this machinery: $(uE)(X)$ is the set of elements which *possibly* belong to X because they fulfill the same set of properties fulfilled by some element which actually is in X; on the other hand $(lE)(X)$ is the set of elements which *necessarily* belong to X since there are no elements outside X which fulfill the same set of properties. This modal interpretation will be analysed in great details in Part III.

If X is an element of the Approximation Space $\mathbf{AS}(U/E)$, then $X = (uE)(X) = (lE)(X)$: its description is "perfect". Topologically speaking, the *boundary* $\mathbb{B}(X) = (uE)(X) \cap -(lE)(X)$ is empty in this case. Actually, a boundary is a region of doubt: if $x \in \mathbb{B}(X)$, then we can say nothing certain about the membership of x in X. We cannot say either that x is certainly (necessarily) in X, or that x has certainly nothing to do with X: in fact it *could* belong to X, since it is indiscernible from some element of X; but actually it does not.

Therefore if one wants to grasp this situation one has to generalize the classical two-valued characteristic function turning it into a three-valued function. But this generalisation works in dependence on the separation properties fulfilled by the topological space $\langle U, \mathbf{AS}(U/E) \rangle$.

In the "concrete" setting, this depends on the granularity of our knowledge which, in turn, depends on the level of accuracy of our information with respect to the given set of objects. We have the best separation properties when $\mathbf{AS}(U/E) = \wp(U)$. In this case the resulting topology is the *discrete* topology (that is Hausdorff and completely disconnected) and one can single out each element of U. In a sense, we have enough basic properties for "naming", or "labeling", any single object which, indeed, is isolated by a singleton of $\wp(U)$. On the

contrary, when all the objects are indiscernible, one obtains the *trivial* topology: $\mathbf{AS}(U/E) = \{\emptyset, U\}$.

However, usually intermediate cases will be given in which some elements can be singularly "named", while others cannot be singled out by means of the information at our disposal: in general in $\langle U, E \rangle$ some equivalence classes are *singletons* while others are not.

Let us denote the family of the equivalence classes that are singleton by B^*, and with P^* the family of the equivalence classes that have cardinality strictly greater than 1. B^* and P^* do not have the same logical role in the construction of a rough set system. In fact the elements in B^* are "exact" in nature and they should enjoy the two principles of Classical Logic reflecting bivalence and exactness: the *excluded middle* and the *contradiction principle*. Indeed, given a set X and an open (basic, of course) singleton $\{s\}$, either $\{s\}$ is included in $(lE)(X)$ or it is included in $-(uE)(X)$. On the contrary any basic open set with at least two elements may be included in the boundaries of at least two different sets. Therefore, if there are no singletons in $\mathbf{AS}(U/E)$, then there are at least two sets X such that $(uE)(X) = U$ and $(lE)(X) = \emptyset$ (think of any set which picks up exactly one element out of any basic open set). From the point of view of Approximation Space theory, such an X is called an *undefinable set*. Obviously, if X is undefinable then $\mathbb{B}(X) = U$.

Thus we may think that for any Rough Set System there are two distinct *local* logical behaviours: one is classical and localized on $B = \bigcup B^*$, whereas the other one, localized on $P = \bigcup P^*$, is purely three-valued. It is the combination of these local behaviours that defines the overall logical features of the system. It follows that the construction of $RS(U/E)$ will depend on the parameter B (or P, which is the same because $B = -P$).

Moreover, in $RS(U/E)$ any rough set induced by an element of the Approximation Space $\mathbf{AS}(U/E)$, must have a particular logical behaviour too: such an element corresponds to an exactly definable subset of U, hence, again, it should fulfill Classical Logic, but within a logical environment which might be three-valued.

Thus we have two levels of local logical behaviours: one is related to the internal definition of rough sets, the other deals with the global logical properties of Rough Set Systems.

The first completely depends on the parameter B (or P). These parameters cannot be recovered from the "geometrical" shape of the

Approximation Space $\mathbf{AS}(U/E)$.[21] It follows that an inspection of the atoms is unavoidable in order to define $RS(U/E)$. Because the information provided by this inspection does not have any lattice-theoretic content, we call B and P *external parameters* or *empirical parameters* and say that they are able to distinguish the *external classical local behaviour* within an Approximation Space.

On the contrary, we can analyse the logico-algebraic structure of $RS(U/E)$ from a pure abstract point of view. In fact, also in this case we have to use a particular parameter, but curiously enough, though this parameter is induced by B, nevertheless it is definable in $RS(U/E)$ by means of a mere lattice-theoretic definition. For this reason we call it an *internal parameter* and we shall see that it distinguishes the *internal classical local behaviour* within a Rough Set System.

In this Chapter we shall analyse both local behaviours by exploiting the mathematical notions that best manage the concept of "being locally the case that", namely *Grothendieck topologies* and *Lawvere-Tierney operators*.

5   Equivalence Classes, Abstraction and Meaning

5.1   Types, Tokens and Abstract Points

So far we have seen that the notion of an equivalence class is exploited in order to account for some abstraction mechanisms in concept formation. It is time to investigate this move in some more details and try to embed it in a more general setting.

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[21] At least we have to compare the cardinality of U with the number of atoms of $\mathbf{AS}(U/E)$. More precisely, if the cardinality of the universe U is n and $\mathbf{AS}(U/E)$ has n atoms, then any atom is a singleton. It immediately follows that $B = U$ and $P = \emptyset$ and we can straightforwardly construct $RS(U/E)$ by means of the techniques described in Part I. If $n \geq 2$ and $\mathbf{AS}(U/E)$ has $n - 1$ atoms, then there are $n - 2$ atoms with cardinality 1 and one atom with cardinality 2. Using this information, it is possible to define a lattice $\mathbf{L}$ *isomorphic* to the rough set system $RS(U/E)$: we can randomly choose an atom x of $\mathbf{AS}(U/E)$ and use it as it were the parameter B, since the elements of x are immaterial in the construction of $\mathbf{L}$. But if $n \geq 3$ we obtain $\mathbf{L} = RS(U/E)$ only by chance, because in order to have identically the equality, we should inspect $\mathbf{AS}(U/E)$, discover the unique non singleton atom and use it as parameter. If $\mathbf{AS}(U/E)$ has $n - 2$ atoms for $n \geq 4$, we cannot even know how many singletons there are.

First of all an equivalence class stands for a *type* of objects (*tokens*). The problem "types *vs* tokens" is a long stated issue in philosophy. In Modern Age we found it since the *querelle* between Locke and Berkeley. In fact, Berkeley doubted that there could be a *generic* representative x of a concept C. That is, Berkeley claimed that if x is a representative of a concept C, then x itself is a *specific* entity belonging to that concept. For example, it is impossible to think of a "generic triangle" without thinking of a "particular triangle" (equilateral or right-angled, and so on). Thus any attempt to provide a generic instance of a type collapses into a mere token.

This problem entered modern logics through Frege's and Russells's works. The "extensional solution" to the problem is to consider equivalence classes $[x]_{\equiv}$ (for some equivalence relation $\equiv$) and choosing a representative y of $[x]_{\equiv}$. Since $y \in [x]_{\equiv}$, this solution is in accordance with Berkeley's vision. The choice of y instead of another element of $[x]_{\equiv}$, is just a matter of *convention* and this relative freedom is supported by the well-known isomorphism theorems and their corollaries, to the extent that $\equiv$ is a congruence relation with respect to the operations we are interested in (independence of the operations of the particular choice, in the quotient structure). But we can illustrate a different and more refined solution, borrowing some techniques from *Pointless Topology*. This solution, in a sense, meets both Locke's and Berkeley's conception.

In Pointless Topology, a family of open subsets of a topological space is considered from a very abstract point of view. In fact, we forget points. What is taken into account is just the abstract algebraic structure, called a *frame*, induced on the family of open sets by their fundamental relationships so that the most relevant relationship between frames is that of *homomorphism*. The algebraic structure of a frame has the following intuitive motivation:

1. Any *usual* open set a of a topological space $\langle U, \Omega(U) \rangle$ (where $\Omega(U)$ is the family of open subsets of U) collects all the objects in U which are near each other with respect to the vicinity relation that is set by the topology $\Omega(U)$. If the nearness relation is interpreted as "similar behaviour with respect to a property $\mathcal{P}$", then a may be thought of as the extension $[\![\mathcal{P}]\!]$ of $\mathcal{P}$.

   Moreover, we are not speaking of generic properties, but of "observable properties". Observable properties fulfill the following principles, that are independent of the nature of points:

2. One can check any disjunction of observations. That is, if x is an element of our universe and $\mathcal{P}$ is an observable property, one can affirm that $x \in \llbracket \mathcal{P} \rrbracket$ (x is in the extension of $\mathcal{P}$, x enjoys $\mathcal{P}$), even if $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \vee ... \vee \mathcal{P}_n \vee ....$ Hence we allow infinite disjunctions, because the first positive observation about $\mathcal{P}(x)$ stops the search. In particular we can check $\bigvee \emptyset$, that is the smallest property.

3. We can check only a finite amount of observations, hence we allow only finite conjunctions of properties. In particular we can consider $\bigwedge \emptyset$, that is the largest property.[22]

4. In order to check $\mathcal{P} \vee (\mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge ... \wedge \mathcal{Q}_i)$, we have to check $(\mathcal{P} \vee \mathcal{Q}_1) \wedge (\mathcal{P} \vee \mathcal{Q}_2) \wedge ... \wedge (\mathcal{P} \vee \mathcal{Q}_i)$; from above it follows that we allow that disjunction distributes only over finite conjunctions.

5. Dually, conjunction distributes over arbitrary disjunctions.

Indeed these are the axioms for any frame of open subsets of a topological space. Generalizing them and forgetting points, we say that:

Definition 5.1. *A* frame *is a lattice bounded by a top element and a bottom element, with infinite disjunctions, finite conjunctions and corresponding distributive laws.*

Henceforth, any frame **F** can be considered as a system of observable properties *tout court, without any reference to points.*

Starting from this consideration, in Pointless Topology a radical phenomenological point of view is assumed: we only perceive properties while substances (objects, points) are "noumena".

This interpretation of Pointless Topology, first appeared in Computer Science literature, within the researches on Denotational Semantics of programming languages:

"*Intuitively, the idea of a computable property is simply this: we have a uniform procedure that, given (a code for) x, tells us within a finite time that*" $\mathcal{P}(x)$" holds, whenever that is true. Of course this is just the idea of semi-decidability.

(...) *a specification of an object (say a program) is a (finite or countable) list of properties that the object is to satisfy. In view of our identification*

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[22]$\bigvee \emptyset$ is the smallest element of the set $\{x : \text{for all } y \in \emptyset, y \leq x\} = \{x : (y \in \emptyset)$ implies $(y \leq x)\}$. But this set equals U, because the premise of the implication is false. Thus $\bigvee \emptyset$ is the smallest element of U. Dually for $\bigwedge \emptyset$.

of properties with open sets, this means that what is specified is always a countable intersection of open sets [more precisely] the computable properties will, rather, be the basic open sets and effective unions of them.

(...) If we really think of the (basic) open sets of a space as the fundamental properties of interest in the space, then, presumably points having the same neighborhood should not be distinguished. We thus require spaces to have the T_0 separation property.

(...) A more radical position would be that, since we can be concerned only with the (ascertainable/computable) properties of points, points should be treated as logical constructions out of properties. Points, in this approach, will be mere 'bundles of properties'." (Smyth [1983]).

We now shall exploit this point of view in order to suggest a possible solution to the "generic element" problem, above discussed. In fact we shall construct "generic elements" in three steps, starting from an abstract frame.

STEP 1 [From Denotational Semantics]:

- Consider just what you effectively have: the phenomena (or observations) and their relations that arrange phenomena into a frame **F**. Thus, consider **F** as an abstract structure, not a family of sets of objects.

- So, any $a \in \mathbf{F}$ is a property whose extension is filled by unknown "noumena".

STEP 2 [From Locke's Naming Principle]:

- Since our thought retains that qualities (that is, "properties" in our terminology) cannot subsist "sine re substante" (without some-thing underlying it), we can assume that qualities define substances. This means that those "noumena" whose qualities go always together (i.e. are indiscernible) must be named by a single name (cf. Locke).

- Under this point of view, we have the equation SUBSTANCE= BUNDLE OF PROPERTIES.

STEP 3 [From Pointless Topology]:

- Bundles of properties define abstract points. Hence ABSTRACT POINTS are particular CHARACTERISTIC FUNCTIONS of their QUALITIES. Thus any abstract point p is represented by a particular functions $\hat{p} : \mathbf{F} \longrightarrow \{0, 1\}$:

- Any function $\hat{p}$ has the intended meaning:

$$\hat{p}(a) = \begin{cases} 1 & \text{if } p \text{ satisfies } a \\ 0 & \text{otherwise} \end{cases}$$

- How to guarantee this intended meaning? Clearly $\hat{p}$ must preserve all the relationships between phenomena, hence all frame operations. For instance if p satisfies the complex property "a and b", then p must satisfy *both* property a *and* property b; this means that $\hat{p}(a \wedge b) = \hat{p}(a) \wedge \hat{p}(b)$. Similarly for $\vee$, the null property $\perp$ and the top property $\top$. In other words $\hat{p}$ must be a $\{0, 1\} - homomorphism$ between $\mathbf{F}$ and $\mathbf{2}$ (the two element Sierpinski frame $\langle \{0, 1\}, 0 \leq 1 \rangle$. Thus,

$$\text{SUBSTANCES} = \mathcal{HOM}(\mathbf{F}, \mathbf{2}).$$

   (the set of $\{0, 1\}-$homomorphisms from $\mathbf{F}$ to $\mathbf{2}$).

- If $\hat{p} \in \mathcal{HOM}(\mathbf{F}, \mathbf{2})$, then the inverse image of its true-kernel (that is, the inverse image of 1 along $\hat{p}$) is a principal filter $\uparrow p = \{x \in \mathbf{F} : p \leq x\}$ generated by a *co-prime* element p of $\mathbf{F}$.
   A *co-prime element* is an element belonging to the set $\mathcal{J}(\mathbf{F}) = \{a \in \mathbf{F} : \forall S \subseteq \mathbf{F}((a \leq \bigvee S) \Rightarrow (a \geq s \text{ for some } s \in S))\}$. For this reason, co-prime elements are also called *join-irreducible* elements: they cannot be reached by means of a disjunction of strictly smaller elements. Now, notice that if $a \lneq c, b \lneq c, c = a \vee b$ and $\hat{p}(c) = 1$ but neither $\hat{p}(a) = 1$ nor $\hat{p}(b) = 1$, then h is not a homomorphism, because in the Sierpinski frame $\mathbf{2}$, $x \vee y = 1$ if and only if $x = 1$ or $y = 1$. But if $\hat{p}(a) = 1$ or $\hat{p}(b) = 1$, then c is not the least element of $\hat{p}^{-1}(1)$, because $a \lneq c$ or because $b \lneq c$. Henceforth, the element generating $\hat{p}^{-1}(1)$ is always join-irreducible, for any homomorphism $\hat{p}$ belonging to $\mathcal{HOM}(\mathbf{F}, \mathbf{2})$.

- Since there is a bijection between $\{0,1\}$-homomorphisms and co-prime elements, we can consider $\mathcal{J}(\mathbf{F})$ as the set $\mathbf{pt}(\mathbf{F})$ of abstract points of $\mathbf{F}$:

$$\mathbf{pt}(\mathbf{F}) = \mathcal{J}(\mathbf{F}).$$

- Now, for any property $a \in \mathbf{F}$, we have to know the abstract points belonging to its (abstract) extension. Since any such extension is a subset of abstract points, to obtain this information we have to apply some function, ϕ, mapping $\mathbf{F}$ to $\wp(\mathbf{pt}(\mathbf{F}))$. This function, again, must preserve the intended meaning of the abstract points. That is, ϕ must be dual to the construction of the abstract points. Thus, the definition of ϕ is the following:

$$\phi : \mathbf{F} \longrightarrow \wp(\mathbf{pt}(\mathbf{F})) : \phi(a) = \{p \in \mathcal{J}(\mathbf{F}) : p \text{ fulfills } a\}$$
$$= \{p \in \mathcal{J}(\mathbf{F}) : a \in \hat{p}^{-1}(1)\}$$
$$= \{p \in \mathcal{J}(\mathbf{F}) : p \leq a\}$$

Hence $\phi(a) = \{p \in \mathcal{J}(\mathbf{F}) : a \in\, \uparrow p\}$.

If $\mathbf{F}$ is finite, then ϕ is always an isomorphism between $\mathbf{F}$ and $\phi(\mathbf{F})$ and this procedure is essentially the core of Birkhoff's duality result for finite distributive lattices.

Here is a simple, but clarifying, example.

Example 5.1. Pointless topology and abstract points
Consider the following frame $\mathbf{F}$:
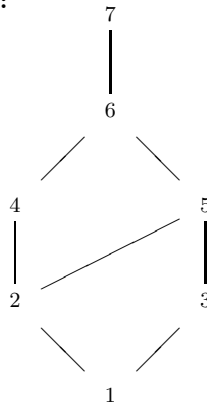
One can notice, for instance, that the map $\hat{p}$ such that $\hat{p}(2) = \hat{p}(4) = \hat{p}(5) = \hat{p}(6) = \hat{p}(7) = 1$ and $\hat{p}(1) = \hat{p}(3) = 0$ is a $\{0,1\}$-homomorphism from $\mathbf{F}$ to $\mathbf{2}$. The pre-image $\hat{p}^{-1}(1) = \{2,4,5,6,7\}$ is a principal filter of $\mathbf{F}$. The element generating $\hat{p}^{-1}(1)$ is 2. Along this way, we find that the set of the elements generating the true-kernels of all the $\{0,1\}$-homomorphisms from $\mathbf{F}$ to $\mathbf{2}$, $\mathbf{pt}(\mathbf{F})$, is

```
          2
          |
          |
          4              3
           \            /
            \          /
               7
```
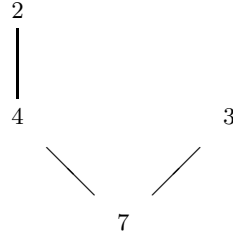
Notice that on $\mathbf{pt}(\mathbf{F})$, the order $\preceq$ is given by $x \preceq y$ if $\uparrow x \subseteq \uparrow y$.

Now let us apply the map $\phi : \mathbf{F} \longrightarrow \wp(\mathbf{pt}(\mathbf{F}))$; for instance to the element 5:

$\phi(5) = \{p \in \mathbf{pt}(\mathbf{F}) : p \text{ fulfills } 5\} = \{p \in \mathbf{pt}(\mathbf{F}) : 5 \in \uparrow p\} = \{p \in \mathbf{pt}(\mathbf{F}) : p \leq 5\} = \{2,3\} = \uparrow_{\preceq} \{2,3\}$, where $\uparrow_{\preceq}$ denotes an ordered filter with respect to the order $\preceq$. Then the application of ϕ to $\mathbf{F}$, $\phi(\mathbf{F})$, gives

```
                {2, 3, 4, 7}
                     |
                     |
                 {2, 3, 4}
                  /      \
                 /        \
           {2, 4}          {2, 3}
             |        \       |
             |         \      |
           {2}          \   {3}
              \          /
               \        /
                   ∅
```
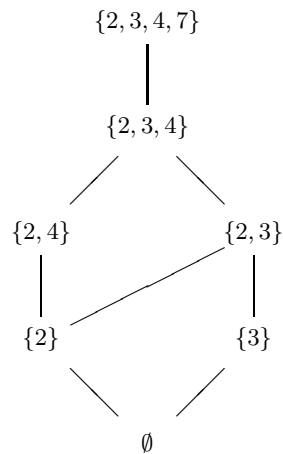
This lattice is easily seen to be isomorphic to $\mathbf{F}$. Moreover $\phi(\mathbf{F}) = \{\uparrow_{\preceq} X : X \subseteq \mathbf{pt}(\mathbf{F})\}$ ($\phi(\mathbf{F})$ is the Alexandrov topology over the preordered set $\langle \mathbf{pt}(\mathbf{F}), \preceq \rangle$ – cf. Mathematical toolkit 16.4).

From the above construction, it follows that co-prime elements are the generic elements we were looking for. They represent abstract points because they represent the basic properties from frame $\mathbf{F}$, because the principal filters that they generate in $\mathcal{J}(\mathbf{F})$ are the topological basic open sets of the spatial (i.e. with points) topological space

$\langle \mathbf{pt}(\mathbf{F}), \phi(\mathbf{F}) \rangle$. In fact, this procedure is called the *spatialisation* of a frame $\mathbf{F}$, $\mathcal{SPAT}(\mathbf{F})$.

It is worth emphasizing that given a generic frame $\mathbf{F}$, it is not guaranteed that $\phi(\mathbf{F})$ is isomorphic to $\mathbf{F}$. It could be only homomorphic. If ϕ is an isomorphism, then $\mathbf{F}$ is said to be *spatial*; this means that it was really the abstraction of a topological space (with points). Otherwise we have a somewhat interesting situation in which there are less abstract points than properties, so that abstract points are not enough to separate or to distinguish, properties: there are at least two properties with the same (abstract) extension. In this case inseparable properties collapse *via* the application ϕ. This may happen only if $\mathbf{F}$ is infinite (for instance a Boolean algebra without isolated points).

Dually, if $\mathbf{F}$ is the frame of open subsets of a topological space $\langle U, \mathbf{F} \rangle$, surely $\phi(\mathbf{F})$ and $\mathbf{F}$ are isomorphic, since $\mathbf{F}$ is "a priori" spatial, but $\langle U, \mathbf{F} \rangle$ and $\langle \mathbf{pt}(\mathbf{F}), \phi(\mathbf{F}) \rangle$ might not be *homeomorphic*. This happens when there are less abstract points than concrete points. In this case $\mathbf{F}$ does not have sufficient properties to separate all "concrete" points. In other words, the original space U is too reach in points: there are at least two distinct points x, x' fulfilling exactly the same properties. Then, the result of the spatialisation is a more essential copy of $\langle U, \mathbf{F} \rangle$, because unseparable points collapse *via* a function η from concrete points to abstract points – that is, from U to $\mathbf{pt}(\mathbf{F})$ – naturally induced by ϕ (i.e. η is a function $U \longmapsto \mathbf{pt}(\mathbf{F})$ preserving the isomorphism ϕ). Then, it should be clear why the resulting space is called a *soberification* of the given one – see Johnstone [1982]. In most cases the soberification of a topological space τ is homeomorphic with its T_0-ification.[23]

We shall see these concepts at work in Part I and Part II.

In any case, through the isomorphism ϕ, a co-prime element is not only a property, but also an element of its own "virtual extension": if $a \in \mathbf{pt}(\mathbf{F})$, then $a \in \phi(a)$, because $a \leq a$. Call it the *generic representative* of the property a. If $a \notin \mathbf{pt}(\mathbf{F})$, then it is a property definable by means of more elementary properties (because a is join-reducible) and its "virtual extension" is made up by their generic representatives. This solves, in a precise mathematical sense, Berkeley's issue.[24]

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[23]Not always, anyway: see Johnstone [1981].

[24]A thorough study on these topics can be found in Pagliani [1992], while a variant of this construction, specifically dealing with Locke's "generic objects", is discussed in Santambrogio [1985].

5.2   Abstract Points and Meaning

In semiotic terms, an abstract point is a "cultural unity", a bundle of cultural properties, not a physical object. In fact, references to physical objects should be avoided in semiotics.

According to U. Eco,

"Let us suppose that I point out a cat, saying |this is a cat|. Everyone would agree that the sentence "the object I have pointed out is a cat" is true, or rather, it is true the sentence "the percept I have pointed out at point in time x was a cat" (...). But in order to make the above sentences verified as true, I must translate them in the following way: "the percept connected with my pointing out at moment x, represents a concrete instance of a perceptive type conceptually defined in such a way that the **properties** *belonging to the perceptive model systematically correspond to the* **semantic properties** *of the semema* cat *and in such a way that both sets of properties usually represent the same meanings". At this point, the cat-referent* **is no longer a mere physical object**. *It has already been transformed into a* **semiotic entity**.
(...) In other terms, both the word |cat| and the percept or object ||cat || culturally stay for the same semema". (Eco [1975], page 221. Emphasis by the author).

5.3   Abstract Points and Rough Sets

Let us sum up the approach proposed above, from the point of view of Rough Set Theory.

In the case of an Approximation Space $\mathbf{AS}(U/E)$, the co-prime elements are the atoms, that is, the elements of U/E. Hence any abstract point e happens to be an equivalence class. Because it is an atom, there are not generic elements strictly smaller than e. It follows that the open set of abstract points $\phi(e)$ is the singleton $\{e\}$, so that all the elements of e collapse into the abstract point e, *via* the function η induced on U by ϕ. It follows that any element of the equivalence class e can be legitimately taken as a representative of e.

This could resemble the extensional solution. Indeed e, *qua* equivalence class, is the *extension* of a basic property $\mathcal{P}$. But, *qua* element of the frame $\mathbf{AS}(U/E)$, from the above discussion it *is* also an abstract property. However, *qua* abstract point it is a generic representative of the basic property $\mathcal{P}$ and of the elements of e, that collapse into e itself,

as well. Therefore we should intend both an equivalence class and its representatives as descriptive *labels* of a basic property. Such labels describe a basic property using a language made up of the attributes, A, the values, V, and by the conjunction $\wedge$. On the contrary, any non-atomic element of $\mathbf{AS}(U/E)$ is described by a disjunction of labels.

Therefore, when we apply the operators (lE) and (uE) to a set X, we virtually apply labels, predetermined by the Information System $\mathbf{I}$, to describe X using the linguistic apparatus at our disposal. In other words, by means of the approximation operations we interpret a set X exploiting the qualitative (or intensional) part of the language of the Information System.

So, we should regard an Approximation Space as a linguistic basis expressing properties induced by a primary observation of the ambient space.[25]

This is the reason why we have introduced Approximation Spaces as background spaces. Now one can add a qualification: background *cultural* spaces, that is, spaces with structures depending on theoretical assumptions.

Pointless topology is a powerful research field which makes it possible to model a pure phenomenological approach to data analysis. Indeed it will guide our discussion throughout Part I when we shall analyse perception systems and in Part II when we shall develop the logico-topological and logico-algebraic interpretations of Rough Set Systems.

6 Rough Sets and Logic

As far as now we have seen that Rough Set Systems have an intrinsic algebraic structure. This structure may be made into different logically interpretable algebras. "Logically interpretable" means that we have language-oriented operations like "or" and "and", language-oriented

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[25]This operation of abstraction and labelling, is central also in the context of morphogenesis. In fact, any object must be equipped with a regulation system, in order to survive in a complex but limited ambient space. This regulation system is based on some mechanism for comparing the forces operated by other objects of the ambient space. This implies that such a comparison is possible. And it is possible when objects share some common field of forces with the same nature.

Therefore an abstraction mechanism is unavoidable that substitutes the same label for different objects. (cf. Bruter [1974], pp. Ch. II, §1)

constants like "definitely yes", "definitely no", and "nothing certain (up to particular subset of the universe of discourse)" and, finally, sentence modalizers like "not", "it is necessary that" and "it is possible that".

We have seen the Rough Set interpretation of the above linguistic machinery. Now, in order to go towards a full logical picture of rough sets, we have to answer the following question: "What are the relationships between empirical constructions like concrete Rough Set Systems and universal constructions like logical systems?" Which amounts to asking "What is a Rough Set Logic?" Let us try to answer.

To some extent, Logic is a rationalisation of empirical activities such as exploration and understanding of space-time relationships, or such as mathematical creation or, in general, the linguistic appropriation of reality (may it be physical or conceptual). Scope and effect of this rationalisation may be illustrated by the following example. Anyone who is familiar with mathematical research can easily confess that nobody obtains results by thinking straightforwardly in terms of lemmata and main propositions. That is, no one creates mathematics by planning "ex ante" a linear development of his/her own thoughts as it is idealized by logical calculi and as we find in its formal exposition.[26] The clean and rational exposition of a result is a linguistic operation applied "ex post", which has not so much relation with its conceptual history and which has two main purposes:

- Communication

- Systematic check of the inference steps

In turn, "communication" means at least: making the result "receivable" and understandable using standard techniques and protocols that are shared and acceptable within the scientific community.[27] Anyway, once deposits of this linguistic apparatus, that we call "Logic", have been properly abstracted, they become active tools for deeper analyses of the given results, for new discoveries, for making different fields communicate each other, or for communicating with artifacts that use symbolic languages.

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[26] Using terms from Natural Calculus, we might say that no one proves anything by means of normal proofs, except for trivial cases.

[27] This has clear advantages but can generate mystifications: acknowledged symbols or even concepts does not necessarily transform any content into a scientific result (see, for instance, Suchman [1993] and some examples from Sokal & Bricmont [1988]).

The extent to which what we have called "empirical activities" is *purely* empirical, is not known to us. We hardly may decide if Logic is just an "ex post" rationalisation of empirical activities or if empirical activities are guided "ex ante" by logical laws too. In any case, apart from any phylogenetic consideration, we can observe that Logic is eminently a linguistic activity: by definition one cannot work with Logic without a language. Neither one can decide if Logic is dialectic in nature (as in Plato or Hegel) or if it is categorial (as in Aristotle, Kant and in modern Logic).

In Indian thought, for instance, Logic definitely is derived from linguistic studies and has a dialectic flavour. Indian grammarians were precursor of logicians in that they discovered universals of language long before the development of Logic as a discipline. And this achievement was the result of empirical investigations on Sanskrit (and other languages). The linguistic school founded between the fifth and the second centuries BC developed over time until its connection with the *Navya-nyaya*, that is, classical Indian Logic which began with Gangesh Upadhyaya in the thirteenth century A.D.

But again, as one can deduce from the discussion in Subsection 3, and as it was pointed out by a distinguished specialist in Indian Logic, this does not prove, of course, that logical principles depend on linguistic structures, because linguistic structures themselves may depend on a deeper structures of thinking or "being" (see Staal [1960]). Nevertheless, it is an empirical and historical matter of fact that Indian scholars developed advanced abstract logical concepts by studying linguistic structures and their actual dialogical uses. Indeed, the dialogical basis of Indian logical thought is particularly evident in some central concepts like that of "paksa" which, roughly speaking, is a sort of connection between a given hypothesis and a prototypical example or counterexample of the hypothesis itself (used, therefore, both in dialogical reasoning, reasoning by cases and in indirect proofs).

Moreover, we do not like to stop our discussion with this open question about the "a priori" *vs* "a posteriori" nature of the logical structure of language or of the linguistic structure of Logic. We must try and venture a workable hypothesis. This hypothesis goes along a "side" interpretation of those phenomena that have been studied both within genetic epistemology (Piaget) and cognitive linguistic (Vygotskij). Infancy is characterised by the effort to have access to language.

We need this effort since we are not born equipped with a formed language. Therefore, language is something definitely different from our senses: whereas senses are tightly coupled with our environment, the activity of developing a linguistic competence induces a permanent distance between human beings and any determined environment. This distance, experienced since infancy, opens the possibility of abstraction and *History* and transforms an *environment* into a *World* (see Agamben [1981]). Otherwise stated, the ontogenetic effort to access language, induces a sort of coding mechanism which separates reality from its representation.[28] Then, the typical universal framework of Logic is, in this respect, the crystallisation of these linguistic mechanisms, the crystallisation of tools that we use to live in a meaningful world.

With a shift of plane, we may therefore intend our logico-algebraic interpretation of rough sets as a process aimed at the linguistic comprehension of a given "reality" and Rough Sets Logic as the crystallisation of the linguistic apparatus that we empirically use in this process to deal with this "reality" in a meaningful way. To summarize, we shall have:

$$\mathbf{AS}(U/E) \longrightarrow RS(U/E) \longrightarrow \mathbf{LM}$$

$$\mathbf{AS}(U'/E') \longrightarrow RS(U'/E') \longrightarrow \mathbf{LM}' \qquad RSL$$
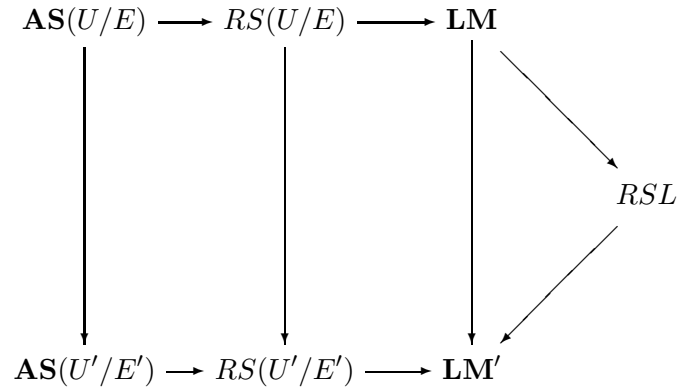
Abstraction levels

$$\mathbf{AS}(U/E) \xrightarrow{1st} RS(U/E) \xrightarrow{2nd} \mathbf{LM} \xrightarrow{3rd} RSL$$

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[28]Moreover, according to De Kerckhove, the phonetically complete Greek-Roman alphabet induced a double coding mechanism (from signs to sounds and from sounds to concepts) that opened the possibility to Western abstraction and rationalisation attitude (see De Kerckhove [1990]).

where **LM** and **LM**$'$ are instances of a class of logico-algebraic models. Any instance is derived from concrete Rough Set Systems, $RS(U/E)$ and $RS(U'/E')$. In turn these systems are derived from particular Approximation Spaces **AS**(U/E) and **AS**(U'/E'), and, thereafter, from particular Information Systems (the "given reality"). Thus both **LM** and **LM**$'$ have their own local use in the process of interpretation of the specific features of **AS**(U/E) and, respectively, **AS**(U'/E') . But Rough Set Logic RSL is complete with respect to **LM**, **LM**$'$ and all the other instances of that class of algebraic models. Completeness pays a price in terms of capability to grasp the specific relations living in specific Information Systems, Approximation Spaces and Rough Set Systems. But with completeness we gain the possibility to "reuse" our discoveries in RSL, throughout different instances of Information Systems.

Although we are aware that this argument does not write the word "end" on the problem of the meaning of a Rough Set Logic, nevertheless we think that it may help the reader positioning this logic within the framework of the present book.

7   Concluding Remarks

It is virtually impossible to end the above discussion with a full stop. Therefore we shall address the reader to a series of topics encompassing general philosophical issues connected with the above discussion and that constitute open research fields.

First of all, Phenomenology-Logic relationships are (although not exclusively) embedded in a research field known as "Formal Ontology". Formal ontology is inspired mainly by Husserl's work and filtered by a series of researches on terminological logic in expert systems (see Brachman et al. [1985]). However Formal Ontology may be framed into different philosophical streams, such as nominalism, conceptualism or realism. An excellent philosophical accounts of Formal Ontology can be found in N. Cocchiarella's papers.

According to a common understanding, Formal Ontology deals with basic conceptual (*a priori*) distinctions and relations such as:

(1) Part-whole, (2) Dependence, (3) Integrity, (4) Identity.

The above distinctions link Formal Ontology with a series of disciplines such as:

(a) Mereology, (b) Topology, (c) Morphology, (d) Logic and general

mixed research fields, like Natural Language Processing, Knowledge Representation, Speech Recognition and Pattern Recognition.

In turn, Mereology (and Ontology), Topology, Morphology and Logic have well-established links with Rough Set Theory and its generalisations and extensions.

It is almost impossible to quote all these connections. In this book we shall develop a part of them.

From the point of view of Rough Set Theory, just to be strictly confined to the above mentioned topics, Lech Polkowski's and Andrzej Skovron's scientific work is a fundamental reference for "Rough Mereology", Ivo Düntsch and co-workers have obtained a number of results about Spatial Reasoning, while Ewa Orłowska is a reference scholar for the relation-algebraic approach to a number of the above research themes (not to mention her pioneering and fundamental researches on Rough Sets and their connections with modal and non-classical Logics and Algebraic Logic).

A large body of Chinese researchers – lead by T. Y. Lin and, more recently, Y. Y. Yao – is emerging in Rough Set Theory, from the mainland, the USA and Canada, which proposes interesting generalisations both from theoretical and application points of view.

Finally we have to mention James F. Peters' researches on the connections between Rough Sets and Neuro Computing and applications of rough set based information granules in a number of fields.

We repeat that this is, by no means, a complete list. However it may serve for useful searches in the WEB.

Part I

A Mathematics
of Perception

Chapter 1

Observations, Noumena and Phenomena

1.1 Foreword

Despite the name, Rough Set Theory relies on a well defined mathematical ground. This, of course, is what one would like to obtain from any sort of formal and analytic approach to "cognitive" problems. But, surprisingly enough, besides the required rigour, Rough Set Theory shares a number of common features with old and new theories belonging to widely different traditions and fields.

And "surprise" is not just a rhetoric if one thinks of the peculiar "practical" problem this theory is originated from. At times, this theory appears as a particular case of more comprehensive approaches, while in other cases it appears as a generalization of well established theories.

The latter case will be evident in the logico-algebraic analysis of Rough Set Theory (see Part II). The former case may be observed when dealing with the very beginning of Rough Set Theory which is based on the concept of a *classification* of entities by means of their observed properties.

From this point of view Rough Set Theory happens to arise from a particular data analysis approach. Its peculiarity is synthesized as follows:

- Data are analysed *statically* at a given point in time of a possibly evolving observation activity.

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- As a consequence, the analysed data provides us only with an *approximated* picture of the domain of interest (because typically we do not have a complete set of observed properties, at a certain point in time).

Therefore, in order to frame Rough Set Theory we shall discuss three basic topics:

1. What are the basic formalisable relationships between "entities" and their "observable properties".

2. What may be intended of "dynamic observation".

3. In what sense Rough Set Theory is based on flat cuts of branching sets of dynamic observations.

The status of an observation system at a certain point in time is essentially a triple $\mathbf{P} = \langle G, M, \Vdash \rangle$, that we call a *Property system*, were G is a set of objects, M a set of properties and $\Vdash \subseteq G \times M$ is intended to be a fulfillment relation. From the concept of an "observation" we shall define a family of basic "perception constructors" mapping sets of objects into sets of properties, called *intensional constructors*, decorated with "i" and sets of properties into sets of objects, called *extensional constructors*, decorated with "e". After that we show that some pairs of constructors from opposite sides, namely $\langle \langle e \rangle, [i] \rangle$ and $\langle \langle i \rangle, [e] \rangle$ fulfill *adjunction properties*. That is, one behaves in a particular way with respect to properties if and only if the other behaves in a mirror-like way with respect to objects. Hence, adjunction properties state a sort of "dialectic" relationship, or mutual relationship, between perception operators, that is what exactly we want in view of our "phenomenological" approach.

Basic constructors generally do not admit inverse operations which from an intensional characterisation of a set of objects X, $[i](X)$ or $\langle i \rangle(X)$ make it possible to determine backwards X by means of constructors decorated with "e".

Indeed, what we have is a set of properties Y obtained through some means of observation, like $\langle i \rangle$ or $[i]$. In a sense Y is a "proxy" of some set of objects that we have to fit into a conceptual pattern.

Fortunately adjunction properties make combinations of basic constructors into generalised *upper approximation* and *lower approximation* operators. A lower approximation of a set X of objects gives its

best description from below, in the sense that it is the largest sub-set of X which can be described by means of our observations. An upper approximation of X gives the best description form above, in the sense that it is the smallest superset of X which can be described by means of our observations. More precisely, we shall see that we can "phenomenologically" approximate X in the following way:

$$\langle e\rangle([i](X)) \subseteq X \subseteq [e](\langle i\rangle(X)).$$

However, this is just the first step, because the approximation operators $\langle e\rangle[i]$ and $[e]\langle i\rangle$ are not continuous, that is, they exhibit "jumps" in the presence of set-theoretical operations. Therefore we synthesize the structuring properties of a *P-system* into a new relational system $\langle G, G, R_{\mathbf{P}}\rangle$ where $R_{\mathbf{P}}$ is a relation between objects – hence no longer between objects and properties – embedding the relevant informational patterns of $\mathbf{P}$. This way we can define a second level informational structure in which adjointness makes second level approximation operators fulfill nice and more refined properties. Also, in this way we shall be able to account for more general forms of Information Systems, called "*Attribute systems*".

Since a symmetrical synthesis may be done with respect to properties (or attributes), given an Information (property or attribute) System $\mathbf{S}$ we shall derive, on the basis of informational criteria:

- A family $\mathbf{G^S}$ of operators manipulating sets of objects

- A family $\mathbf{M^S}$ of operators manipulating sets of properties

- A family $\mathbf{I^S}$ of operators transforming sets of objects into sets of properties

- A family $\mathbf{E^S}$ of operators transforming sets of properties into sets of objects

A *Multi-agent pre-topological Perception System* will be therefore defined as a structure:

$$\langle G, M, \{\mathbf{G^{S_k}}\}_{k\in K}, \{\mathbf{M^{S_j}}\}_{j\in J}, \mathbf{I^S}, \mathbf{E^S}\rangle$$

where any $\mathbf{S}_k$ is an Information System on the set of objects G and set of properties or attributes M_k possibly distinct from M, any $\mathbf{S}_j$ is an Information System on the set of properties (or attributes) M and

set of objects G_j possibly distinct from G, $\mathbf{I^S} : \wp(G) \longmapsto \wp(M)$ and $\mathbf{E^S} : \wp(M) \longmapsto \wp(G)$, where for any set X, $\wp(X)$ denotes the *powerset* of X (set of all sets).

This generalisation makes it possible to manipulate sets of objects (of properties, attributes) by subsequently applying informational criteria induced by different Information Systems. Notice that any of the above families is possibly empty.

Eventually we shall focus on a particular Perception System $\langle G, \emptyset, \{\Box, \Diamond\}, \emptyset, \emptyset, \emptyset \rangle$, were $\Box$ and $\Diamond$ are sorts of topological interior, respectively closure operators, defined on the basis of rather intuitive informational criteria. Such a Perception System is what we call a "Classical Approximation System".

1.2   Formal Relationships Between "Noumena" and "Phenomena"

When we are given a set of "cognitive data" (meaning data that we want to organize in some cognitive manner in order to make them emancipate from the status of "data" to the status of "knowledge"), the very first approach is to organize them in categories, on the ground of some patterns depending on those properties through which they manifest to the observer.

Otherwise stated, data are to be considered, from a methodological point of view, as "*noumena*" whose initial links with our sensitiveness are given by their *observable properties* or "*phenomena*".

"Noumenon", the singular of "noumena", means "to be perceived by mind". In any phenomenological approach to reality, *noumena* cannot be directly named because they receive an ontological and conceptual status only through the filter of *phenomena*. In other words, although we can assume that observables properties are projections of *noumena*, by pulling back the conceptual elaboration of these observables we obtain just *phenomena* while *noumena* remain as philosophical marks with a double meaning: (a) *phenomena* are not void constructions but, indeed, they "speak" of something; (b) *phenomena* are not that "something" but, in a sense, approximations whose limits depend on our categorisation capabilities.

Actually, this picture is coherent with the Aristotelian point of view that something exists beneath *phenomena*: a *substance* (from the

Latin "*substantia*", literally: something that stays below) or an *essence*. Otherwise any pull-back from *phenomena* to *noumena* would be vacuous and we should deal only with observable properties (indeed, this is the point of Berkeley's radical idealism – see Introduction).

This is, clearly, a restrictive meaning to the philosophical term "phenomenon". Indeed, in a sense, it is close to what in classical German philosophy is called "*Erscheinung*" (manifestation) in contraposition to "*Schein*" (appearance): since a "*Schein*" does not presume an essence ("*Wesen*"), it may be a fallacy ("*Betrug*"), while an "*Erscheinung*" is the manifestation of a property of an essence, the only means through which we are aware of it: "*Das Wesen muss erscheinen*" ("Any being must manifest" – see G. W. F. Hegel, *Wissenschaft der Logik*, Meiner, Leipzig). However, the phenomenological approach does not permit an essence to be presumed independently of any phenomenon, so that phenomena are, so to say, 'cognitive proxies' for essences.

We do not claim that things actually run in this way. First of all, phenomenology is much more complex an approach. Second, "observable property" is a tricky concept. Actually, in view of the limiting results of recursion theory (see Frame 4.4) we shall assume that "observable property" means a property observable within a finite interval of time ("finite observation").

What we have roughly described, is just a methodological approach providing us with the conceptual framework which is able to suggest the mathematical tools that we have to adopt.

Shortly, we apply the term "phenomenon" to "what appears" to an observer through a "conceptual grid". Therefore, within the limits of this text a *phenomenon* will be understood as an *external manifestation* whose relationship with a substantial reality is the result of an interpretation process within some "conceptual co-ordinates" that are presumed to be the *explicans* of the essence underlying (*sub-stante*) the observed phenomenon (we shall have other occasions for better explaining this point).

Observation is a dynamic process aimed at getting more and more information about a domain. The larger is the information, the finer is the picture that we have about the elements of the domain. Using topological concepts, an observation process makes it possible to move from a trivial topology on the domain, in which everything is indistinguishable, to a topology in which any single element is sharply separable
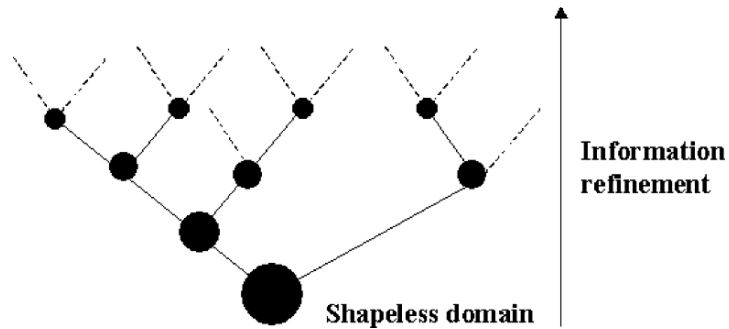
Figure 1.1: A process of differentiation via observations
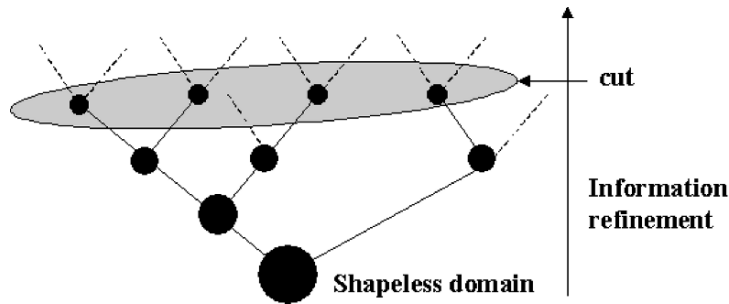
Figure 1.2: A slice of an observation process

from all the other elements of the domain (say a discrete topology or a Hausdorff space). In this case we can "name"or "label" each single element of the domain by means of its characteristic properties.

However, under this consideration, observation is an asymptotic process. What usually happens is that at a certain point in time we stop our observation process, at least temporarily, and analyse the stock of pieces of information collected so far. In a sense we consider a "flat slice" of the observation process (see Figures 1.1 and 1.2).

1.2.1   Property Systems – *P-Systems*

This information slice provides us essentially with a set of observed properties which, we have assumed, are induced by some entities.

Therefore, our information slice is basically composed by:

- A set G of 'objects'

- A set M of 'observable properties'

- A relation between G and M, that we denote, by now, with the symbol $\Vdash$

The symbol G is after the word *"Gegenstand"* – plural: *'Gegenstände'* – a term used in classic German philosophy to denote 'what stays before the cognitive subject', that is an *uninterpreted object* (so that *'Gegenstand'* is different, in German, from *'Object'*, that is something identified by means of its qualities after an interpretation activity). However, members of the set G will be usually referred to as 'points', a neutral term with a bit of a geometrical flavour, or as 'items' or 'entities'.

On the other side, the symbol M is after the German term *'Merkmale'*, meaning 'signs', 'properties' or 'qualities'. However, the members of the set M will be mostly referred to as 'observables' or as '(observable) properties'.

We shall assume that the members of M are *binary* properties, that is, 'yes-no' properties. In other words, we shall not deal, by now, with properties having graded or fuzzy answers. Therefore, given $a \in G$ and $b \in M$ we shall say that if $a \Vdash b$, then a enjoys the property b, or that a induces the *phenomenon* or *observable property* b.

REMARKS. With the term "property" we do not intend to mean necessarily a *single* property. In fact the term "property" implicitly refers to some "level of abstraction" so that it could refer to a *set* or *synthesis* of "more primitive" properties.

Finally, the cognitive subject is, first of all, an observer, so that she is not interested in points or in properties as such, but she will focus on the relationships between points and observables so that all the ingredients do not have a real meaning if they stand alone. A corollary of this conclusion is that we have also to assume that $\Vdash$ *is defined for all the elements of G* and M. This could be questionable. Indeed if we set an experiment in which a hypothetical property A is not detected because no object fulfills it, then this amounts to saying that all objects do not fulfill A or that any experiment for A failed (maybe we have to fix or

refine some instrumentation). However from an informational point of view A carries no information at all because it cannot help making any distinction among objects. Symmetrically, if an object g does not fulfill any property, then it is a "non-object" from a phenomenological point of view (literally, an "*un-wesen*", a "non-being" in German classical philosophy).

To conclude, the mathematical structure we shall start with is a triple $\langle G, M, \Vdash \rangle$, where G and M are sets and $\Vdash$ is a binary relation between the elements of these sets, that is, a subset of the Cartesian product $G \times M$ such that the domain of $\Vdash$ is the whole of G and the range is the whole of M.

REMARKS. Since a collection of observations is supposed to be finite, if not otherwise stated it is assumed that we deal with *finite sets*. However, in general we shall underline if a result strictly depends on this assumption.

Everything is coded in the following definition:

Definition 1.2.1 (Property system). *A triple $\langle G, M, \Vdash \rangle$ where G and M are finite sets, $\Vdash \subseteq G \times M$ is a relation such that for all $g \in G$ there is $m \in M$ such that $\langle g, m \rangle \in \Vdash$, and for all $m \in M$ there is $g \in G$ such that $\langle g, m \rangle \in \Vdash$, is called a* property system *or a* P-system.
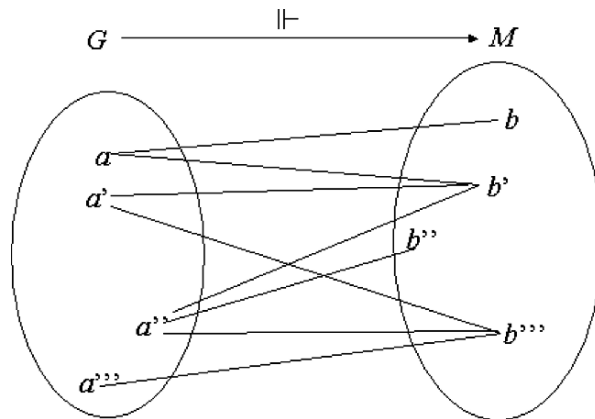
Figure 1.3: A *P-system*

From now on we assume that **P** always denotes such a triple $\langle G, M, \Vdash \rangle$.

1.2.1.1 Functional Property Systems

It has been assumed that the properties we shall deal with admit just yes/no answers. Simple examples of a *P-system* arise when the relation $\Vdash$ is a *functional relation* according to the following definition:

Definition 1.2.2 (Functional relation)**.** *Let $R \subseteq A \times B$ be a binary relation defined for all the elements of A, then R is said to be* functional *if $\langle a, b \rangle \in R$ and $\langle a, b' \rangle \in R$ implies $b = b'$.*

Thus R is a functional relation if any element of A is associated with a *unique* element of B. A *P-system* such that $\Vdash$ is a functional relation will be called a *functional P-system*, or an *FP-system* (see Figure 1.4).

A number of *P-systems* may be represented in a functional setting. Suppose we are associating each student of Ballygunge Science College to her home zip code. Clearly, any student will be mapped onto just one value (e.g. *700.103* or *700.105* or ...), although two students may have the same zip code.

REMARKS. Notice that strictly speaking in this example the elements of M are not zip codes but properties like "700.103_zip-coded", "700.105_zip-coded" and so on, while $\Vdash$ is a deterministic link.

Anyway, sometimes by abuse of language, we shall call $\Vdash$ with the collective name (or type) of the properties in M, if any.
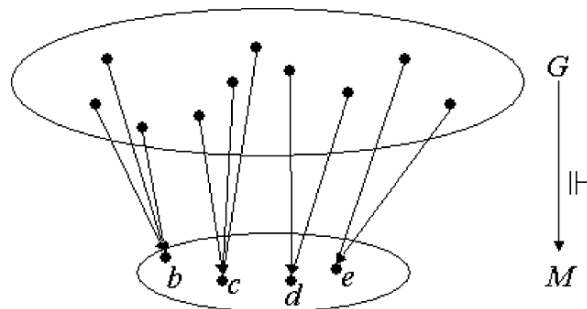
Figure 1.4: A functional *P-system*

FP-systems exhibit a high classification power, as we shall see in a while.

1.2.1.2   Relational Property Systems

In other cases, on the contrary, some elements of G must be mapped onto more elements of M and we shall speak of *relational P-systems* at large, or *RP-systems*.

This happens for instance when we do not have enough information to uniquely determine a value for a property m of an element g, so that the value is indeterministically assigned among a set of possibilities, obtaining a reading such as "the value of g for property $\Vdash$ is m_1 **or** m_2 **or** ..." (for instance, "Bob is eight or ten years old").

We have also cases in which multiple choices are obliged. For instance if $\Vdash$ is the property "*having_son*" and at least one g has two sons m_1 and m_2, then $\langle G, M, \Vdash \rangle$ is surely an *RP-system*. But in this case the reading is "the value of g for property $\Vdash$ is m_1 **and** m_2".

Although the two situations may be given, at a first sight, the same formal shape, by no means they are equivalent and we shall not deal with the disjunctive (former) case unless otherwise specified.

Finally, we obtain $RP - systems$ if we merge more $FP - systems$ together. As a matter if fact, this is the most relevant case and it is tantamount to evaluating objects against a set of non mutually incompatible properties.
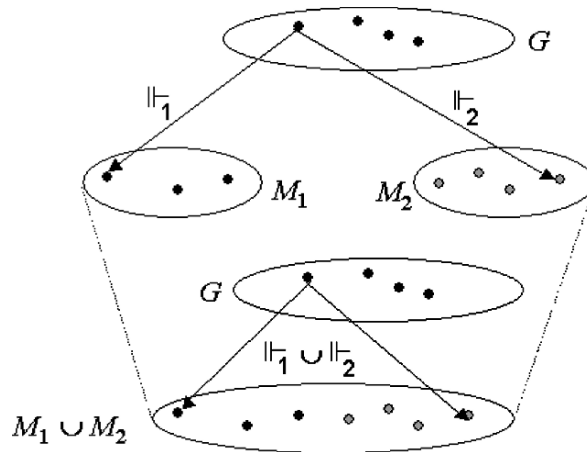
Figure 1.5: Merging two functional *P-systems*

1.2.1.3 Dichotomic Property Systems

An important kind of *RP-system* is given by the so-called *dichotomic P-systems*. In such systems any property is coupled with a complementary property.

Definition 1.2.3 (Dichotomic property system). *Let* $\mathbf{P} = \langle G, M, \Vdash \rangle$ *be a P-system. We say that* $\mathbf{P}$ *is a* dichotomic *P-system, or DP-system, if for all* $p \in M$ *there is* $\overline{p} \in M$ *such that for all* $g \in G$, $g \Vdash p$ *if and only if* $g \not\Vdash \overline{p}$. $\overline{p}$ *is called the* complementary property *of* p.

We shall see that dichotomic systems enjoy particular features since the presence of mutually incompatible properties renders this kind of systems a high classification power. Indeed *FP-systems* and *DP-systems* are closely linked together and we shall see that *RP-systems* may be reduced to suitable *DP-systems* and *FP-systems*.

1.2.2 Attribute Systems – *A-Systems*

Very often data are represented by means of multi-valued matrices of type $\langle G \times At \times V \rangle$, where At is a set of attributes, V is a set of values, namely the union of a family of sets $\{V_a\}_{a \in At}$, where V_a is the set of possible values for the attribute a, and, finally, the entries of the matrix is given by the family of functions $\{a : G \longmapsto V_a\}_{a \in At}$ or $\{a : G \longmapsto \wp(V_a) - \{\emptyset\}\}_{a \in At}$. In the first case we obtain a so-called *Deterministic Attribute System* or *A-system*, in the second a *Non-deterministic Attribute System* or *nd-A-system*.[1]

Definition 1.2.4 (Attribute system). *(A-system)*
Any structure of the form $\langle G, At, \{V_a\}_{a \in At}, \rangle$, *where* G, At *and* V_a *are sets and for each* $a \in At$, $a : G \longmapsto V_a$ *is a function, is called a* deterministic Attribute System *or* A-system. *If* $a : G \longmapsto \wp(V_a) - \{\emptyset\}$, *then the triple will be called a* non-deterministic Attribute System *or* nd-A-system.

Later on in this Chapter we shall see how to translate an *A-system* into a *P-system*.

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[1] Usually Attribute Systems come along with an evaluation function $v : G \times At \longmapsto V$ or, if non-deterministic, $v : G \times At \longmapsto \wp(V)$. However our equivalent presentation is more comfortable because kernels of a_i, k_{a_i}, are important while the kernel of v makes a poor sense.

Example 1.2.1. Property and Attribute Systems
Here are the **basic examples** in this Chapter: a *P-system* $\mathbf{P} = \langle G, M, \Vdash \rangle$, an *FP-system* $\mathbf{F} = \langle G, M', \hat{f} \rangle$ and an *A-system* $\mathbf{A} = \langle G, At, \{V_a\}_{a \in At} \rangle$ where $G = \{a, a', a'', a'''\}, M = \{b, b', b'', b'''\}, M' = \{m, m', m''\}, At = \{A, A', A''\}, V_A = \{0, 1, 3\}, V_{A'} = \{b, c, f\}, V_{A''} = \{\alpha, \delta\}$ and the evaluation functions are given by the following matrices (here $vl \subseteq G \times At \times \bigcup_{a \in At} \{V_a\}$):

$\Vdash$	b	b'	b''	b'''	$\hat{f}$	m	m'	m''	vl	A	A'	A''
a	1	1	0	0	a	1	0	0	a	1	b	α
a'	0	1	0	1	a'	0	1	0	a'	0	c	α
a''	0	1	1	1	a''	1	0	0	a''	1	b	α
a'''	0	0	0	1	a'''	0	0	1	a'''	3	f	δ

Evidently, $\Vdash$ is not a map because, for instance, both $\langle a', b' \rangle \in \Vdash$ and $\langle a', b''' \rangle \in \Vdash$ but $b' \neq b'''$. On the contrary $\mathbf{F}$ is clearly functional.

1.3   Functional *P-Systems* and Conceptualisation

Our starting point was the assumption that we have a conceptual awareness of perceptive events only if we are able to classify our "*percepta*" is some way (see the Introduction).

If $\langle G, M, \Vdash \rangle$ is an *FP-system*, we are in a privileged position for achieving this goal. Indeed if we read back or, more precisely, if we *pull-back* the map $\Vdash$, we obtain an equivalence relation E, so that any element of G will belong to one and just one equivalence class modulo E, without ambiguity or borderline situations. This is a perfect case of a classification.
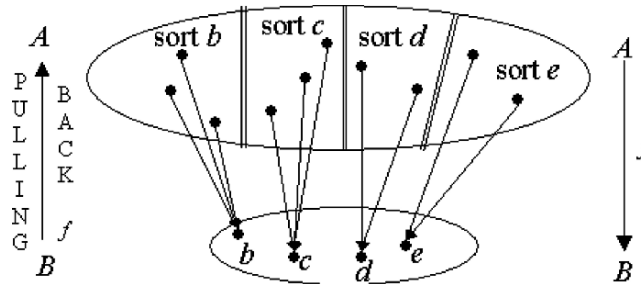
Figure 1.6: A *B*-valued classification of *A*

In the presence of *RP-systems* we obtain less sharp classifications. More precisely, the sharpness degree depends on the way we consider the interplay between relations and inverse relations and on the nature of the *RP-system* we are dealing with (see later).

1.3.1 Categorizing Through Functional *P-systems*

It is clear that any function $f : A \longmapsto B$ induces a classification of the elements of A by pulling back f itself. Indeed, any $b \in B$ may be viewed as a property-value which makes it possible to classify the elements of A by gathering together in a *sort* all those $x \in A$ such that x assumes value b via the evaluation f, that is, all x such that $f(x) = b$. This interpretation of a function f is sometimes described by saying that f is a *property B-evaluated on A*. Using the previous example, suppose B is the set of possible values of the property "zip code", b means "700-103" and A is a set of students. Then the sort $\{x \in A : f(x) = b\}$ is that of students with residence's zip code = 700.103.

Actually it is already affirmed that we prefer to see "700.103_zip-coded" itself as a particular property. In view of this preference we shall say that f is a *B-evaluated classification of A* (see Figure 1.6).
Because of an analogy taken from agriculture, any sort is sometimes called a "*fibre*" (or, also, a "*stalk*") and the family of sorts is called the *stalk space* (or *espace etalé*) of f. The elements of a fibre are called "*germs* (of a function)" and, clearly, the relation which links the elements in each fibre together is the kernel k_f. In Figure 1.7 the set A is split into four fibers or stalks. Or, to put it in another way, the elements of A are gathered around four distinct stalks.

The formal operation which makes it possible to associate each member of B with a fibre of A is $f^{\leftarrow}$: looking back pointwise from
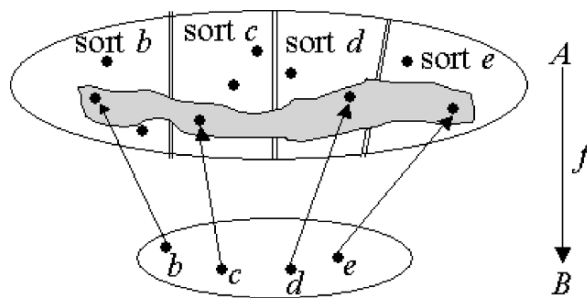
Figure 1.7: A section

B to A through f. Indeed, the stalk space of f equals $\{f^{\leftarrow}(\{b\})\}_{b \in B} = \{f^{\leftarrow}(\{f(a)\})\}_{a \in A}$.

If, instead of the domain, we focus on the codomain of f, we can interpret this function as a way to list or parameterize (some of) the elements of B, through the elements of A. In view of this interpretation it is natural to associate with each property from B a representative in A of each equivalence class modulo the kernel k_f, by choosing a member from each sort. The set of these representatives is called a *cross section* of f (see Figure 1.7).

Therefore, a cross section of f is an isomorphic copy of the image of f, Im_f, in the domain of f. More precisely, what is described in Figure 1.7 is the image of a particular type of function defined as follows:

Definition 1.3.1 (Section)**.** *Let $f : A \longmapsto B$ be a function. Then a morphism $s : B \longmapsto A$ is called a* section *of f, if $s \circ f = 1_B$, that is, if the following diagram commutes (meaning that the two paths from the origin to the target coincide):*

$$
\begin{array}{ccc}
 & A & \\
{}^{s}\nearrow & & \searrow{}^{f} \\
B & \xrightarrow[\;\;1_B\;\;]{} & B
\end{array}
$$
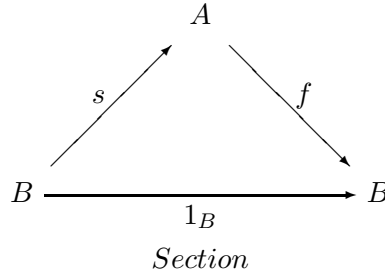
Section

The notion of a section has its own dual:

Definition 1.3.2 (Retraction)**.** *Let $f : A \longmapsto B$ be a function. Then a morphism $r : B \longmapsto A$ is called a* retraction *of f, if $f \circ r = 1_A$, that is, if the following diagram commutes:*

$$
\begin{array}{ccc}
 & B & \\
{}^{f}\nearrow & & \searrow{}^{r} \\
A & \xrightarrow[\;\;1_A\;\;]{} & A
\end{array}
$$
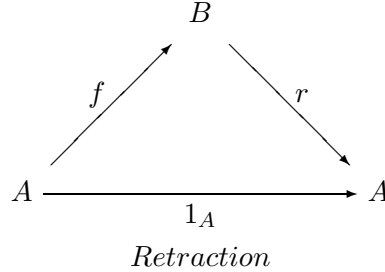
Retraction

Sections are also called "co-retractions". From the above diagrams, we immediately deduce that if s is a section of f, then f is a retraction of s; and vice-versa. Moreover, it is not difficult to verify that f does not have any section if it is not *onto B* (otherwise, how would it be possible to obtain 1_B?). Intuitively if there is a $b \in B$ that is not f-image of any $a \in A$, then b would be associated with a void sort (that is odd, although not completely a non-sense – see remarks about Pointless Topology in the Introduction). However, it is not required for f to be injective. Vice-versa, a function f does not have any retraction if f is not *into B*. In fact, if $f(a) = f(a') = b$, for $a \neq a'$, then any morphism from B to A either maps b onto a and forgets a', or it maps b onto a' and forgets a, because of the uniqueness of image. However, it is not necessary for f to be onto.

Since these properties will be used throughout this Chapter, we should highlight them:

$$\text{If a function f has a section, then f is surjective} \qquad (1.3.1)$$

$$\text{If a function f has a retraction, then f is injective} \qquad (1.3.2)$$

From the above discussion, it follows that for any function $f : A \longmapsto A$, f_o is a section with retraction f^o.[2]

Given a cross section S of A, another natural maneuver is to group around the members of S the elements of A belonging to the same fibre or stalk.

Let us specify the mathematics to accomplish this.

If $r : A \longmapsto B$ is a retraction of a function $h : B \longmapsto A$, then $r \circ h$ is an idempotent endomorphism in A: $(r \circ h) \circ (r \circ h) = r \circ (h \circ r) \circ h = r \circ 1_B \circ h = r \circ h$.

It follows that if $s : B \longmapsto A$ is a section of $f : A \longmapsto B$, then $f \circ s$ is an idempotent endomorphism in A, assumed s is onto, because f is a retraction of s (see Figure 1.8).

Clearly, if $a = s(b)$, then $s(f(a)) = s(f(s(b))) = s(1_B(b)) = s(b) = a$, so that a is a fixed point of the endomorphism $f \circ s$. Moreover, $(f \circ s)(a') = a$ for all a' such that $f(a) = f(a')$, since $s(f(a')) = s(f(a)) = a$. Henceforth, this endomorphism makes it possible to group any element x of A around a representative, $s(f(x))$, of its own fibre $f^{\leftarrow}f^{\rightarrow}(\{x\})$.

---

[2]For the meaning of f_o and f^o the reader is addressed to Mathematical toolkit 16.2.
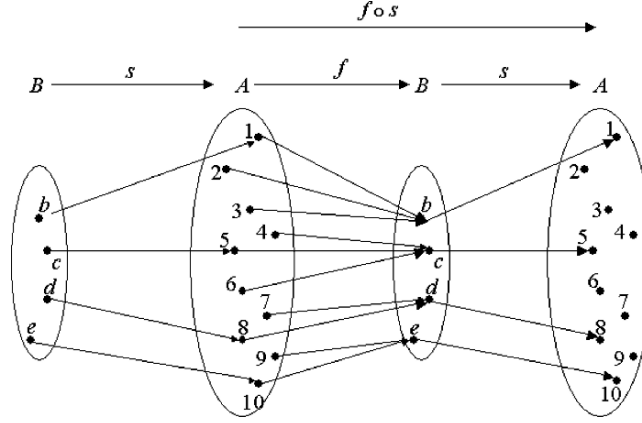
Figure 1.8: Retraction + section = idempotent endomorphism

This endomorphism is a trace of f and s in A and its core is the notion of a "kernel". Indeed, a section of f is obtained by peaking-up an element from any equivalence class $[x]_{k_f}$ of the kernel. In a sense, we first use the elements of B as "proxies" of the equivalence classes of k_f (setting b as a "proxy" of $[a]_{k_f}$, when $[a]_{k_f} = f^{\leftarrow}(\{b\})$) and then we apply the Isomorphism Theorem.

We shall extensively use the notions of a retraction and a section in the sequel. Now we want to show that in turn these notions are special cases of a more fundamental concept: a *divisor*.

Indeed, from elementary arithmetic we know that the solution to the equation $6 \times x = 18$ is given by dividing 18 by 6. Similarly, in the world of morphisms we can have solutions to equations of type $f \circ x = h$ or $x \circ f = h$, for given f and h. Let us then call x a *right divisor* of h by f, in the first equation, and a *left divisor* of h by f, in the second.

Definition 1.3.3 (Left divisor). *Let $f : A \longmapsto B$, $g : B \longmapsto C$ and $h : A \longmapsto C$ be three functions. Then, g is called a* right divisor of h *by f and f is called a* left divisor of h *by g if $h = f \circ g$, that is, if the following diagram commutes:*
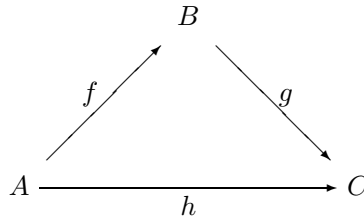
Clearly, things are interesting when we are given f and h and we have to find a right divisor; or the other way around, when we are given h and g and have to find a left divisor.

The first problem is also called the "determination problem". Indeed, if we find a right divisor of h by f, then we can say that h is determined by f. For instance a mapping h between a set of people and their residence districts, may be determined by a mapping f between people and their residence ZIP codes, by means of a correspondence g between ZIP codes and districts. We can see that a map g is a right divisor of h by f only if for all $a, a' \in A, f(a) = f(a')$ implies $h(a) = h(a')$. Moreover, if f is an isomorphisms, then it is straightforward to see that $f^{-1} \circ h$ is the unique right divisor of h by f.

The second problem is called the "choice problem". For instance, if h is as above and g maps the set of candidates onto the districts they want to represent, then a left divisor f is given by choosing a candidate through an election. One can see that f is a left divisor of h by g only if for all $a \in A$, there exists a $b \in B$ such that $h(a) = g(b)$. Moreover, if g is an isomorphisms, then $h \circ g^{-1}$ is the unique divisor of h by g.

The proof of all the above assertions is left as an exercise.

It is immediate to verify that retractions and sections are instances of right and, respectively, left divisors.

Example 1.3.1. Sections and retracts

Suppose $A = \{a, a'\}$, $B = \{b\}$ and $f : A \longmapsto B$ is such that $f(a) = f(a') = b$. If f has a retraction r, then for any $x \in A, r(f(x)) = x$. But we have just two possibilities: either $r(b) = a$ or $r(b) = a'$. In the first case $r(f(a')) = a \neq a'$, while in the second $r(f(a)) = a' \neq a$. Hence f does not have any retract. Intuitively, we cannot have a retraction r if r has to map a smaller set into a larger set.

Suppose now $g : B \longmapsto A$ is such that $g(b) = a$. If g has a section s, then $g(s(x)) = x$, any $x \in B$. But there is just a map from A to B and it is f so that $g(f(x)) = g(b) = x$ only if $x = a$, while $g(f(x)) \neq x$ if $x = a'$. It follows that g does not have any section. Intuitively, a section has to map a smaller set into a larger set. On the contrary, g has a retraction and it is f. Indeed, $f(g(b)) = f(a) = b$. For the same reason, g is a section of f.[3]

---

[3]For a further example, look below at Excursus 1.1: function c^o is a retraction of function c_o which, in turn, is a section of c^o. Indeed, c_o selects an element out of each equivalence class $[x]_{\kappa_{C^o}} = [x]_{\kappa_c}$.

Exercise 1.1. *Let A and B be two sets.*

(A) *Let $f : A \longmapsto B$. How many sections does f have, if any?*

(B) *Let $f : A \longmapsto B$. How many retracts does f have, if any?*

(C) *Prove that $f : A \longmapsto B$ has a section only if it is epi.*

(D) *Prove that $f : A \longmapsto B$ has a retraction, only if it is into.*

1.4   Categorizing Through Relational *P-Systems*

It is observed that if we are given an *FP-system* we obtain sharply delimited categories in a rather straightforward way. Indeed, the notion of a "kernel" (or of a "fibred product") and those of a "section" and a "retraction" provide us with the necessary machinery to compute clusters of objects.

On the contrary, if we deal with relational *P-systems*, as we have indeed to do, it hardly results in sharp classifications, at least without a proper manipulation of the given *P-system* that, in turn, may be or may not be an appropriate maneuver. It follows that the identity relation in the definition of left and right divisors must be weakened to an inequality relation "$\geq$" or "$\leq$".

Therefore, to deal with generic cases we need a more subtle mathematical machinery and here below we give an initial intuitive picture of what we have to explore.

Such a machinery is based on the notion of an "approximation". However, this notion depends on another one. Indeed, we cannot speak of "approximation" without comparing a result with a goal and this goal depends on the *granularity* level of its definition.

For instance, if we are to retrieve documents, we enter a series of keywords. The more complete is this series, the smaller the number of documents which are retrieved (provided we are using an "and" search), since more specialized is our query, or, otherwise stated, since finer is its grain. We can use a ratio to measure the *precision*, or dually the *approximation* rate of search results, namely the number of relevant documents retrieved over the number of documents retrieved. Clearly

the finer is the grain of our query the higher is the precision. But this measure is also affected by the granularity of the documents stored in the system. One should notice first that a precision equal 1 could mean nothing from a cognitive ergonomic point of view. In fact suppose we have to catalogue a library. If we decide that we have only one basic grain, i.e. the entire library, any query will result in one and only answer whose precision ratio is, trivially, 1. Of course, this is an extreme case. However it drives our attention to the fact that "precision" alone measures little and it must be related also to the cardinality of the entire stock of the stored pieces of information which are supposed to be the atomic relevant data. Clearly, the larger is this stock the less affected is "precision".

So we decide, presumably, to refine the granularity of the stored pieces of information by lowering them to the level of single books. Again the same issue may apply to the sub-level of Books, and so on to Sections, Paragraphs, Pages, Lines and Words. Where to stop is a matter of application-domain, capability, time, space and cognitive ergonomy.

In sum, precision/approximation depend both on query and document granularity, that is, on granularity of objects and properties.

In general we face a situation in which queries refer to a set of possible answers and not to single objects. Otherwise we would not have "queries" but "selections" (the realm of sections and retractions).

In other words, the world of objects is made of finer grains than that of modeling or interpretation activities, in principle.

So we can distinguish an *extensional granulation*, related to objects, and an *intensional granulation*, related to properties, and assume that the extensional granulation is finer than the intensional one. Thus when we have to determine a point on the extensional scale by means of the intensional ruler, we hardly will be able to get a precise determination.

In the example of Figure 1.9 we have to reduce a continuum of points, represented on an extensional scale, to a series of discrete intervals, represented on an intensional scale. The point x can be approximated by a best approximation from above, viz. 5.5, and a best approximation from below, viz. 5. But in order to be able to have such "best approximations" the intensional granulation and its relationships with the extensional granulation must fulfill non trivial properties.
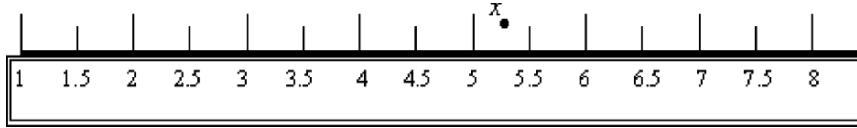
Figure 1.9: Scaling and approximating

First of all we need an order to give the term "best" a meaning. But at what level of abstraction?

To be sure, if the fulfillment relation $\Vdash$ is an injective functional relation then we know that we can recover each single element of G with a retraction without the need of any order.

REMARKS. The reader should agree that this is an unrealistic situation. In fact if there were a 1-1 relationships between substances and properties then properties would just be perfect copies of objects. Of course, someone could claim that the best model of a cat is a cat, but this makes a poor meaning in any modeling activity. Indeed, modeling means abstracting and abstracting means losing or renouncing some quality or characteristic.

If $\Vdash$ is a surjective functional relation, we have one or more sections and if G is ordered in some way then we can apply this order to select the "best" (with respect to the given order) section. For instance if $m \in M$ is a query and $X = \Vdash^{\leftarrow} (\{m\})$ then X is a set of candidate results of the query m. If the elements of X are ordered in some way, we can use this order to choose, for instance, the least or the largest element of X, if any.

Symmetrically, if M were ordered, we could make any fulfillment relation into a function by choosing for each $g \in G$ the "best" element fulfilled by g and, then, find retractions or sections.

However if G and M are not ordered then we have to change strategy, because $\Vdash$ is seldom a surjective or injective functional relation. In a sense it is an event which happens only if we are able to completely reduce a structure to a simpler one.

Having this picture in mind, the reader is invited to notice what follows. From a phenomenological point of view we cannot assume that either G or M are ordered. Indeed all objects are at a peer level because they come into our awareness by means of their manifested properties. Similarly, properties are at a peer level because they come

into our observations just because they are manifested by unknown objects.

To be sure, this genuine phenomenological approach can be questioned and, actually, it happens to be too extreme in several situations.

However, methodologically we decided to assume that the only relationships between objects are those induced by the fulfillment relation $\Vdash$ and such relationships are grouping relations so that we can compare subsets of objects (or properties) but not, directly, objects (or properties). In other words, we assume that there is no relation (hence any order) either between objects or between properties, if not otherwise stated. It follows that the result of the phenomenological activity is, a "type" not a "token", at least in this case.[4] Therefore, we shall move from the level of pure *P-systems* $\langle G, M, \Vdash \rangle$ to that of *Perception systems* $\langle \wp(G), \wp(M), \phi_i \rangle$ where ϕ_i is a map from $\wp(G)$ to $\wp(M)$ or from $\wp(M)$ to $\wp(G)$. It follows that we have to manipulate some kinds of subsets of the domain and the codomain of $\Vdash$ which embed enough ordering features to compute approximations (see Figure 1.10).

1.4.1 Types and Approximation

Suppose ϕ is an isotonic function which maps subsets of a set (type) A into subsets of a set (type) B. Suppose $Y \subseteq B$ and $\phi(X) \supseteq Y$, for X a subset of A. Then we can say that X approximates Y from above *via* ϕ. The smallest of these Xs can therefore be thought of as a "best approximation from above" *via* ϕ, because its image is the closest to Y.

In order to obtain such a best approximation, if any, we should take $\bigcap \phi^{\leftarrow}(\uparrow Y)$, where $\uparrow Y = \{Y' \subseteq B : Y' \supseteq Y\}$. In fact $\bigcap \phi^{\leftarrow}(\uparrow Y) = \bigcap \{X : \phi(X) \supseteq Y\}$.

Dually, if we take $\bigcup \phi^{\leftarrow}(\downarrow Y)$, where $\downarrow Y = \{Y' \subseteq B : Y' \subseteq Y\}$, we should obtain a "best approximation from below" of Y, if any, *via* ϕ, because $\bigcup \phi^{\leftarrow 1}(\downarrow Y) = \bigcup \{X : \phi(X) \subseteq Y\}$.

---

[4]Indeed, this is consistent with the fact that, according to Aristotle, we cannot reproduce reality, but just interpret it through a "mimesis" (reproducing) effort. For a different mathematical approach arriving at the same conclusion, see Frame 4.2.
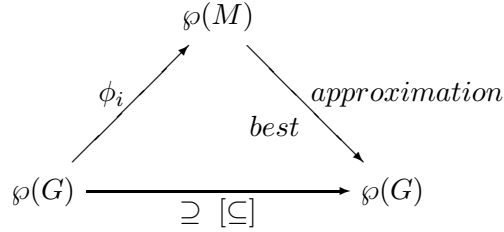
Figure 1.10: Approximation deals with types

To be sure, this approach is successful only if $\phi(\bigcap \phi^{\leftarrow}(\uparrow Y)) \supseteq Y$ and, dually, $\phi(\bigcup \phi^{\leftarrow}(\downarrow Y)) \subseteq Y$. So we now shall examine the conditions under which the above operations are admissible and behave as expected.

First, the mathematical machinery that has to be used shall be presented, in an abstract way. Subsequently, in Section 2.1 this machinery will actually be applied to functions between the two powersets $\wp(G)$ and $\wp(M)$ and order relations will turn into subset relation on $\wp(\wp(G))$ and $\wp(\wp(M))$ (this is a reason why it is more comfortable to present the basic techniques in a more abstract setting).

1.4.2   Divisors and Residuals

Terminology and Notation

Since in the next subsections we shall deal with maps which happen to be functors between categories (in some contexts), we shall denote them with Greek lower symbols.

Let us start with the problem of how to approximate a function from below and from above by means of a pair of other functions.

If $\mathbf{A} = \langle A, \leq_A \rangle$, $\mathbf{B} = \langle B, \leq_B \rangle$ and $\mathbf{C} = \langle C, \leq_C \rangle$ are partially ordered sets and three maps $\phi : \mathbf{A} \longmapsto \mathbf{C}$, $\lambda : \mathbf{A} \longmapsto \mathbf{B}$ and $\varrho : \mathbf{B} \longmapsto \mathbf{C}$ are given, we can have the following notable situations:

(i) For all $a \in A, (\lambda \circ \varrho)(a) \leq_C \phi(a)$ (which shall be denoted as $\lambda \circ \varrho \leq_C \phi$).

(ii) For all $a \in A, (\lambda \circ \varrho)(a) \geq_C \phi(a)$ (which shall be denoted as $\lambda \circ \varrho \geq_C \phi$).

The following diagram may be helpful:
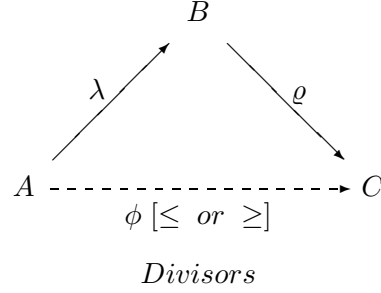
Divisors

First of all, we recall that $(\lambda \circ \varrho)(x) = \varrho(\lambda(x)) = \varrho\lambda(x)$.

Now, if (i) is satisfied and ϱ is isotone, then we say that ϱ is an *upper-divisor* of ϕ by λ and that λ is a *lower-divisor* of ϕ by ϱ – notice that no restriction on λ is imposed.

Usually, in literature the terms "left" and "right" divisor are more common and we have kept traces of this fact in the symbols λ (after "left") and ϱ (after "right"). But as we develop our discourse this terminology can generate misunderstanding as it will be clear a few paragraphs below. If (ii) is satisfied, then we have to reverse the terminology as to the partial order direction and call ϱ a *lower-divisor* of ϕ by λ, provided it is isotone, and λ an *upper-divisor* of ϕ by ϱ. Obviously, if we reverse the order of the two categories, then we have to swap the terms "upper" and "lower".

Now we can ask: under what conditions do three mappings fulfill conditions (i) and (ii)? under what conditions such divisors exist?

Let us start with the following result about isotonicity. Indeed, if we hope to be able to define two approximating maps, at least one of them should have a regular (isotone) behaviour.

The intuition behind the following *Lemma* is simple and clear: if any principal order ideal (filter) in B has a corresponding order ideal (filter) in A, *via* ϕ, then the order of A is paralleled by that of B *via* ϕ.

Lemma 1.4.1. *Let ϕ be a mapping between two preordered sets **A** and **B**. Then ϕ is isotone iff for all $b \in B$, $\phi^{\leftarrow}(\downarrow b)$ is either empty or is an ideal of **A**, iff $\phi^{\leftarrow}(\uparrow b)$ is either empty or is a filter of **A**.*

Proof. Assume ϕ is isotone. Suppose $\phi^{\leftarrow}(\downarrow b) \neq \emptyset$ for $b \in B$. Then if $a \in \phi^{\leftarrow}(\downarrow b)$ and $a' \leq_A a$ we have, by hypothesis, $\phi(a') \leq_B \phi(a) \leq_B b$

so that $a' \in \phi^{\leftarrow}(\downarrow b)$. Suppose, conversely, that for any $b \in B$, $\phi^{\leftarrow}(\downarrow b)$ is either empty or an ideal in A. For each $a \in A$ we have trivially $\phi(a) \leq_B \phi(a)$ whence $a \in \phi^{\leftarrow}(\downarrow \phi(a))$. But by hypothesis $\phi^{\leftarrow}(\downarrow \phi(a))$ is an ideal of A. It follows that $a' \leq_A a \Rightarrow a' \in \phi^{\leftarrow}(\downarrow \phi(a))$; henceforth $\phi(a') \leq_B \phi(a)$.   QED

Now, if we compare two subsets X and Y of an ordered structure and $X \subseteq Y$ then the minimal elements of X are greater than or equal to the minimal elements of Y. We use this simple fact in the following *Proposition*:

Proposition 1.4.1. *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *be partially ordered sets and let* $\phi : \mathbf{A} \longmapsto \mathbf{C}$, $\lambda : \mathbf{A} \longmapsto \mathbf{B}$ *and* $\varrho : \mathbf{B} \longmapsto \mathbf{C}$ *be maps. Then,*

1. *Given ϕ and λ, $\lambda \circ \varrho \geq_C \phi$ if and only if $\forall x \in B, \lambda^{\leftarrow}(\downarrow x) \subseteq \phi^{\leftarrow}(\downarrow \varrho(x))$, provided ϱ is isotone.*

2. *Given ϕ and λ, $\lambda \circ \varrho \leq_C \phi$ if and only if $\forall x \in B, \lambda^{\leftarrow}(\uparrow x) \subseteq \phi^{\leftarrow}(\uparrow \varrho(x))$, provided ϱ is isotone.*

3. *Given ϕ and ϱ, there is λ such that $\lambda \circ \varrho \leq_C \phi$ if and only if $\forall x \in A, \varrho^{\leftarrow}(\downarrow \phi(x)) \neq \emptyset$.*

4. *Given ϕ and ϱ, there is λ such that $\lambda \circ \varrho \geq_C \phi$ if and only if $\forall x \in A, \varrho^{\leftarrow}(\uparrow \phi(x)) \neq \emptyset$.*

Proof. (**1**) ($\Rightarrow$) Assume $\lambda \circ \varrho \geq_C \phi$. If $\lambda^{\leftarrow}(\downarrow x) = \emptyset$, then the conclusion is trivial. Otherwise for each $a \in \lambda^{\leftarrow}(\downarrow x)$ we have $\lambda(a) \leq_B x$ and from the isotonicity of ϱ and hypothesis, we obtain $\phi(a) \leq_C \varrho(\lambda(a)) \leq_C \rho(x)$. It follows that $a \in \phi^{\leftarrow}(\downarrow \varrho(x))$. ($\Leftarrow$) For all $a \in A$ we have $a \in \lambda^{\leftarrow}(\downarrow (\lambda(a))) \subseteq \phi^{\leftarrow}(\downarrow (\varrho(\lambda(a))))$. Whence $\phi(a) \leq_C \rho(\lambda(a))$. (**2**) Dually.
(**3**) If $\lambda \circ \varrho \leq_C \phi$, then for all $a \in A$, $\varrho(\lambda(a)) \leq_C \phi(a)$ so that $\lambda(a) \in \varrho^{\leftarrow}(\downarrow (\phi(a)))$. It follows that for all $a \in A$, $\varrho^{\leftarrow}(\downarrow (\phi(a))) \neq \emptyset$. Conversely, if the last inequality holds, then we can define a map $\lambda : A \longmapsto B$ by associating with each $a \in A$ a chosen element belonging to $\varrho^{\leftarrow}(\downarrow (\phi(a)))$. Obviously, for all $a \in A, \varrho(\lambda(a)) \leq_C \phi(a)$. (**4**) Is obtained dually.   QED

Now, let us specialize the above mechanisms to the case in which $\mathbf{C} = \mathbf{A}$ and ϕ is the identity map $1_{\mathbf{A}}$:

$$B$$

$$\lambda \qquad \varrho$$

$$A \dashrightarrow A$$
$$1_A \ [\leq \ or \ \geq]$$

$$Divisors$$
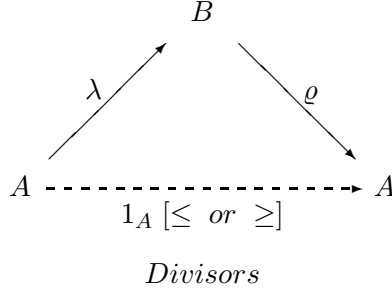
In this case we have:

Proposition 1.4.2. *Let* $\mathbf{A}$, $\mathbf{B}$ *be partially ordered sets and let* $\varrho :$ $\mathbf{B} \longmapsto \mathbf{A}$ *be isotone. Then,*

1. *There exists* $\lambda : \mathbf{A} \longmapsto \mathbf{B}$ *such that* $\lambda \circ \varrho \leq_A 1_A$ *if and only if given any principal order ideal* $\downarrow x$ *of* $\mathbf{A}$, $\varrho^{\leftarrow}(\downarrow x)$ *is an order ideal of* $\mathbf{B}$.

2. *There exists* $\lambda : \mathbf{A} \longmapsto \mathbf{B}$ *such that* $\lambda \circ \varrho \geq_A 1_A$ *if and only if given any principal order filter* $\uparrow x$ *of* $\mathbf{A}$, $\varrho^{\leftarrow}(\uparrow x)$ *is an order filter of* $\mathbf{B}$.

Proof. (**1**) is an immediate corollary of the above *Proposition* 1.4.1 and *Lemma* 1.4.1, while (**2**) is obtained dually from (**1**).   QED

As will be stated later on in *Definition* 1.4.1, if hypothesis 1 of *Proposition* 1.4.2 holds, ρ is called an *upper residual* of λ and λ is called a *lower residual* of ρ. If hypothesis 2 holds we reverse the terminology. So, in order to have such kind of weak divisors we need a sort of *continuity* between ordered structures. Analogously, in order to have derivatives we need continuity. A function $f : X \longmapsto Y$ is continuous if given any open subset $A \subseteq Y$, the pre-image $f^{\leftarrow}(A)$ is an open subset of X. In this case we can define the derivative function of f (up to well-known pathological functions). Upper and lower residuals are sorts of derivative functions and order filters (ideals) play the role of open sets.

The last step is to further specialize the above result to the case that both residuals are isotone maps, hence to embed the above results into the category of partially ordered sets. It is exactly this specialization which constitutes the core of the fundamental notion of an *adjunction* (and eventually we definitely forget the terms "right" and "left").

Proposition 1.4.3. *Let* **A** *and* **B** *be partially ordered sets,* ϕ *a functor (i.e. an isotone map) between* **A** *and* **B***. Then, the following conditions are equivalent:*

1. *(a) There exists a functor* $\psi : \mathbf{B} \longmapsto \mathbf{A}$ *such that* $\phi \circ \psi \geq_A 1_A$ *and* $\psi \circ \phi \leq_B 1_B$.
   (a′) For all $b \in B, \phi^{\leftarrow}(\downarrow b)$ *is a principal order ideal of* **A***.*

2. *(b) There exists a functor* $\vartheta : A \longmapsto B$ *such that* $\vartheta \circ \phi \geq_B 1_B$ *and* $\phi \circ \vartheta \leq_A 1_A$.
   (b′) For all $a \in A, \vartheta^{\leftarrow}(\uparrow a)$ *is a principal filter of* B.

Proof. (**1**) $(a \Rightarrow a')$. Suppose (a) holds. Then from *Propositions* 1.4.1 and 1.4.2 we obtain for all $b \in B, \emptyset \subsetneq \phi^{\leftarrow}(\downarrow b) \subseteq \downarrow \psi(b)$. Suppose now $a \in\downarrow \psi(b)$, then $a \leq_A \psi(b)$ so that $\phi(a) \leq_B \phi(\psi(b)) \leq_B 1_B(b) = b$. It follows that $a \in \phi^{\leftarrow}(\downarrow \psi(b))$ and, thereafter, that for all $b \in B$, $\emptyset \subsetneq \phi^{\leftarrow}(\downarrow b) =\downarrow \psi(b)$. $(a' \Rightarrow a)$ Conversely, if (a') holds, then for all $b \in B$ there exists $a \in A$, clearly unique, such that $\phi^{\leftarrow}(\downarrow b) =\downarrow a$. From the above *Lemma* 1.4.1 it follows that ϕ is isotone. Clearly for any given b such an a is unique and we can set a mapping $\psi : B \longmapsto A; \psi(b) = a$. Since for all $b \in B$, $\psi(b) \in\downarrow \psi(b) = \phi^{\leftarrow}(\downarrow b)$, we immediately obtain that for all $b \in B$, $\phi(\psi(b)) \leq_B b$. Moreover, since for all $a \in A$, $a \in \phi^{\leftarrow}(\downarrow \phi(a)) =\downarrow (\psi(\phi(a)))$, we immediately obtain that for all $a \in A$, $\psi(\phi(a)) \geq_A a$.
(**2**) *Mutatis mutandis* the symmetric theorem with respect to ordering holds, too, by switching the partial order direction and swapping ideals and filters (we shall qualify such a kind of theorems as "*order-dual*").
<div align="right">QED</div>

The reader must take care of the fact that we have two levels of duality. The first swaps the partial order ($\leq$ into $\geq$ and vice-versa). The second swaps the order of application of the functions ($\phi \circ \psi$ into $\psi \circ \phi$, and the other way around) and the position of their domain and codomains. This is the reason why we do not use terms such "right residual", "dual residual", and the like (by the way, we notice that in usual literature given a map ϕ, the upper residual is denoted by ϕ_* and the lower residual is denoted by ϕ^*).

REMARKS. From now on **O** and **O′** shall denote two arbitrary partially ordered sets **O** = $\langle O, \leq \rangle$ and, respectively, **O′** = $\langle O', \leq' \rangle$.

Now we encode the existence preconditions, as specialized to partially ordered sets, in the following definition.

Definition 1.4.1. *Let* $\mathbf{O}$ *and* $\mathbf{O}'$, *be two preordered sets. Then,*

1. *A map* $\phi : \mathbf{O} \longmapsto \mathbf{O}'$ *is* upper-residuated *if for any principal ideal* $\downarrow p'$ *of* $\mathbf{O}'$, $\phi^{\leftarrow}(\downarrow p')$ *is a principal ideal of* $\mathbf{O}$. *In this case the map* $\phi^{+} : \mathbf{O}' \longmapsto \mathbf{O} : \phi^{+}(p') = max(\phi^{\leftarrow}(\downarrow p'))$ *is called the* upper-residual *of* ϕ.

2. *A map* $\phi : \mathbf{O} \longmapsto \mathbf{O}'$ *is* lower-residuated *if for any principal filter* $\uparrow p'$ *of* $\mathbf{O}'$, $\phi^{\leftarrow}(\uparrow p')$ *is a principal filter of* $\mathbf{O}$. *In this case the map* $\phi^{-} : \mathbf{O}' \longmapsto \mathbf{O} : \phi^{-}(p') = min(\phi^{\leftarrow}(\uparrow p'))$ *is called the* lower-residual *of* ϕ.

In other terms $\phi^{+}(p')$ is the generator of the principal ideal $\phi^{\leftarrow}(\downarrow p')$ that exists if ϕ is residuated. Dually for $\phi^{-}(p')$. From the above results we trivially have that if ϕ is upper (lower) residuated then it is isotone.

Proposition 1.4.3 proves that ϕ^{+} and ϕ^{-} behave in the required way, that is, for all $x \in O'$, $\phi\phi^{+}(x) \leq x$ and $\phi\phi^{-}(x) \geq x$. It is pretty evident that if ϕ^{+} is upper residual of ϕ then ϕ is lower residual of ϕ^{+}, that is, $\phi = (\phi^{+})^{-}$ and, dually, if ϕ^{-} exists then $\phi = (\phi^{-})^{+}$. Otherwise stated, lower and upper residuals are mutually defined notions. This is what is stated by the fundamental notion of a *Galois adjunction* which now comes onto stage.

1.4.3   Galois Adjunctions and Galois Connections

An overlook at Mathematical toolkit 16.3 is recommended.

Definition 1.4.2 (Adjointness relation)**.** *Let* $\sigma : \mathbf{O} \longmapsto \mathbf{O}'$ *and* $\iota : \mathbf{O}' \longmapsto \mathbf{O}$ *be two maps. Then we say that* ι *and* σ *fulfill an* adjointness relation *if the following holds:*

$$\forall p \in O, \forall p' \in O', \iota(p') \leq p \ \ if \ and \ only \ if \ \ p' \leq' \sigma(p) \qquad (1.4.3)$$

If the above conditions hold, then σ *is called the* upper adjoint *of* ι *and* ι *is called the* lower adjoint *of* σ. *This fact is denoted by*

$$\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O} \qquad (1.4.4)$$

and we shall say that the two maps form an adjunction *between* **O**
and **O**′. *If the two partial orders are understood we shall denote the*
adjunction with $\iota \dashv \sigma$, *too.*

We use the two symbols ι and σ for mnemonic reasons: ι stays for
"inferior" (=lower) and σ stays for "superior" (=upper).

When an adjointness relationship holds between two partial ordered
structures we say that the pair $\langle \sigma, \iota \rangle$ forms a *Galois adjunction* or an
axiality. This name is after the notion of a *Galois connection* which is
defined by means of a similar but covariant condition where, indeed, ι
and σ are antitone:

$$\forall p \in O, \forall p' \in O', \iota(p) \geq' p' \; if \; and \; only \; if \; \; p \leq \sigma(p') \qquad (1.4.5)$$

We read this fact by saying that the pair $\langle \sigma, \iota \rangle$ forms a *Galois connection*
or a *polarity*. Clearly, a Galois connection is a Galois adjunction with
the right structure turned into its opposite. Moreover, the notion of a
Galois adjunction is completely symmetric:

Proposition 1.4.4. *For any pair of order preserving maps* $\sigma : \mathbf{O} \longmapsto$
$\mathbf{O}'$ *and* $\iota : \mathbf{O}' \longmapsto \mathbf{O}$,

1. $\langle \sigma, \iota \rangle$ *is a polarity between* $\mathbf{O}^{\mathbf{op}}$ *and* $\mathbf{O}'$ *if and only if* $\langle \sigma, \iota \rangle$ *is an*
   axiality between $\mathbf{O}$ *and* $\mathbf{O}'$.

2. $\mathbf{O}' \dashv^{\iota, \sigma} \mathbf{O}$ *iff* $\mathbf{O}^{\mathbf{op}} \dashv^{\sigma, \iota} \mathbf{O}'^{\mathbf{op}}$.

Proof. Trivial, by easy inspection.   QED

Proposition 1.4.5. *Let it be* $\mathbf{O} \dashv^{\iota, \sigma} \mathbf{O}'$ *and* $\mathbf{O}' \dashv^{\iota', \sigma'} \mathbf{O}''$. *Then*
$\mathbf{O} \dashv^{\iota \circ \iota', \sigma' \circ \sigma} \mathbf{O}''$.

Proof. Trivial, by easy inspection of the ordering using the fact that
composition of isotonic maps are isotonic.   QED

REMARKS. So, axiality turns into polarity if we turn the right structure
upside-down. However, it must be emphasized that we must take some
care if the two structures (the two categories) coincide.

In literature about Category Theory, lower adjoints are usually called
"left adjoint", while upper adjoints are called "right adjoints". But, as
we have anticipated, here the position of the category does not matter.

Indeed, as noticed in [Gierz et al. 1980], this terminology is ambiguous and in our context this ambiguity could generate confusion because we shall meet situations in which operations must be concatenated in a precise order (as in the case of divisors or in the case of composition of relations).

Proposition 1.4.6. *Let* $\sigma : \mathbf{O} \longmapsto \mathbf{O}'$ *and* $\iota : \mathbf{O}' \longmapsto \mathbf{O}$ *be mappings,* $p \in O$ *and* $p' \in O'$. *Then,*

    1. $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O}$ *if and only if* $\sigma\iota(p') \geq' p'$ *and* $\iota\sigma(p) \leq p$, *and both* ι *and* σ *are isotone.*

    2. $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O^{op}}$ *if and only if* $\sigma\iota(p') \geq' p'$ *and* $\iota\sigma(p) \geq p$ *and both* ι *and* σ *are antitone.*

Proof. (**1**) ($\Rightarrow$) If ι is a lower adjoint, from (1.4.3) we obtain $\iota\sigma(p) \leq p$ from $\sigma(p) \leq' \sigma(p)$. Dually, if σ is an upper adjoint, we obtain $p' \leq' \sigma\iota(p')$ from $\iota(p') \leq \iota(p')$. Moreover if $x' \leq p'$ then by transitivity $x' \leq \sigma\iota(p')$, whence $\iota(x') \leq \iota(p')$ and symmetrically for σ. ($\Leftarrow$) if $p' \leq' \sigma(p)$ we obtain, from isotonicity of ι, $\iota(p') \leq \iota\sigma(p)$. But for hypothesis $\iota\sigma(p) \leq p$, so that $\iota(p') \leq p$. Similarly $p \geq \iota(p')$ implies $\sigma(p) \geq' \sigma\iota(p') \geq' p'$.
(**2**) Trivially from (1) and the fact that the right category is $\mathbf{O^{op}}$.   QED

Corollary 1.4.1. *Let* $f : A \longmapsto B$ *be a map. Then* $\langle \wp(A), \subseteq \rangle \dashv^{f^{\rightarrow},f^{\leftarrow}} \langle \wp(B), \subseteq \rangle$.

Proof. Evidently, both $f^{\leftarrow}$ and $f^{\rightarrow}$ are isotone. Moreover, suppose $X \subseteq B$, $x \in f^{\rightarrow}(f^{\leftarrow}(X))$ and $x \notin X$. Then there are $a \in A$ and $x' \in X$ such that $f^{\rightarrow}(\{a\}) \supseteq \{x, x'\}$, so that f is not a function. It follows that necessarily $f^{\rightarrow}(f^{\leftarrow}(X)) \subseteq X$. If $y \in Y \subseteq A$ but $y \notin f^{\leftarrow}(f^{\rightarrow}(Y))$ then for no $b \in B, f(y) = b$. It follows that f is not defined over all A. Thus necessarily $Y \subseteq f^{\leftarrow}(f^{\rightarrow}(Y))$ because we have assumed by default that f is a total function. From *Proposition* 1.4.6 we conclude the proof.   QED

So adjoint pairs behave exactly the residuation-like way described in order to define approximating maps. And, indeed, now we show the links between residuation and adjointness.

Proposition 1.4.7. *Let* $\sigma : \mathbf{O} \longmapsto \mathbf{O}'$ *and* $\iota : \mathbf{O}' \longmapsto \mathbf{O}$ *be two mappings. Then the following conditions are equivalent:*

1. $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O}$.

2. σ *is isotone and* $\iota(p') = min(\sigma^{\leftarrow}(\uparrow p'))$, *for all* $p' \in \mathbf{O}'$ *(1-min).*

3. ι *is isotone and* $\sigma(p) = max(\iota^{\leftarrow}(\downarrow p))$, *for all* $p \in \mathbf{O}$ *(1-max).*

The proof is immediate from *Proposition* 1.4.3 and *Proposition* 1.4.6.

Example 1.4.1. Adjointness

Conditions $\iota\sigma(x) \leq x$ and $\sigma\iota(y) \geq y$, on the one side, and isotonicity of both ι and σ, on the other side, must hold in order for a pair of functions $\langle \iota, \sigma \rangle$ to be an adjoint pair. Here is an extreme counterexample. Let $\mathbf{A} = \langle A = \{1,2,3\}, 1 \leq 3, 2 \leq 3, 1 \leq 1, 2 \leq 2, 3 \leq 3 \rangle$, $\mathbf{B} = \langle B = \{a,b,c\}, a \leq b, a \leq a, b \leq b, c \leq c \rangle$ and let $f : A \longmapsto B; f(1) = c, f(2) = a, f(3) = b$, $g : B \longmapsto A; g(a) = 2, g(b) = 3, g(c) = 1$. Then, g is isotone, $g = f^{-1}$, g is monic and epic (so is f). Hence for all $x \in A$, $gf(x) = x$ and for all $y \in B$, $fg(y) = y$. However g and f do not form a Galois adjunction because f is not isotone. Indeed, although for all $x \in B$, $g(x) = max(f^{-1}(\downarrow x))$ and $g(x) = min(f^{-1}(\uparrow x))$, nonetheless, neither $min(g^{-1}(\uparrow 1))$ nor $max(g^{-1}(\downarrow 3))$ exist.

As a further extreme situation let us compare the following two pairs of functions:

Non adjoint maps g' is lower adjoint of f
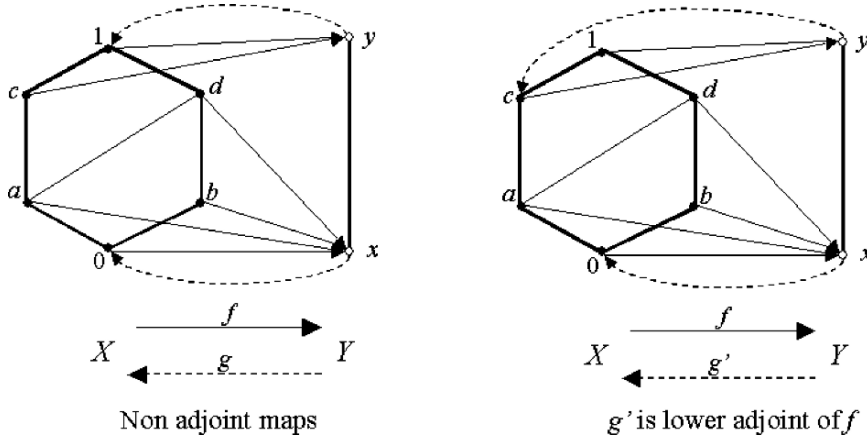
We can notice what follows: although both f and g are homomorphisms, they do not form a pair of adjoint maps. Indeed $min(f^{-1}(\uparrow y)) = min\{c,1\} = c \neq 1 = g(y)$. Also, $max(f^{-1}(\downarrow x)) = max\{0,a,b,d\} = d \neq 0 = g(x)$ (we can also notice that $y \leq y = f(c)$ while $g(y) = 1 \not\leq c$). Thus neither $f \dashv g$ nor $g \dashv f$ hold. However, since they both are homomorphisms they must have both upper and lower adjoints.

Indeed, $g' \dashv f$. Moreover set $g'' : Y \longmapsto X$ such that $g''(y) = 1$ and $g''(x) = d$. Then $f \dashv g''$. Now, set $f' : X \longmapsto Y$ such that $f'(a) = y$ if $a = 1$ and $f'(a) = x$ otherwise. Then $g \dashv f'$. Finally, if $f'' : X \longmapsto Y$ is such that $f''(a) = x$ if $a = 0$ and $f''(a) = y$ otherwise, then $f'' \dashv g$.

Exercise 1.2. *Consider Example 1.4.1. Find the upper adjoint of f' and the lower adjoint of f'', if any.*

1.4.4   Algebraic Properties of Adjoint Maps

Adjoint maps enjoy very interesting properties which are fundamental in the present investigation.

One of the most important features connected with Galois adjunctions is, in particular, that given two complete lattices $\mathbf{L}$ and $\mathbf{L}'$, if a map $\phi : \mathbf{L} \longmapsto \mathbf{L}'$ preserves joins, then it has an upper residual ϕ^+ and this upper residual is a meet-preserving map $\mathbf{L}' \longmapsto \mathbf{L}$. Dually, if a map $\phi : \mathbf{L} \longmapsto \mathbf{L}'$ preserves meets, then it has a lower residual ϕ^- and this lower residual is a join-preserving map $\mathbf{L}' \longmapsto \mathbf{L}$. The idea here is that formulae $(1 - min)$ and $(1 - max)$ above define $infs$ and $sups$ in lattices. Categorically stated, upper adjoints preserve *limits*, while lower adjoints preserve *colimits*. We prove the above statements by means of two Lemmata.

Lemma 1.4.2. *Let $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O}$ hold. Then σ preserves all the existing* infs *and ι preserves all the existing* sups.

Proof. Let $P \subseteq O$. Suppose $p = \bigwedge P$. Since σ is order preserving and $p \leq x, \forall x \in P$, we have $\sigma(p) \leq \sigma(x)$, for all $x \in P$. If t is a lower bound of $\sigma^{\rightarrow}(P)$, then for all $x \in P, \sigma(x) \geq t$ and $x \geq \iota(t)$. It follows that any lower bound of $\sigma^{\rightarrow}(P)$ is mapped by ι onto a lower bound of P. But p is the greatest of such lower bounds. Therefore $p \geq \iota(t)$ for all such t. It follows that $\sigma(p) \geq' t$, for all lower bounds t of $\sigma^{\rightarrow}(P)$. Thus, $\sigma(p)$ is the greatest among the lower bounds of $\sigma^{\rightarrow}(P)$, that is, $\sigma(p) = \bigwedge \sigma^{\rightarrow}(P)$. But $p = \bigwedge P$ so that we have $\sigma(\bigwedge P) = \bigwedge \sigma^{\rightarrow}(P)$.

Dually one can prove that ι preserves sups.   QED

Lemma 1.4.3. *Let $\mathbf{L}$ be a complete lattice and $\mathbf{O}$ a partially ordered set. Then,*

1. *Any map $\phi : \mathbf{L} \longmapsto \mathbf{O}$ which preserves all infs has a lower adjoint, ϕ^-.*

2. Any map $\phi : \mathbf{L} \longmapsto \mathbf{O}$ which preserves all sups has an upper adjoint, ϕ^+.

Proof. (**1**) Since $\mathbf{L}$ is a complete lattice, we can define a morphism $\phi^- : \mathbf{O} \longmapsto \mathbf{L}$ by means of formula $\phi^-(p) = \bigwedge \phi^{\leftarrow}(\uparrow p)$, which is a specialization to complete lattices of formula $(1 - min)$ of *Proposition* 1.4.7. Clearly, ϕ^- is isotone (see the proof for the isotonicity of ι in *Proposition* 1.4.7). If now $p \leq \phi(l)$, then $l \in \phi^{\leftarrow}(\uparrow p)$, so that $\phi^-(p) = \bigwedge \phi^{\leftarrow}(\uparrow p) \leq l$. Hence, $p \leq \phi(l)$ implies $\phi^-(p) \leq l$. Conversely, assume that $\phi^-(p) \leq l$. Clearly $\phi^-(p) \leq l$ implies $\bigwedge \phi^{\leftarrow}(\uparrow p) \leq l$, by definition of ϕ^-. Moreover, since ϕ preserves all infs, it is monotone. Hence, $\phi(\bigwedge \phi^{\leftarrow}(\uparrow p)) \leq \phi(l)$. But since ϕ preserves infs, we obtain: $\phi(\bigwedge \phi^{\leftarrow}(\uparrow p)) = \bigwedge \phi(\phi^{\leftarrow}(\uparrow p)) \geq \bigwedge(\uparrow p) = p$ (notice that the inequality "$\geq$" does not turn into equality unless ϕ is surjective). (**2**) Dually.   QED[5]

To sum up, it is clear that in order to have a lower adjoint ϕ^- (an upper adjoint ϕ^+), a map ϕ is required to fulfill something weaker than being a lattice homomorphism: ϕ has to be a $\wedge$-preserving map (ϕ has to be a $\vee$-preserving map). And, vice-versa, in order to preserve *infs* – i.e. limits – (to preserve *sups* – i.e. co-limits) for a map ϕ between a complete lattice and a partially ordered set, ϕ must be isotone and must have a lower adjoint (an upper adjoint).

At this point we can list a number of important properties of adjoint maps between complete lattices.

Proposition 1.4.8. *Let $\mathbf{L}' \dashv^{\iota,\sigma} \mathbf{L}$ and $\mathbf{L}' \dashv^{\varepsilon,\varsigma} \mathbf{L}^{op}$ hold. Let $x', y' \in L'$ and $x, y \in L$. Then the following holds:*

1. *(a) σ is multiplicative; (b) ι is additive.*

2. *(a) $\sigma(x \vee y) \geq' \sigma(x) \vee' \sigma(y)$; (b) $\iota(x' \wedge' y') \leq \iota(x') \wedge \iota(y')$.*

3. *(a) ς and ε are anti-additive maps between $\mathbf{L}$ and $\mathbf{L}'$.*

4. *(a) $\varepsilon(x' \vee' y') \geq \varepsilon(x') \vee \varepsilon(y')$; (b) $\varsigma(x \wedge y) \geq' \varsigma(x) \wedge' \varsigma(y)$.*

5. *(a) $\iota\sigma(x \vee y) \geq \iota\sigma(x) \vee \iota\sigma(y)$; (b) $\sigma\iota(x' \vee' y') \geq' \sigma\iota(x') \vee' \sigma\iota(y')$.*

---

[5]Implicitly we have also proved that $\phi(\phi^-(p)) \geq p$. In fact, since $x \leq \phi(y)$ implies $\phi(y) \in \uparrow x$ and, therefore, $y \in \phi^{\leftarrow}(\uparrow x)$, we immediately obtain that $\phi(\phi^-(p)) \in \uparrow p$ and $\phi^-(p) \in \phi^{\leftarrow}(\uparrow p)$, so that $\bigwedge \phi^{\leftarrow}(\uparrow p) = min(\phi^{\leftarrow}(\uparrow p))$. In other words, $\bigwedge \phi^{\leftarrow}(\uparrow p)$ belongs to $(\phi^{\leftarrow}(\uparrow p))$, and, consequently it is its minimum.

6. *(a) $\iota\sigma(x \wedge y) \leq \iota\sigma(x) \wedge \iota\sigma(y)$; (b) $\sigma\iota(x' \wedge' y') \leq' \sigma\iota(x') \wedge' \sigma\iota(y')$.*

7. *(a) $\varepsilon\varsigma(x \wedge y) \leq \varepsilon\varsigma(x) \wedge \varepsilon\varsigma(y)$; (b) $\varsigma\varepsilon(x' \wedge' y') \leq' \varsigma\varepsilon(x') \wedge' \varsigma\varepsilon(y')$.*

8. *(a) $\varepsilon\varsigma(x \vee y) \geq \varepsilon\varsigma(x) \vee \varepsilon\varsigma(y)$; (b) $\varsigma\varepsilon(x' \vee' y') \geq' \varsigma\varepsilon(x') \vee' \varsigma\varepsilon(y')$.*

9. *(a) $\iota = \iota\sigma\iota$, $\sigma = \sigma\iota\sigma$, $\varepsilon = \varepsilon\varsigma\varepsilon$, $\varsigma = \varsigma\varepsilon\varsigma$, (b) $\sigma\iota$, $\iota\sigma$, $\varepsilon\varsigma$ and $\varsigma\varepsilon$ are idempotent.*

10. *$\iota\sigma(0) = 0$, $\sigma\iota(1') = 1'$, $\varepsilon\varsigma(1) = 1$, $\varsigma\varepsilon(1') = 1'$.*

Proof. (**1**) From *Proposition* 1.4.2. (**2**) (a) $x \vee y \geq x$ and $x \vee y \geq y$, hence, from isotonicity, $\sigma(x \vee y) \geq' \sigma(x)$ and $\sigma(x \vee y) \geq' \sigma(y)$, so that $\sigma(x \vee y) \geq' \sigma(x) \vee' \sigma(y)$. (b) $x' \wedge' y' \leq x'$ and $x' \wedge' y' \leq y'$, hence, from isotonicity, $\iota(x' \wedge' y') \leq \iota(x')$ and $\iota(x' \wedge' y') \leq \iota(y')$, so that $\iota(x' \wedge' y') \leq \iota(x') \wedge \iota(y')$. (**3**) ς maps limits of $\mathbf{L}^{op}$, hence co-limits of $\mathbf{L}$, hence *sups*, into limits of $\mathbf{L}'$, that is, *infs*. Similarly, ε maps co-limits of $\mathbf{L}'$, hence *sups* of $\mathbf{L}'$, into co-limits of $\mathbf{L}^{op}$, hence *infs* of $\mathbf{L}$. (**5**), (**6**), (**7**) and (**8**) are proved directly from the previous points, eventually with the aid of *Proposition* 1.4.6. (**9**) (a) Because $\iota(p') = \iota(p')$ and, in view of *Proposition* 1.4.6, $\iota\sigma(p) \leq p$, we have $\iota\sigma(\iota(p')) \leq \iota(p')$. Vice-versa, since ι is monotone and $p' \leq \sigma\iota(p')$, we obtain $\iota(p') \leq \iota\sigma(\iota(p'))$. It follows that $\iota(p') = \iota\sigma\iota(p')$. Dually one can prove the remaining equations. (b) From these equations one can straightforwardly prove $\sigma\iota = \sigma\iota\sigma\iota$ and $\iota\sigma = \iota\sigma\iota\sigma$ and the other equations. (**10**) From the decreasing property of $\iota\sigma$ we obtain $\iota\sigma(0) \leq 0$, dually from the increasing property of $\sigma\iota$, $\varsigma\varepsilon$ and $\varepsilon\varsigma$ we have the other equations. QED

It is must be emphasized that the above properties about ε and ς are valid if we fix $\mathbf{L}$ so that the limits of $\mathbf{L}^{op}$ are the co-limits of $\mathbf{L}$. Otherwise $\mathbf{L}^{op}$ as such is a lattice with its own *infs* and *sups* and the normal adjointness properties hold.

Now we have a good stock of results in order to "implement" a sufficiently large body of useful operators, namely those operators which will constitute the backbone of all the present story.

Definition 1.4.3. *Let $\phi : \mathbf{O} \longmapsto \mathbf{O}$ be an operator on a partially ordered set and $\vartheta : \mathbf{L} \longmapsto \mathbf{L}'$ be an operator between two lattices. Then,*

1. *ϕ is a* projection *operator on $\mathbf{O}$ if and only if it is isotone and idempotent.*

2. *A projection operator is a* closure operator *if and only if it is increasing.*

3. *A projection operator is an* interior operator *if and only if it is decreasing.*

4. ϑ *is a* modal operator *if and only if it is normal and additive.*

5. ϑ *is a* co-modal operator *if and only if it is co-normal and multiplicative.*

6. *If ϑ is a closure operator then it is* topological *if and only if it is modal.*

7. *If ϑ an interior operator, then it is* topological *if and only if it is co-modal.*

8. ϑ *is an* anti-modal operator *if and only if it is anti-normal and anti-additive.*

9. ϑ *is an* anti-co-modal operator *if and only if it is anti-co-normal and anti-multiplicative.*

10. ϑ *is a* complementation operator *if and only if it is an anti modal and anti-co-modal operator.*

Then from *Proposition* 1.4.6 we immediately obtain:

Corollary 1.4.2. *Let $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O}$ and $\mathbf{O}' \dashv^{\varepsilon,\varsigma} \mathbf{O}^{op}$ hold (hence the latter is a Galois connection between $\mathbf{O}$ and $\mathbf{O}'$). Then,*

1. *(a) $\sigma\iota$ is a closure operator on $\mathbf{O}'$; (b) $\iota\sigma$ is an interior operator on $\mathbf{O}$.*

2. *(a) $\varsigma\varepsilon$ is a closure operator on $\mathbf{O}'$; (b) $\varepsilon\varsigma$ is a closure operator on $\mathbf{O}$.*

Notice that none of these operators needs to be topological.

Moreover given the above adjointness situations we can underline what follows:

- σ is half of a co-modal operator: it lacks co-normality.

- ι is half of a modal operator: it lacks normality.

- ϵ and ς are half of an anti-modal operator: they lack anti-normality.

REMARKS. Modal operators are also called *possibility* operators and co-modal operators are also called *necessity* operators while anti-modal operators are called *sufficiency* operators. We shall adopt this more intuitive terminology in Section 2.2. Notice that for an operator ϑ to be a modality we do not require ϑ to be an endomorphism. That is, our notions apply to generic morphisms between two possibly distinct lattices.

Excursus 1.1 (*Closure systems and adjointness*).

It should be noted that given a function $f : A \longmapsto B$, with f^o we denote the *corestriction* $A \longmapsto Im_f$ of f to Im_f. Dually, with f_o we shall denote the inclusion $Im_f \longmapsto B$ of Im_f into B.

1. Let $f : \mathbf{L} \longmapsto \mathbf{L}$, for $\mathbf{L}$ a lattice. With $f(\mathbf{L})$ we shall denote Im_f equipped with the order inherited from $\mathbf{L}$. It can be shown:

   (a) f is a projection if and only if f^o is a retraction of $\mathbf{L}$ on $f(\mathbf{L})$ with coretraction f_o.

   (b) f is a closure operator if and only if $\mathbf{L} \dashv^{f^o, f_o} f(\mathbf{L})$ if and only if $f = \iota\sigma$ and $\mathbf{S} \dashv^{\iota, \sigma} \mathbf{L}$, for some $\mathbf{S}$.

   (c) f is an interior operator if and only if $f(\mathbf{L}) \dashv^{f_o, f^o} \mathbf{L}$ if and only if $f = \sigma\iota$ and $\mathbf{S} \dashv^{\iota, \sigma} \mathbf{L}$, for some $\mathbf{S}$.

   Indeed, remember that f can be decomposed into $f^o f_o$. Now suppose f is a closure operator, then $f^o f_o \geq 1_{\mathbf{L}}$ because, trivially, $f^o f_o = f$. Conversely, if $f^o \dashv f_o$ then $f^o f_o \geq 1_{\mathbf{L}}$. If $f = \iota\sigma$ and $\mathbf{S} \dashv^{\iota, \sigma} \mathbf{L}$ then f is idempotent. Moreover $\iota\sigma \geq 1_{\mathbf{L}}$ for adjunction properties, so that f is a closure operator. In a dual manner we prove point 1c.

2. Let $\mathbf{L}$ be a complete lattice and $\mathtt{C}(\mathbf{L})$ be the set of all families of subsets of $\mathbf{L}$ which are closed under arbitrary meets. $\langle \mathtt{C}(\mathbf{L}), \subseteq \rangle$ is a partially ordered set and if $S \in \mathtt{C}(\mathbf{L})$ then S is said to be a *closure system*.

   As an example of a closure system, consider a lattice $\mathbf{L}$ with minimum 0. Then the family of all ideals of $\mathbf{L}$, $\mathscr{I}(\mathbf{L})$, is a closure system such that any element $I \in \mathscr{I}(\mathbf{L})$ is given by a closure operator $c_I(A) = \downarrow \bigvee A$, any $A \subseteq L$. The reader will notice that this fact will lies at the core of our definition of closure operators on Information Systems. Another example: Let $\mathbf{G}$ be a group and $Sub(\mathbf{G})$ the family of subgroups of $\mathbf{G}$. Then $Sub(\mathbf{G})$ is a closure system and for $S \in Sub(\mathbf{G})$ the corresponding closure operator is given by $c_S(A) = \langle A \rangle$, where $\langle A \rangle$ is the subgroup generated by A.

3. More in general, we can associate closure systems and closure operators by exploiting the fact that a section of a function f selects an element out of each equivalence class modulo the kernel of f, and that f_o is a coretraction, hence a section. Indeed, we can associate with each closure operator c over $\mathbf{L}$ a member of $\mathtt{C}(\mathbf{L})$ just by taking the family $\phi(c) = \{x : x = c(x)\}$. The association ϕ turns to be an order isomorphism between the set of closure operators pointwise ordered and $\mathtt{C}(\mathbf{L})^{op}$.

   The inverse map ϕ^{-1} associates with any closure system $S \in \mathtt{C}(\mathbf{L})$ the upper adjoint of the inclusion map $in : S \longrightarrow L$ followed by in itself. The

upper adjoint of *in* is clearly given by the formula $in^+(x) = min(\uparrow x \cap S) = \bigwedge(\uparrow x \cap S)$, all $x \in L$. Therefore, ϕ^{-1} associates with any closure system S the closure operator c_S which induces S itself, by means of the formula $c_S(x) = \bigwedge\{y \in S : x \leq y\}$, any $x \in L$. Figures 1.11 and 1.12 depict an illustrative example:
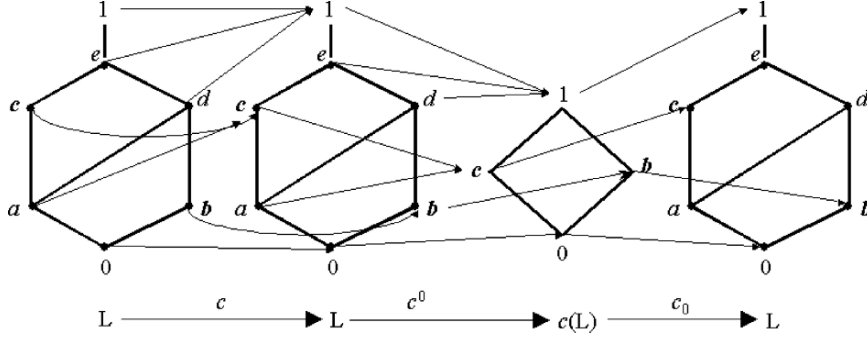
Figure 1.11: Decomposition of a closure operator
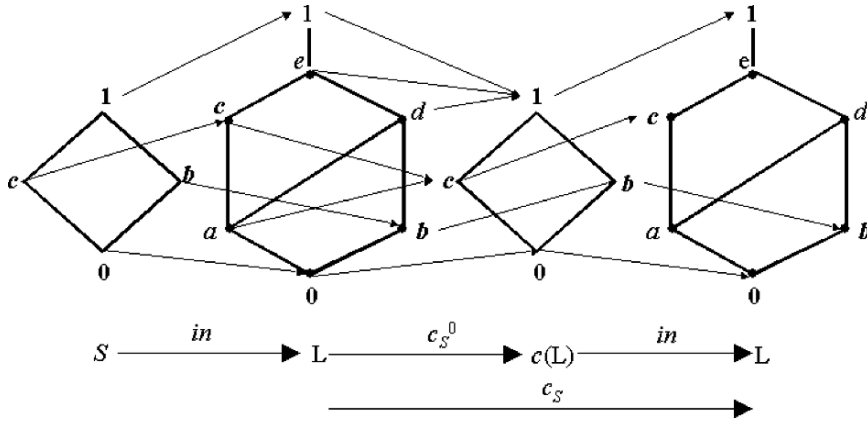
Figure 1.12: Composition of a closure operator

In the next paragraph we shall see situations in which the above operators turn into topological operators and/or complete modal, co-modal and anti-modal operators. These situations lie at the core of our arguments about "perception operators". But we must immediately notice that the lack of properties involving top and bottom elements is quite obvious since they depend on the nature of $\mathbf{O}, \mathbf{O}'$ and $\mathbf{L}, \mathbf{L}'$. On the contrary, the lack of properties concerning preservation of operations

may be partially amended when we restrict domains to the families of fixed points of the operators $\sigma\iota, \iota\sigma, \varepsilon\varsigma$ and $\varsigma\varepsilon$.

To this end the following two results are fundamental:

Lemma 1.4.4. *Let* $\phi : \mathbf{O} \longmapsto \mathbf{O}$ *be a map. Then,*
(a) If ϕ *is closure then* $\mathbf{O} \dashv^{\phi^o, \phi_o} Im_\phi$; *(b) if* ϕ *is interior then* $Im_\phi \dashv^{\phi_o, \phi^o} \mathbf{O}$.

Proof. (a) We remind that for any map f, $f_o = 1_{Im_f}, f^o = f, f = f^o \circ f_o$ and if f is idempotent then $f_o \circ f^o = 1_{Im_f}$. Therefore, since ϕ is idempotent $\phi_o \circ \phi^o = 1_{Im_f}$ and since it is increasing $\phi^o \circ \phi_o \geq 1_O$. Moreover, ϕ is monotone so is ϕ^o, while ϕ_o is monotone *qua* identity map. Hence the thesis follows from *Proposition* 1.4.6. (b) By duality. Anyway the following direct proof may be of some interest. Because ϕ is interior $\phi(x) = \phi_o \phi^o(x) \leq x$ any $x \in O$. Hence if $y \leq \phi^o(x)$, since ϕ_o is monotonic we have $\phi_o(y) \leq \phi_o \phi^o(x) \leq x$, and by transitivity $\phi_o(y) \leq x$. Conversely if $\phi_o(y) \leq x$ then $\phi^o \phi_o(y) \leq \phi^o(x)$. But $\phi^o \phi_o(y) = 1_{Im_f}(y) = y$. Thus $y \leq \phi^o(x)$. QED

Corollary 1.4.3. *Let* $\phi : \mathbf{L} \longmapsto \mathbf{L}$ *be a map. Then,*

1. *If* ϕ *is closure then (a)* ϕ^o *is additive, (b)* $\phi_o(Im_\phi)$ *is closed under meets, (c)* $\mathbf{Sat}_\phi(L) =_{def} \langle Im_\phi, \wedge, \sqcup, 1 \rangle$, *where for all* $x, y \in Im_\phi$, $x \sqcup y = \phi(x \vee y)$, *is a lattice; (d) If, in addition,* ϕ *is surjective, then it is topological.*

2. *If* ϕ *is interior then (a)* ϕ^o *is multiplicative, (b)* $\phi_o(Im_\phi)$ *is closed under joins, (c)* $\mathbf{Sat}_\phi(L) =_{def} \langle Im_\phi, \sqcap, \vee, 0 \rangle$, *where for all* $x, y \in Im_\phi$, $x \sqcap y = \phi(x \wedge y)$, *is a lattice. (d) If, in addition,* ϕ *is surjective, then it is topological.*

Proof. (**1**) (a) Assume ϕ is closure. Then from *Lemma* 1.4.4 ϕ^o is lower adjoint. Hence ϕ^o is additive. (b) In turn, ϕ_o is upper adjoint, thus it is multiplicative. But $\phi_o = 1_{Im_\phi}$ so that meets in Im_ϕ must coincide with meets in $\mathbf{L}$. (c) Finally, notice that $1 \in Im_\phi$; now, consider the set of upper bounds $\{a, b\}^u$. Clearly the least upper bound (join) of a and b, $a \sqcup b$, is $\bigwedge \{a, b\}^u$. If $\{a, b\}^u = \emptyset$ then $\bigwedge \{a, b\}^u = 1$. Otherwise suppose $x = \bigwedge \{a, b\}^u$. Then x exists in Im_ϕ because it is closed under meets. We claim that $x = \phi(a \vee b)$. By definition, $x \geq a, x \geq b$ so that $x \geq a \vee b$. Since $x \in Im_\phi$, $x = \phi(z)$ for some $z \in L$. Hence $\phi(z) \geq a \vee b$. For isotonicity $\phi\phi(z) \geq \phi(a \vee b)$ and for idempotence

$\phi(z) \geq \phi(a \vee b)$, that is, $x \geq \phi(a \vee b)$. But $a, b \leq a \vee b$ and since ϕ is inflationary $a \vee b \leq \phi(a \vee b)$. It follows that $\phi(a \vee b) \in \{a, b\}^u$. Moreover, $\phi(a \vee b) \in Im_\phi$. Thus $x \leq \phi(a \vee b)$. (d) If ϕ is surjective, then it coincides with ϕ^o. (**2**) may be proved dually. QED

REMARKS. If ϕ is an operator on a lattice **L** and $\phi_o(Im_\phi)$ is closed under meets (under joins), then one can say that Im_ϕ *inherits meets* (*joins*) from **L**.

Notice that if ϕ is closure then joins in **L** and joins in Im_ϕ may differ. Hence, although for all $x \in L, \phi(x) = \phi^o(x)$ and ϕ^o is additive, nonetheless ϕ in general is not additive and $\phi_o(Im_\phi)$ could be not closed under joins. Moreover from the above results we deduce that ϕ is additive if joins in **L** and joins in Im_ϕ coincide. The same remarks hold for interior maps and meets.

These results together with *Corollary* 1.4.2 give the following proposition (where we have just to notice that by turning $\mathbf{L}^{op}$ upside-down interiors turns into closures and joins into meets):

Proposition 1.4.9. *Let* $\mathbf{L}' \dashv^{\iota,\sigma} \mathbf{L}$ *and* $\mathbf{L}' \dashv^{\varepsilon,\varsigma} \mathbf{L}^{op}$ *hold. Then:*

1. $\mathbf{Sat}_{\iota\sigma}(\mathbf{L}) = \langle Im_{\iota\sigma}, \sqcap, \vee, 0 \rangle$, *where for all* $x, y \in Im_{\iota\sigma}$, $x \sqcap y = \iota\sigma(x \wedge y)$, *is a lattice.*

2. $\mathbf{Sat}_{\sigma\iota}(\mathbf{L}') = \langle Im_{\sigma\iota}, \wedge', \sqcup, 1' \rangle$, *where for all* $x, y \in Im_{\sigma\iota}$, $x \sqcup y = \sigma\iota(x \vee' y)$, *is a lattice.*

3. $\mathbf{Sat}_{\varsigma\varepsilon}(\mathbf{L}') = \langle Im_{\varsigma\varepsilon}, \wedge', \sqcup, 1' \rangle$, *where for all* $x, y \in Im_{\varsigma\varepsilon}$, $x \sqcup y = \varsigma\varepsilon(x \vee' y)$, *is a lattice.*

4. $\mathbf{Sat}_{\varepsilon\varsigma}(\mathbf{L}) = \langle Im_{\varepsilon\varsigma}, \wedge, \sqcup, 1 \rangle$, *where for all* $x, y \in Im_{\varepsilon\varsigma}$, $x \sqcup y = \varepsilon\varsigma(x \vee y)$, *is a lattice.*

Exercise 1.3. *Give a direct proof of Proposition 1.4.9 [Hints: start with* $\iota\sigma(x \vee y) \geq \iota\sigma(x) \vee \iota\sigma(y) = x \vee y$*].*

Until now we have dealt with the situation in which a function can be approximated from below or from above by means of a pair of adjoint maps. Now let us focus on a case in which the pair of adjoint maps provides us with the *same* information as the two identity maps. That is, when the two residuals are divisors of the two identity maps.

We have already met this situation in Section 1.3 where given a mapping ϕ from A to B, a mapping ψ from B to A was called a *section*

of ϕ, if $\psi \circ \phi = 1_B$, that is, for any $b \in B, \phi(\psi(b)) = b$. Dually, if $\phi \circ \psi = 1_A$, then ψ was called a *retraction* of ϕ.

Also, it is proved that in order to have a section, ϕ must be surjective, while in order to have a retraction it must be injective. Thus it is straightforward to show that the lower adjoint of a surjective mapping ϕ is a retraction of ϕ and the upper adjoint of an injective mapping ψ is a section of ψ (of course, provided they exists).

Proposition 1.4.10. *Let* $\mathbf{O}$ *and* $\mathbf{O}'$ *be partially ordered sets. Let* $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O}$, $p' \in \mathbf{O}'$ *and* $p \in \mathbf{O}$. *Then the following are two sets of equivalent statements:*

1. *(a) σ is surjective; (b) $\iota(p') = min(\sigma^{\leftarrow}(\{p'\}))$; (c) $\sigma\iota(p') = p'$; (d) ι is injective.*

2. *(a') σ is injective; (b') $\sigma(p) = max(\iota^{\leftarrow}(\{p\}))$; (c') $\iota\sigma(p) = p$; (d') ι is surjective.*

Proof. (**1**) $(a) \Rightarrow (b)$: $\sigma\iota(p') = \sigma(min(\sigma^{\leftarrow}(\uparrow p')))$. But since σ is monotonic, $\sigma(min(X)) = min(\sigma^{\rightarrow}(X))$, any X, so that $\sigma(min(\sigma^{\leftarrow}(\uparrow p'))) = min(\sigma^{\rightarrow}\sigma^{\leftarrow}(\uparrow p'))$. Moreover, since σ is surjective, $\sigma^{\rightarrow}\sigma^{\leftarrow}(\uparrow p') = \uparrow p'$. Hence $min(\sigma^{\rightarrow}\sigma^{\leftarrow}(\uparrow p')) = min(\uparrow p') = p'$. Therefore, $\iota(p') \in \sigma^{\leftarrow}(\{p'\})$ (notice that this relation holds because of surjectivity of σ, while generally, just $\iota(p') \in \sigma^{\leftarrow}(\uparrow p')$ holds). Therefore, since $p' = min(\uparrow p')$ and ι is monotone, we obtain $\iota(p') = min(\sigma^{\leftarrow}(\{p'\}))$.

$(b) \Rightarrow (c)$: Because $\iota(p') \in \sigma^{\leftarrow}(\{p'\})$, we immediately have $\sigma\iota(p') = p'$, for all $p' \in \mathbf{O}'$.

$(c) \Rightarrow (d)$: Since, for hypothesis, $\sigma\iota(p') = p'$, all $p' \in \mathbf{O}'$, we have that ι is a section of σ. Hence, as we have already seen, ι must be injective.[6]

$(d) \Rightarrow (a)$: From $\iota\sigma\iota = \iota$ we obtain $\iota\sigma\iota(x) = \iota(x)$. But since ι is injective, for hypothesis, we must deduce $\sigma\iota(x) = x$. Thus, σ is a retraction, whence it is surjective. (**2**) Dually. QED

An adjunction $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O}$ such that $\iota\sigma = 1_O$, is called a *Galois insertion* or *Galois embedding* or a *reflection*. If $\sigma\iota = \mathbf{1}_{O'}$ then it is called a *co-reflection*.

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[6]Moreover, since ι is a section of σ, σ is a retraction of ι. Hence σ must be surjective. Thus, as a side effect we have also proved $(c) \Rightarrow (a)$.

Example 1.4.2.

About the maps depicted in Example 1.4.1, the reader will note that $g'f(1) = c \lneq 1$ because f is not injective while $fg'(y) = y, fg'(x) = x$ because f is surjective.

It is worth noticing that from $f'' \dashv g \dashv f'$ we can obtain the composition $(f'' \circ g) \dashv (f' \circ g)$. Indeed $(f'' \circ g)(a) = 0$ if $a = 0$, $(f'' \circ g)(a) = 1$ otherwise and $(f' \circ g)(a) = 0$ if $a = 1$, 0 otherwise and we can readily check that the adjointness properties hold.

Chapter 2

Concrete and Formal Information Constructions

2.1 Concrete and Formal Observation Spaces

Now let us come back to the observation systems.

First of all, let us make an inventory of the basic actions one is allowed to do in a generic *P-system* $\langle G, M, \Vdash \rangle$:

- Any point of G may be uniquely associated with a subset of M (collecting the observed properties of the given point).

- Any property of M may be associated with a subset of G (collecting the points manifesting the given property).

It seems to be a really poor stock of actions. Can we mine meaningful information by means of the mathematical tools so far introduced? Can one define a "conceptual geometry" on G and M using this machinery?

It will be seen that observation systems together with the mathematical machinery so far introduced are sufficient to define a rather complex and comprehensive theory.

We shall start with an '*obs*ervation function' $obs : G \longmapsto \wp(M)$, defined by setting:

$$b \in obs(a) \Leftrightarrow a \Vdash b. \tag{2.1.1}$$

Technically, *obs* is what is called a *constructor* because it builds-up a set from a point. Obviously, for each point a, $obs(a)$ is $\{b \in M : a \Vdash b\}$, that is, the set of properties that are observed of a. We shall call $obs(a)$ the 'intension of a'. In fact, any element a appears through the series

of its observable properties, so that $obs(a)$ is actually the *intensional description* of a. The intension of a point a is, therefore, nothing but its description through the observable properties listed in M. We shall also say that if $a \Vdash b$ (i.e. if $b \in obs(a)$), then b is an observable property *connected* with a and that a belongs to the *field* of b.

By means of the observation function obs we obtain the family

$$\mathcal{OBS}(G) = \{obs(a) : a \in G\}$$

We shall call $\langle M, \mathcal{OBS}(G) \rangle$ a *formal observation space*. In pure mathematical terms, obs is an indexing function. Indeed, in $obs(a)$ the symbol a may be considered a mere aseptic index for a family of observables. This is coherent with our phenomenological approach because we can avoid the symbol a playing the ambiguous role of a pre-phenomenological essence.

Also, in mathematical terms this construction is reversible because observables may be conceived as indices for sets of points. This is coherent with the fact that a genuine phenomenological approach does not collapse into a sort of idealism. To avoid this danger, we can deal with a "*sub*stance function" $sub : M \longmapsto \wp(G)$ defined by setting:

$$a \in sub(b) \Leftrightarrow a \Vdash b. \qquad (2.1.2)$$

This symmetry reflects the intuition that a point can be *intensionally* conceived of as the set of properties it is connected with, just as a property may be *extensionally* conceived of as the set of points belonging to its field. Dually to obs, given a property $b \in M$, $sub(b) = \{a \in M : a \Vdash b\}$, so that $sub(b)$ is the 'extension', or the field, of b (see Figure 2.1 below).

The link between these two functions is the above mentioned relation $\Vdash$, because for all $a \in G$ and $b \in M$,

$$a \in sub(b) \Leftrightarrow b \in obs(a) \Leftrightarrow a \Vdash b \qquad (2.1.3)$$

Obviously, we can dually introduce the notion of a *concrete observation space* by coupling the set G with the family $\mathcal{SUB}(M) = \{sub(b) : b \in M\}$.

REMARKS. obs (sub) takes in input a point (an observable) and outputs a subset of M (of G). In fact, obs and sub are equivalent to the following characteristic functions, for $g \in G, p \in M$:

$$g(p) = \begin{cases} 1 & \text{if } g \Vdash p \\ 0 & \text{otherwise} \end{cases} \qquad p(g) = \begin{cases} 1 & \text{if } g \Vdash p \\ 0 & \text{otherwise} \end{cases}$$
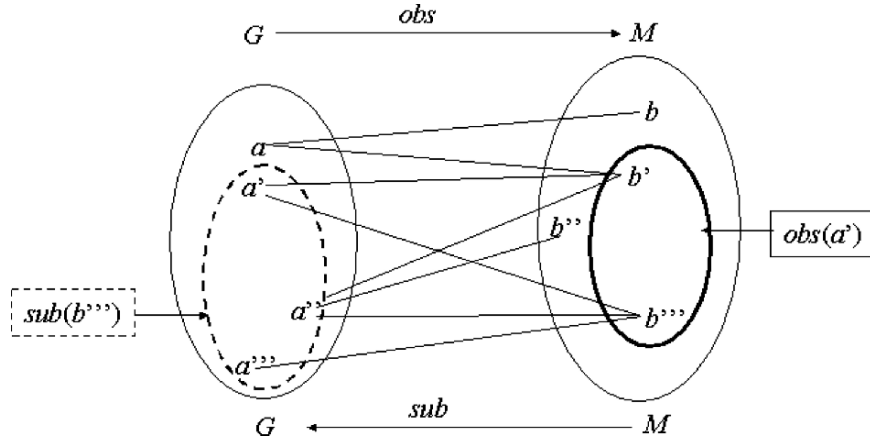
Figure 2.1: A Observations and *Substances*

2.1.1   Observations and Partial Observations

As we have already noticed, the above picture is coherent with a phenomenological approach because that way we do not disregard *noumena* although we do not refer to them in a direct manner, in our conceptual construction. In a sense, we are applying Husserl's method called "*epochè*". According to this method substances do not suddenly disappear at once as claimed by extreme idealism (cf. again Berkeley's argument reported in the Introduction), but all essentialist considerations are "enclosed within a pair of brackets", or "excluded" (*Ausschaltung*), meaning that we have to suspend any prescriptive judgement about the what of an object, because an object can be referred to only through a phenomenological medium.

We now notice that since the set M is given and fixed, any *P-system* will provide only with *partial* observations of the members of G. This limitation means that a single point x possibly fails to be uniquely described by its intension $obs(x)$. Indeed, we can have distinct points x and y such that $obs(x) = obs(y)$ (i.e. *obs* should fail to be injective). Obviously, if we change the set M of observables or if we enlarge it, we can eventually obtain a situation where $obs(x) \neq obs(y)$ whenever $x \neq y$.[1] Generally, the larger is M, the finer is the description of any

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[1]We shall see that in a topological interpretation this means that the filter of neighborhoods of x converges to x.

point from a fixed set G. Anyway, even if we reach descriptions such that $obs(x) \neq obs(y)$, for any $x \neq y$, it cannot be said that the combination of properties $obs(x)$ captures the *essence* of x. Indeed, if we enlarge G, we can have, again, a new point $z \neq x$ such that $obs(x) = obs(z)$.

Finally, if only finite observations are considered, we should, in principle, deal only with positive observations. In fact, negative observations are not that sound if only finitely many observations are allowed (see Frame 4.4).

The above remarks can be synthesized by saying that the precision of the intentional description of any object strictly depends on the particular *P-system* at hand. This, again, will avoid any essentialism, and we stressed this position by saying that the symbol x in the notation $obs(x)$ is just an index of a family of members of M.

However, we shall also say that $obs(x)$ is an intensional *approximation* of a 'partially describable' member x of G and claim that if $obs(x) = obs(y)$, then x and y cannot be *discerned* by means of the observable approximating properties (or "partial descriptions") at hand, so that x and y will be said to be *indiscernible* in the given *P-system*.

Indeed, if obs fails to be injective then we know that it cannot have a retraction and this means that the identity $1_{\wp(G)}$ cannot be determined by means of the properties at our disposal (that is, the subsets of M mapped by obs), so that a "loss of identity" literally happens.

If x and y are indiscernible they will collapse into the same intentional description.

This is what happens in pointless topology, where two "concrete" points may collapse into a single abstract point, that is, into a single bundle of properties (cf. Introduction).

We shall see how to compute an indiscernibility relation from a *P-system*. Before that, we have to introduce the basic operators acting on *P*-systems. Some of them will be directly used in the following paragraphs, some others will be used later on in derived *P*-systems. Finally, some of them are introduced for the sake of completeness and for making it possible to have a wider look at the topic.

Actually, a number of operators can be defined on *P*-systems. But we require at least three basic intuitive characteristic:

Mandatory requirements for operators on observation systems
All the operators must be defined by using the constructors "obs" and "sub"
The operators must make it possible to define approximation operators by means of adjunction properties
The operators must not be defined just in order to technically fit the above points, but, on the contrary they must have an intuitive reading

2.1.2 Relations and Galois Adjunctions

Since $\Vdash$ is a relation we need a stock of definitions and properties about these mathematical objects. Therefore, the reader is invited to go through Mathematical toolkit 16.5.

REMARKS. For reasons which will be explained later, from now on we shall adopt the following symbols[2]:

Relational operator	Symbol
$R(\cdots)$	$\langle R^{\smile} \rangle$
$R^{\smile}(\cdots)$	$\langle R \rangle$
$R \Longrightarrow (\cdots)$	$[R^{\smile}]$
$R^{\smile} \Longrightarrow (\cdots)$	$[R]$
$R^{\smile} \Longleftarrow (\cdots)$	$[[R^{\smile}]]$
$R \Longleftarrow (\cdots)$	$[[R]]$

We are going to prove the basic properties of the operations induced by binary relations. First we need a Lemma.

Lemma 2.1.1. *Let $R \subseteq A \times B$. Then for any $X \subseteq A, Y \subseteq B, a \in A, b \in B$:*

1. $b \in [R^{\smile}](X)$ iff $\langle R \rangle(b) \subseteq X$ iff $R^{\smile}(b) \subseteq X$.

2. $a \in [R](Y)$ iff $R(a) \subseteq Y$ iff $\langle R^{\smile} \rangle(a) \subseteq Y$.

3. $a \in [[R]](Y)$ iff $Y \subseteq R(a)$ iff $Y \subseteq \langle R^{\smile} \rangle(a)$.

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[2]It can seem rather odd that relational operators defined by means of $R^{\smile}$ turn into symbols decorated with R and vice-versa and, even worst, this is not the case for $[[\alpha]]$ operators. However there are several reasons depending on both logic and relational algebra. The logical reasons will be explained in Frame 4.13 of the present Part, while the other reasons will be explained in Frame 15.18.2 of Part III.

4. $b \in [[R^\smile]](X)$ iff $X \subseteq R^\smile(b)$ iff $X \subseteq \langle R \rangle(b)$.

5. (a) $[[R]](\emptyset) = A$, (b) $[[R^\smile]](\emptyset) = B$, (c) $\langle R^\smile \rangle(\emptyset) = \langle R \rangle(\emptyset) = \emptyset$.
   (d) $[R^\smile](A) = B$, (e) if R is onto, $\langle R^\smile \rangle(A) = B$, $[R^\smile](\emptyset) = \emptyset$.
   (f) $[R](B) = A$, (g) if $R^\smile$ is onto, $\langle R \rangle(B) = A$, $[R](\emptyset) = \emptyset$.

6. If X and Y are singletons, then (a) $\langle R^\smile \rangle(X) = [[R^\smile]](X)$;
   (b) $\langle R \rangle(Y) = [[R]](Y)$.

7. (a) If R is onto, $[R^\smile](X) \subseteq \langle R^\smile \rangle(X)$; (b) If $R^\smile$ is onto $[R](Y) \subseteq \langle R \rangle(Y)$.

8. If R is a functional relation then $[R](Y) = \langle R \rangle(Y)$.

9. If $R^\smile$ is a functional relation then $[R^\smile](X) = \langle R^\smile \rangle(X)$.

10. If R is an isomorphism relation, then $\langle R^\smile \rangle(X) = [R^\smile](X)$ and
    $[R](Y) = \langle R \rangle(Y)$.

Proof. (**1**) By definition $b \in [R^\smile](X)$ iff $\forall a(\langle a, b \rangle \in R \Rightarrow a \in X)$ iff $R^\smile(b) \subseteq X$ iff $\langle R \rangle(b) \subseteq X$. (**2**) By symmetry. (**3**) and (**4**) from (1) and (2) by swapping the position of the relations $\in$ and R. (**5**) (a), (b) and (c) are obvious. (d) For any $b \in B$, either $\langle a, b \rangle \in R$ for some $a \in A$ or the premise of the implication defining the operator $[R^\smile]$ is false. (e) If R is surjective then for all $a \in A$ there is a $b \in B$ such that $\langle a, b \rangle \in R$. Moreover, in $[R^\smile](\emptyset)$ the premise is true for some element while the conclusion is false; hence the consequence is always false. Similar proofs for (f) and (g). (**6**) Applied on singletons the definitions of $[[\alpha]]$ and $\langle \alpha \rangle$ operators trivially coincide, for $\alpha = R$ or $\alpha = R^\smile$. (**7**) For all $b \in B$, $b \in [R^\smile](X)$ iff $R^\smile(b) \subseteq X$ iff (for isotonicity of $R(\ldots)$) $R(R^\smile(b)) \subseteq R(X)$. But $b \in R(R^\smile(b))$. Hence $b \in R(X) = \langle R^\smile \rangle(X)$. Symmetrically for $[R]$ and $\langle R \rangle$. (**8**) If R is a functional relation, by definition $R^\smile$ is onto, thus from point (7) $[R](Y) \subseteq \langle R \rangle(Y)$ for any $Y \subseteq B$. Suppose $x \in \langle R \rangle(Y)$ and $x \notin [R](Y)$. Then there are $y \in Y$ and $y' \notin Y$ such that $\langle x, y \rangle \in R$ and $\langle x, y' \rangle \in R$. Hence R is not functional. (**9**) Just an instance of (8). (**10**) Straightforwardly from (8) and (9) by noticing that if R is an injective functional relation then $R^\smile$ is a functional relation too.   QED

Proposition 2.1.1. *Consider the partially ordered sets* $\mathbf{A} = \langle \wp(A), \subseteq \rangle$ *and* $\mathbf{B} = \langle \wp(B), \subseteq \rangle$. *Let* $R \subseteq A \times B$ *be any relation,* $\hat{f} \subseteq A \times B$ *any*

functional relation and $\hat{g} \subseteq A \times B$ any isomorphism relation. Then the following holds:

1. (a) $\mathbf{A} \dashv^{\langle R^{\smile}\rangle, [R]} \mathbf{B}$; (b) $\mathbf{B} \dashv^{\langle R\rangle, [R^{\smile}]} \mathbf{A}$.

2. (a) $\mathbf{A} \dashv^{[[R^{\smile}]], [[R]]} \mathbf{B}^{op}$; (b) $\mathbf{B}^{op} \dashv^{[[R]], [[R^{\smile}]]} \mathbf{A}$.

3. (a) $\mathbf{A} \dashv^{\langle \hat{f}^{\smile}\rangle, \langle \hat{f}\rangle} \mathbf{B}$; (b) $\mathbf{B} \dashv^{\langle \hat{g}\rangle, \langle \hat{g}^{\smile}\rangle} \mathbf{A}$ *and* $\mathbf{A} \dashv^{\langle \hat{g}^{\smile}\rangle, \langle \hat{g}\rangle} \mathbf{B}$.

Proof. (**1**) (a) $\langle R\rangle(Y) \subseteq X$ iff $R^{\smile}(y) \subseteq X, \forall y \in Y$ iff (from *Lemma* 2.1.1.(1)) $Y \subseteq [R^{\smile}](X)$. (b) By symmetry. (**2**) $X \subseteq [[R]](Y)$ iff $\forall x \in X(Y \subseteq R(x))$ (from *Lemma* 2.1.1.(3)), iff $\forall x \in X, \forall y \in Y(y \in R(x))$ iff $\forall y \in Y(X \subseteq R^{\smile}(y))$ iff $\forall y \in Y(y \in [[R^{\smile}]](X))$ iff $Y \subseteq [[R^{\smile}]](X)$ (in view of *Lemma* 2.1.1.(4)). (**3**) Either directly from *Lemma* 2.1.1.(8) and (9) or from point (1) above and *Corollary* 1.4.1 by noticing that if $\hat{f}$ is a functional relation then $\langle \hat{f}^{\smile}\rangle = f^{\rightarrow}$ and $\langle \hat{f}\rangle = f^{\leftarrow}$.   QED

The latter statement says nothing else than if R is just a renaming, then subsets of elements with a given name correspond exactly with subsets of the same elements renamed.

Notice that if $R^{\smile}$ is onto then from 2.1.1.(7) $\langle R^{\smile}\rangle(X) \subseteq Y$ implies $X \subseteq \langle R\rangle(Y)$ but the reverse implication holds only if R is functional. Symmetrically, if R is onto then $X \subseteq \langle R\rangle(Y)$ implies $[R^{\smile}](X) \subseteq Y$ but the reverse implication holds only if $R^{\smile}$ is functional.

Example 2.1.1. Relations, maps and functions

Consider the relation $\Vdash$ of our basic Example 1.2.1.

We can compute some example of neighborhoods: $\Vdash (a) = \{b, b'\}$, $\Vdash (a') = \{b', b'''\}$, $\Vdash (\{a, a'\}) = \{b, b', b'''\} = \Vdash (a) \cup \Vdash (a')$. One can verify that $a \in \Vdash^{\smile} (\{b, b''\})$ implies $\Vdash (a) \cap \{b, b''\} = \{b, b'\} \cap \{b, b''\} = \{b\} \neq \emptyset$; and vice-versa.

Here below we depict the converse relation $\Vdash^{\smile}$ together with the two compositions $\Vdash \otimes \Vdash^{\smile}$ and $\Vdash^{\smile} \otimes \Vdash$:

$\Vdash^{\smile}$	a	a'	a''	a'''	$\Vdash \otimes \Vdash^{\smile}$	a	a'	a''	a'''	$\Vdash^{\smile} \otimes \Vdash$	b	b'	b''	b'''
b	1	0	0	0	a	1	1	1	0	b	1	1	0	0
b'	1	1	1	0	a'	1	1	1	1	b'	1	1	1	1
b''	0	0	1	0	a''	1	1	1	1	b''	0	1	1	1
b'''	0	1	1	1	a'''	0	1	1	1	b'''	0	1	1	1

The two compositions can be computed because $\Vdash \subseteq G \times M$ and $\Vdash^{\smile} \subseteq M \times G$.

One can easily verify that $\langle x, y\rangle \in \Vdash \otimes \Vdash^{\smile}$ if and only if there is a "path" from x to y, that is, if there is an $m \in M$ such that $\langle x, m\rangle \in \Vdash$ and $\langle m, y\rangle \in \Vdash^{\smile}$ (symmetrically for $\Vdash^{\smile} \otimes \Vdash$). For instance, $\langle a, b'\rangle \in \Vdash$ and $\langle b', a'''\rangle \in \Vdash^{\smile}$ so that $\langle a, a'''\rangle \in \Vdash \otimes \Vdash^{\smile}$. Since $\Vdash = \Vdash^{\smile\smile}$, and $\Vdash \otimes \Vdash^{\smile} \supsetneq \Delta_A$, we have that $\Vdash^{\smile}$ is surjective

but it is not functional. Indeed, both $\langle b', a \rangle$ and $\langle b', a' \rangle$ belong to $\Vdash$ so that $\langle a, a' \rangle \in \Vdash \otimes \Vdash^{\smile}$ although $a \neq a'$. In the same manner we can verify that $\Vdash$ is surjective too but not functional.

On the contrary the relation $\hat{f}$ of Example 1.2.1 is functional because $\hat{f}^{\smile} \otimes \hat{f} = \Delta_{M'}$.

Exercise 2.1. *Prove that for any relation R:* $-(R(X)) = -\{y : \forall x(\langle x, y \rangle \in R \Rightarrow x \notin X)\}$.

We claim that now we have all the ingredients to start deliberating on observation systems.

2.2   The Basic Phenomenological Constructors

It is noticed that we must lift from the level of tokens to the level of types (i.e. sets of tokens). Therefore, now we have to extend the constructors *sub* and *obs* from tokens to types, and call such extensions "basic phenomenological constructors" or "perception constructors" induced by a *P-system*.

We qualify these constructors as "basic" because they are defined by means of elementary formulae of the form $\exists x(\alpha(\ldots, x, \ldots) \ \& \ \beta(\ldots, x, \ldots))$ or $\forall x(\alpha(\ldots, x, \ldots) \Rrightarrow \beta(\ldots, x, \ldots))$, where α and β are either $\in$, *obs* or *sub*.

These constructors will make it possible to define different kinds of structures over $\wp(G)$ and $\wp(M)$. Since such structures are defined as extensions of the two functions *obs* and *sub* and since, in turn, these two functions are linked by the relation (2.1.3), it is clear that any structurisation on points will have a dual structurisation on observables, and vice-versa. In a precise sense, the pair $\langle sub, obs \rangle$ defines parallel constructions of structures connecting the elements of G with each other and structures connecting the elements of M.

Essentially all the basic constructors will coincide with the operators definable from binary relations which have been introduced at the beginning of the previous Section. In a few cases we shall deal with new operators.

Basically, we have four types of extensions of *obs* and *subs* to a subset Z of G or M:

- Existential extension: it is defined by existentially quantifying the membership formula "$x \in Z$".

- Universal extension: it is defined by universally quantifying the membership formula "$x \in Z$".

- Co-existential extension: it is defined by universally quantifying the fulfillment formula "$x \Vdash y$", or, equivalently, $x \in sub(y), y \in obs(x)$.

- Co-universal extension: it is defined by existentially quantifying the fulfillment formula "$x \Vdash y$", or, equivalently, $x \in sub(y), y \in obs(x)$.

REMARKS. In the present Chapter we shall basically deal with existential and co-existential extensions. Some discussion shall be reserved for universal extensions. We shall not deal with co-universal extensions because, for the time being, they have a rather odd interpretation in terms of perception systems.

Definition 2.2.1 (Basic phenomenological constructors). *Let* $\mathbf{P} = \langle G, M, \Vdash \rangle$ *be a P-system. Then:*

- $\langle e \rangle : \wp(M) \longmapsto \wp(G)$;
  $\langle e \rangle(Y) = \{a \in G : \exists b(b \in Y \ \& \ a \in sub(b))\} = \langle \Vdash \rangle(Y)$.

- $[e] : \wp(M) \longmapsto \wp(G)$;
  $[e](Y) = \{a \in G : \forall b(a \in sub(b) \Rightarrow b \in Y)\} = [\Vdash](Y)$.

- $[[e]] : \wp(M) \longmapsto \wp(G)$;
  $[[e]](Y) = \{a \in G : \forall b(b \in Y \Rightarrow a \in sub(b))\} = [[\Vdash]]$.

- $\langle i \rangle : \wp(G) \longmapsto \wp(M)$;
  $\langle i \rangle(X) = \{b \in M : \exists a(a \in X \ \& \ b \in obs(a))\} = \langle \Vdash^{\smile} \rangle(X)$.

- $[i] : \wp(G) \longmapsto \wp(M)$;
  $[i](X) = \{b \in M : \forall a(b \in obs(a) \Rightarrow a \in X)\} = [\Vdash^{\smile}](X)$.

- $[[i]] : \wp(G) \longmapsto \wp(M)$;
  $[[i]](X) = \{b \in M : \forall a(a \in X \Rightarrow b \in obs(a))\} = [[\Vdash^{\smile}]](X)$.

An intuitive interpretation of the above functions is in order. A constructor is decorated with 'e' when its application gives an *extent*. We can verify that $\langle e \rangle$ is the "existential" or "disjunctive" extension of *sub* while $[[e]]$ is its "universal" or "conjunctive" extension.

Obviously, $\langle e \rangle = sub^{\rightarrow} = obs^{\leftarrow}$. We leave as an exercise the proof that $[e]$ is the dual of $\langle e \rangle$, hence it is the "co-existential extension" of *sub* (the dual of $[[e]]$ will be discussed "*en passant*" later on).

Given $Y \subseteq M$, the set $[[e]](Y)$ collects the points which fulfill *at least all* the properties from Y (and, possibly, other properties outside Y), while $\langle e \rangle (Y)$ gives the set of points which fulfill *at least one* property from Y (and possibly other properties outside Y). Finally, $[e](Y)$ – sometimes called the "*restriction of Y along* $\Vdash^\smile$" – collects the points which fulfill *at most all* the properties from Y (possibly not all the properties in Y, but no property outside Y).

The same considerations apply symmetrically to the operators decorated with 'i' (because the result of these functions is an *intent*). Indeed $\langle i \rangle$ and $[[i]]$ are the disjunctive and, respectively, conjunctive extensions to subsets of G of the function *obs*; clearly $\langle i \rangle = obs^\rightarrow = sub^\leftarrow$. More precisely, $\langle i \rangle (X)$ collects the set of properties which are fulfilled *at least by one* point of X, while $[[i]](X)$ collects the set of properties which are fulfilled *at least by all* the points of X, that is, the properties *shared at least by all* the points of X. Finally, $[i](X)$, the "co-existential extension" of *obs*, gives the set of properties which are fulfilled *at most by all* of the points in X. In particular, $[i](\{x\})$ is the set of properties which uniquely characterise x (see Figure 2.2 below).

We summarize these remarks in the following table:

	... property/ies in Y	**... point/s in X**
at least one ...	$\langle e \rangle (Y)$	$\langle i \rangle (X)$
at least all ...	$[[e]](Y)$	$[[i]](X)$
at most all ...	$[e](Y)$	$[i](X)$

The above basic constructors are just particular interpretations of the operators defined in the previous Section. The following correspondence table may be helpful.

P-systems reading	*Relational system* reading	*Relational operations*
$\langle e \rangle$	$\langle \Vdash \rangle$	$\Vdash^\smile (\dots)$
$[e]$	$[\Vdash]$	$\Vdash^\smile \Longrightarrow (\dots)$
$\langle i \rangle$	$\langle \Vdash^\smile \rangle$	$\Vdash (\dots)$
$[i]$	$[\Vdash^\smile]$	$\Vdash \Longrightarrow (\dots)$
$[[e]]$	$[[\Vdash]]$	$\Vdash \Longleftarrow (\dots)$
$[[i]]$	$[[\Vdash^\smile]]$	$\Vdash^\smile \Longleftarrow (\dots)$

Figure 2.2: Basic constructors derived from a *P-system*

Therefore, from *Proposition* 2.1.1 it is clear that the following adjointness properties hold in any *P-system* $\mathbf{P} = \langle G, M, \Vdash \rangle$:

(a) $\mathbf{M} \dashv^{\langle e \rangle, [i]} \mathbf{G}$; (b) $\mathbf{G} \dashv^{\langle i \rangle, [e]} \mathbf{M}$;

(c) $\mathbf{M} \dashv^{[[e]], [[i]]} \mathbf{G^{op}}$; (d) $\mathbf{G} \dashv^{[[i]], [[e]]} \mathbf{M^{op}}$.

It follows, immediately, that the boxed operators (i.e. $[i]$ and $[e]$) are co-modal, the diamond operators (i.e. $\langle i \rangle$ and $\langle e \rangle$) are modal while the double boxed operators (i.e. $[[e]]$ and $[[i]]$) are anti-modal.

2.2.1   A Modal Reading of the Basic Constructors

At this point a modal reading of the basic constructors is in order. Thus, let us read these constructors by means of operators taken from extended forms of modal logic, namely, *possibility*, *necessity* and *sufficiency*. We shall see in Part III that such a reading is also supported by the fact that these operators, indeed, may play the role of semantic counterparts of modal operators. The proposed interpretation comes straightforwardly from *Proposition* 2.1.1. We prefer to give a few explanations and present the matter in a tabular and pictorial form. For the sake of completeness, in this section we shall deal also with the operators dual of $[[i]]$ and, respectively, $[[e]]$:

- $\langle\langle i \rangle\rangle(X) =_{def} -[[i]](-X)$, for $X \subseteq G$, the "co-universal extension" of *obs*

- $\langle\langle e \rangle\rangle(Y) =_{def} -[[e]](-Y)$, for $Y \subseteq M$, the "co-universal extension" of *sub*

The matter will be presented just for intensional operators.

Operator	$x \in X$	$b \in B$	Example reading
$\langle i \rangle (X) = B$	If $x \in X$ then it is **possible** that x enjoys elements in B	b is enjoyed by some element collected in X	There are examples of elements in X that enjoy b
$[i] (X) = B$	To enjoy elements in B it is **necessary** to be in X	b is enjoyed by at most all the element of X	There are not examples of elements enjoying b that are not in X
$[[i]] (X) = B$	To enjoy elements in B it is **sufficient** to be in X	b is enjoyed by at least all the elements of X	There are not examples of elements of X that do not enjoy b
$\langle\langle i \rangle\rangle (X) = B$	It is **possible** that some element outside X does not enjoy elements in B	b is not enjoyed by some element outside X	There are examples of elements in $-X$ not enjoying b

A pictorial description of the above modal reading is shown in Figures 2.3 and 2.4.

In Formal Concept Analysis $[[i]](X)$ gives the "intent" of the set X, that is, as already noticed, the set of properties that are shared by all the elements from X or, otherwise stated, the set of properties that characterise X as a whole. Symmetrically, $[[e]](Y)$ gives the set of objects that share all the properties listed in Y, hence $[[e]](Y)$ is the "extent" of the set of properties Y. On this basis, we shall give a mathematical proof of a well-known philosophical law which concerns extensions and intentions of concepts, the aforementioned "Loy de Port Royal" which states that intension and extension are contravariant. Indeed the more a set is characterised (or defined), the less is the number of its elements (for example, a generic inhabitant of India fulfills less properties than a Bengali, because a Bengali is an Indian with further properties, for instance "living in Bengal").
See Frame 4.9 for further details.

Exercise 2.2.
(A) Consider $A = \{a, a', a''\}, B = \{b, b', b'', b'''\}$. Let $R \subseteq A \times B$ be given by the following table:

R	b	b'	b''	b'''
a	1	0	1	0
a'	0	1	0	0
a''	0	0	1	1

Compute:

(1) $\langle\langle i\rangle\rangle(\{a, a'\})$, (2) $[[i]](\{a, a''\})$, (3) $[i](\{a, a'\})$.

(B) Prove that in any P-system $\langle\langle i\rangle\rangle(X) = (-\Vdash)(-X)$.
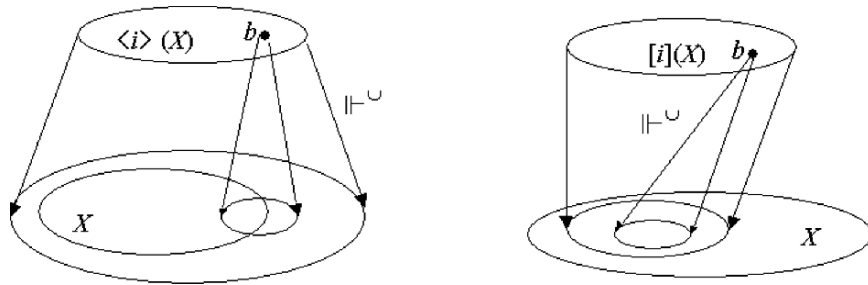
Figure 2.3: Possibility and Necessity
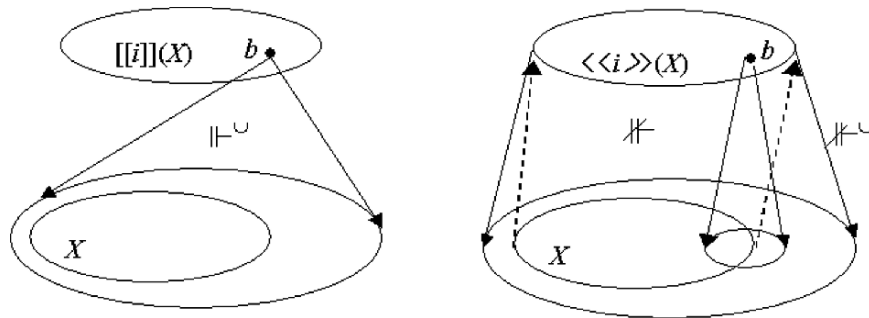
Figure 2.4: Sufficiency and dual sufficiency

The above functions are linked by some fundamental relationships. First recall that our operators are defined on Boolean algebras of type $\langle \wp(A), \cap, \cup, A, -, \emptyset\rangle$, so that negation coincides with set-theoretical complementation.

Thus, let us say that if an operator Op^o is obtained by negating all of the defining subformulas in an operator Op, and by further applying

the contraposition law according to negations (or, equivalently, by first
putting the definition in disjunctive normal form), then Op^o and Op are
said to be *opposite* or *orthogonal* (to each other), or "o" in symbols.[3] If
$Op^d(X) = \sim Op(\sim X)$ then Op^d is called the *dual* of Op and we denote
the relation of duality with "d". Eventually, Op_1 "od" Op_2 means that
the operator Op_2 is obtained by taking the opposite of the dual of Op_1.
Then we have (for intensional operators):

	$[i]$	$[[i]]$	$\langle i \rangle$	$\langle\langle i \rangle\rangle$
$[i]$	$=$	o	d	od
$[[i]]$	o	$=$	od	d
$\langle i \rangle$	d	od	$=$	o
$\langle\langle i \rangle\rangle$	od	d	o	$=$

The same happens of extensional operators. The details are left as
exercises.

However, from now on we shall no longer deal with $\langle\langle i \rangle\rangle$ and $\langle\langle e \rangle\rangle$.
About the remaining operators, let us observe that the internal struc-
ture of their definitions splits them into three classes. Moreover their
domains split each class into two sub-classes:

	Internal structure	Domain
$\langle e \rangle$	$\exists \&$	$\wp(M)$
$\langle i \rangle$	$\exists \&$	$\wp(G)$
$[e]$	$\forall \Rightarrow$	$\wp(M)$
$[i]$	$\forall \Rightarrow$	$\wp(G)$
$[[e]]$	$\Rightarrow \forall$	$\wp(M)$
$[[i]]$	$\Rightarrow \forall$	$\wp(G)$

In the above table by means of the sequence $\Rightarrow \forall$ we denote the fact
that the universally quantified formula which appears in the definition
of the operator has the relation $\in$ in the premise and the relation $\Vdash$
(in the guise of *sub* or *obs*) in the conclusion. On the contrary in $\forall \Rightarrow$-
formulas we have the other way around. This is another way to state
that $\forall \Rightarrow$ and $\Rightarrow \forall$ formulas are opposite to each other.

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[3]So, for instance, if $\alpha \Rightarrow \beta$ appears in a defining formula, of Op, then in Op^o we
have $\sim \beta \Rightarrow \sim \alpha$.

We can easily observe that functions decorated with e and functions decorated with i are symmetric with respect to the relation $\Vdash$, and we denote this fact with (s), while box-functions and diamond-functions are formally dual (d). It follows that box-functions decorated with e (with i) are symmetric-dual (sd) to diamond-functions decorated with i (with e):

	$\langle e \rangle$	$\langle i \rangle$	$[e]$	$[i]$	$[[e]]$	$[[i]]$
$\langle e \rangle$	$=$	s	d	sd	od	ods
$\langle i \rangle$	s	$=$	sd	d	ods	od
$[e]$	d	sd	$=$	s	o	os
$[i]$	sd	d	s	$=$	os	o
$[[e]]$	od	ods	o	os	$=$	s
$[[i]]$	ods	od	os	o	s	$=$

Obviously, symmetric functions fulfill the same formal properties, opposite functions fulfill opposite properties, while dual and symmetric-dual operators fulfill dual properties, as we can see from the following result.

The reader will notice that functions $[[e]]$ and $[[i]]$ behave like $[e]$ and $[i]$, respectively, but with the inversion of the relation $\subseteq$. The intuitive reason, discussed above, is the "Loy de Port Royal", while the formal reason is the polarity property between $[[e]]$ and $[[i]]$.

Example 2.2.1. Basic formal constructors

Consider again our basic Example 1.2.1. About system **P**, we can see, for instance, that $\langle i \rangle(a') = \{b', b'''\}$, $\langle i \rangle(a) = \{b, b'\}$, and $\langle i \rangle(\{a, a'\}) = \{b, b', b'''\}$ which shows that $\langle i \rangle(a) \cup \langle i \rangle(a') = \langle i \rangle(\{a\} \cup \{a'\})$ and, moreover, that $\langle i \rangle(a) \subseteq \langle i \rangle(\{a, a'\})$ (instances of the **sup-distributivity** and, respectively, **monotonicity** of $\langle i \rangle$).

Thus, if $x \in \{a, a'\}$ it is **possible** that $x \Vdash b$ or $x \Vdash b'$ or $x \Vdash b'''$. However, $\langle e \rangle(b) \neq \{a'\}$, $\langle e \rangle(b') \neq \{a'\}$ and $\langle e \rangle(b''') \neq \{a'\}$. It follows that $[i](a') = \emptyset$, that is tantamount to saying that there are not properties uniquely characterising a'. On the contrary, $\langle e \rangle(b) = \{a\}$ so that b is a **characteristic** property of a, as well as b'' is a characteristic property of a''. But if we consider a, a' and a'' together, we get a "synergy" result: $[i](\{a, a', a''\}) = \{b, b', b''\}$ because if $x \in \{b, b', b''\}$ then $\Vdash^{\smile}(\{x\}) \subseteq \{a, a', a''\}$, and only in this case (notice that $[i](a') \cup [i](a) \cup [i](a'') = \{b, b''\} \subseteq [i](\{a\} \cup \{a'\} \cup \{a''\})$). i.e. non sup-distributivity of $[i]$; on the contrary, $[i](\{a, a''\} \cap \{a'', a'''\}) = [i](a'') = \{b''\} = \{b', b''\} \cap \{b'', b'''\} = [i](\{a, a''\}) \cap [i](\{a'', a'''\})$ (i.e. **inf-distributivity** of $[i]$)). So, in order to fulfill properties in $\{b, b', b''\}$ it is **necessary** to belong to $\{a, a', a''\}$. In fact, $a''' \notin \{a, a', a''\}$ and a''' does not fulfill any element of $\{b, b', b''\}$. However, the condition is not sufficient, because, for instance, $a \in \{a, a', a''\}$ but $a \nVdash b''$.

On the opposite side, we can easily see that $\{a'''\} = [e](\{b'', b'''\})$.

Finally, $\{b', b'''\} = [[i]](\{a', a''\})$ because b' and b''' are the properties shared by a' and a''. Thus, it is **sufficient** to belong to $\{a', a''\}$ in order to fulfill the properties in $\{b', b'''\}$. However, it is not necessary, because, for instance $a''' \notin \{a', a''\}$ but $a''' \Vdash b'''$.

Moreover, this example shows that $[[i]](\{a, a'\}) \cap [[i]](\{a', a''\}) = \{b'\} = [[i]](\{a, a'\} \cup \{a', a''\})$ whereas, for instance, $[[i]](\{a\} \cap \{a'\}) = [[i]](\emptyset) = \{b, b', b'', b''', b''''\} \supseteq \{b, b', b'''\} = [[i]](\{a\}) \cup [[i]](\{a'\})$ (which is also an instance of the fact that $[[i]]$ – and $[[e]]$ – is **antitone**).

To end, notice the following instances of the **adjoint properties** of the basic constructors:

adj.1) $\langle i \rangle(\{a', a'''\}) = \{b', b'''\} \subseteq \{b', b'', b'''\}$; $\{a', a'''\} \subseteq \{a', a'', a'''\} = [e](\{b', b'', b'''\})$.

adj.2) $\langle e \rangle(\{b, b'\}) = \{a, a'\} \subseteq \{a, a', a''\}$; $\{b, b'\} \subseteq \{b, b', b''\} = [i](\{a, a', a''\})$.

ax.1) $\{b'\} \subseteq \{b', b'''\} = [[i]](\{a', a''\})$; $[[e]](\{b'\}) = \{a, a', a''\} \supseteq \{a', a''\}$.

ax.2) $\{a''\} \subseteq \{a''\} = [[e]](\{b', b''\})$; $[[i]](\{a''\}) = \{b', b'', b'''\} \supseteq \{b', b''\}$.

Exercise 2.3. *Without using any adjointness property, but only logical and set-theoretical means, prove the distributive features of $[i], \langle i \rangle,$ $[e], \langle e \rangle, [[i]]$ and $[[e]]$ according to the theses of Proposition 1.4.8 (Hints: for instance, to prove that the operator $\langle \alpha \rangle$ distributes over unions, for $\alpha = i$ or $\alpha = e$, proceed as follows: start with the fact that $\exists x(x \in X \cap Y) \Rightarrow \exists x(x \in X) \,\&\, \exists x(x \in Y)$ and use the logical law $(\forall x(x \in X) \vee \forall x(x \in Y)) \Rightarrow \forall x(x \in X \cup Y)$).*

2.3 Formal Operators on Points and on Observables

Sequences of constructors with alternate decoration provide a number of operators on $\wp(G)$ and $\wp(M)$. We shall deal with a restricted set of possibilities. In fact, we shall use sequences of adjoint operators. Indeed axiality says that if one operator lowers an element then the coniugate operator lifts it, and vice-versa, so that by combining them either we obtain the maximum of the lowering elements or the minimum of the lifting elements of a given argument.

Definition 2.3.1 (Formal perception operators)**.** *Let $\langle G, M, \Vdash \rangle$ be a P-system. Then:*

- $int : \wp(G) \longmapsto \wp(G); int(X) = \langle e \rangle([i](X))$.

- $cl : \wp(G) \longmapsto \wp(G); cl(X) = [e](\langle i \rangle(X))$.

- $est : \wp(G) \longmapsto \wp(G); est(X) = [[e]]([[i]](X)).$
- $\mathcal{A} : \wp(M) \longmapsto \wp(M); \mathcal{A}(Y) = [i](\langle e \rangle(Y)).$
- $\mathcal{C} : \wp(M) \longmapsto \wp(M); \mathcal{C}(Y) = \langle i \rangle([e](Y)).$
- $\mathcal{ITS} : \wp(M) \longmapsto \wp(M); \mathcal{ITS}(Y) = [[i]]([[e]](Y)).$

As like as the basic constructors decorated by i and e, these oper-
ators can be classified on the basis of two distinct criteria, viz. their
defining formulae and their domain:

	Logical class	Domain
$\mathcal{C}$	$\exists\forall$	$\wp(M)$
int	$\exists\forall$	$\wp(G)$
$\mathcal{A}$	$\forall\exists$	$\wp(M)$
cl	$\forall\exists$	$\wp(G)$
$\mathcal{ITS}$	$\forall\forall$	$\wp(M)$
est	$\forall\forall$	$\wp(G)$

The domain-based classification is immediate. As for the classification
based on the logical structure, notice that $\mathcal{C}(Y) = \langle i \rangle([e]\,(Y))$ and that
$\langle i \rangle$ is a $\exists\&$-function while $[e]$ is a $\forall \Rightarrow$-function, so that $\mathcal{C}$ is a $\exists\forall$-
operator. Dually for $\mathcal{A}$ which is defined by means of the sequence $[i]\langle e \rangle$.
The argument for int and cl is the same. Finally, notice that both
est and $\mathcal{ITS}$ are defined by means of a sequence of two universally
quantified conditions ($\Rightarrow \forall$-operators) (see Figure 2.5).

From the above classification it follows that the two $\forall\exists$-operators are
symmetric with respect to their domains and co-domains, as like as the
two $\exists\forall$-operators, while $\forall\exists$-operators and $\exists\forall$-operators are symmetric-
dual. Therefore, we obtain a symmetry-duality table that is formally
equivalent to the symmetry-duality table for basic constructors:

	$\mathcal{C}$	int	$\mathcal{A}$	cl	$\mathcal{ITS}$	est
$\mathcal{C}$	$=$	s	d	sd		
int	s	$=$	sd	d		
$\mathcal{A}$	d	sd	$=$	s	$=$	
cl	sd	d	s	$=$		
$\mathcal{ITS}$					$=$	s
est					s	$=$
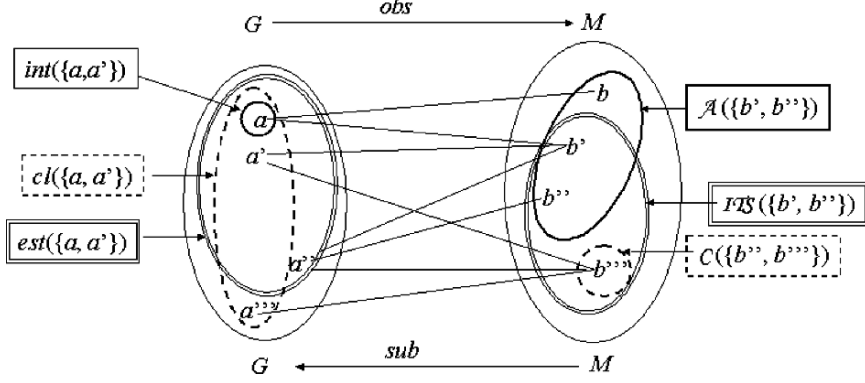
Figure 2.5: Basic formal perception operators

Proposition 2.3.1. *In any P-system $\langle G, M, \Vdash \rangle$, for any $X \subseteq G, Y \subseteq M$, $a \in G$, $b \in M$:*

1. *$b \in \mathcal{A}(Y)$ iff $\Vdash^{\smile}(b) \subseteq \Vdash^{\smile}(Y)$, iff $\langle e \rangle(b) \subseteq \langle e \rangle(Y)$.*

2. *$a \in cl(X)$ iff $\Vdash(a) \subseteq \Vdash(X)$, iff $\langle i \rangle(a) \subseteq \langle i \rangle(Y)$.*

3. *$a \in int(X)$ iff $\langle i \rangle(a) \cap [i](X) \neq \emptyset$.*

4. *$b \in \mathcal{C}(Y)$ iff $\langle e \rangle(b) \cap [e](Y) \neq \emptyset$.*

5. *$a \in est(X)$ iff $[[i]](X) \subseteq \Vdash(a)$, iff $\forall m \in M((\forall x \in X(x \Vdash m)) \Rightarrow a \Vdash m)$.*

6. *$b \in \mathcal{ITS}(Y)$ iff $[[e]](Y) \subseteq \Vdash^{\smile}(b)$, iff $\forall g \in G((\forall y \in Y(g \Vdash y)) \Rightarrow g \Vdash b)$.*

Proof. (**1**) By definition $b \in \mathcal{A}(Y)$ iff $b \in [i](\langle e \rangle(Y))$. Hence from *Lemma* 2.1.1.(1), $b \in \mathcal{A}(Y)$ iff $\langle e \rangle(b) \subseteq \langle e \rangle(Y)$ iff $\Vdash^{\smile}(b) \subseteq \Vdash^{\smile}(Y)$. (**2**) By symmetry. (**3**) $a \in int(X)$ iff $a \in \langle e \rangle([i](X))$ iff $a \in \Vdash^{\smile}(\{b : \Vdash^{\smile}(b) \subseteq X\})$, iff $\Vdash(a) \cap \{b : \Vdash^{\smile}(b) \subseteq X\} \neq \emptyset$, iff $\langle i \rangle(a) \cap [i](X) \neq \emptyset$. (**4**) By symmetry. (**5**) The first equivalence comes from *Lemma* 2.1.1.(4) and the definition of "*est*". Moreover, $[[i]](X) = \{m : X \subseteq \Vdash^{\smile}(m)\} = \{m : \forall x \in X(x \Vdash m)\}$. Henceforth $[[i]](X) \subseteq \Vdash(a)$ iff $a \Vdash m$ for all m such that $x \Vdash m$, for any member x of X. (**6**) By symmetry.   QED

It is worth noticing, at once, that $a \in est(X)$ if and only if a fulfills *at least all* the properties that are shared *by all* the elements of X. In

this sense $est(X)$ is the *extent* of the set of properties that characterises X as a whole. Symmetrically, $b \in \mathcal{ITS}(Y)$ if and only if b is fulfilled by *at least all* the objects that enjoy *all* the properties from Y. In this sense $\mathcal{ITS}(Y)$ is the *intent* of the set of objects that are characterised by Y as a whole.

In order to understand the meaning of the other operators, let us recall that the elements of M can be interpreted as "*formal neighborhoods*". In fact, in topological terms a neighborhood of a point x is a collection of points that are linked with x by means of some *nearness* relation. Since a member b of M is associated, *via* $\Vdash^{\smile}$ with a subset X of G, b may be intended as a 'proxy' of X itself. Thus if X is a *concrete* neighborhood of a point x, then b may be intended as a *formal* neighborhood of x, on the basis of the observation that the nearness relation represented by X states that two points are close to each other if they both fulfill property b.

This interpretation will be developed in details in Part III.

By now we use it to justify the symbol *int*. Indeed the usual definition tells us that for any subset X of G, a point a belongs to the interior of X if and only if there is a neighborhood of a included in X. In this formal framework we cannot verify directly if a formal neighborhood b of a is included in a set of points X. Indeed, we have to check: first if b is a formal neighborhood of a, that is, $a \Vdash b$ (hence, if $a \in \langle e \rangle (b)$); second if the *extension* of b, $\langle e \rangle (b)$, is included in X. From the adjunction properties, $\langle e \rangle (b) \subseteq X$ if and only if $\{b\} \subseteq [i](X)$. The conclusion is that a belongs to the interior of X if and only if $a \in \langle e \rangle (b)$ for b belonging to $[i](X)$. To sum up:

$$
\begin{aligned}
interior(X) &= \{a : \exists b (a \in \langle e \rangle (b) \ \& \ b \in [i](X))\} \\
&= \langle e \rangle ([i](X)) = int(X) \quad\quad (2.3.4)
\end{aligned}
$$

Otherwise stated, $int(X)$ is the extension of all the neighborhoods m such that $\langle e \rangle (m) \subseteq (X)$.

Much in the same line, we justify the symbol *cl*. In fact for any subset X of G, a belongs to the *closure* of X if and only if any neighborhood of a has non empty intersection with X. Thus $a \in cl(X)$ if and only if for all $b \in \Vdash (a)$ (that is, for all formal neighborhoods of a) $\Vdash^{\smile} (b) \cap X \neq \emptyset$. Therefore, in view of point 3 of Mathematical toolkit 16.5, $\{b : a \Vdash b\}$(i.e. $\langle i \rangle (a)$) must be included in $\langle i \rangle (X)$ and, from *Proposition*

2.1.1.(2), this holds if and only if $a \in [e]\langle i\rangle(X)$. To sum up:

$$
\begin{aligned}
closure(X) &= \{a : \forall b (a \in \langle e\rangle(b) \Rightarrow (X \cap \langle e\rangle(b) \neq \emptyset))\} \\
&= [e]\,(\langle i\rangle(X)) = cl(X) \qquad\qquad\qquad (2.3.5)
\end{aligned}
$$

To put it another way, $cl(X)$ is the extension of all and only those neighborhoods m such that $\langle e\rangle(m) \cap (X) \neq \emptyset$.

Example 2.3.1. Basic formal operators – I

We still refer to our basic Example 1.2.1.

Let us try and compute some instances of basic formal operators on system **P**:

1. Extensional operators:

$int(\{a, a'\}) = \langle e\rangle[i](\{a, a'\}) = \langle e\rangle(\{b\}) = \{a\}$; $int(\{a''\}) = \langle e\rangle(\{b''\}) = \{a''\}$.

$cl(\{a, a'\}) = [e]\langle i\rangle(\{a, a'\}) = [e](\{b, b', b'''\}) = \{a, a', a'''\}$; $cl(\{a, a''\}) = [e](M) = G$.

$est(\{b', b''\}) = [[i]][[e]](\{b', b''\}) = [[i]](\{a''\}) = \{b', b'', b'''\}$.

$int(int(\{a, a'\})) = int(\{a\}) = \langle e\rangle[i](\{a\}) = \langle e\rangle(\{b\}) = \{a\} = int(\{a, a'\})$.

$cl(cl(\{a, a'\})) = cl(\{a, a', a'''\}) = [e]\langle i\rangle(\{a, a', a'''\}) = [e](\{b, b'b'''\}) = \{a, a', a'''\}$

$= cl(\{a, a'\})$.

$est(est(\{b', b''\})) = est(\{b', b'', b'''\}) = [[i]][[e]](\{b', b'', b'''\})$

$= [[i]](\{a''\}) = \{b', b'', b'''\} = est(\{b', b''\})$.

Thus, we have examples of the fact that int is **decreasing** while cl and est are **increasing** and all of them are **idempotent**. Moreover, one can see that $int(\{a, a'\})$ $\cup int(\{a''\}) = \{a, a''\} \subseteq \{a, a', a''\} = int(\{a, a'\} \cup \{a''\})$ and $cl(\{a, a'\}) \cap cl(\{a, a''\})$ $= \{a, a', a'''\} \supseteq \{a\} = cl(\{a, a'\} \cap \{a, a''\})$.

2. Intensional operators:

$\mathcal{A}(\{b, b'\}) = [i]\langle e\rangle(\{b, b'\}) = [i](\{a, a', a''\}) = \{b, b', b''\}$.

$\mathcal{C}(\{b'', b'''\}) = \langle i\rangle[e](\{b'', b'''\}) = \langle i\rangle(\{a''\}) = \{b''\}$.

$\mathcal{ITS}(\{b', b''\}) = [[i]][[e]](\{b', b''\}) = [[i]](\{a''\}) = \{b', b'', b'''\}$.

$\mathcal{A}(\mathcal{A}(\{b, b'\})) = \mathcal{A}(\{b, b', b''\}) = [i]\langle e\rangle(\{b, b', b''\}) = [i](\{a, a', a''\}) = \{b, b', b''\} = \mathcal{A}(\{b, b'\})$.

$\mathcal{C}(\mathcal{C}(\{b'', b'''\})) = \mathcal{C}(\{b'''\}) = \langle i\rangle[e](\{b'''\}) = \langle i\rangle(\{a'''\}) = \{b'''\} = \mathcal{C}(\{b'', b'''\})$.

$\mathcal{ITS}(\mathcal{ITS}(\{b', b''\})) = \mathcal{ITS}(\{b'''\}) = [[i]][[e]](\{b', b'', b'''\}) = [[i]](\{a''\}) =$

$\{b', b'', b'''\} = \mathcal{ITS}(\{b', b''\})$.

Thus, this is also an example of the fact that $\mathcal{C}$ is **decreasing** while $\mathcal{A}$ and $\mathcal{ITS}$ are **increasing** and all of them are **idempotent**.

Exercise 2.4. *Prove that:*

(A) $int(X) = \bigcup\{\langle e\rangle(m) : \langle e\rangle(m) \subseteq X\} = \langle e\rangle\{m : \langle e\rangle(m) \subseteq X\}$.

(B) $cl(X) = [e]\{m : \langle e\rangle(m) \cap X \neq \emptyset\}$.

(C) $cl(X) = [e]\bigcap\{-\{m\} : X \subseteq -\langle e\rangle(m)\}$.

(D) $cl(X) = [e]\bigcap\{Z : X \subseteq [e](Z)\}$.

2.3.1 Algebraic Properties of Formal Perception Systems

From the adjunction properties of the basic constructors we deduce, in view of *Corollary* 1.4.2 (and Example 2.3.1), that in a *P-system* $\mathbf{P} = \langle G, M, \Vdash \rangle$ the following hold:

Interior operators	$int, \mathcal{C}$
Closure operators	$cl, \mathcal{A}, est, \mathcal{ITS}$

In view of the observation after *Corollary* 1.4.2 one easily notices that none of the above operators needs to be topological.

Definition 2.3.2. *Let* $\mathbf{P} = \langle G, M, \Vdash \rangle$ *be a P-system. Then we define the following families of fixpoints of the operators induced by* $\mathbf{P}$*:*

1. $\Omega_{int}(\mathbf{P}) = Im_{int} = \{X \subseteq G : int(X) = X\}$ – *the extensional open subsets of* $\mathbf{P}$*.*

2. $\Gamma_{cl}(\mathbf{P}) = Im_{cl} = \{X \subseteq G : cl(X) = X\}$ – *the extensional closed subsets of* $\mathbf{P}$*.*

3. $\Gamma_{est}(\mathbf{P}) = Im_{est} = \{X \subseteq G : est(X) = X\}$ – *the extents of* $\mathbf{P}$*.*

4. $\Omega_{\mathcal{A}}(\mathbf{P}) = Im_{\mathcal{A}} = \{Y \subseteq M : \mathcal{A}(Y) = Y\}$ – *the intensional open subsets of* $\mathbf{P}$*.*

5. $\Gamma_{\mathcal{C}}(\mathbf{P}) = Im_{\mathcal{C}} = \{Y \subseteq M : \mathcal{C}(Y) = Y\}$ – *the intensional closed subsets of* $\mathbf{P}$*.*

6. $\Gamma_{\mathcal{ITS}}(\mathbf{P}) = Im_{\mathcal{ITS}} = \{Y \subseteq M : \mathcal{ITS}(Y) = Y\}$ – *the intents of* $\mathbf{P}$*.*

TERMINOLOGY AND NOTATION. If X belongs to one of the above families for an operator $Op \in \{cl, est, \mathcal{C}, \mathcal{ITS}, int, \mathcal{A}\}$, then X is called $Op - saturated$ (or "saturated", if the operator Op is understood). It is worth mentioning that in pointfree topology extensional operators are called "concrete" , while intensional operators are called "formal".

Proposition 2.3.2. *Let* $\mathbf{P} = \langle G, M, \Vdash \rangle$ *be a P-system. Then:*

1. $\mathbf{Sat}_{int}(\mathbf{P}) = \langle \Omega_{int}(\mathbf{P}), \cup, \wedge, \emptyset, G \rangle$, *where* $\bigwedge_{i \in I} X_i = int(\bigcap_{i \in I} X_i)$, *is a complete lattice.*

2. $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P}) = \langle \Omega_{\mathcal{A}}(\mathbf{P}), \vee, \cap, \emptyset, M \rangle$, *where* $\bigvee_{i \in I} Y_i = \mathcal{A}(\bigcup_{i \in I} Y_i)$, *is a complete lattice.*

3. $\mathbf{Sat}_{cl}(\mathbf{P}) = \langle \Gamma_{cl}(\mathbf{P}), \vee, \cap, \emptyset, G \rangle$, *where* $\bigvee_{i \in I} X_i = cl(\bigcup_{i \in I} X_i)$, *is a complete lattice.*

4. $\mathbf{Sat}_{\mathcal{C}}(\mathbf{P}) = \langle \Gamma_{\mathcal{C}}(\mathbf{P}), \cup, \wedge, \emptyset, M \rangle$, *where* $\bigwedge_{i \in I} Y_i = \mathcal{C}(\bigcap_{i \in I} Y_i)$, *is a complete lattice.*

5. $\mathbf{Sat}_{est}(\mathbf{P}) = \langle \Gamma_{est}(\mathbf{P}), \cap, \vee, est(\emptyset), G \rangle$, *where* $\bigvee_{i \in I} X_i = est(\bigcup_{i \in I} X_i)$, *is a complete lattice.*

6. $\mathbf{Sat}_{\mathcal{ITS}}(\mathbf{P}) = \langle \Gamma_{\mathcal{ITS}}(\mathbf{P}), \cap, \vee, \mathcal{ITS}(\emptyset), M \rangle$, *where* $\bigvee_{i \in I} Y_i = \mathcal{ITS}(\bigcup_{i \in I} Y_i)$, *is a complete lattice.*

Proof. Much work has already be done in *Proposition* 1.4.9. We just need to justify the choice of top and bottom elements. To this end, remember that in any *P-system* both $\Vdash$ and $\Vdash^{\smile}$ are onto. Hence in view of *Lemma* 2.1.1.(5). $int(G) = \langle e \rangle [i](G) = \langle \Vdash \rangle [\Vdash^{\smile}](G) = \langle \Vdash \rangle(M) = G$, and analogously for the other operators. The only difference is for $\mathcal{ITS}$ and *est* because $[[\Vdash]](\emptyset) = G$ but $[[\Vdash^{\smile}]](G) = \{m : \Vdash^{\smile}(m) = G\} \supseteq \emptyset$; dually for $[[\Vdash]][[\Vdash^{\smile}]](\emptyset)$.   QED

From the definitions in the above Proposition it follows that the partial order between saturated subsets is the order which is inherited from the structure they are derived from. Thus, for instance, we shall have $\langle \Gamma_{\mathcal{C}}(M), \subseteq \rangle$.

Now we shall discuss the isomorphic relationships between formal operators. This result will help us to have a complete picture about the relationships between formal perception operators. Particularly we shall see that three types of relationships must be distinguished: (a) symmetry, (b) duality, (c) isomorphism.

First, we need a Lemma:

Lemma 2.3.1. *Let* $\mathbf{P} = \langle G, M, \Vdash \rangle$ *be a P-system. Then for all* $X \subseteq G, Y \subseteq M,$

$X \in \Omega_{int}(\mathbf{P})$ *iff*	$X \in \Gamma_{cl}(\mathbf{P})$ *iff*	$X \in \Gamma_{est}(\mathbf{P})$ *iff*
$X = \langle e \rangle(Y')$	$X = [e](Y')$	$X = [[e]](Y')$
$Y \in \Omega_{\mathcal{A}}(\mathbf{P})$ *iff*	$Y \in \Gamma_{\mathcal{C}}(\mathbf{P})$ *iff*	$Y \in \Gamma_{\mathcal{ITS}}(\mathbf{P})$ *iff*
$Y = [i](X')$	$Y = \langle i \rangle(X')$	$Y = [[i]](X')$

for some $Y' \subseteq M, X' \subseteq G$.

Proof. If $X = \langle e \rangle(Y')$ then $X = \langle e \rangle[i]\langle e \rangle(Y')$, from *Proposition* 1.4.8.(9). Therefore, by definition of *int*, $X = int(\langle e \rangle(Y')) = int(X)$. Vice-versa, if $X = int(X)$, then $X = \langle e \rangle[i](X)$. Hence, $X = \langle e \rangle(Y')$ for $Y' = [i](X)$. The other cases are proved in the same way, by exploiting the appropriate equations of *Proposition* 1.4.8.(9).   QED

Corollary 2.3.1. *Let $\mathbf{P} = \langle G, M, \Vdash \rangle$ be a P-system. Then,*

1. *$\langle e \rangle$ is an isomorphism between $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$ and $\mathbf{Sat}_{int}(\mathbf{P})$.*

2. *$[i]$ is an isomorphism between $\mathbf{Sat}_{int}(\mathbf{P})$ and $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$.*

3. *$[e]$ is an isomorphism between $\mathbf{Sat}_{\mathcal{C}}(\mathbf{P})$ and $\mathbf{Sat}_{cl}(\mathbf{P})$.*

4. *$\langle i \rangle$ is an isomorphism between $\mathbf{Sat}_{cl}(\mathbf{P})$ and $\mathbf{Sat}_{\mathcal{C}}(\mathbf{P})$.*

5. *$[[i]]$ is an anti-isomorphism between $\mathbf{Sat}_{est}(\mathbf{P})$ and $\mathbf{Sat}_{\mathcal{ITS}}(\mathbf{P})$.*

6. *$[[e]]$ is an anti-isomorphism between $\mathbf{Sat}_{\mathcal{ITS}}(\mathbf{P})$ and $\mathbf{Sat}_{est}(\mathbf{P})$.*

7. *The set-theoretic complementation is an anti-isomorphism between $\mathbf{Sat}_{cl}(\mathbf{P})$ and $\mathbf{Sat}_{int}(\mathbf{P})$ and between $\mathbf{Sat}_{\mathcal{C}}(\mathbf{P})$ and $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$.*

Proof. Let us notice, at once, that the proof for an operator requires the proof for its adjoint operator. Then, let us prove (**1**) and (**2**) together. First, let us prove bijection for $\langle e \rangle$ and $[i]$. From *Lemma* 2.3.1 the codomain of $\langle e \rangle$ is $\Omega_{int}(\mathbf{P})$ and the codomain of $[i]$ is $\Omega_{\mathcal{A}}(\mathbf{P})$. Moreover, for all $X \in \Omega_{int}(\mathbf{P}), X = \langle e \rangle[i](X)$ and for all $Y \in \Omega_{\mathcal{A}}(\mathbf{P}), Y = [i]\langle e \rangle(Y)$. From the adjunction properties we have:

(i) $\langle e \rangle$ is surjective on $\Omega_{int}(\mathbf{P})$ and (ii) $[i]$ is injective from $\Omega_{int}(\mathbf{P})$.
(iii) $\langle e \rangle$ is injective from $\Omega_{\mathcal{A}}(\mathbf{P})$ and (iv) $[i]$ is surjective onto $\Omega_{\mathcal{A}}(\mathbf{P})$. Moreover, if $[i]$ is restricted to $\Omega_{int}(\mathbf{P})$, then its codomain is the set $H = \{Y : Y = [i](X) \ \& \ X \in \Omega_{int}(\mathbf{P})\}$. Clearly, $H \subseteq \Omega_{\mathcal{A}}(\mathbf{P})$. In turn, if

$\langle e \rangle$ is restricted to $\Omega_{\mathcal{A}}(\mathbf{P})$, then its codomain is the set $K = \{X : X = \langle e \rangle(Y)$ & $Y \in \Omega_{\mathcal{A}}(\mathbf{P})\}$. Clearly $K \subseteq \Omega_{int}(G)$. Therefore, (i) and (iii) give that $\langle e \rangle$ is bijective if restricted to $\Omega_{\mathcal{A}}(\mathbf{P})$, while (ii) and (iv) give that $[i]$ is a bijection whenever restricted to $\Omega_{int}(\mathbf{P})$.[4]

Now it is to show that $\langle e \rangle$ and $[i]$ preserve joins and meets. For $\langle e \rangle$ we proceed as follows: (v) $\langle e \rangle(\bigvee_{i \in I}(\mathcal{A}(Y_i))) =_{def} \langle e \rangle(\mathcal{A}(\bigcup_{i \in I}(\mathcal{A}(Y_i))))$. But $\langle e \rangle \mathcal{A} = \langle e \rangle$, from *Lemma* 2.3.1.(9). Moreover, $\langle e \rangle$ distributes over unions. Hence the right side of (v) equals to $\bigcup_{i \in I}\langle e \rangle(\mathcal{A}(Y_i))$. But in view of *Proposition* 2.3.2, the union of extensional open subsets is open and from *Lemma* 2.3.1 $\langle e \rangle(\mathcal{A}(Y_i))$ belongs to $\Omega_{int}(\mathbf{P})$ indeed, so that the right side of (v) turns into $int(\bigcup_{i \in I}\langle e \rangle(\mathcal{A}(Y_i))) =_{def} \bigvee_{i \in I}\langle e \rangle(\mathcal{A}(Y_i))$.
(vi) $\langle e \rangle(\bigwedge_{i \in I}\mathcal{A}(Y_i)) = \langle e \rangle(\bigcap_{i \in I}[i]\langle e \rangle(Y_i))$. Since $[i]$ distributes over intersections, the right side of (vi) turns into $\langle e \rangle[i](\bigcap_{i \in I}\langle e \rangle(Y_i)) = int(\bigcap_{i \in I}\langle e \rangle(Y_i))$. But $\langle e \rangle = \langle e \rangle \mathcal{A}$, so that the last term is exactly $\bigwedge_{i \in I}\langle e \rangle(\mathcal{A}(Y_i))$. Since $[i]$ is the inverse of $\langle e \rangle$, *qua* isomorphism, we have that $[i]$ preserves meets and joins, too.

As to (**3**) and (**4**), the results come by symmetry.

(**5**) and (**6**) As in the above proof (we can optimize a passage by noticing that $[[e]]$ and $[[i]]$ are both upper and lower adjoints) and by the fact that in polarities the right category is reversed upside down with respect to the order (hence, with respect to the lattice operations). (**7**) By duality between the operators.   QED

It should be noticed carefully that points 5 and 6 of *Corollary* 2.3.1 definitely code the "Loy de Port-Royal".

As we have seen, $int(X)$ and $cl(X)$ translates the usual topological definitions of an interior and, respectively, a closure of a set $X \subseteq G$, into the language provided by observation systems.

Therefore, from *Corollary* 2.3.1, we immediately have that given $Y \subseteq M$, $\mathcal{A}(Y) = Y$ expresses the fact that Y is a "formal open subset" and $\mathcal{C}(Y) = Y$ means that Y is a "formal closed subset" (Figure 2.6).

---

[4]As side results, we have: (a) $\Omega_{\mathcal{A}}(\mathbf{P}) = H$ and (b) $\Omega_{int}(\mathbf{P}) = K$. This is not surprising, because if $Y \in \Omega_{\mathcal{A}}(\mathbf{P})$ then $Y = [i]\langle e \rangle(Z)$ for some $Z \subseteq M$ and $\langle e \rangle(Z) \in \Omega_{int}(\mathbf{P})$, any $Z \subseteq M$. Vice-versa, if $X \in \Omega_{int}(\mathbf{P})$, then $X = \langle e \rangle(Z)$. Hence $[i](X) = [i]\langle e \rangle(Z)$ belongs to $\Omega_{\mathcal{A}}(\mathbf{P})$. Symmetrically for (b).
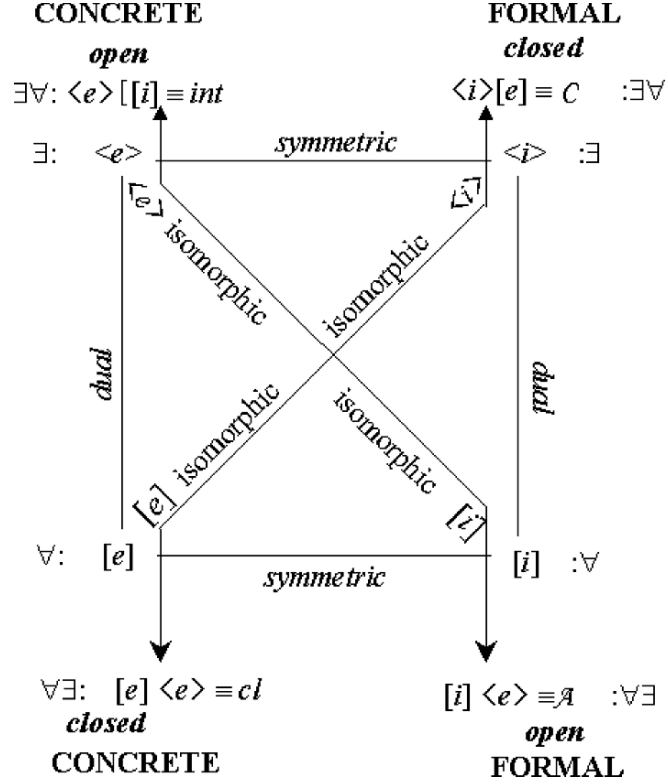
Figure 2.6: Concrete and formal topological operators

All the same, once more the reader must pay attention to the fact that although int and $\mathcal{C}$ are interior operators, they are not, in general, topological interior operators, since it is not guaranteed that they preserve intersections. This can readily be verified by considering the fact that, from the very definition of int, we have:

$$int(X) = \langle e \rangle \{m : \langle e \rangle(m) \subseteq X\}, \forall X \subseteq G. \qquad (2.3.6)$$

so that an open set may be a union of other open sets but it is not guaranteed that the intersection of open sets is open.

Similarly,

$$cl(X) = [e]\{m : \langle e \rangle(m) \cap X \neq \emptyset\}, \forall X \subseteq G. \qquad (2.3.7)$$

so that it is not guaranteed that cl, $\mathcal{A}$, est and $\mathcal{ITS}$ preserve unions (see Part III for details on this topic).

However we have a simple, although notable, result about this topic.

Proposition 2.3.3. *Let θ and ϕ be two dual basic operators. Then,*

1. *θ is a closure operator if and only if ϕ is an interior operator.*
2. *θ is topological if and only if ϕ is topological.*

Proof. (**1**) Trivially, since complementation reverses the order. (**2**) Suppose θ is additive, then $\phi(X \cap Y) = -\theta - (X \cap Y) = -\theta(-X \cup -Y) = -(\theta(-X) \cup \theta(-Y)) = -\theta(-X) \cap -\theta(-Y) = \phi(X) \cap \phi(Y)$. Dually for the opposite implication. QED

We end this Section with a useful characterisation suggested by equation 2.3.6

Proposition 2.3.4. *Let $\mathbf{P}$ be a P-system. Then for any $X \subseteq G$,*

$$int(X) = \bigcup\{\langle e \rangle(m) : \langle e \rangle(m) \subseteq X\}$$

Hence, for obvious reasons, we call the family $\{\langle e \rangle(m) : \langle e \rangle(m) \subseteq X\}_{X \subseteq G}$ the *base* of $\Omega_{int}(\mathbf{P})$. It is left to the reader to prove that equations 2.3.6 and 2.3.7 hold and that *Proposition* 2.3.4 derives from 2.3.6.

Example 2.3.2. Basic perception operators – II

Let us now visualise the lattices of saturated sets (from the basic Example 1.2.1):
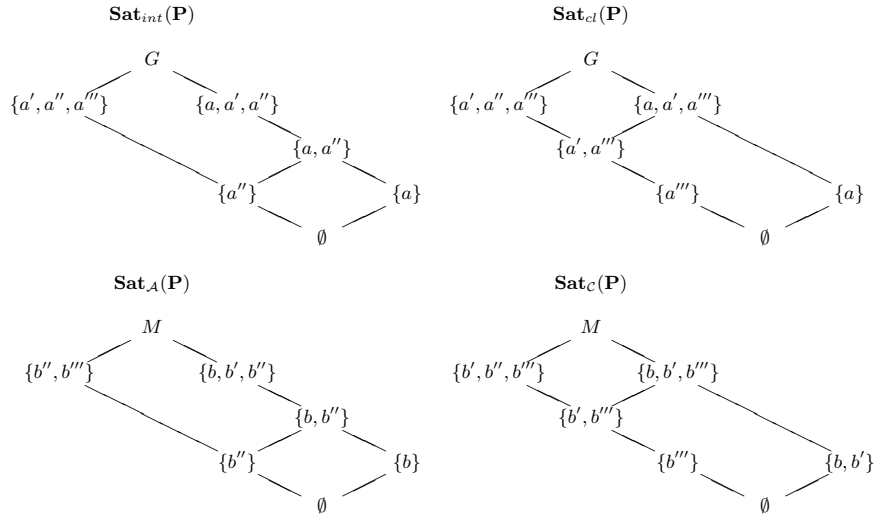
Notice, for instance, that $\{b\} = [i](\{a\})$ and $\{b'', b'''\} = [i](\{a', a'', a'''\})$. Indeed, $[i]$ is an isomorphism from $\mathbf{Sat}_{int}(\mathbf{P})$ to $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$. Otherwise stated, given a set

X in $\mathbf{Sat}_{int}(\mathbf{P})$, the corresponding element Y of $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$ is the set of properties which uniquely characterises X. Vice-versa, recall that $\mathcal{A} = [i]\langle e \rangle$ and, in fact, symmetrically, $\langle e \rangle$ is an isomorphism from $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$ to $\mathbf{Sat}_{int}(\mathbf{P})$ (for example, $\langle e \rangle(\{b, b''\}) = \{a, a''\}$). Hence, given a set Y in $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$, the corresponding element in $\mathbf{Sat}_{int}(\mathbf{P})$ is the set of objects uniquely characterised by Y. Dually, $\langle i \rangle$ is an isomorphism between $\mathbf{Sat}_{cl}(\mathbf{P})$ and $\mathbf{Sat}_{\mathcal{C}}(\mathbf{P})$ (for instance, $\langle i \rangle(\{a\}) = \{b, b'\}$) and $[e]$ is the inverse of $\langle i \rangle$ (e. g. $[e](\{b, b'\}) = \{a\}$).

Moreover we can verify that the set-theoretic complementation is an anti-isomorphism between $\mathbf{Sat}_{cl}(\mathbf{P})$ and $\mathbf{Sat}_{int}(\mathbf{P})$ and between $\mathbf{Sat}_{\mathcal{C}}(\mathbf{P})$ and $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$. For instance $-\{a\} = \{a', a'', a'''\}$ and $-\{a, a''\} = \{a', a'''\}$, $-\{b', b''\} = \{b, b''\}$, and so on.

Let us verify some examples of operations on these lattices.
$\{a\} \vee \{a'''\} = \{a, a', a'''\} = cl(\{a\} \cup \{a'''\})$. In fact $\mathbf{Sat}_{cl}(\mathbf{P})$ is not closed under unions, but the join, of X and Y in $\mathbf{Sat}_{cl}(\mathbf{P})$ is given by $cl(X \cup Y)$. Dually, $\{a, a', a''\} \wedge \{a', a'', a'''\} = \{a''\} = int(\{a, a', a''\} \cap \{a', a'', a'''\})$, because $\mathbf{Sat}_{int}(\mathbf{P})$ is not closed under intersections but the meet of X and Y in $\mathbf{Sat}_{int}(\mathbf{P})$ is given by $int(X \cap Y)$.

So, notice that cl is not additive because $cl(X \cup Y) \neq cl(X) \cup cl(Y)$ (for instance $cl(\{a\}) \cup cl(\{a'''\}) = \{a\} \cup \{a'''\} = \{a, a'''\} \neq cl(\{a, a'''\}))$. Similarly, int is not multiplicative, because intersections of fixpoints of int generally are not fixpoints of int.

On the contrary, $cl^o : \wp(\{a, a', a'', a'''\}) \longmapsto \Gamma_{cl}$ is additive because $cl^o(X \cup Y) = cl(X \cup Y) = X \vee Y = cl^o(X) \vee cl^o(Y)$, and $int^o : \wp(\{a, a', a'', a'''\}) \longmapsto \Omega_{int}$ is multiplicative because $int^o(X \cap Y) = int(X \cap Y) = X \wedge Y = int^o(X) \wedge int^o(Y)$ (for instance, $cl^o(\{a\}) \vee cl^o(\{a'''\}) = \{a, a', a'''\} = cl^o(\{a, a'''\}))$.

One can now easily find symmetrical examples for $\mathbf{Sat}_{\mathcal{A}}(\mathbf{P})$ and $\mathbf{Sat}_{\mathcal{C}}(\mathbf{P})$.

Exercise 2.5.
(A) *Using the equation $[i]\langle e \rangle[i] = [i]$, prove that for all $Y, Y' \subseteq \Omega_{\mathcal{A}}(\mathbf{P})$, $\langle e \rangle(Y) = \langle e \rangle(Y')$ implies $Y = Y'$.*
(B) *Just using logical transforms of connectives and quantifiers prove: $[[i]](A \cap B) \supseteq [[i]](A) \cup [[i]](B)$ and $[[i]](A \cup B) = [[i]](A) \cap [[i]](B)$.*
(C) *In Proposition 2.1.1 it is claimed that $b \in [[i]](X)$ iff $X \subseteq \langle e \rangle(b)$. In view of this claim is it possible to derive, $Y \subseteq [[i]](X)$ iff $X \subseteq \langle e \rangle(Y)$?*
(D) *Prove: $[[i]](X) = \bigcap\{\langle e \rangle(m) : X \subseteq \langle e \rangle(m)\}$.*
(E) *Just using logical transforms, prove $[e](Y) = -\langle e \rangle(-Y)$.*
(F) *Prove that $\langle\langle i \rangle\rangle(X) = \{b \in M : \exists a(a \notin X \ \& \ a \not\Vdash b)\}$.*
(G) *Prove that $[[i]]$ and $[i]$ are orthogonal (that is, they arise by negating all the subformulas of the other).*
(H) *Compute all the elements of $\Omega_{\mathcal{A}}(\mathbf{P})$, $\Gamma_{\mathcal{C}}(\mathbf{P})$, $\Gamma_{cl}(\mathbf{P})$ and $\Omega_{int}(\mathbf{P})$.*
(I) *Prove that cl and int are dual.*

2.3.2 Multi-Agent Pre-Topological Approximation Systems

We wonder whether int, cl or est and $\mathcal{ITS}$ have a proper informational and conceptual interpretation. Now we give an answer as to int and cl while a discussion for est and $\mathcal{ITS}$ is reserved for the future.

Given $X \subseteq G$ it is known that $[e]\langle i\rangle(X) \supseteq X$ and $\langle e\rangle[i](X) \subseteq X$. We can interpret these relationships by saying that:

(ua) cl is an *upper approximation* of the identity map on $\wp(G)$.

(la) int is a *lower approximation* of the identity map on $\wp(G)$.

More precisely, since $\langle i\rangle(X) = min([e]^{\leftarrow}(\uparrow X)) = min\{X' \subseteq G : [e](X') \supseteq X\}$, we have that $[e]\langle i\rangle(X)$ (i.e. $cl(X)$) is the *best approximation from above* to X *via* function $[e]$.

Dually, $[i](X) = max(\langle e\rangle^{\leftarrow}(\downarrow X)) = max\{X' \subseteq G : \langle e\rangle(X') \subseteq X\}$. Hence, $\langle e\rangle[i](X)$ (i.e. $int(X)$) is the *best approximation from below* to X, *via* function $\langle e\rangle$.[5]

If $\langle i\rangle$ is injective (or, equivalently, $[e]$ is surjective), then we can exactly reach X from above by means of $[e]$. The element that must be mapped is, indeed, $\langle i\rangle(X)$. Dually, if $[i]$ is injective (or $\langle e\rangle$ is surjective), then we can exactly reach X from below by means of $\langle e\rangle$ applied to $[i](X)$.

Given a *P-system* $\mathbf{P}$, let us define the following families of operators manipulating sets of objects: $\mathbf{G^P} = \{int, cl\}$. On this basis we can introduce the notion of a *Multi-agent pre-topological Approximation Systems* (*mpAP* for short):

$$\langle G, M, \{\mathbf{G^{P_k}}\}_{k \in K}\rangle \text{ or, shortly, } \langle G, \{\mathbf{G^{P_k}}\}_{k \in K}\rangle$$

where any $\mathbf{P}_k$ is an Information System on the same set of objects G. This generalisation makes it possible to manipulate sets of objects by subsequently applying informational criteria induced by different Information Systems.

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[5]The pairs of concepts $\langle least, upper\rangle$ and $\langle greatest, lower\rangle$ are essential. Indeed, dealing with upper approximations is too vague, since we can have a plenty of upper approximations with no upper bound (dually for lower approximations). Exactly for the same reasons we deal with least upper bounds, greatest lower bounds, least fixed points, least common multiples, greatest common divisors and so forth. In this duality, the notion of a *limit* (or *co-limit*) is embedded.

Particularly, the structure $\langle G, int, cl \rangle$ will be called a *Basic pre-topological Approximation System.*

By carrying on the same construction on properties (or attributes) and setting $\mathbf{M^P} = \{\mathcal{A}, \mathcal{C}\}$ we obtain a Multi-agent pre-topological co-Approximation System $\langle M, G, \{\mathbf{M^{P_j}}\}_{j \in J} \rangle$ or, shortly, $\langle M, \{\mathbf{M^{P_j}}\}_{j \in J} \rangle$ and by merging the two spaces we obtain the notion of a *Multi-agent pre-topological Perception System*:

Definition 2.3.3. *Let* $\mathbf{P}$ *be a P-system, then the structure:*

$$\langle G, M, \{\mathbf{G^{P_k}}\}_{k \in K}, \{\mathbf{M^{P_j}}\}_{j \in J}, \mathbf{I^P}, \mathbf{E^P} \rangle,$$

where any $\mathbf{P}_k$ *is an Information System on the set of objects* G *and set of properties or attributes* M_k *possibly distinct from* M, *any* $\mathbf{P}_j$ *is an Information System on the set of properties (or attributes)* M *and set of objects* G_j *possibly distinct from* G, $\mathbf{I^P} : \wp(G) \longmapsto \wp(M)$ *and* $\mathbf{E^P} : \wp(M) \longmapsto \wp(G)$ *are maps, is called a* Multi-agent pre-topological Perception System.

Notice that any of the above families may be empty.

Particularly, if $card(k) = card(j) = 1$ we shall adopt the term *(Single-agent) pre-topological Perception System*.

Chapter 3

Pre-Topological and Topological Approximation Operators

3.1 Information, Concepts and Formal Operators

So far we have listed a number of well-defined mathematical instruments that act on either sides of abstraction we are dealing with, that is, points and properties. But what is the informational interpretation of the above machinery? What is a plausible philosophical interpretation of all these mathematical results? Are they able to provide points (i.e. noumena) with a proper informational and conceptual structure based on their manifested properties (i.e. phenomena)?

First of all, from *Corollary* 2.3.1 we can say that the operators $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{ITS}$ give the intensional (or formal) images of the extensional structures that are defined by *int*, *cl* and *est* on G and, symmetrically, through *int*, *cl* and *est* we obtain extensional (or concrete) images of the formal structures that $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{ITS}$ define on M. However, this imaging is not a mere mirroring, since intensional structures are not definable without the extensional structures, and the other way around. This is a plausible point of view under any non-mechanical approach to cognitive acts.

Thus we maintain that *P-systems* equipped with the basic constructors and the basic operators are good starting points.

But we have not yet answered the question above. On the contrary, another one is to be added.

We know that *int* and *cl* provide us with lower and upper approximations of any set $X \subseteq G$. But are we really happy with this machinery? The answer is "yes and no". Yes, because we have found a mathematically sound way to deal with approximations which respects a reliable intuitive interpretation. No, because both *int* and *cl* are discontinuous operators because *int* is not multiplicative and *cl* is not additive, so that we have to face "jumps" which can be too wide and make us miss information.

In order to solve the above issue, we must again start noticing that any answer and solution depends on the nature of the *P-system* at hand. Generally, the nature of points is not really important. More important is the nature of properties. And, even more important is the nature of the operator supposed to better represent the basic perception act.

Once this choice has been made we can easily define higher level structures. In fact, suppose Op is our hypothetical choice. Then, given a *P-system* $\mathbf{P} = \langle G, M, \Vdash \rangle$, the family of Op-saturated subsets of G immediately induces a first order derivative *P-system* $\mathbf{P}' = \langle G, G, R \rangle$ where $R \subseteq G \times G$ is a binary relation derived from the Op-saturated structure we started with (the same apply to any family of symmetric Op-saturated subsets of M, of course).

As usual we have two basic derivatives: an "existential" and a "universal" derivative:

- (U) $\langle g, g' \rangle \in R$ if for all Op-saturated set O, $g \in O$ implies $g' \in O$

- (E) $\langle g, g' \rangle \in R$ if there exists an Op-saturated set O, such that $g, g' \in O$

A variant is obtained by substituting the implication in (U) with a bi-implication.

Moreover, since $\mathbf{P}'$ is a *P-system*, we can compute its saturated structures and obtain second order derivatives of $\mathbf{P}$, using relations induced by other operators if convenient. And so on. However, we wont exploit relations directly defined by means of Op-saturated sets, but we shall derive such kind of new relations from a conceptual analysis in our *phenomena-noumena* framework.

All we have now to do is to carefully decide the starting point and understand the informational content of the derivatives.

3.1.1 Choosing the Initial Perception Act

So far we have assumed that a "phenomenological system" is a set of objects coupled with a set of attributes or properties through which we are aware of them: "data + attributes" (*A-systems*) or "data + properties" (*B-systems*), this is what we supposedly are given. That is, it has been assumed that original sources of knowledge are systems of information of either of these sorts. Probably this is not "the" way one interacts with the world (it is surely too much an Aristotelian beginning), but without any doubts a number of information is presented this way.

P-systems are special cases of *A-systems* in which $V = \{0, 1\}$ – up to a proviso that we shall detail later on. This fact will be underlined by giving *A-systems* the more general term introduced by Z. Pawlak: *Information Systems* or the term *Multivalued Contexts* introduced by R. Wille. However, since in turn, as we shall see, *A-systems* are equivalent to particular *P-systems*, we shall continue to use the present terminology and collect both *P-systems* and *A-systems* under the term "Information Systems".

We also have assumed that our first act of knowledge is a grouping act, a sort of "data abstraction". This can basically be performed in two opposite ways: either collect around an object g the elements which fulfills at least all the properties (or attribute-values) of g, or the elements fulfilling at most all the properties (or attribute-values) of g. Otherwise stated, in the first case we collect the objects which are characterised at least as g by the properties (attributes) at hand, while in the second case we collect the objects which are characterised at most as g. However if we consider attribute-values instead of properties, the two conditions collapse (see later on).

Moreover notice that the grouping rule just asserted does not imply any form of symmetry. Indeed, g' could manifest all the properties of g but also additional properties that are not manifested by g. To put it another way, we are claiming that our basic grouping act is not based on the notion "to manifest *exactly* the same properties", but on the notion "to manifest *at least* (or *at most*) the same properties". Indeed, the set of properties which are manifested by g, is the attracting phenomenon around which we form our perception. Thus from an analytical point of view we have just to focus on these properties and not to take into account additional properties. In the present paragraph it will be seen

that the former notion is subsumed by the latter. That is, if we define this way the basic "cells" of our categorisation process, a wider range of cases shall be covered.

However, it is worth discussing the notion of "sameness" between properties.

Indeed, one can legitimately ask: "What does 'same' mean? Does it mean 'equal' or may I consider some more subtle concept like 'equal up to some tolerated difference'? You know, we have experimental errors and other sorts of noises, or sometimes an ϵ difference is not important".

Yes, all of the above are perspicuous and acceptable interpretations.

The answer of Classical Rough Set Theory (from now on CRST) is: "same" means "identical". But, we have to add immediately, "identical" up to the way our data are collected and presented. Indeed, CRST must not be interpreted as a witness of "rigidity". It should be such, if we had some ingenuously realistic point of view for which we are "tabulae rasae" ready to be impressed by external data. On the contrary we know that any "external datum" is filtered by a number of "intentional" co-ordinates, so that any datum is actually an "intentional datum". Hence any attribute is manifested just by means of some more or less artificial "experimental system". Therefore a datum belongs to an Information System because of and to the extent to which it is within the framework of a "conceptual scaling":

"*A quality, as such, is never an object of observation. We can see that a thing is blue or green, but the quality of being blue and the quality of being green are not things that we see; they are products of logical reflections.*" (Charles. S. Peirce, *The Fixation of Belief*, Popular Science Monthly, 12, November 1877, pp. 1–15).

Thus although we are considering "identical values", nonetheless we are allowed to consider as identical, for instance, the values "3" and "4" – i.e. the two properties "taking value 3" and "taking value 4" – if they belong to the same conceptual interval (for instance if they represent the distance in millimeters from the center of a target of two different shots during a target-shooting game and we are merely interested if the shot is within the center).

Therefore, the scale we use is a matter of presentation of data, while the real objection to the "same-as-identical" assumption, is related to its interpretation and affects the level of granularity of the resulting conceptual structure. In fact variants and extensions of CRST have been proposed and the still increasing literature on rough sets presents

a good repertoire of solutions in the neighborhoods of the basic aspects of the classical theory.

Now let us come back to our "perception cells".

We have basically two choices, as we know (a third is a "neutral" choice):

Let $\langle G, At, \{V_a\}_{a \in At} \rangle$ be an *Attribute System.* Then for all
  $g \in G$ define:

$Q_g = \{g' : \forall a \in At, \forall x \in V_a((a(g) = x) \Rrightarrow (a(g') = x))\}$ –
  quantum of information at g.

Let $\langle G, M, \Vdash \rangle$ be a *Property System.* Then for all $g \in G$ define:

$Q_g^{\uparrow} = \{g' : \forall m \in M(m \in \langle i \rangle(g) \Rrightarrow m \in \langle i \rangle(g'))\}$ – the *quantum
  of information at g.*

$Q_g^{\downarrow} = \{g' : \forall m \in M(m \in \langle i \rangle(g') \Rrightarrow m \in \langle i \rangle(g))\}$ – the
  co-quantum of information at g.

Remember that the two names g and g' are not part of the information we have: they just stand for the observation of some phenomena. Therefore we maintain that an object g' must be associated with the same "category" as g if g' manifests *all* the phenomenological properties that are manifested by g. That is, we are not able to perceive g without perceiving g'.

This assumption reflects the idea "g is perceived together with g' whenever g' manifests *at least* the same properties as g". Indeed, if $\langle i \rangle(g)$ and $\langle i \rangle(g')$ are the set of *positive* properties fulfilled by g and, respectively, g' and $\langle i \rangle(g) \subseteq \langle i \rangle(g')$, then g' surely belongs to the extension of the set of properties $\langle i \rangle(g)$. For example, if someone manifests the property "to be Italian", whenever we consider what is perceivable by means of the same property we must include *any* Italian, with no regards to any additional properties they can manifest (for instance, "to be Roman" or "to have a degree").

Therefore, given (the properties manifested by) an object g the perception cell organised around g should be $Q_g^{\uparrow}$ which should be referred to as the "*minimum perceptibilium at location g*", because it is not possible to perceive g without perceiving (the manifestations of) the other members of that "perception parcel". Therefore, we call such

perception parcel a *quantum of perception* or a *quantum of information at g*. This terminology is not from CRST. Instead we drew our inspiration from Bell [1983] and this term expresses a sort of programme that may be epitomized by the slogan "**any information is a quantum of information**".

Objects in the sense of object-oriented programming, for instance, are very sophisticated "quanta of information". Here we have more basic "quanta of information".

The reader is invited to notice that the name "quantum" is not suggested by a vague analogy living in the Hyperuranium: on the contrary, as it is seen in Frame 4.6.1, under certain circumstances quanta of information happen to be particular *quanta at a location* in the sense of Bell [1983] and quanta at a location, in turn, are connected with ortholattices. Eventually, ortholattices are connected with Quantum Logic.

As to quanta of information from a *Property System*, we can elaborate a little further (*Attribute Systems* shall be resumed later on).

Proposition 3.1.1. *Let* $\mathbf{P} = \langle G, M, \Vdash \rangle$ *be a P-system and* $g, g' \in G$. *Then,*

1. $Q_g^{\uparrow} = est(g)$.

2. $g' \in Q_g^{\uparrow}$ *iff for all* $X \in \Gamma_{est}, g \in X \Rightarrow g' \in X$.

3. $g' \in Q_g^{\uparrow}$ *iff for all* $m \in M, g \in \langle e \rangle(m) \Rightarrow g' \in \langle e \rangle(m)$, *iff* $\langle i \rangle(g) \subseteq \langle i \rangle(g')$.

4. $g' \in Q_g^{\uparrow}$ *iff* $g \in Q_{g'}^{\downarrow}$ *iff* $g \in cl(g')$.

Proof. The proofs are based on *Proposition* 2.1.1.(6) and *Proposition* 2.3.1.(5).

(**1**) Indeed, $g' \in est(g)$ iff $\Vdash (g') \supseteq [[i]](g)$. But $[[i]](g) = \langle i \rangle(g) = \Vdash (g)$, whence $g' \in est(g)$ if and only if $m \in \Vdash (g) \Rightarrow m \in \Vdash (g')$ if and only if $g' \in Q_g^{\uparrow}$.

(**2**) ($\Leftarrow$) Suppose $X \in \Gamma_{est}$ and $g \in X \Rightarrow g' \in X$. Then $\Vdash (g) \supseteq [[i]](X)$ implies $\Vdash (g') \supseteq [[i]](X)$. Since this happens for all *est*-saturated X, it happens for $[[i]](g)$ too and we trivially obtain $\Vdash (g') \supseteq \Vdash (g)$, so that $g' \in Q_g^{\uparrow}$. ($\Rightarrow$) If $g' \in Q_g^{\uparrow}$ then $\Vdash (g') \supseteq \Vdash (g)$. If, moreover, $g \in X$, for $X = est(X)$, then $\Vdash (g) \supseteq [[i]](X)$. By transitivity, $\Vdash (g') \supseteq [[i]](X)$, whence $g' \in X$ too.

(**3**) Indeed, $g \in \langle e \rangle(x)$ if and only if $g \Vdash x$ if and only if $x \in \Vdash (g)$. Hence for all $m \in M, g \in \langle e \rangle(m) \Rrightarrow g' \in \langle e \rangle(m)$ if and only if for all $m \in M, m \in \langle i \rangle(g) \Rrightarrow m \in \langle i \rangle(g')$.

(**4**) So, $g' \in Q_g^{\uparrow}$ if and only if $\langle i \rangle(g) \subseteq \langle i \rangle(g')$ if and only if $g \in Q_{g'}^{\downarrow}$ if and only if $g \in cl(g')$. $\hspace{2cm}$ QED

Corollary 3.1.1. *In any P-system $\langle G, M, \Vdash \rangle$, for all $g \in G$, $est(g) = \{g' : g \in cl(g')\}$.*

Indeed, this result, a trivial consequence of the above *Proposition*, formally states, in view of *Proposition* 2.3.1.(2), that g' is perceived together with g if and only if it fulfills at least all the properties fulfilled by g. The above intuition is highlighted by the following observation: a quantum of perception at a location g is the *universal extension* of function *sub* to the set of properties $\langle i \rangle(g)$ fulfilled by g. In turn, since we start with a singleton $\{g\}$, $\langle i \rangle(g)$ is both a universal and an existential extension of function *obs*. Hence,

$$Q_g^{\uparrow} = [[e]]\langle i \rangle(g) = [[e]][[i]](g) = \{g' : g \in cl(g')\}.$$

However, this is true of singletons, but $\langle i \rangle$ and $[[i]]$ reveal their nature of opposite dual operators when applied to arbitrary sets and this difference openly works when we have to decide how to combine elementary cells or, more precisely, how to move from grouping-maneuvers around a single object to grouping maneuvers around two or more objects.

In this case, too, we have essentially two kinds of choice: universal extensions from singletons to a generic set X and existential extensions.

As to existential extensions we have two alternatives which are compatible with our basic choice:

$$\boxed{Q_X^{\cup} = \bigcup_{x \in X} Q_x^{\uparrow} \quad \bigg| \quad Q_X^{\oplus} = [[e]]\langle i \rangle(X)}$$

As to universal extensions we briefly discuss only the following alternative:

$$Q_X^{\otimes} = [[e]][[i]](X) = est(X).$$

The superscript $\otimes$ underlines the fact that in $est(X)$ we consider the properties which glue the elements of X together. Otherwise stated, we extract from $\Vdash (X)$ the set of properties P' which are shared by all the elements of X. Then we apply the symmetric process to P'. Thus, according to this universal extension, an object g belongs to $Q_X^{\otimes}$ if g

fulfills all the properties fulfilled by all the elements of X. In fact, the universal extension of quanta of information is the basic mechanism for the construction of formal concepts in Formal Concept Analysis (see Frame 4.9).

In $Q_X^\oplus$ we add together all the properties fulfilled by the elements of X, as if X were a single object perceived through the properties in $\langle i \rangle(X)$. Subsequently we collect all the elements g such that $\langle i \rangle(g) \supseteq \langle i \rangle(X)$.

In $Q_X^\cup$ we consider X as an actual set where all elements carry a specific information.

For the very reasons discussed so far, as initial perception acts we shall adopt $Q_g^\uparrow$ and $Q_X^\cup$, which, moreover, makes a uniform treatment of both P-*systems* and A-*systems* possible.

Moreover, notice that $Q_X^\oplus$ is not an additive operator. Indeed, for any $A, B \subseteq G$, $Q_{A \cup B}^\oplus = Q_A^\oplus \cap Q_B^\oplus$ (the operator $\langle i \rangle$ distributes over $\cup$, but the operator $[[e]]$ turns $\cup$ into $\cap$).

On the contrary, $Q_X^\cup$ is trivially additive (notice that we obtain $Q_X^\cup$ by simply swapping the two quantifiers in $Q_X^\oplus$).

TERMINOLOGY AND NOTATION. Since by default we shall deal just with quanta of information and additive extensions of quanta of information – from now on also *i-quanta* – we shall generally avoid "$\uparrow$" and $\cup$" to qualify Q_g or Q_X. On the contrary, whenever we need to explicitly distinguish the Information Systems which induce the quanta of information we are dealing with, we shall use the name of the system as an exponent (e.i. $Q_g^{\mathbf{P}}$, $Q_X^{\mathbf{S}}$, etc.). Moreover, if $\mathbf{O} = \langle X, R \rangle$, we set $-\mathbf{O} = \langle X, -R \rangle$ and $\mathbf{O}^\smile = \langle X, R^\smile \rangle$.

From now on, if $\mathbf{A}$ denotes an A-*system*, then it is understood that $\mathbf{A} = \langle G, At, V \rangle$. Similarly, if $\mathbf{P}$ denotes a P-*system*, then it is understood that $\mathbf{P} = \langle G, M, \Vdash \rangle$. With $\mathbf{S}$ we shall denote either of these two kinds of systems.

We can ask whether these are the only meaningful combinations of phenomenological operators to start with. Within the limit of the present approach the answer is "yes". Clearly other operators could be adopted, with different informational meaning and the reader will be able to try and find other interesting combinations.

However, our choice is powerful and expressive enough to make the development of a number of theoretical and technical features possible.

Example 3.1.1.

Consider the *P-system* **P** of Example 1.2.1. Let us compute some quanta of information of **P**: $Q_a = \{a\}, Q_{a'} = \{a', a''\}, Q_{a''} = \{a''\}, Q_{a'''} = \{a', a'', a'''\}, Q_{\{a,a'\}} = \{a, a', a''\}$ and so on additively. It is worth noticing that $cl(\{a, a'''\})$ is $\{a, a', a'''\}$ and $int(\{a', a''\}) = \{a''\}$, while $Q_{\{a,a'''\}} = \{a, a'''\}, Q_{\{a',a''\}} = \{a', a,'' a'''\}$, so that Q_X is definitely an operator different from both cl and int (from int because Q is increasing, too).

Now let us compute some instances of the operator $Q^{\oplus}$:
$Q_a^{\oplus} = [[e]]\langle i \rangle(\{a\}) = [[e]](\{b, b'\}) = \{a\} = Q_a$, and so on for any application of $Q^{\oplus}$ to singletons. On the contrary, $Q_{\{a,a'\}}^{\oplus} = [[e]]\langle i \rangle(\{a, a'\}) = [[e]](\{b', b'''\}) = \{a', a''\} = \emptyset$.

More in general notice that if $x \in Q_Y^{\oplus}$ then $\langle i \rangle(x) \supseteq \langle i \rangle(Y)$. It follows that for all $y \in Y, \langle i \rangle(y) \subseteq \langle i \rangle(x)$, so that $x \in Q_y$. Thus $x \in Q_Y^{\oplus}$ implies $x \in Q_Y$. The converse implication does not hold, though, because if $x \in Q_Y$ then $\exists y \in Y$ such that $x \in Q_y$, while $x \in Q_Y^{\oplus}$ if and only if $x \in \bigcap_{y \in Y} Q_y$.

3.1.2 Information Quantum Relational Systems

Now that we have chosen the basic mechanisms (basis and step) leading from atomic perceptions (or "elementary perception cells") to complex perception, let us analyse what kinds of relation arise between elements of G from these grouping-maneuvers.

Let us then set the following definition:

Definition 3.1.1 (Information Quantum Relational System)**.** *Let* **S** *be an Information System over a set of points G. Let R be a binary relation on G. We say that R is induced by* **S** *whenever the following holds, for all $g, g' \in G$:*

$$\langle g, g' \rangle \in R \text{ iff } g' \in Q_g.$$

We call R the information quantum relation *– or* i-quantum relation *or* quantum relation *in short – induced by* **S** *and it will be denoted as $R_{\mathbf{S}}$. Moreover,* **Q**(**S**) *will denote the relational system $\langle G, R_{\mathbf{S}} \rangle$, which is called the* Informational Quantum Relational System *– IQRS, or* Quantum Relational System *in short, induced by* **S***.*

Since $g' \in Q_g$ says that g' fulfills at least all the properties fulfilled by g, then $\langle g, g' \rangle \in R_{\mathbf{S}}$ has the same meaning.

REMARKS. Notice that $\langle G, R_{\mathbf{S}} \rangle$ is a square *P-system* $\langle G, G, R_{\mathbf{S}} \rangle$ and we shall liberally use this fact.

Clearly the properties of i-quantum relations depend on the patterns of objects induced by the given systems. However, they uniformly fulfill some basic properties. In view of the additivity property of generalised quanta, we can confine our attention to i-quanta at a location.

Lemma 3.1.1. *In any Information System* **S** *over a set G, for all $g, g', g'' \in G$:*

1. *(a) $g \in Q_g$ (q-reflexivity); (b) $g'' \in Q_{g'}$ & $g' \in Q_g \Rightarrow g'' \in Q_g$ (q-transitivity).*

2. *If* **S** *is an A-system, or a functional or dichotomic P-system then $g' \in Q_g$ implies $g \in Q_{g'}$ (AFD-q-symmetry).*

Proof. (**1**) The two statements are obvious consequences of transitivity and reflexivity of the relation $\subseteq$. Notice that antisymmetry does not hold because of the obvious fact that $g' \in Q_g$ and $g \in Q'_g$ does not imply $g' = g$. (**2**) Now, let **S** be an *A-system*. Suppose $g' \in Q_g$ and $a(g) \neq x$, then $a(g) = x'$ for some $x' \neq x$ so that $a(g') = x'$, because $g' \in Q_g$, whence $a(g') \neq x$ too. Therefore, $g' \in Q_g$ implies $g \in Q_{g'}$, so that the induced relation is also symmetric. If **S** is a functional *P-system* we trivially obtain the proof from the fact that $\Vdash (g)$ is a singleton. Finally, if **S** is dichotomic and $g' \in Q_g$, then g' fulfills at least the same properties as g. Now, if $g' \Vdash p$ while $g \nVdash p$, then $g \Vdash \overline{p}$ (where $\overline{p}$ is a complementary copy of p), but $g' \nVdash \overline{p}$, since it fulfills p. Hence we cannot have $\Vdash (g) \subseteq \Vdash (g')$, whence $g' \notin Q_g$. Contradiction.   QED

So notice that in *A-systems* the universal quantification over values hides a bi-implication because the set of attribute-values of g' and that of g must coincide in order to have $g' \in Q_g$.

As an immediate consequence of the above result we have:

Proposition 3.1.2. *Let* **S** *be an Information System. Then:*

1. *The i-quantum relation $R_{\mathbf{S}}$ induced by* **S** *is a preorder.*

2. *If* **S** *is an A-system or an FP or DP system, then $R_{\mathbf{S}}$ is an equivalence relation.*

3. *If* **S** *is an FP-system then $R_{\mathbf{S}} = \Vdash \otimes \Vdash^{\smile}$ and $g' \in Q_g^{\uparrow}$ iff $g' \in Q_g^{\downarrow}$ iff $g' \in [g]_{k_{\Vdash}}$.*

Proof. We have to prove statement (3), only. In view of *Proposition* 3.1.1.(4) we just need to show that if $\hat{\Vdash}$ is a map then for all $x \in G$, $cl(\{x\}) = [x]_{\kappa_{\hat{\Vdash}}}$. From *Proposition* 2.3.1 $a \in cl(\{a'\})$ if and only if $\langle i \rangle(\{a\}) \subseteq \langle i \rangle(\{a'\})$. Therefore, if $\hat{\Vdash}$ happens to be a map, we have that $a \in cl(\{a'\})$ if and only if $\langle i \rangle(\{a\}) = \langle i \rangle(\{a'\})$, since exactly one value is admitted. We can conclude that for all $x \in G$, $cl(\{x\}) = \{x' : \Vdash (x') = \Vdash (x)\} = \Vdash \otimes \Vdash^{\smile} (\{x\}) = [x]_{\kappa_{\hat{\Vdash}}}$. QED

From the above results we immediately obtain some interesting consequences about *FP-systems*:

Corollary 3.1.2. *Let* **P** *be a functional P-system. Then,*

(a) *cl is a topological closure operator.*
(b) *int is a topological interior operator.*

Proof. From *Proposition* 3.1.1.(4) and *Proposition* 3.1.2.(3) we have that $cl(x) = [x]_{\kappa_{\hat{\Vdash}}}$. But $\kappa_{\hat{\Vdash}}$ is the kernel of $\hat{\Vdash}$ and the kernel of a function is a congruence. It follows by induction that $cl(X) \cup cl(Y) = [X]_{\kappa_{\hat{\Vdash}}} \cup [Y]_{\kappa_{\hat{\Vdash}}} = [X \cup Y]_{\kappa_{\hat{\Vdash}}} = cl(X \cup Y)$. Hence cl is additive. Since int is dual of cl, from *Proposition* 2.3.3 we immediately obtain that int is multiplicative. QED

Example 3.1.2. Information quanta and information quantum relations
Consider the *P-system* **P** of Example 1.2.1. Here below the i-quantum relations $R_{\mathbf{P}}$ and $R_{\mathbf{Q}(\mathbf{P})}$ are displayed:

$R_{\mathbf{P}}$	a	a'	a''	a'''	$R_{\mathbf{Q}(\mathbf{P})}$	a	a'	a''	a'''
a	1	0	0	0	a	1	0	0	0
a'	0	1	1	0	a'	0	1	0	1
a''	0	0	1	0	a''	0	1	1	1
a'''	0	1	1	1	a'''	0	0	0	1

Thus $\langle a', a'' \rangle \in R_{\mathbf{P}}$ because $a'' \in Q_a$, but the opposite does not hold ($a' \notin Q_{a''}$).

We remind that by definition we have $\mathbf{Q}(\mathbf{P}) = \langle G, G, R_{\mathbf{P}} \rangle$, $\mathbf{Q}(\mathbf{Q}(\mathbf{P})) = \langle G, G, R_{\mathbf{Q}(\mathbf{P})} \rangle$, and so on.

It is easy to verify that both of the above relations are reflexive and transitive, and $R_{\mathbf{P}}^{\smile} = R_{\mathbf{Q}(\mathbf{P})}$. Moreover, one can see that, for instance, $Q_{a'}^{\mathbf{Q}(\mathbf{P})} = \{a', a'''\}$ or $Q_{a'''}^{\mathbf{Q}(\mathbf{P})} = \{a', a'', a'''\}$. Indeed we have that $a' \in Q_{a'''}$ and $a''' \in Q_{a'}^{\mathbf{Q}(\mathbf{P})}$, or $a'' \in Q_{a'}$ whereas $a' \in Q_{a''}^{\mathbf{Q}(\mathbf{P})}$, and so on.

As to the functional system **F**, we can trivially verify that $Q_x^{\mathbf{Q}(\mathbf{F})} = \{g : f(g) = f(x)\}$, any $x \in G$. For instance, $Q_a^{\mathbf{Q}(\mathbf{F})} = \{a, a''\}$. Thus $R_{\mathbf{F}} = k_f$.

Passing to the *A-system* $\mathbf{A}$, we have: $Q_a^{\mathbf{A}} = \{a, a''\} = Q_{a''}^{\mathbf{A}}, Q_{a'}^{\mathbf{A}} = \{a'\}, Q_{a'''}^{\mathbf{A}} = \{a'''\}$. Thus the resulting i-quantum relation $R_{\mathbf{A}} = \{\langle a, a''\rangle, \langle a'', a\rangle, \langle a, a\rangle, \langle a'', a''\rangle, \langle a', a'\rangle, \langle a''', a'''\rangle\}$ is an equivalence relation.

Exercise 3.1.
Prove that for any preorder $\mathbf{O} = \langle A, R\rangle$, $\langle R\rangle(X) = [[-R]][[-R^{\smile}]](X) = Q_X^{\cup} \supseteq Q_X^{\oplus}$.

We list some results in terms of IQRSs. Since i-quantum relations are preorders, it is useful to prove some general facts about this kind of relations:

Proposition 3.1.3. *Let* $\mathbf{O} = \langle X, R\rangle$ *be any preordered set. Then for any* $x, y \in X$ *the following are equivalent:*

1	2	3	4	5
$y \in R(x)$	$R(y) \subseteq R(x)$	$x \in Q_y^{\mathbf{O}}$	$x \in R^{\smile}(y)$	$y \in Q_x^{\mathbf{O}^{\smile}}$

Proof. ($1 \Longleftrightarrow 2$) $y \in R(x)$ iff $\langle x, y\rangle \in R$. Suppose $\langle y, y'\rangle \in R$. Since R is transitive, then $\langle x, y'\rangle \in R$, too, so that $R(x) \supseteq R(y)$. Conversely, since R is reflexive, $y \in R(y)$ holds. Thus if $R(x) \supseteq R(y)$ then $y \in R(x)$. All the other equivalences are obvious consequences or even definitions.   QED

Corollary 3.1.3. *Let* $\mathbf{S}$ *be any Information System over a set* G. *Then the following are equivalent:*

1	2	3	4	5	6
$g' \in Q_g^{\mathbf{S}}$	$g' \in R_{\mathbf{S}}(g)$	$g \in Q_{g'}^{\mathbf{Q}(\mathbf{S})}$	$g' \in Q_g^{\mathbf{Q}(\mathbf{Q}(\mathbf{S}))}$	$g \in R_{\mathbf{S}}^{\smile}(g')$	$g \in Q_{g'}^{-\mathbf{S}}$

Proof. ($1 \Longleftrightarrow 2$) is just a definition. ($2 \Longleftrightarrow 3 \Longleftrightarrow 4$) Now, $g' \in R_{\mathbf{S}}(g)$ iff $R_{\mathbf{S}}(g') \subseteq R_{\mathbf{S}}(g)$ iff $g \in Q_{g'}^{\mathbf{Q}(\mathbf{S})}$ iff $g \in R_{\mathbf{Q}(\mathbf{S})}(g')$ iff $R_{\mathbf{Q}(\mathbf{S})}(g) \subseteq R_{\mathbf{Q}(\mathbf{S})}(g')$ iff $g' \in Q_g^{\mathbf{Q}(\mathbf{Q}(\mathbf{S}))}$. ($2 \Longleftrightarrow 5$) is trivial. ($5 \Longleftrightarrow 6$) is obtained from trivial set-theoretic considerations, ($X \subseteq Y$ iff $-Y \subseteq -X$). QED

Particularly, $Q_g^{\mathbf{Q}(\mathbf{S})} = Q_g^{\downarrow}$. Moreover, the second set of equivalences shows that IQRSs of level higher than 1 do not provide us with any further information.

Corollary 3.1.4. *Let* **S** *is an A-system, or an FP-system or a DP-system over a set* G, $g, g' \in G$, $X \subseteq G$. *Then the following are pairs of equivalent expressions:*

$g' \in Q_g^{\mathbf{S}}; g' \in Q_g^{\mathbf{Q(S)}}$	$R_{\mathbf{S}}(X); R_{\mathbf{Q(S)}}(X)$

Moreover, since a *P-system* is a generic relational system we have that all facts valid for *P-systems* are valid for any relational system.

It is noticed that the notion of a quantum of information is asymmetric for *P-systems*, because if g' fulfills strictly more properties than g, we have $g' \in Q_g$ but $g \notin Q_{g'}$.

On the contrary it is symmetric in the case of *A-systems* and dichotomic or functional *P-systems*.

However dichotomic and functional *P-systems* seem to belong to a rather odd kind of objects, since "in nature" they are not that frequent. On the contrary, in view of the formal properties they share with *A-systems* we should expect that they are likely to arise from natural transforms of *A-systems* into *P-systems*, as we are going to see, indeed.

In order to be manipulated by the operators we have introduced so far, *A-systems* must be reshaped. Namely, we have to transform attributes into properties. Obviously, this transformation must preserve the expressive capability of the original *A-system*. More precisely, the quanta of information induced by the original *A-system* must coincide with the quanta of information induced by the resulting *P-system*.

Hence, we need a notion to compare the expressive (or discriminatory) capabilities of different information systems over the same set of objects.

3.2 Comparing Perception Systems

First of all we should ask whether it is possible to compare two quanta of information Q_g and $Q_{g'}$. At first sight we would say that Q_g is finer than $Q_{g'}$ if $Q_g \subseteq Q_{g'}$. However, this intuition works for *P-systems*, but it does not work for *A-systems* because if $Q_g \subseteq Q_{g'}$ then $Q_{g'} \subseteq Q_g$. Thus a non trivial comparison of quanta of information require a specialized notion of a quantum of information.

Definition 3.2.1 (Relativised quanta of information).

- *Let* **A** *be an A-system. The* quantum of information *of g relative to a subset $A \subseteq At$ is defined as:* $Q_g \upharpoonright A = \{g' \in G : \forall a \in A, \forall x \in V_a((a(g) = x) \Rightarrow (a(g') = x))\}$.

- *Let* **P** *be a P-system. The* quantum of information *of g with respect to a subset $A \subseteq M$ is defined as:* $Q_g \upharpoonright A = \{g' \in G : \forall a \in A(g \Vdash a \Rightarrow g' \Vdash a)\}$.

Now we can introduce the well-known notion of a *functional dependence*.

Definition 3.2.2 (Functional dependence and informational equivalence). *Let* **S** *be an Information System. Let $A, A' \subseteq At$ (or $A, A' \subseteq M$), $g \in G$.*

1. *We say that A' functionally depends on A at g, in symbols $A \mapsto_g A'$, if for all $g' \in G$, $g' \in Q_g \upharpoonright A \Rightarrow g' \in Q_g \upharpoonright A'$ (that is, if $Q_g \upharpoonright A \subseteq Q_g \upharpoonright A'$).*

2. *We say that A' functionally depends on A, in symbols $A \mapsto A'$, if for all $g \in G$, $A \mapsto_g A'$.*

3. *If $A \mapsto A'$ and $A' \mapsto A$, we say that A and A' are informationally equivalent, $A \cong_I A'$ (thus, $A \cong_I A'$ if for all $g \in G$, $Q_g \upharpoonright A = Q_g \upharpoonright A'$).*

So, a set of attributes (properties) A' functionally depends on a set of attributes (properties) A if the quanta of information induced by A are sharper than those induced by A'. Therefore, A' depends on A if A has a higher discriminatory capability than A'. Indeed, the highest discrimination capability is given when $Q_g = \{g\}$ for each $g \in G$, that is, when the i-quantum relation reduces to the diagonal relation.

Terminology and Notation. In what follows, given a *P-system* **P** and $X \subseteq M$ the symbol $\Vdash\upharpoonright X$ will denote the relation $\Vdash$ with co-domain restricted to X. Moreover with **P** $\upharpoonright X$ we shall denote the subsystem $\langle G, X, \Vdash\upharpoonright X \rangle$ and, for the sake of uniformity, if **P** is an *A-system* and $X \subseteq At$, with **P** $\upharpoonright X$ we shall denote the subsystem $\langle G, X, \{V_a\}_{a \in X} \rangle$.

The following statement formalises the above intuitions with respect to i-quantum relations:

Proposition 3.2.1. *Let* **S** *be an Information System. Let* $A, A' \subseteq At$
$(A, A' \subseteq M)$ *such that* $A \mapsto A'$*. Then* $R_{(\mathbf{A} \upharpoonright A)} \subseteq R_{(\mathbf{A} \upharpoonright A')}$*.*

Proof. The proof is immediate. Suppose $A \mapsto A'$. Then for all $g \in G$,
$Q_g \upharpoonright A \subseteq Q_g \upharpoonright A'$, so that $\langle g, g' \rangle \in R_{(\mathbf{A} \upharpoonright A)}$ implies $\langle g, g' \rangle \in R_{(\mathbf{A} \upharpoonright A')}$.
$$\text{QED}$$

From this discussion it follows that we can naturally extend the notion
of a functional dependence in order to cover the case in which we must
compare two sets X and X' of properties or attributes from two distinct
(property or attribute) systems **S** and **S'** over the same set of points G.
Therefore, if we compare the entire set of properties (or attributes) of
S with the entire set of properties (or attributes) of **S'**, we can extend
the notion of an "informational equivalence" from sets of properties
(attributes) to entire systems:

Definition 3.2.3 (I-equivalence)**.** *Let* **S** *and* **S'** *be Information Systems
over the same set of points* G*. Let* S *and* S' *be the sets of attributes
(properties) of* **S** *and, respectively,* **S'***.*

1. *We say that* **S** *and* **S'** *are* informationally equivalent*, in symbols*
   S $\cong_I$ **S'***, if and only if for any* $g \in G, Q_g \upharpoonright S = Q_g \upharpoonright S'$*.*

2. *If, moreover, there is a 1-1 mapping* $h : S \longmapsto S'$ *such that for
   all* $g, g' \in G$ *and for all* $m \in S$*,* $g' \in Q_g \upharpoonright m$ *if and only if
   * $g' \in Q_g \upharpoonright h(m)$*, then we say that* **S** *and* **S'** *are* informationally
   isomorphic*, in symbols* **S** $\approx_I$ **S'***.*

It is easy to verify that **S** $\cong_I$ **S'** if for any set of properties (attributes)
X of **S**, there is a set of properties (attributes) X' of **S'** such that
$X \mapsto X'$ and vice-versa.

TERMINOLOGY AND NOTATION. From now on, whenever needed, an
operator Op will be decorated by a superscript to underline the system Op
refers to − for instance $int^{\mathbf{P}}$, $int^{\mathbf{P'}}$ and so on.

Not surprisingly, since $Q^{\uparrow}$ and cl are symmetric with respect to infor-
mational power, informational equivalence tells something about the
behaviour of cl and int:

Proposition 3.2.2. *Let* **P** $\cong_I$ **P'***. Then for all* $x \in G$*,* $cl^{\mathbf{P}}(x) =
cl^{\mathbf{P'}}(x)$*. If* $cl^{\mathbf{P}}$ *and* $cl^{\mathbf{P'}}$ *are topological, then* $cl^{\mathbf{P}}(X) = cl^{\mathbf{P'}}(X)$ *and
* $int^{\mathbf{P}}(X) = int^{\mathbf{P'}}(X)$*, for any* $X \subseteq G$*.*

Proof. Suppose $cl^{\mathbf{P}}(x) \neq cl^{\mathbf{P}'}(x)$. Then there is $g \in G$ such that, say, $g \in cl^{\mathbf{P}}(x)$ and $g \notin cl^{\mathbf{P}'}(x)$. It follows that $\Vdash (g) \subseteq \Vdash (x)$ but $\Vdash' (g) \not\subseteq \Vdash' (x)$. Thus $x \in Q^{\mathbf{P}}(g)$ and $x \notin Q^{\mathbf{P}'}(g)$, so that $\mathbf{P} \not\cong_I \mathbf{P}'$. Further, if $cl^{\mathbf{P}}(x) = cl^{\mathbf{P}'}(x)$ for any $x \in G$ and both closure operators are additive, then by easy induction we obtain that $cl^{\mathbf{P}}(X) = cl^{\mathbf{P}'}(X)$ for any $X \subseteq G$. Moreover, suppose $int^{\mathbf{P}}(X) \neq int^{\mathbf{P}'}(X)$. Then $-int^{\mathbf{P}}(X) \neq -int^{\mathbf{P}'}(X)$, so that $cl^{\mathbf{P}}(-X) \neq cl^{\mathbf{P}'}(-X)$ – contradiction.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ QED

The above reasoning for generic subsets does not hold if either $cl^{\mathbf{P}}$ or $cl^{\mathbf{P}'}$ is not topological because in this case the equality between $cl^{\mathbf{P}}$ and $cl^{\mathbf{P}'}$ is guaranteed just for singletons so that $cl^{\mathbf{P}}(-X) \neq cl^{\mathbf{P}'}(-X)$ is not in general a contradiction (notice that we can have $\mathbf{P}$ and $\mathbf{P}'$ such that $int^{\mathbf{P}}(x) \neq int^{\mathbf{P}'}(x)$ but $\mathbf{P} \cong_I \mathbf{P}'$).

REMARKS. The above proposition underlines that our notion of an "informational equivalence" suffers from some limitations. Therefore, the relation $\cong_I$ is far to be considered the "best" way to compare Information Systems. Nonetheless it is very useful to our purpose. Anyway, a connected notion is given in Frame 4.2.

It should be stressed that we can compare not only the informational behaviour of the same point with respect to two different sets of properties (attributes), but we can also compare the behaviours of two different points with respect to the same set of properties (attributes).

Definition 3.2.4. *Let* $\mathbf{S}$ *be an Information System,* $X \subseteq M$ *(or* $X \subseteq At$*) and* $g, g' \in G$.

1. *We say that* g *is an* X-*specialization of* g' *(or that* g' *is an* X-*approximation of* g*), in symbols* $g' \preccurlyeq_X g$*, if and only if the following condition holds:*

$$\forall p \in G(g' \in Q_p \restriction X \Rightarrow g \in Q_p \restriction X).$$

2. *We say that* g *is a* specialization *of* g'*,* $g' \preccurlyeq g$*, if and only if* $g' \preccurlyeq_M g$ *(*$g' \preccurlyeq_{At} g$*).*

Since from q-reflexivity $x \in Q_x$, if $g' \preccurlyeq_X g$ then $g \in Q_{g'} \restriction X$, so that $g' \preccurlyeq_X g$ says that g fulfills at least all the properties from X that are

fulfilled by g' (clearly, if we are dealing with an A-*system* then $g' \preccurlyeq_X g$ implies $g \preccurlyeq_X g'$). Therefore, $g' \preccurlyeq g$ implies $\langle g', g \rangle \in R_{\mathbf{S}}$. Conversely, if $\langle g', g \rangle \in R_{\mathbf{S}}$ then $g \in Q_{g'}$. Hence $g' \in Q_x$ implies $g \in Q_x$, any $x \in G$, from q-transitivity. It follows that the two relations $\preccurlyeq$ and $R_{\mathbf{S}}$ coincide. In fact they are instances of the notion of a *specialization preorder* which is discussed in Frame 4.3 as related to topological spaces. In what follows we shall construct a topology $\Omega_Q(\mathbf{S})$ on G such that its specialization preorder will indeed be $\preccurlyeq$ (that is, $R_{\mathbf{S}}$).

Obviously, if $\mathbf{S}$ is an A-*system* or an FP or a DP system then $\preccurlyeq$ is an equivalence relation.

Example 3.2.1. I-dependence and i-equivalence
In our basic Example 1.2.1 let $A = \{b, b'\}$ and $B = \{b'', b'''\}$. Then $Q_{a''} \restriction A = \{a, a', a''\}$ while $Q_{a''} \restriction B = \{a''\}$. It follows that $B \mapsto_{a''} A$. On the contrary, $Q_{a'} \restriction A = \{a, a', a''\}$ and $Q_{a'} \restriction B = \{a', a'', a'''\}$ are not comparable. Hence $B \mapsto A$ does not hold.

Let us now compare the Information Systems $\mathbf{P}$, $\mathbf{F}$ and $\mathbf{A}$. We can notice what follows:

(a) $\mathbf{A} \not\cong_I \mathbf{P}$ because $Q_a^{\mathbf{A}} = \{a, a''\}$ while $Q_a^{\mathbf{P}} = \{a\}$. Neither $\mathbf{P} \mapsto \mathbf{A}$ because $Q_{a'}^{\mathbf{P}} = \{a', a''\}$ while $Q_{a'}^{\mathbf{A}} = \{a'\}$.
(b) $\mathbf{F} \cong_I \mathbf{A}$, because for all $g \in G, Q_g^{\mathbf{A}} = Q_g^{\mathbf{F}}$. However $\mathbf{A}$ and $\mathbf{F}$ are not i-isomorphic because, for instance, $Q_{a'}^{\mathbf{F}} \restriction \{m\} = G$ while $Q_{a'}^{\mathbf{A}} \restriction \{A\} = G$, for no $A \in At$.

Now we exhibit an example of two informationally equivalent P-*systems*, $\mathbf{P}$ and $\mathbf{P}'$ such that in general $int^{\mathbf{P}} \neq int^{\mathbf{P}'}$:

$\mathbf{P}$	a	b	c	$\mathbf{P}'$	a	b	c	d
1	1	0	1	1	1	0	1	0
2	1	1	0	2	0	1	0	1
3	0	1	1	3	0	1	1	0

We can trivially verify that for any $g \in G$, $Q_g^{\mathbf{P}} = Q_g^{\mathbf{P}'} = \{g\}$, so that $\mathbf{P} \cong_I \mathbf{P}'$. However, $int^{\mathbf{P}}(\{1\}) = \emptyset$ while $int^{\mathbf{P}'}(\{1\}) = \{1\}$ and $int^{\mathbf{P}}(\{2\}) = \emptyset \neq int^{\mathbf{P}'}(\{2\}) = \{2\}$. (Notice that neither $cl^{\mathbf{P}}$ nor $cl^{\mathbf{P}'}$ are topological – for instance $cl^{\mathbf{P}}(\{1, 2\}) = \{1, 2, 3\} \not\supseteq cl^{\mathbf{P}}(\{1\}) \cup cl^{\mathbf{P}}(\{2\}) = \{1, 2\}$ and the same happens for $cl^{\mathbf{P}'}$.
In the next example we show two informationally equivalent P-*systems*, $\mathbf{P}$ and $\mathbf{P}'$ such that $cl^{\mathbf{P}'}$ is topological but $cl^{\mathbf{P}}$ is not:

$\mathbf{P}$	a	b	c	$\mathbf{P}'$	a	b	c	d
1	1	0	0	1	1	0	0	0
2	1	1	0	2	1	1	0	0
3	0	0	1	3	0	0	1	0
4	0	1	1	4	0	0	1	1

The reader is invited to check that $\mathbf{P} \cong_I \mathbf{P}'$. However, whereas $cl^{\mathbf{P}'}$ is topological – for instance $cl^{\mathbf{P}'}(\{2\}) \cup cl^{\mathbf{P}'}(\{3\}) = \{1,2\} \cup \{3\} = cl^{\mathbf{P}'}(\{1,2,3\})$ – $cl^{\mathbf{P}}$ is not: $cl^{\mathbf{P}}(\{2\}) \cup cl^{\mathbf{P}}(\{3\}) = \{1,2,3\} \subsetneq \{1,2,3,4\} = cl^{\mathbf{P}}(\{1,2,3\})$.

3.3   Higher Level Operators

Let $\mathbf{S}$ be an Information System and let $\mathbf{Q}(\mathbf{S}) = \langle G, G, R_{\mathbf{S}} \rangle$ be its induced IQRS.

What kinds of patterns of data can we collect by applying our operators to these derivative systems?

First, notice that in IQRSs there is no longer the distinction between objects and properties and intension or extension. Therefore it is better we change our symbols and notation and come back to the original one, once more:

The operator	defined as	turns into
$\langle i \rangle$	$\langle i \rangle(X) = \{g : \exists g'(g' \in X \ \& \ \langle g', g \rangle \in R_{\mathbf{S}})\} = \{g : R_{\mathbf{S}}^{\smile}(g) \cap X \neq \emptyset\}$	$\langle R_{\mathbf{S}}^{\smile} \rangle$
$\langle e \rangle$	$\langle e \rangle(X) = \{g : \exists g'(g' \in X \ \& \ \langle g, g' \rangle \in R_{\mathbf{S}})\} = \{g : R_{\mathbf{S}}(g) \cap X \neq \emptyset\}$	$\langle R_{\mathbf{S}} \rangle$
$[i]$	$[i](X) = \{g : \forall g'(\langle g', g \rangle \in R_{\mathbf{S}} \Rightarrow g' \in X)\} = \{g : R_{\mathbf{S}}^{\smile}(g) \subseteq X\}$	$[R_{\mathbf{S}}^{\smile}]$
$[e]$	$[e](X) = \{g : \forall g'(\langle g, g' \rangle \in R_{\mathbf{S}} \Rightarrow g' \in X)\} = \{g : R_{\mathbf{S}}(g) \subseteq X\}$	$[R_{\mathbf{S}}]$

Let us call the above operators decorated by $R_{\mathbf{S}}$ "quantum operators" (notice that in this context $[[R_{\mathbf{S}}]]$ and $[[R_{\mathbf{S}}^{\smile}]]$ are not quantum operators).

Quantum operators behave in a very particular way, because, we remind, they fulfill adjoint properties. Namely $\langle R_{\mathbf{S}} \rangle \dashv [R_{\mathbf{S}}^{\smile}]$ and $\langle R_{\mathbf{S}}^{\smile} \rangle \dashv [R_{\mathbf{S}}]$.

Actually, the following results apply to any preorder, too.

Proposition 3.3.1. *Let* $\mathbf{Q}(\mathbf{S}) = \langle G, G, R_{\mathbf{S}} \rangle$ *be a IQRS. Let* O_i *and* O_j *be any two adjoint quantum operators from the set* $\{\langle R_{\mathbf{S}}^{\smile} \rangle, \langle R_{\mathbf{S}} \rangle, [R_{\mathbf{S}}^{\smile}],$ $[R_{\mathbf{S}}]\}$. *Then:*
(a) $O_i O_j = O_j$; *(b)* $O_j O_j = O_j$; *(c) the fixpoints of* O_i *and* O_j *coincide.*

Proof. (a) (i) In view of *Proposition* 2.3.1.(2), for all $g \in G$ and $X \subseteq G$, $g \in [R_{\mathbf{S}}]\langle R_{\mathbf{S}}^{\smile}\rangle(X)$ iff $\langle R_{\mathbf{S}}^{\smile}\rangle(g) \subseteq \langle R_{\mathbf{S}}^{\smile}\rangle(X)$, iff $R_{\mathbf{S}}(g) \subseteq R_{\mathbf{S}}(X)$ iff (from *Proposition* 3.1.3) $g \in R_{\mathbf{S}}(X)$ iff $g \in \langle R_{\mathbf{S}}^{\smile}\rangle(X)$. One can prove $g \in [R_{\mathbf{S}}^{\smile}]\langle R_{\mathbf{S}}\rangle(X)$ iff $g \in \langle R_{\mathbf{S}}\rangle(X)$ similarly.

(ii) In view of *Proposition* 2.3.1.(3), $g \in \langle R_{\mathbf{S}}\rangle[R_{\mathbf{S}}^{\smile}](X)$ iff $\langle R_{\mathbf{S}}^{\smile}\rangle(g) \cap [R_{\mathbf{S}}^{\smile}](X) \neq \emptyset$, iff there is g' such that $a \in R_{\mathbf{S}}^{\smile}(g')$ and $g' \in [R_{\mathbf{S}}^{\smile}](X)$. But $g' \in [R_{\mathbf{S}}^{\smile}](X)$ iff $R_{\mathbf{S}}^{\smile}(g') \subseteq X$ and $a \in R_{\mathbf{S}}^{\smile}(g')$ iff (again from *Proposition* 3.1.3) $R_{\mathbf{S}}^{\smile}(g) \subseteq R_{\mathbf{S}}^{\smile}(g')$ so that we must have $R_{\mathbf{S}}^{\smile}(g) \subseteq X$. Hence, $g \in \langle R_{\mathbf{S}}\rangle[R_{\mathbf{S}}^{\smile}](X)$ iff $g \in [R_{\mathbf{S}}^{\smile}](X)$.

(b) From point (a) and *Proposition* 1.4.8.(9), $O_j O_j = O_i O_j O_i O_j = O_i O_j = O_j$.

(c) Let $X = O_j(X)$. Then using point (a) $O_i(X) = O_i O_j(X) = O_j(X) = X$.   QED[1]

In view of these results we can prove a number of properties.

Corollary 3.3.1. *Let* $\mathbf{Q}(\mathbf{S})$ *be an IQRS. Then,*

$Q_{(\ldots)}^{\mathbf{S}} = \langle R_{\mathbf{S}}^{\smile}\rangle = cl^{\mathbf{Q}(\mathbf{S})}$	$Q_{(\ldots)}^{\mathbf{Q}(\mathbf{S})} = \langle R_{\mathbf{S}}\rangle = \mathcal{A}^{\mathbf{Q}(\mathbf{S})}$	$[R_{\mathbf{S}}^{\smile}] = int^{\mathbf{Q}(\mathbf{S})}$	$[R_{\mathbf{S}}] = \mathcal{C}^{\mathbf{Q}(\mathbf{S})}$

Definition 3.3.1. *Let* $\mathbf{S}$ *be an Information System. With* $\Omega_Q(\mathbf{S})$ *we shall denote the family* $\{Q_X^{\mathbf{S}} : X \subseteq G\}$. *With* $\mathbf{Q}_n(\mathbf{S})$ *we denote the n-nested application of the functor* $\mathbf{Q}$ *to* $\mathbf{S}$.

Corollary 3.3.2. *Let* $\mathbf{S}$ *be an Information System,* $\mathbf{H}$ *an A-system or FP-system or DP-system,* $\mathbf{F}$ *an FP-system. Then,*

1. $\langle R_{\mathbf{Q}_n(\mathbf{S})}^{\smile}\rangle = \langle R_{\mathbf{Q}_{n+1}(\mathbf{S})}\rangle$ *and* $[R_{\mathbf{Q}_n(\mathbf{S})}^{\smile}] = [R_{\mathbf{Q}_{n+1}(\mathbf{S})}]$, $n \geq 0$.

2. $\Omega_Q(\mathbf{Q}_n(\mathbf{S})) = \Gamma_{cl}(\mathbf{Q}_{n+1}(\mathbf{S})) = \Gamma_{\mathcal{C}}(\mathbf{Q}_{n+1}(\mathbf{S}))$, $n \geq 0$.

3. $\Omega_{int}(\mathbf{Q}_n(\mathbf{S})) = \Omega_{\mathcal{A}}(\mathbf{Q}_n(\mathbf{S})) = \Gamma_{cl}(\mathbf{Q}_{n+1}(\mathbf{S})) = \Gamma_{\mathcal{C}}(\mathbf{Q}_{n+1}(\mathbf{S}))$, $n \geq 1$.

4. $\Gamma_{cl}(\mathbf{Q}_n(\mathbf{S})) = \Gamma_{\mathcal{C}}(\mathbf{Q}_n(\mathbf{S})) = \Omega_{int}(\mathbf{Q}_{n+1}(\mathbf{S})) = \Omega_{\mathcal{A}}(\mathbf{Q}_{n+1}(\mathbf{S}))$, $n \geq 1$.

5. $\Omega_{int}(\mathbf{Q}_n(\mathbf{S})) = \Omega_{\mathcal{A}}(\mathbf{Q}_n(\mathbf{S})) = \Omega_Q(\mathbf{Q}_n(\mathbf{S}))$, $n \geq 1$.

---

[1]We must remark that if we start with arbitrary relations and not preorders (hence from arbitrary topologies and not from Alexandrov topologies), things can run a different way. For a more general investigation in this topic, see Ghilardi & Meloni [1991].

6. $\Omega_{int}(\mathbf{Q}_n(\mathbf{H})) = \Omega_{\mathcal{A}}(\mathbf{Q}_n(\mathbf{H})) = \Gamma_{cl}(\mathbf{Q}_n(\mathbf{H})) = \Omega_Q(\mathbf{Q}_n(\mathbf{H}))$
$= \Gamma_{\mathcal{C}}(\mathbf{Q}_n(\mathbf{H}))$, $n \geq 1$.

7. $\Omega_Q(\mathbf{Q}_n(\mathbf{H})) = \Omega_Q(\mathbf{Q}_{n+1}(\mathbf{H}))$, $\Omega_Q(\mathbf{Q}_n(\mathbf{F})) = \Omega_{int}(\mathbf{Q}_n(\mathbf{F}))$ $n \geq 0$.

8. $\mathbf{Sat}_{int}(\mathbf{Q}_n(\mathbf{S}))$ *and* $\mathbf{Sat}_{cl}(\mathbf{Q}_n(\mathbf{S}))$, $\mathbf{Sat}_{\mathcal{A}}(\mathbf{Q}_n(\mathbf{S}))$ *and* $\mathbf{Sat}_{\mathcal{C}}$ $(\mathbf{Q}_n(\mathbf{S}))$, $\mathbf{Sat}_{\mathcal{A}}(\mathbf{Q}_n(\mathbf{S}))$ *and* $\mathbf{Sat}_{int}(\mathbf{Q}_n(\mathbf{S}))$, $\mathbf{Sat}_{\mathcal{C}}(\mathbf{Q}_n(\mathbf{S}))$ *and* $\mathbf{Sat}_{cl}(\mathbf{Q}_n(\mathbf{S}))$, *are pairwise antisomorphic,* $n \geq 0$.

Proof. (**1**) By easy induction from *Corollary* 3.3.1. (**2**), (**3**) and (**4**) immediately follow by recalling that in $\mathbf{Q}(\mathbf{S})$ adjoint operators have the same fixpoints and *Corollary* 3.3.1. (**5**) From the previous points, $\Omega_{\mathcal{A}}(\mathbf{Q}_3(\mathbf{S})) = \Omega_{\mathcal{A}}(\mathbf{Q}(\mathbf{S}))$ and $\Omega_{\mathcal{A}}(\mathbf{Q}_3(\mathbf{S})) = \Gamma_{cl}(\mathbf{Q}_2(\mathbf{S})) = \Omega_Q(\mathbf{Q}(\mathbf{S}))$: Thus $\Omega_{\mathcal{A}}(\mathbf{Q}(\mathbf{S})) = \Omega_Q(\mathbf{Q}(\mathbf{S}))$ and by induction the thesis. (**6**) and (**7**) Directly from (1)-(3) and the fact that $\mathbf{Q}_n(\mathbf{H}) = \mathbf{Q}_{n+1}(\mathbf{H})$, $n \geq 1$ because of symmetry of $R_{\mathbf{H}}$. The second equation is given by *Corollary* 3.1.2 and an inductive extension of *Proposition* 3.3.3 below. (**8**) From *Proposition* 3.1.3, $\mathcal{A}$ and int have the same fixpoints as well as $\mathcal{C}$ and cl. Moreover from $cl = R_{\mathbf{S}}(\ldots)$ and $\mathcal{A} = R_{\mathbf{S}}^{\smile}(\ldots)$ we have the entire thesis.   QED

It is obvious that in practice we shall deal just with n in the range $0 - 1$, because $\mathbf{Q}_3(\mathbf{S}) = \mathbf{Q}_1(\mathbf{S})$.

Lemma 3.3.1. *For any* P-system $\mathbf{P}$, (a) $\Omega_{int}(\mathbf{P}) \subseteq \Omega_Q(\mathbf{P})$; (b) $\Gamma_{cl}(\mathbf{P}) \subseteq \Omega_{int}(\mathbf{Q}(\mathbf{P}))$.

Proof. (a) Assume $X \subsetneq Q_X$. Thus we must have some x such that $x \notin X$ and $x \in Q_X$. Thus there is $g \in X$ such that $\Vdash (x) \supsetneq\Vdash (g)$, so that for all $m \in M$ such that $g \Vdash m$ surely $\Vdash^{\smile} (m) \nsubseteq X$. It follows that $g \notin int^{\mathbf{P}}(X)$ and, hence, $int^{\mathbf{P}}(X) \subsetneq X$. (b) We remind that $x \in cl^{\mathbf{P}}(X)$ iff $\langle i \rangle(x) \subseteq \langle i \rangle(X)$ iff $\Vdash^{\smile} (x) \subseteq\Vdash^{\smile} (X)$. Moreover, if $x \in X, \langle i \rangle(x) \subseteq \langle i \rangle(X)$. Now suppose $X \neq int^{\mathbf{Q}(\mathbf{P})}(X)$. Then there is $x \in X$ such that $R_{\mathbf{P}}(x) \nsubseteq X$. Hence $\{y : x \in Q_y^{\mathbf{P}}\} \nsubseteq X$. Thus $\{y : \langle i \rangle(y) \subseteq \langle i \rangle(x)\} \nsubseteq X$. This means that there is $g \notin X$ such that $\langle i \rangle(g) \subseteq \langle i \rangle(x) \subseteq \langle i \rangle(X)$ so that $\Vdash^{\smile} \langle i \rangle(g) \subseteq\Vdash^{\smile} \langle i \rangle(x) \subseteq\Vdash^{\smile} \langle i \rangle(X)$. It follows that $X \subsetneq cl^{\mathbf{P}}(X)$.   QED

Corollary 3.3.3. *For all* $X \subseteq G$ *in a P-system,* $int(X) \subseteq Q_X$.

Proof. From the above Lemma we have $int(X) = Q_{int(X)}$, all $X \subseteq G$. But $int(X) \subseteq X$ and $Q_{(\ldots)}$ is monotonic.   QED

Corollary 3.3.4. *Let* **S** *be an Information System. Then,*

1. $Q^{\mathbf{S}}_{(...)}, Q^{\mathbf{Q(S)}}_{(...)}, \langle R_{\mathbf{S}} \rangle$ *and* $\langle R_{\mathbf{S}}^{\smile} \rangle$ *are topological closure operators and their images are closed under intersections.*

2. $[R_{\mathbf{S}}]$ *and* $[R_{\mathbf{S}}^{\smile}]$ *are topological interior operators and their images are closed under unions.*

3. $int^{\mathbf{Q(S)}}$ *and* $\mathcal{C}^{\mathbf{Q(S)}}$ *are topological interior operators;* $cl^{\mathbf{Q(S)}}$ *and* $\mathcal{A}^{\mathbf{Q(S)}}$ *are topological closure operators.*

Proof. (**1**) From *Corollary* 3.3.1, we have that $Q^{\mathbf{S}}_{(...)}, Q^{\mathbf{Q(S)}}_{(...)}, \langle R_{\mathbf{S}} \rangle$ and $\langle R_{\mathbf{S}}^{\smile} \rangle$ are closure operators. Moreover, since they are lower adjoints between $\langle \wp(G), \subseteq \rangle$ and itself they preserve colimits, that is, unions. Finally, from *Proposition* 2.3.2 their images are closed under intersections. (**2**) Again from *Corollary* 3.3.1, $[R_{\mathbf{S}}]$ and $[R_{\mathbf{S}}^{\smile}]$ are interior operators. Moreover, as they are lower adjoints between $\langle \wp(G), \subseteq \rangle$ and itself they preserve limits, that is, intersections. Finally from *Proposition* 2.3.2 their images are closed under unions. (**3**) Trivially, $int^{\mathbf{Q(S)}} = \langle R_{\mathbf{S}} \rangle [R_{\mathbf{S}}^{\smile}] = [R_{\mathbf{S}}^{\smile}]$, thus from the previous point we have the result. The other statements are proved in a similar way. QED

Corollary 3.3.5 (I-quantum systems)**.** *Let* **S** *be an Information System. Then,*

1. $\mathbf{Sat}_Q(\mathbf{S}) = \langle \Omega_Q(\mathbf{S}), \cup, \cap, \emptyset, G \rangle$ *is a distributive lattice, called the* I-quantum system *(or IQS) induced by* **S**.

2. $\mathbf{Sat}_Q(\mathbf{Q(S)}) = \langle \Omega_Q(\mathbf{Q(S)}), \cup, \cap, \emptyset, G \rangle$ *is a distributive lattice, called the* co-I-quantum system *(or co-IQS) induced by* **S**.

3. *The set theoretical complement is an antisomorphism between* $\mathbf{Sat}_Q(\mathbf{S})$ *and* $\mathbf{Sat}_Q(\mathbf{Q(S)})$.

4. $\langle G, \Omega_Q(\mathbf{S}) \rangle$ *and* $\langle G, \Omega_Q(\mathbf{Q(S)}) \rangle$ *are topological spaces, where the interior operators are* $int^{\mathbf{Q}_2(\mathbf{S})}$ *and, respectively,* $int^{\mathbf{Q}_1(\mathbf{S})}$.

5. $\mathbf{Sat}_{int}(\mathbf{Q(S)}), \mathbf{Sat}_{cl}(\mathbf{Q(S)}), \mathbf{Sat}_{\mathcal{A}}(\mathbf{Q(S)})$ *and* $\mathbf{Sat}_{\mathcal{C}}(\mathbf{Q(S)})$, *equipped with the set-theoretical operations, are distributive lattices.*

6. *If* **S** *is a preordered set (that is,* $G = M$ *and* $R \subseteq G \times G$ *is a preorder), then* $\Omega_Q(\mathbf{S}) = \Omega_{int}(\mathbf{S})$.

Proof. (**1**) We know that the operator $Q_{(\ldots)}$ is additive. Thus $\Omega_Q(\mathbf{S})$ is closed under unions. From *Proposition* 3.3.4 it is closed under intersections too. Moreover, since $\Omega_Q(\mathbf{S})$ is a (finite) lattice of sets $\mathbf{Sat}_Q(\mathbf{S})$ inherits distributivity from the corresponding property of unions and intersections. (**2**) Since $\mathbf{Q}(\mathbf{S})$ is a *P-system* the above considerations apply to this structure. (**3**) From *Corollary* 3.1.3 we know that $R_{-\mathbf{S}} = R_{\breve{\mathbf{S}}} = R_{\mathbf{Q}(\mathbf{S})}$, so that we obtain immediately the thesis. (**4**) Any family of open sets of a topological space enjoys distributivity of arbitrary unions over finite intersections and of intersection over arbitrary unions. Moreover, from *Corollary* 3.3.2.(2), $\Omega_Q(\mathbf{S}) = \Gamma_{cl}(\mathbf{Q}(\mathbf{S}))$. But from 3.3.2.(4) $\Gamma_{cl}(\mathbf{Q}(\mathbf{S})) = \Omega_{int}(\mathbf{Q}_2(\mathbf{S}))$. Finally, from 3.3.2.(5) we have the result for co-IQ systems. (**5**) From *Proposition* 2.3.2 and *Corollary* 3.3.4. (**6**) Obvious, since in this case $\mathbf{S} = \mathbf{Q}(\mathbf{S}')$ for some Information System $\mathbf{S}'$.
<div align="right">QED</div>

Proposition 3.3.2 (I-quantum equivalence relations and Boolean algebras). *Let $\mathbf{S}$ be an Information system. If $R_\mathbf{S}$ is an equivalence relation, then $\mathbf{Sat}_Q(\mathbf{S})$ is a Boolean algebra.*

Proof. We show that if $R_\mathbf{S}$ is an equivalence relation then any element Q_X of $\Omega_Q(\mathbf{S})$ is complemented by $\overline{Q_X} = \bigcup\limits_{z \notin Q_X} Q_z$. First, let us prove that $\overline{Q_X} \cup Q_X = G$. In fact for all $g \in G$ if $g \notin Q_X$ then $g \in \overline{Q_X}$ because $g \in Q_g$ (q-reflexivity). Now we prove that $\overline{Q_X} \cap Q_X = \emptyset$. Assume $z \notin Q_X$. We have just to prove that if $z' \in Q_z$ then $z' \notin Q_X$. So let $z' \in Q_z$. From q-symmetry, $z \in Q_{z'}$. So, if there is an $x \in X$ such that $z' \in Q_x$ we have $z \in Q_x$ too (from q-transitivity), hence $z \in Q_X$. Contradiction.
<div align="right">QED</div>

Corollary 3.3.6. *Let $\mathbf{S}$ be an Information system. Then, if $\mathbf{S}$ is an A-system, or a dichotomic or a functional P-system, then:*

1. *$\mathbf{Sat}_Q(\mathbf{S})$ is a Boolean algebra.*

2. *$\mathbf{Sat}_{int}(\mathbf{Q}(\mathbf{S}))$, $\mathbf{Sat}_{cl}(\mathbf{Q}(\mathbf{S}))$, $\mathbf{Sat}_\mathcal{A}(\mathbf{Q}(\mathbf{S}))$ and $\mathbf{Sat}_\mathcal{C}(\mathbf{Q}(\mathbf{S}))$ are Boolean algebras.*

Proof. (**1**) comes straightforwardly from the previous *Proposition*. (**2**) is a consequence of (**1**) and *Proposition* 3.3.2.(6).
<div align="right">QED</div>

We shall recover the above results from a different point of view in Section 3.4 below. By now, the following result about the family of co-prime elements of $\mathbf{Sat_Q(S)}$ is worth mentioning:

Lemma 3.3.2. *Let* **S** *be an Information System. Then for any* $X \in \mathcal{J}$ *(*$\mathbf{Sat_Q(S)}$*),* $X = Q_g$ *for some* $g \in X$*.*

Proof. Trivial from the additive definition of the operator Q and its increasing property.   QED

Lemma 3.3.3. *Let* **P** *be a P-system and* $g \in G$*. Then* $Q_g = \bigcap\{\langle e\rangle(m) : m \in \langle i\rangle(g)\}$*.*

Proof. Indeed, $x \in Q_g$ iff $\langle i\rangle(x) \supseteq \langle i\rangle(g)$ iff $x \in \langle e\rangle(m), \forall m \in \langle i\rangle(g)$.   QED

Proposition 3.3.3. *Let* **P** *be a P-system such that cl (int) is topological. Then* $\mathbf{Sat}_Q(\mathbf{P}) = \mathbf{Sat}_{int}(\mathbf{P})$*.*

Proof. We have seen in *Lemma* 3.3.1 that $\Omega_{int}(\mathbf{P}) \subseteq \Omega_Q(\mathbf{P})$. Now we need just to show that if $X \in \mathcal{J}(\mathbf{Sat}_Q(\mathbf{P}))$ then $X = int(X)$. The proof is immediate: if int is topological then it is multiplicative and since for all $m \in M$, $int(\langle e\rangle(m)) = \langle e\rangle(m)$ (from *Lemma* 2.3.1), in view of the above *Lemma* 3.3.3 we have the result.   QED

If we compare this result with *Corollary* 3.3.2.(7) we can note that *P-systems* such that *int* and *cl* are topological behave like functional systems.

REMARKS. Do not confuse *A-systems* in which $V = \{0, 1\}$ (i.e. binary *A-systems*) with *P-systems*. Indeed they are formally similar – and we exploit this fact when convenient – but, conceptually, they are different objects, because in a *P-system* 1 and 0 are "yes" and "no" answers, respectively, to questions concerning fulfillment of properties, while in binary *A-systems* they are the attribute-values that objects can be assigned. Hence, for instance, if in a binary *A-system* for any object g, $vl(\langle g, A\rangle) = 0$ this means that attribute A meaningfully applies to all objects and that they uniformly take value 0 at A. On the contrary, if this happens of a property m in a *P-system*, then we can conclude that m is not fulfilled by any object. This is the proviso we promised to discuss a few pages back.[2]

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[2]We can question whether a property m which is not fulfilled by any object is meaningful or useless. Clearly, m is not able to characterise objects in the sense that it does not have any discriminatory ability. However, the same happens of a property m' which is fulfilled by all the objects. But keeping or discharging all such m or m' has important consequences on the behaviour of the "Concept Lattices" operators. Indeed, if we keep m, then surely $[[e]][[i]](\emptyset) = \emptyset$, while if we discharge m' then surely $[[i]][[e]](\emptyset) = \emptyset$.

At this point we can end the subsection with an analogue of the duality between distributive lattices and preorders in the context of i-quantum relations and *P-systems*.

Proposition 3.3.4 (Duality between preorders and Information Systems)**.**

1. Let $\mathbf{O} = \langle G, R \rangle$ be a preorder, then there is an Information System $\mathbf{I(O)}$ over G such that $R_{\mathbf{I(O)}} = R$ (hence, $\mathbf{Q(I(O))} \cong_I \mathbf{O}$).

2. Let $\mathbf{S}$ be an Information System. Then $\mathbf{I(Q(S))} \cong_I \mathbf{S}$, where $\mathbf{I}$ is the operator defined in point (1).

Proof. (**1**) Trivially, $R_{\mathbf{O}} = R^{\smile}$ and $R_{\mathbf{Q(O)}} = R$. Hence $\mathbf{Q(Q(O))} \cong_I \mathbf{O}$. Thus, the required operator $\mathbf{I}$ is $\mathbf{Q}$.[3] (**2**) Since $\mathbf{Q(S)}$ is a preorder, from the previous point we have $\mathbf{Q(Q(Q(S)))} \cong_I \mathbf{Q(S)}$ so that trivially $\mathbf{Q(Q(S))} \cong_I \mathbf{S}$.   QED

Example 3.3.1. Quantum relational systems
Consider the Information Systems of Example 1.2.1. Here below we display the lattices $\mathbf{Sat}_Q(\mathbf{P})$ and $\mathbf{Sat}_Q(\mathbf{A})$:
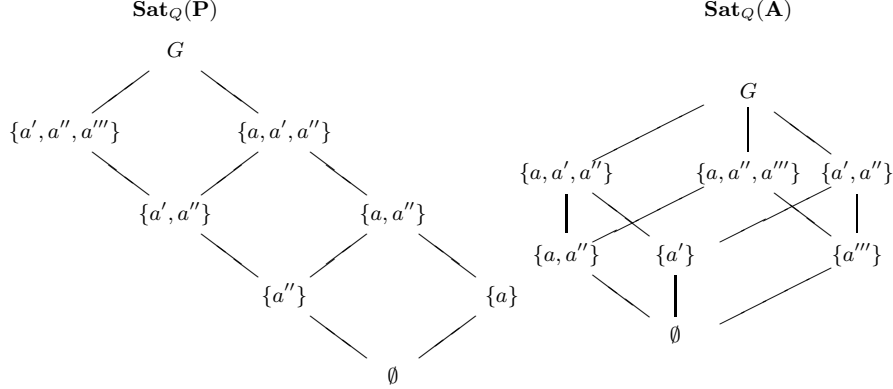
It is easy to verify that both of them are distributive lattices and that, moreover, $\mathbf{Sat}_Q(\mathbf{A})$ is a Boolean algebra. Indeed, for instance, the element $Q^{\mathbf{A}}_{\{a,a'\}} = \{a, a', a''\}$ is complemented by the element $\bigcup_{z \notin Q^{\mathbf{A}}_{\{a,a'\}}} Q^{\mathbf{A}}_z = Q^{\mathbf{A}}_{\{a'''\}} = \{a'''\}$.

If we compare $\mathbf{Sat}_Q(\mathbf{P})$ and $\mathbf{Sat}_{int}(\mathbf{P})$ we notice that the only difference is the lack of $\{a', a''\}$ in $\mathbf{Sat}_{int}(\mathbf{P})$, because $int(\{a', a''\}) = \{a''\}$. On the contrary, for any singleton $\{x\} \in \mathbf{Sat}_Q(\mathbf{P})$, $x = Q_{\{x\}} = R_{\mathbf{P}}(\{x\}) = int(\{x\})$. The family of co-prime elements $\mathcal{J}(\mathbf{Sat}_Q(\mathbf{P}))$ is $\{\{a''\}, \{a\}, \{a', a''\}, \{a', a'', a'''\}\}$ and we can see that

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[3]For a more conceptual proof see Frame 4.6.3.

$\{a''\} = \langle e \rangle (b''), \{a\} = \langle e \rangle (b), \{a', a'', a'''\} = \langle e \rangle (b''')$ while for no $m \in M, \{a', a''\} = \langle e \rangle (m)$ and $\langle e \rangle (b') \notin \mathcal{J}(\mathbf{Sat}_Q(\mathbf{P}))$. So we cannot either recover $\mathcal{J}(\mathbf{Sat}_Q(\mathbf{P}))$ via $\mathcal{J}(\mathbf{Sat}_{int}(\mathbf{P}))$ or $\mathcal{J}(\mathbf{Sat}_{int}(\mathbf{P}))$ via $\mathcal{J}(\mathbf{Sat}_Q(\mathbf{P}))$.

One can notice that $\{a', a''\} = \{a', a'', a'''\} \cap \{a, a', a''\}$. Nonetheless pay attention that $\mathbf{Sat}_Q(\mathbf{P})$ generally is not the closure under intersections of $\mathcal{J}(\mathbf{Sat}_{int}(\mathbf{P}))$. For instance, consider a *P-system* **S** such that $G = \{a, b, c, d, e\}$, $M = \{A, B, C\}$ and $\Vdash (a) = \{A\}, \Vdash (b) = \{A, B\}, \Vdash (c) = \{A, B, C\}, \Vdash (d) = \{C\}, \Vdash (e) = \{A, C\}$. Then $Q^{\mathbf{S}}_{\{b, e\}} = \{b, c, e\}$ but $\mathcal{J}(\mathbf{Sat}_{int}(\mathbf{S})) = \{\{b, c\}, \{c, d, e\}, \{a, b, c, e\}, G, \emptyset\}$ so that $\{b, c, e\}$ is not an intersection of elements of $\mathcal{J}(\mathbf{Sat}_{int}(\mathbf{S}))$.

The reader can straightforwardly verify that $\Omega_Q(\mathbf{P}) = \Gamma_{cl}(\mathbf{Q}(\mathbf{P}))$. Indeed in view of *Proposition* 3.3.2 given an Information System **S**, the index n must be greater than or equal to 1 in order to have $\Omega_{int}(\mathbf{Q}_n(\mathbf{S})) = \Gamma_{cl}(\mathbf{Q}_{n+1}(\mathbf{S})) = \Omega_Q(\mathbf{Q}_n(\mathbf{S}))$. If $n = 0$, $\Omega_Q(\mathbf{Q}_0(\mathbf{S})) = \Gamma_{cl}(\mathbf{Q}_1(\mathbf{S}))$. On the contrary, $\Gamma_{cl}(\mathbf{Q}_1(\mathbf{S}))$ is $\Omega_{int}(\mathbf{Q}_0(\mathbf{S}))$ plus some missed elements which are the difference between $\Omega_Q(\mathbf{S})$ and $\Omega_{int}(\mathbf{S})$. In our example the missed element is $\{a', a''\}$ which in $\mathbf{Q}(\mathbf{P})$ equals $cl(\{a', a''\})$. On the contrary, in **P** $int(\{a', a''\}) = \emptyset$.

Finally, in view of *Corollary* 3.3.5.(4) let us verify that if $\Omega_Q(\mathbf{P})$ is considered as the family of open subsets of a topological space on G, then the corresponding interior operator is $int^{\mathbf{Q}_2(\mathbf{P})}$: $int^{\mathbf{Q}_2(\mathbf{P})}(\{a'', a'''\}) = \langle e \rangle (\{a'''\}) = \{a'''\} \in \Omega_Q(\mathbf{P})$. Notice that $Q^{\mathbf{P}}_{\{a'', a'''\}} = \{a', a'', a'''\}$.

Exercise 3.2. *Prove that i-quanta are closed under intersections, without using the properties of adjoint functors but just transitivity of i-quantum relations.*

3.4 Transforming Perception Systems

Now we are equipped with a sufficient machinery in order to transform *A-systems* into informationally equivalent *P-systems*.

Let **A** be an *A-system*. The basic step is derived from the observation that any attribute a is actually a set of properties, namely the set of admissible attribute values for a. Thus we start with associating each attribute a with the family $\mathcal{N}(a) = \{a_v\}_{v \in V_a}$. We set $\mathcal{N}(At) = \bigcup_{a \in At} \mathcal{N}(a)$. For each value v, a_v is the property "taking value v for attribute a". We call this transform "scale nominalisation".

Now let us set a relation $\Vdash^{\mathcal{N}}$ as:

$$g \Vdash^{\mathcal{N}} a_v \text{ if and only if } a(g) = v, \text{ all } g \in G, a \in At, v \in V_a.$$

The resulting system, $\mathcal{N}(\mathbf{A}) = \langle G, \mathcal{N}(At), \Vdash^{\mathcal{N}} \rangle$, is called the "*nominalisation of* **A**". $\mathcal{N}(\mathbf{A})$ will be called a *nominal A-system* or *NA-system*. Obviously, the following holds:

Proposition 3.4.1. *Let* **A** *be an A-system. Then* $\mathcal{N}(\mathbf{A})$ *is a P-system.*

REMARKS. Nominalising is tantamount to setting a ternary relation $R \subseteq G \times At \times \bigcup_{a \in At} V_a$. In fact, $\Vdash^{\mathcal{N}}$ is a binary relation $G \times (At \times \bigcup_{a \in At} V_a)$ − indeed, $a(g) = v$ is equivalent to $g(\langle a, v \rangle) = 1$.

Moreover, if we formally consider *P-systems* as binary *A-systems*, we can also nominalise *P-systems*. But in this case we have a further property:

Proposition 3.4.2. *Let* **P** *be a P-system. Then* $\mathcal{N}(\mathbf{P})$ *is a dichotomic system.*

Proof. This is obvious, because for any property p, the nominalisation $\mathcal{N}(p) = \{p_1, p_0\}$ forms a pair of complementary properties, since for all $g \in G$, $g \Vdash^{\mathcal{N}} p_1$ if and only if $g \Vdash p$ and $g \Vdash^{\mathcal{N}} p_0$ if and only if $g \nVdash p$. QED

If nominalisation of *P-systems* produces dichotomic systems, nominalisation of dichotomic systems does not give rise to any further result.

Proposition 3.4.3. *If* **S** *is an A-system, or a dichotomic or functional P-system, then* $\mathcal{N}(\mathbf{S}) \cong_I \mathbf{S}$.

Proof. Let $\langle p, \overline{p} \rangle$ be a pair of complementary properties in **S**. After nominalisation we shall obtain two pairs $\mathcal{N}(p) = \{p_1, p_0\}$ and $\mathcal{N}(\overline{p}) = \{\overline{p_1}, \overline{p_0}\}$. Clearly, for any $g \in G$, $g \Vdash p$ in **P** if and only if $g \Vdash^{\mathcal{N}} p_1$ in $\mathcal{N}(\mathbf{S})$. But $g \Vdash^{\mathcal{N}} p_1$ if and only if $g \nVdash^{\mathcal{N}} p_0$ if and only if $g \Vdash^{\mathcal{N}} \overline{p_0}$. Conversely, $g \Vdash \overline{p}$ if and only if $g \Vdash^{\mathcal{N}} p_0$ if and only if $g \nVdash^{\mathcal{N}} p_1$ if and only if $g \Vdash^{\mathcal{N}} \overline{p_1}$.

     As to *A-systems*, let us prove that for any $g \in G$, $Q_g \upharpoonright At = Q_g \upharpoonright \mathcal{N}(At)$. Indeed, if $g' \in Q_g \upharpoonright At$, then $a(g) = x$ if and only if $a(g') = x$, all $a \in At$. Therefore for any $x \in \mathcal{N}(a)$, $g \Vdash a_x$ if and only if $g' \Vdash a_x$, whence $g' \in Q_g \upharpoonright \mathcal{N}(a)$. Moreover, $g' \nVdash a_{x'}$ for any other $x' \neq x$, so that the reverse implication holds too.

     If **S** is functional and $g' \in Q_g \upharpoonright M$ then $g \Vdash m$ if and only if $g' \Vdash m$, since $\hat{\Vdash}(g) = \hat{\Vdash}(g') = m$. Thus the proof runs as above.   QED

As usual, we shall also write $\Vdash^{\mathcal{N}(\mathbf{S})}$ if we have to refer to a particular Information System **S**.

Recalling that for any A-*system* A, $\mathcal{N}(\mathbf{A})$ is not only a P-*system* but it is still an A-*system* with $At = \{0, 1\}$, we obtain the following corollary:

Corollary 3.4.1. *Let* $\mathbf{S}$ *be an Information System. Then* $\mathcal{N}(\mathbf{S}) \cong_I \mathcal{N}(\mathcal{N}(\mathbf{S}))$.

Proof. If $\mathbf{S}$ is a P-*system* then $\mathcal{N}(\mathbf{S})$ is a dichotomic system. If $\mathbf{S}$ is an A-*system* then $\mathcal{N}(\mathbf{S})$ is a binary A-*system*. In both cases from *Proposition* 3.4.3 $\mathcal{N}(\mathcal{N}(\mathbf{S})) \cong_I \mathcal{N}(\mathbf{S})$. QED

Corollary 3.4.2. *If* $\mathbf{A}$ *is an A-system then there is a dichotomic system* $\mathbf{D}$ *such that* $\mathbf{D} \cong_I \mathbf{A}$.

Proof. Since $\mathcal{N}(\mathbf{A})$ is a P-*system*, from *Proposition* 3.4.2 $\mathcal{N}(\mathcal{N}(\mathbf{A}))$ is dichotomic. But from *Proposition* 3.4.3 and *Corollary* 3.4.1 $\mathbf{A} \cong_I \mathcal{N}(\mathbf{A}) \cong_I \mathcal{N}(\mathcal{N}(\mathbf{A}))$. QED

As a by-product we again obtain Proposition 3.1.2.(2), which states that i-quantum relations induced by A-*systems*, dichotomic and functional P-*systems* are equivalence relations, and *Corollary* 3.3.6.(2). Notice that *Proposition* 3.4.1, as well as *Corollary* 3.1.2.(2), relies on the fact that we are dealing with deterministic Information Systems so that if g and g' take the same attribute value for a, say v, then g and g' cannot also take v' and, respectively, v'' for a, such that $v' \neq v''$.

Now let us see how different perception operators relate to each other in transformed systems.

Proposition 3.4.4. *Let* $\mathbf{A}$ *be an A-system. Then for all* $X \subseteq G$, (a) $int^{\mathcal{N}(\mathbf{A})}(X) \subseteq Q_X^{\mathcal{N}(\mathbf{A})} \subseteq cl^{\mathcal{N}(\mathbf{A})}(X)$; (b) $int^{\mathcal{N}(\mathbf{A})}(X) \subseteq Q_X^{\mathbf{A}} \subseteq cl^{\mathcal{N}(\mathbf{A})}(X)$.

Proof. It is to notice that from *Proposition* 3.4.3, $\mathcal{N}(\mathbf{A}) \equiv_I \mathbf{A}$. Therefore we have just to prove (a). But $R_{\mathbf{A}} = R_{\mathcal{N}(\mathbf{A})}$. Hence $R_{\mathcal{N}(\mathbf{A})}$ is an equivalence relation (from *Proposition* 3.1.2.(2)). Now, if $g \in Q_X^{\mathcal{N}(\mathbf{A})}$ then there is $x \in X$ such that $\langle i \rangle^{\mathcal{N}(\mathbf{A})}(x) = \langle i \rangle^{\mathcal{N}(\mathbf{A})}(g)$. But $x \in cl^{\mathcal{N}(\mathbf{A})}(X)$, because cl is increasing, thus $\langle i \rangle^{\mathcal{N}(\mathbf{A})}(x) \subseteq \langle i \rangle^{\mathcal{N}(\mathbf{A})}(X)$. It follows that $\langle i \rangle^{\mathcal{N}(\mathbf{A})}(g) \subseteq \langle i \rangle^{\mathcal{N}(\mathbf{A})}(X)$, too, so that $g \in cl^{\mathcal{N}(\mathbf{A})}(X)$. Moreover, since $\mathcal{N}(\mathbf{A})$ is a P-*system*, from *Corollary* 3.3.3 we have $int^{\mathcal{N}(\mathbf{A})}(X) \subseteq Q_X^{\mathcal{N}(\mathbf{A})}$. QED

Example 3.4.1. Nominalisation

Here are some examples of nominalisation.

Let us nominalise the Information Systems $\mathbf{A}$ and $\mathbf{P}$ of Example 1.2.1:

$\Vdash^{\mathcal{N}(\mathbf{A})}$	A_0	A_1	A_3	A'_b	A'_d	A'_f	A''_α	A''_δ	$\Vdash^{\mathcal{N}(\mathbf{P})}$	b_1	b_0	b'_1	b'_0	b''_1	b''_0	b'''_1	b'''_0
a	0	1	0	1	0	0	1	0	a	1	0	1	0	0	1	0	1
a'	1	0	0	0	1	0	1	0	a'	0	1	1	0	0	1	1	0
a''	0	1	0	1	0	0	1	0	a''	0	1	1	0	1	0	1	0
a'''	0	0	1	0	0	1	0	1	a'''	0	1	0	1	0	1	1	0

So, $\mathcal{N}(A) = \{A_0, A_1, A_3\}$, $\mathcal{N}(b) = \{b_1, b_0\}$ and so on. It is evident that, for instance, $a \in Q_{a''}^{\mathcal{N}(\mathbf{A})}$ and $a'' \in Q_a^{\mathcal{N}(\mathbf{A})}$. But the same happens already in $\mathbf{A}$. Indeed, $Q_a^{\mathbf{A}} = \{a, a''\} = Q_{a''}^{\mathbf{A}}$. On the contrary, while $Q_{a'}^{\mathbf{P}} = \{a', a''\}$ we have $Q_{a'}^{\mathcal{N}(\mathbf{P})} = \{a'\}$. In fact $a'' \in Q_{a'}^{\mathbf{P}}$ because it fulfills all the properties fulfilled by a' (i.e. b' and b''') plus the additional property b''. But in $\mathcal{N}(\mathbf{P})$ this latter fact prevents a''' from belonging to $Q_{a'}^{\mathcal{N}(\mathbf{P})}$, because property b'' splits into the pair $\langle b''_0, b''_1 \rangle$ and $a' \Vdash^{\mathcal{N}(\mathbf{P})} b''_0$ while $a'' \Vdash^{\mathcal{N}(\mathbf{P})} b''_1$, which are mutually exclusive possibilities.

If we further nominalise $\mathcal{N}(\mathbf{P})$ we split $\langle b''_0, b''_1 \rangle$ into $\langle b''_{0_1}, b''_{0_0}, b''_{1_1}, b''_{1_0} \rangle$:

$\Vdash^{\mathcal{N}^{\mathcal{N}(\mathbf{P})}}$	$\ldots$	b''_{0_1}	b''_{0_0}	b''_{1_1}	b''_{1_0}	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\ldots$	a'	1	0	0	1	$\ldots$
$\ldots$	a''	0	1	1	0	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

It is obvious that the pairs $\langle b''_{0_1}, b''_{1_0} \rangle$ and $\langle b''_{0_0}, b''_{1_1} \rangle$ give the same information as b''_0 and, respectively, b''_1. In fact the columns are pairwise duplicated in $\mathcal{N}(\mathcal{N}(\mathbf{P}))$ and after simplification we obtain the columns corresponding to the generating properties in $\mathcal{N}(\mathbf{P})$.

It is not difficult to verify that $R_{\mathcal{N}(\mathbf{A})} = R_{\mathbf{A}}$ so that $\mathcal{N}(\mathbf{A}) \cong_I \mathbf{Q}(\mathbf{A})$.

3.5   Topological Approximation Operators

In view of the duality between Information Systems and preorders, we can develop the rest of the theory from a more abstract point of view.

Thus, from now on we shall deal with preordered structures and assume, intuitively, that they represent some information quantum relation system.

Corollary 3.5.1. *Let* $\mathbf{O} = \langle G, G, R \rangle$ *be a preordered set. Let* $X \subseteq G$. *Then:*

The application	is the least fixpoint of	including X
$\langle R \rangle(X)$	$\langle R \rangle$ $[R^{\smile}]$ $\mathcal{A}$ int	
$\langle R^{\smile} \rangle(X)$	$\langle R^{\smile} \rangle$ $[R]$ cl $\mathcal{C}$	
The application	is the largest fixpoint of	included in X
$[R](X)$	$[R]$ $\langle R^{\smile} \rangle$ $\mathcal{C}$ cl	
$[R^{\smile}](X)$	$[R^{\smile}]$ $\langle R \rangle$ int $\mathcal{A}$	

Proof. First notice that from *Proposition* 3.3.1 the fixpoints listed in each row coincide. Now, for didactic purposes, we prove the first two cases by means of two different approaches.

(a) Obviously $\langle R \rangle(X) \supseteq X$ and from idempotence $\langle R \rangle(X)$ is a fixed point of $\langle R \rangle$. Suppose Z is a fixed point of $\langle R \rangle$ and $X \subseteq Z$. From monotonicity $\langle R \rangle(X) \subseteq \langle R \rangle(Z) = Z$. Hence $\langle R \rangle(X)$ is the least fixpoint of $\langle R \rangle$ including X.

(b) From *Proposition* 1.4.6 (but see also Subsection 2.3.2) $[R]\langle R^{\smile} \rangle(X)$ is the smallest image of $[R]$ greater than or equal to X and $[R]$ is idempotent. Thus it is the least fixpoint of $[R]$ which includes X. Hence, from *Proposition* 3.3.1, it is also the least fixpoint of $\langle R^{\smile} \rangle(X)$ including X.

The remaining cases are proved analogously.   QED

Corollary 3.5.2. *Let* $\mathbf{O} = \langle G, G, R \rangle$ *be a preordered set. Then for all* $X \subseteq G$,

(i) $\langle R \rangle(X) = \bigcap \{Z : Z \in \Omega_{\mathcal{A}}(\mathbf{O}) \ \& \ Z \supseteq X\}$;
(ii) $[R](X) = \bigcup \{Z : Z \in \Gamma_{\mathcal{C}}(\mathbf{O}) \ \& \ Z \subseteq X\}$;
(iii) $\langle R^{\smile} \rangle(X) = \bigcap \{Z : Z \in \Gamma_{cl}(\mathbf{O}) \ \& \ Z \supseteq X\}$;
(iv) $[R^{\smile}](X) = \bigcup \{Z : Z \in \Omega_{int}(\mathbf{O}) \ \& \ Z \subseteq X\}$.

Henceforth, for obvious reasons we shall adopt the following terminology:

$\langle R \rangle(X)$	Direct upper R-approximation of X, also denoted by $(uR)(X)$
$\langle R^{\smile} \rangle(X)$	Inverse upper R-approximation of X, also denoted by $(uR^{\smile})(X)$
$[R](X)$	Direct lower R-approximation of X, also denoted by $(lR)(X)$
$[R^{\smile}](X)$	Inverse lower R-approximation of X, also denoted by $(lR^{\smile})(X)$

The information-oriented reading of the above operators is:

$\langle R \rangle(X)$	Set of the elements specialized by some member of X (or, which approximate some member of X)
$\langle R^{\smile} \rangle(X)$	Set of the elements approximated by some member of X (or, which specialize some member of X)
$[R](X)$	Set of the elements specialized only by members of X (or, which approximate only members of X)
$[R^{\smile}](X)$	Set of elements approximated only by members of X (or, which specialize only elements of X)

Particularly we can give an information-oriented interpretation to some combinations of operators:

$[R]\langle R \rangle(X)$	Set of the elements which are specialized only by elements specialized by some member of X ($x \in [R]\langle R \rangle(X)$ only if each element which specializes x is specialized by some member of X)
$[R^{\smile}]\langle R^{\smile} \rangle(X)$	Set of the elements which are approximated only by elements approximated by some member of X ($x \in [R^{\smile}]\langle R^{\smile} \rangle(X)$ only if each element which approximates x is approximated by some member of X)

Besides these operators we add also the interpretation of $[[R]]$ and $[[R^{\smile}]]$:

$[[R]](X)$	Set of the elements specialized by all the members of X (or, which approximate all the members of X)
$[[R^{\smile}]](X)$	Set of the elements approximated by all the members of X (or, which specialize all the members of X)

3.6   Topological Approximation Systems

Given an Information system $\mathbf{S}$, from $\mathbf{Q}(\mathbf{S})$ we can define the following families of operators manipulating sets of objects: $\mathbf{G^S} = \{\langle R_\mathbf{S}\rangle, \langle R_\mathbf{\widetilde{S}}\rangle,$ $[R_\mathbf{S}], [R_\mathbf{\widetilde{S}}]\}$. If $\mathbf{S}$ is a *P-system* then we can add *est* to $\mathbf{G^S}$. A *Multi-agent topological Approximation System* will be therefore defined as a structure:

$$\langle G, G, \{\mathbf{G}^{\mathbf{S}_k}\}_{k\in K}\rangle \text{ or, shortly, } \langle G, \{\mathbf{G}^{\mathbf{S}_k}\}_{k\in K}\rangle$$

where any $\mathbf{S}_k$ is an Information System on the same set of objects G. This generalisation makes it possible to manipulate subsets of objects by subsequently applying informational criteria induced by different Information Systems. Particularly, if $card(K) = 1$ we shall speak of "(*single-agent*) *topological Approximation Systems*.

Notice that if we start with an arbitrary relational structure $\langle X, Y, R\rangle$, we have to deal separately with $\mathcal{A}, \mathcal{C}, int$ and cl, as we have done in the initial part of this Chapter, because fixed points of adjoint functors may fail to coincide, even if $X = Y$.

We end this Chapter by defining some interesting examples of Single-agent topological Approximation Systems.

Definition 3.6.1. *Let* $\mathbf{Q}(\mathbf{S}) = \langle G, G, R_\mathbf{S}\rangle$ *be an IQRS. Then,*

1. $\langle G, [R_\mathbf{S}], \langle R_\mathbf{S}\rangle\rangle$ *– will be called a* Direct Intuitionistic Approximation System.

2. $\langle G, [R_\mathbf{\widetilde{S}}], \langle R_\mathbf{\widetilde{S}}\rangle\rangle$ *– will be called an* Inverse Intuitionistic Approximation System.

3. $\langle G, [R_\mathbf{\widetilde{S}}], \langle R_\mathbf{S}\rangle\rangle$ *– will be called a* Galois Intuitionistic Approximation System.

4. $\langle G, [R_\mathbf{S}], \langle R_\mathbf{\widetilde{S}}\rangle\rangle$ *– will be called a* co-Galois Intuitionistic Approximation System.

The term "intuitionistic" is after the link between preorders and intuitionistic models (see Frame 4.13).

Definition 3.6.2 (Indiscernibilty Space)**.** *Let* $\mathbf{E} = \langle G, G, E\rangle$ *be a P-system such that E is an equivalence relation. Then* $\mathbf{E}$ *is called an* Indiscernibility Space *and shall be denoted by $\langle G, E\rangle$, too.*

Definition 3.6.3 (Pawlak Approximation System)**.** *Let* $\langle G, G, E \rangle$ *be an Indiscernibility Space. Let* $\mathbb{I}_E$ *and* $\mathbb{C}_E$ *be the interior and, respectively, topological operators of the topological space induced by taking* $\{[x]_E : x \in G\}$ *as a subbasis. Then* $\langle G, \mathbb{I}_E, \mathbb{C}_E \rangle$ *is called a* Pawlak Approximation System.

From the above discussion the following statement is obvious:

Proposition 3.6.1. *Let* $\mathbf{E} = \langle G, G, E \rangle$ *be an Indiscernibility Space. Let* $\mathcal{L} \in \{[E], [e]^{\mathbf{E}}, [i]^{\mathbf{E}}, \mathcal{C}^{\mathbf{E}}, int^{\mathbf{E}}\}$ *and* $\mathcal{U} \in \{\langle E \rangle, \langle e \rangle^{\mathbf{E}}, \langle i \rangle^{\mathbf{E}} \mathcal{A}^{\mathbf{E}}, cl^{\mathbf{E}}\}$ *Then the Single-agent topological Approximation System* $\langle G, \mathcal{L}, \mathcal{U} \rangle$ *is a* Pawlak Approximation System.

Proof. From *Corollary* 3.3.1 we know that for any preorder R, for any $X \subseteq G$, $[R](X) = \mathcal{C}(X)$ and $\langle R \rangle(X) = \mathcal{A}(X)$. Moreover, if R is an equivalence relation, $R^{\smile} = R$. It follows that $[R](X) = [R^{\smile}](X) = cal(X) = int(X)$ and $\langle R \rangle(X) = \langle R^{\smile} \rangle(X) = \mathcal{A}(X) = cl(X)$. Therefore, a Pawlak Approximation System is any of the types listed in *Definition* 3.6.1.   QED

But we can prove a further fact. To this end we introduce the notion of an *Approximation Equivalence*, or *a-equivalence* between Single-agent Approximation Systems:

Definition 3.6.4. *Let* $\mathbf{A} = \langle G, \mathbb{I}, \mathbb{C} \rangle$ *and* $\mathbf{A}' = \langle G, \mathbb{I}', \mathbb{C}' \rangle$ *be two Single-agent topological or pre-topological Approximation Systems, with* $\mathbb{I}, \mathbb{I}'$ *interior operators and* $\mathbb{C}, \mathbb{C}'$ *closure operators. Then we say that* $\mathbf{A}$ *and* $\mathbf{A}'$ *are a-equivalent, in symbols,* $\mathbf{A} \cong_a \mathbf{A}'$ *if and only if* $\Omega_{\mathbb{I}}(G) = \Omega_{\mathbb{I}'}(G)$ *and* $\Gamma_{\mathbb{C}}(G) = \Gamma_{\mathbb{C}'}(G)$.

Clearly, by duality one equality implies the other. We use this definition in the following statement:

Proposition 3.6.2. *Let* $\mathbf{S}$ *be an A-system or FP-system or DP-system. Let us set* $\Diamond = \langle R_{\mathbf{S}} \rangle$ *and* $\Box = [R_{\mathbf{S}}]$. *Then:*

1. $\Diamond = \langle R_{\mathbf{S}}^{\smile} \rangle$, $\Box = [R_{\mathbf{S}}^{\smile}]$ *and* $\langle G, \Box, \Diamond \rangle$ *is a Pawlak Approximation System.*

2. *For any* $X \subseteq G$, $\Diamond(X) \subseteq cl^{\mathbf{S}}(X)$ *and* $int^{\mathbf{S}}(X) \subseteq \Box(X)$.

3. *If* S *is an FP-system then* $\langle G, \Box, \Diamond \rangle \cong_a \langle G, int^{\mathbf{S}}, cl^{\mathbf{S}} \rangle$.

Proof. (**1**) Immediate, from the fact, proved in *Proposition* 3.1.2.(3), that $R_{\mathbf{S}}$ in these cases is an equivalence relation. (**2**) Let $g \in \Diamond(X)$. Then there exists a $g' \in X$ such that $\langle i \rangle(g) = \langle i \rangle(g')$. It follows that $\langle i \rangle(g) \subseteq \langle i \rangle(X)$ so that $g \in cl^{\mathbf{S}}(X)$. Therefore, since $\Diamond(-X) \subseteq cl^{\mathbf{S}}(-X)$ we obtain $-\Diamond(-X) \supseteq -cl^{\mathbf{S}}(-X)$, that is, $\Box(X) \supseteq int^{\mathbf{S}}(X)$. (**3**) It is sufficient to use in addition *Proposition* 2.1.1.(3) together with *Proposition* 2.3.3, or the latter Proposition and *Proposition* 3.1.2.(3) which together with *Proposition* 3.1.1.(4) states that $cl^{\mathbf{S}}(g) = [g]_{k_f}$, any $g \in G$.   QED.

Notice that we cannot set $\langle G, \Box, \Diamond \rangle = \langle G, int^{\mathbf{S}}, cl^{\mathbf{S}} \rangle$ instead of $\langle G, \Box, \Diamond \rangle \cong_a \langle G, int^{\mathbf{S}}, cl^{\mathbf{S}} \rangle$, because, actually, the former system is $\langle G, G, \Box, \Diamond \rangle$ while the latter is $\langle G, M, int^{\mathbf{S}}, cl^{\mathbf{S}} \rangle$. However, if $\mathbf{S}$ is an *FP-system*, then Single-agent pre-topological Approximation Systems, Single-agent topological Approximation Systems and Pawlak Approximation Systems induce the same family of fixed points.

Given an Information System $\mathbf{S}$, in $\mathbf{Q}(\mathbf{S})$ the set M coincides with G. Thus Multi-agent topological Approximation Systems are also Multi-agent topological Perception Systems on $\mathbf{Q}(\mathbf{S})$. Further, we can carry on the quantisation transform from the point of view of properties (or attributes) and obtain Multi-agent topological co-Approximation Systems $\langle M, M, \{\mathbf{M}^{\mathbf{S}_j}\}_{j \in J} \rangle$. By merging the two spaces we can define, on an Information System $\mathbf{S}$, the notion of a *Multi-agent topological Perception System*: $\langle G, M, \{\mathbf{G}^{\mathbf{S}_k}\}_{k \in K}, \{\mathbf{M}^{\mathbf{S}_j}\}_{j \in J}, \mathbf{I}^{\mathbf{S}}, \mathbf{E}^{\mathbf{S}} \rangle$, where any $\mathbf{S}_k$ is an Information System on the set of objects G and set of properties or attributes M_k possibly distinct from M, any $\mathbf{S}_j$ is an Information System on the set of properties (or attributes) M and set of objects G_j possibly distinct from G, $\mathbf{I}^{\mathbf{S}} : \wp(G) \longmapsto \wp(M)$.

However, we shall not go further into this topic.

Chapter 4

Frames (Part I)

4.1 Frame – Approximation

Any complete account for an history of the concept of an *approximation* is trivially non-feasible.

   Indeed, approximation is a fundamental concept in a number of fields and it is not likely to try and list even a small amount of them. However, because of its historical role in Mathematics we recall Archimedes' exhaustion method (ca. 287–212 BC). This method is based on the observation that we can compute the area of, for instance, a circle, by means of a series of more and more refined approximations from below and from above.
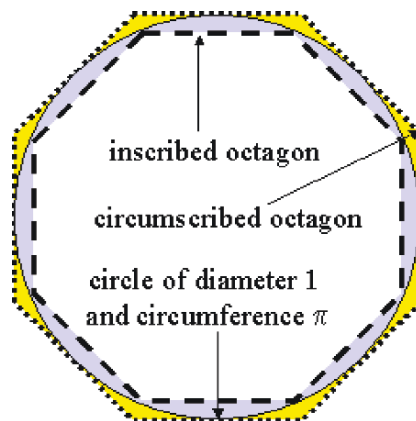
Figure 4.1: Archimedes' exhaustion method

More precisely, Archimedes' main idea was to try and approximate the circle using inscribed (lower approximating) and circumscribed (upper approximating) regular polygons, as shown in Figure 4.1.

However, this method may be traced back to Eudoxus of Cnidos (ca. 400–347 BC) and to the well-known paradoxes by Zeno of Elea (ca. 495–435 BC) accounted by Plato in "Parmenides", which can be considered a first kind of (never ending) approximation reasoning (see Kline [1972]).

4.2   Frame – Classification

An approximation process is a means for classifying objects. Indeed "concrete" items are assigned classifying *types* (i.e. "attribute-values", "properties", "neighborhoods", and so on along all the possible interpretations of M we discussed in this Chapter) and types intensionally approximate collections of objects.

Classification is, obviously, a main topic in many domains and fields, such as document management, information retrieval and so on and the approaches to this topic are impressively numerous.

We just mention a recent contribution to its scientific definition, which is closely related to our approach (in what follows we use our terminology and symbols).

In their valuable book Barwise & Seligman [1997], J. Barwise and J. Seligman describe a *classification* as a *P-system* $\mathbf{A} = \langle G, M, \Vdash \rangle$. The elements of G are called the *tokens* of $\mathbf{A}$ and the elements of M are called the *types* of $\mathbf{A}$, which are used to classify the tokens from G by means of the binary relation $\Vdash$ which determines which token are of which type.

Now, let $\Gamma, \Lambda \subseteq M$. Then a token g is said to satisfy the *sequent* $\langle \Gamma, \Lambda \rangle$ if the following holds:

$$\langle i \rangle(g) \supseteq \Gamma \Rightarrow \langle i \rangle(g) \cap \Lambda \neq \emptyset$$

that is, if g fulfills all the types of Γ then it must be at least of a type from Λ (indeed any logical sequent $\{\gamma_1, \gamma_2, ..., \gamma_n\} \vdash \{\lambda_1, \lambda_2, ..., \lambda_m\}$ is read $\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n \vdash \lambda_1 \vee \lambda_2 \vee ... \vee \lambda_m$).

It follows that g satisfies a constraint $\langle \Gamma, \Lambda \rangle$ if g belongs to at least a type listed in Λ.

The sequent $\langle \Gamma, \Lambda \rangle$ is valid in $\mathbf{A}$, or Γ *entails* Λ in $\mathbf{A}$, $\Gamma \vdash_{\mathbf{A}} \Lambda$, if every token of G satisfies $\langle \Gamma, \Lambda \rangle$. If this is the case, the sequent at hand is called a *constraint* supported by the classification $\mathbf{A}$.

The set of valid constraint in $\mathbf{A}$ is called the *theory* of $\mathbf{A}$, $Th(\mathbf{A})$.

The notion of a constraint makes it possible to describe a number of facts about a classification:

- If both Γ and Λ are singletons, then $\Gamma \models_{\mathbf{A}} \Lambda$ means that Γ logically entails Λ in $\mathbf{A}$.

- $\Gamma \models_{\mathbf{A}}$ means that no tokens are of type Γ in $\mathbf{A}$.

- If $\Gamma, \Lambda \models_{\mathbf{A}}$, then Γ and Λ are mutually exclusive in $\mathbf{A}$.

- If $\models_{\mathbf{A}} \Lambda, \Gamma$, then every token of $\mathbf{A}$ belongs at least to one of the types listed in Λ or Γ.

Further, we can think of a distributed information system as a collection of classification structures. These structures must be related to each other in a proper way in order to make up a system. The required connections are called *infomorphisms*.

Given two classifications $\mathbf{A} = \langle G_{\mathbf{A}}, M_{\mathbf{A}}, \Vdash^{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle G_{\mathbf{B}}, M_{\mathbf{B}}, \Vdash^{\mathbf{B}} \rangle$, an infomorphism between $\mathbf{A}$ and $\mathbf{B}$ is a pair of functions $\langle f_M, f_G \rangle$ such that $f_M : M_{\mathbf{A}} \longmapsto M_{\mathbf{B}}$, $f_G : G_{\mathbf{B}} \longmapsto G_{\mathbf{A}}$ and the following holds, for all $g \in G_{\mathbf{B}}$ and $m \in M_{\mathbf{A}}$:

$$f_G(g) \Vdash^{\mathbf{A}} m \quad iff \quad g \Vdash^{\mathbf{B}} f_M(m)$$

It follows that classification structures plus infomorphism are *Chu spaces* (see Frame 4.12).

In Rough Set Theory this approach was exploited in Skowron et al. [2003] to investigate nets of Information Systems.

We now prove that our notion of an "informational equivalence" (see Section 3.2.3) is an instance of infomorphism.

Let $\mathbf{P} = \langle G, M, \Vdash \rangle$ and $\mathbf{P}' = \langle G, M', \Vdash' \rangle$ be two *P-systems*. Clearly, $\{Q_x \upharpoonright M\}_{x \in G}$ and $\{Q_x \upharpoonright M'\}_{x \in G}$ are two classifications of the objects in G. Thus, let us set $\mathbf{A} = \langle G, \{Q_x \upharpoonright M\}_{x \in G}, \in \rangle$, $\mathbf{B} = \langle G, \{Q_x \upharpoonright M'\}_{x \in G}, \in \rangle$. Moreover, let us set $f_G = Id$ and $f_M(Q_x \upharpoonright M) = Q_{f_G^{-1}(x)} \upharpoonright M' = Q_x \upharpoonright M'$. Then we have:

Proposition 4.2.1. $\mathbf{P} \cong_I \mathbf{P}'$ *if and only if* $\langle f_M, f_G \rangle$ *is an infomorphism.*

Proof. $f_G(y) \in Q_x \upharpoonright M$ iff $y \in Q_x \upharpoonright M$ (since f_G is the identity function) iff $y \in Q_x \upharpoonright M'$ (by $\cong_I$) iff $y \in f_M(Q_x \upharpoonright M)$ (by definition of f_M).

Vice-versa, $y \in Q_x \upharpoonright M$ iff $f_G(y) \in Q_x \upharpoonright M$ iff $y \in f_M(Q_x \upharpoonright M)$ (by infomorphism) iff $y \in Q_x \upharpoonright M'$ (by definition of f_M).   QED

Also, the above machinery has been combined with Formal Concept Analysis to develop a general framework to define and implement *Formal Ontologies*, especially by R. Kent (cf. for instance Kent [2000]).[1]

However, for these developments we address the reader to the following web-sites: "http://www.ontologos.org/" and "http://suo.ieee.org/".

4.3 Frame – Categorizing Through Pointless Topology

In this frame we want to show how classifications may be defined by means of the mathematical tools provided by pointless topology, as depicted in the Introduction. Indeed, we shall see that we have basically to apply the pull-back schema to a particular function that maps points of a topological space onto the so called *spectral space* of the lattice of the open sets of the topological space. Then we shall decompose this function much in the same way described in *Theorem* 16.5.1 of Mathematical toolkit 16.5. We shall need just an additional subtle, but surprisingly intuitive, step. This step aims at analysing the relationships between points, that are induced by the given topology. It is based on the notion of a *specialization preorder*.

Let us recall the definition from Mathematical toolkit 16.4

Definition 4.3.1 (Specialisation preorder). *Given a topological space $\langle X, \Omega(X) \rangle$ and two elements x and x' of X, we say that x is specialized by x'(x' specializes x; x' is a better approximation than x; x' refines x), denoted $x \preceq x'$, if and only if for any open set $O \in \Omega(X)$, $x \in O \Rightarrow x' \in O$. The relation $\preceq$ will be called the* specialization *preorder on X induced by $\Omega(X)$.*

Proposition 4.3.1. *Given a topological space $\langle X, \Omega(X) \rangle$, the specialization preorder induces a partial order $\leqslant$ on the set of abstract points $\mathcal{J}(\Omega(X))$ by setting $P \leqslant P'$ if and only if $\forall p' \in P'$, $\exists p \in P$ such that*

---

[1]A Formal Ontology is "*an explicit specification of a conceptualization,*" as maintained in Gruber [1993] or a sort of "*dictionary or glossary, but with greater detail and structure that enables computers to process its content. An ontology consists of a set of concepts, axioms, and relationships that describe a domain of interest,*" as stated by the IEEE P1600.1 Standard Upper Ontology Working Group.

$p \preceq p'$. *This partial order $\leqslant$ is contravariant with respect to the lattice order of $\Omega(X)$ restricted to $\mathcal{J}(\Omega(X))$.*

Proof. The relation $\leqslant$ is a partial order. In fact, let us suppose that $P \leqslant P'$ and $P' \leqslant P$ but $P \neq P'$. So, suppose $x \in P$ but $x \notin P'$. Since $P' \leqslant P$ there exists $y \in P'$ such that $y \preceq x$. But $y \preceq x$ if and only if for all A, $y \in A \Rightarrow x \in A$. It follows that $x \in P'$, too (contradiction). Henceforth $P \subseteq P'$.   QED

Proposition 4.3.2. *Given a topological space $\langle X, \Omega(X) \rangle$, let us define an equivalence relation $\equiv$ by putting, for any $x, x' \in G$, $x \equiv x'$ if and only if $x \preceq x'$ and $x' \preceq x$ (where $\preceq$ is the specialization preorder induced by $\Omega(X)$). Let us factorise X through $\equiv$. On $X/_{\equiv}$ we inherit a partial order $\sqsubseteq$ from $\preceq$, by setting $[x]_{\equiv} \sqsubseteq [x']_{\equiv}$ if and only if $x \preceq x'$.*

Proof. The relation $\sqsubseteq$ is a partial order. In fact, suppose $[y]_{\equiv} \sqsubseteq [x]_{\equiv}$ and $[x]_{\equiv} \sqsubseteq [y]_{\equiv}$. Then, from $[y]_{\equiv} \sqsubseteq [x]_{\equiv}$ we have $y \preceq x$ and from $[x]_{\equiv} \sqsubseteq [y]_{\equiv}$ we have $x \preceq y$. It follows that $x \equiv y$, so that $[y]_{\equiv} = [x]_{\equiv}$.   QED

Definition 4.3.2 (T_0-**ification**). *Let $\langle X, \Omega(X) \rangle$ be a topological space. By considering the set of order filters of the poset $\langle X/_{\equiv}, \sqsubseteq \rangle$ we obtain another topological space $\langle X/_{\equiv}, \Omega_{\sqsubseteq}(X/_{\equiv}) \rangle$, called the T_0-ification of $\langle X, \Omega(X) \rangle$ (the name is after the property of T_0 spaces in general topology).*

Definition 4.3.3 (**Identification space**). *Let $\langle X, \Omega(X) \rangle$ be a topological space and let E be an equivalence relation on X. Let us define on $X/_E$ a topological space $\langle X/_E, \Omega_I(X/_E) \rangle$ by means of the following equivalence: $\forall O \in \wp(X/_E)$, $O \in \Omega_I(X/_E)$ iff $\mathfrak{nat}^{-1}(O) \in \Omega(X)$, where $\mathfrak{nat}$ is the canonical application $\mathfrak{nat}(x) = [x]_E$, for any $x \in X$. Then $\langle X/_E, \Omega_I(X/_E) \rangle$ is called an identification space of $\langle X, \Omega(X) \rangle$.*

Definition 4.3.4 (**Identification map**). *Let $\langle X, \Omega(X) \rangle$ and $\langle Y, \Omega(Y) \rangle$ be two topological spaces. Then a map $\phi : X \longmapsto Y$ is called an identification map if it is onto and $Y' \in \Omega(Y)$ if and only if $\phi^{\leftarrow}(Y') \in \Omega(X)$.*

Clearly, given an identification space $\langle X/_E, \Omega_I(X/_E) \rangle$, the canonical map $\mathfrak{nat}(x) = [x]_E$ is an identification map from $\langle X, \Omega(X) \rangle$.

Proposition 4.3.3. *The T_0-ification of a topological space $\langle X, \Omega(X) \rangle$ is an identification space.*

Proof. If $O \in \Omega_{\sqsubseteq}(X/_{\equiv})$ then $O =\uparrow_{\sqsubseteq} [x]_{\equiv}$ for some $x \in X$. Thus, $\mathfrak{nat}^{-1}(O) = \mathfrak{nat}^{-1}\{[x']_{\equiv} : [x']_{\equiv} \sqsupseteq [x]_{\equiv}\}$. But since $[x']_{\equiv} \sqsupseteq [x]_{\equiv}$ if and only if $x' \succeq x$, we obtain that $\mathfrak{nat}^{-1}(O) =\uparrow_{\preceq} x$. Therefore $\mathfrak{nat}^{-1}(O) \in \Omega(X)$. Conversely, if $\mathfrak{nat}^{-1}(O) \in \Omega(X)$, then $\mathfrak{nat}^{-1}(O) =\uparrow_{\preceq} x$ for some $x \in X$. It follows that $O = \mathfrak{nat}^{\rightarrow}(\uparrow_{\preceq} x) = \bigcup\{\mathfrak{nat}(x') : x' \succeq x\} = \{[x']_{\equiv} : x' \succeq x\} =\uparrow_{\sqsubseteq} [x]_{\equiv}$. Henceforth, $O \in \Omega_{\sqsubseteq}(X/_{\equiv})$.   QED

Now, following the pointless topology approach, let us define a map from concrete points to abstract points, that is a map $\psi\colon G \longmapsto \mathcal{J}(\Omega(X))$ $: \psi(x) =\uparrow_{\preceq} (x)$. But we know that ψ^{-1} is the partition modulo the kernel of ψ, which is given as the pull-back of ψ along itself.

Proposition 4.3.4. *The functional relation corresponding to the kernel of ψ coincides with $\widehat{\mathfrak{nat}}$, i.e. the functional relation corresponding to the canonical map $\mathfrak{nat}$.*

Proof. We have just seen that $\uparrow_{\preceq} (x) = \widehat{\mathfrak{nat}}^{\smile} (\uparrow_{\sqsubseteq} [x]_{\equiv})$. It follows that $\hat{\psi}^{\smile}(\uparrow_{\preceq} (x)) = [x]_{\equiv}$, that is, $\hat{\psi} \otimes \hat{\psi}^{\smile} = \hat{\kappa}$ (pay attention that $\hat{\psi} \otimes \hat{\psi}^{\smile}$ means "$\hat{\psi}^{\smile}$ after $\hat{\psi}$").   QED

It follows that ψ^{-1} is a homeomorphism between $\langle\mathcal{J}(\Omega(X)),$ $\Omega_{\preceq}(\mathcal{J}(\Omega(X)))\rangle$ and $\langle X/_{\equiv}, \Omega_{\sqsubseteq}(X/_{\equiv})\rangle$, while $\mathfrak{nat}$ is an isomorphism between $\Omega(X)$ and the former two structures.

Moreover, we know that the extension of ψ to $\wp(X)$, ϕ, defined by the equation $\phi(O) = \{P \in \mathcal{J}(\Omega(X)) : O \preceq P\}$, is an isomorphism between $\Omega(X)$ and $\Omega_{\preceq}(\mathcal{J}(\Omega(X)))$ (indeed it is a 1-1 identification map). Hence it is an isomorphism between $\Omega(X)$ and $\Omega_{\sqsubseteq}(X/_{\equiv})$, too.

However, if for two distinct elements O and O', $\phi(O) = \phi(O')$, then $\Omega_{\preceq}(\mathcal{J}(\Omega(X)))$ and $\Omega_{\sqsubseteq}(X/_{\equiv})$ are not homeomorphic with $\Omega(X)$, because the elements that are "duplicated" in $\Omega(X)$ collapse into a single element in the other spaces. This means that we can identify elements that always fulfill the same properties either by means of the T_0-ification technique or the soberification technique, and obtain the same result (we also recall that this is not always true in the infinite case).

$$
\begin{array}{ccc}
 & \Omega(X) & \\
{}^{iso}\nearrow & & \nwarrow{}^{iso} \\
\Omega_{\sqsubseteq}(X/_{\equiv}) & \xleftrightarrow[homeo]{} & \Omega_{\preceq}(\mathcal{J}(\Omega(X)))
\end{array}
$$

For an example see Frame 4.5.

How to interpret this journey from a phenomenological point of view?

"Points" are perceived just through observations of their properties. We know the properties, not the objects. This is what is meant by saying that objects (points) are just "bundles of properties". The equation $\psi(x) = \psi(x')$ or, equivalently, $\mathfrak{nat}(x) = \mathfrak{nat}(x')$ means, indeed, that x and x' "come always together": they manifest the same properties. This equation depends, of course, on the collected information. In other terms it depends on the considered "slice" of the evolving observation process. In fact, if we stop our observations at some point in time t and collect our information, then points do not have any longer the possibility to manifest any eventual difference by means of finer properties. If under the finest observations we have collected, x and x' cannot be distinguished, then both of them are put into the same equivalence class.

4.4   Frame – Observable Properties

When we speak of an "observable property" we must take precautions because we have to define, first, what "observable" means.

Intuitively, a property p is observable of an object g if some perceptive subject s is able to *decide* if it is the case that $g \vDash p$. Thus, the problem is now about the meaning of the verb "to decide". Of course, decision procedures in human beings and decision procedures in mechanical artifacts like computers are not the same (indeed they can seriously diverge). However, since we are trying to define a *formal* framework for some observation logic and, moreover, since we are not able, for the time being, to describe human decision procedures with a sufficiently acceptable precision degree, we shall assume that a property is observable if it is decidable from the point of view of Computability Theory.

4.4.1   Decidable and Semi-Decidable Properties

In view of the previous assumption we can say that property P is decidable, if there is an effective procedure which is able to affirm that $g \vDash P$ holds, for any object g of the domain of discourse. Here we assume that "effective" has the meaning defined by Church's Thesis. Hence it means "Turing computable", "Post computable", "computable

by means of recursive functions", "λ computable" and so on (see any book on Computability Theory, for instance, Manna [1974]). Roughly speaking, in this context "effective" means "computable by means of a mechanical procedure". If this is the case, we also say that $g \vDash P$ is a *solvable problem.*

Unfortunately, it is well-known that there are *unsolvable problems* and, even worst, that such problems are usually important. Here is a limited list of unsolvable problems:

1. Checking mechanically whether an arbitrary program will halt on a given input (the Halting Problem).

2. Printing out all and only the true statements of arithmetic (Gödel's Incompleteness Theorem).

3. Deciding whether a given sentence of first-order logic is valid or not (Church's Theorem).

In some more abstract terms we shall say that a (decision) problem A (over an alphabet Σ) is a subset $A \subseteq \Sigma^*$, where Σ^* is the set of all strings (words) of symbols in Σ:

Definition 4.4.1. *A decision problem A over Σ is decidable if its characteristic function χ_A is computable:*

$$\chi_A(x) = \left\{ \begin{array}{ll} 1 & \textit{if } x \in A \\ 0 & \textit{if } x \notin A \end{array} \right.$$

If a decision problem A is decidable than A is said to be a *recursive set.*

From Church's Theorem above, we know that the set of valid first-order logic sentences, $VFOL$, is not recursive, because there is no mechanical procedure which is able to say, for any first-order logic sentence s, if $s \in VFOL$ or $s \notin VFOL$. Note that $s \notin VFOL$ means, classically, $s \in -VFOL$, where "$-$" is the set theoretic complement with respect to the set FOL of all first-order logic sentences.

There are some subsets of $VFOL$ which are recursive (that is, decidable) such as, for instance, the set of Horn clauses (on which the program language Prolog relies). But the entire set $VFOL$ is not.

However, $VFOL$ fulfills a weaker but still interesting property: there is a mechanical procedure which decides if $s \in VFOL$, though. This procedure is not able to decide if $s \notin VFOL$. Otherwise stated, if s is

valid our procedure notifies it. But if s is not valid our procedure may notify it or run forever.

Thus membership in $VFOL$ is a *semi-decidable* problem and $VFOL$ is said to be *recursively enumerable*. It follows that a set is recursive if both itself and its complement are recursively enumerable.

Gödel's Incompleteness Theorem says that the set of true arithmetic statements is not even recursively enumerable.

In view of the concepts so far defined, we shall say that a (positive) property is "finitely observable" if it is semi-decidable (or, equivalently, if its domain of validity is recursively enumerable).

4.4.2   Decidable Properties, Topology, Domains and Geometric Logic

In a seminal paper M. B. Smyth (see Smyth [1978], but see also Smyth [1992]) faced the problem as how to characterise decidable properties. He introduced an ordering $\sqsubseteq$ on a domain D, which is considered as an information ordering, i.e., $x \sqsubseteq y$ if y contains more information than x. This ordering is, therefore, very close to what we have introduced as a "specialization preorder". A (semi) decidable property P is thought of as an *observable property*. If x has enough information for the property P to hold, i.e., $\chi_{[[P]]}(x) = 1$, and $x \sqsubseteq y$ then y also contains enough information for the property P to hold. If we think of y as a *neighbor* of x (from the point of view of a nearness relation based on the notion "to have more information than ..."), then this intuition leads to the fact that a semi-decidable property is, indeed, a kind of open set, because for any element in it, the set contains also its "neighbors".

As Vickers [1989] puts it:

"An observation must be made in finite time, after a finite amount of work.... A finite observation in itself is neutral in its logical content. It can be used positively, to affirm an assertion (most concretely, the assertion "this observation can me made"), or negatively, to refute an assertion ("this observation can never be made"). However, we shall tend to give the observations an implicit logical content by taking the positive viewpoint.... [Having established] in what circumstances [an] assertion can be affirmed or refuted, let us now ask when it is true or false.... The answer depends on how we classify all the borderline cases .... If we say that for all borderline cases the assertion is false, then "true" means "affirmably

true" ... this means that sets corresponding to truth are open. On the other hand, we may count all the borderline cases as true, so that "true" means "irrefutable" and the sets corresponding to truth are closed."

The logic of affirmative assertions (finite observations) admits some nice properties: arbitrary disjunctions, finite conjunctions, distribution of conjunction over arbitrary disjunctions and distribution of disjunctions over finite conjunctions (intuitively disjunctions may be arbitrary because we can stop the sequence as soon as we observe evidences to affirm a sentence; dually we cannot accept infinite conjunction, because this would imply an infinite amount of observations). Such a logic is known as *Propositional Geometric Logic* (see Frame 4.5).

To be sure, things are a little bit more complex when applied to program semantics. In fact we must embed this intuition into a framework in which also issues related to "finiteness" (compactness) and "approximation by means of finite elements" must be taken into account (for instance for computation purposes real numbers must be limits of their finite – or concrete – approximations). This approach leads to the technical notions of a "Domain" ("complete partial order – cpo", "algebraic cpo" and "powerdomain" – see Abramsky & Jung [1994], Scott [1982] and Vickers [1989]).

4.5   Frame – Finite Observations: The Binary Machine Example

Let us recall three fundamental assumptions we made about observation processes:

- any substance (or "entity", "object" or "point") is a "bundle" of finite observable properties;

- observation is a dynamic activity evolving over time;

- any Information System is a *section* of the results of this dynamic activity at a certain point in time.

Considered from a dynamic point of view, observations have a peculiar geometry that was discussed in Section 5.1 of Introduction which corresponds to a peculiar logic.

Given a *P-system* **P**, a *positive observation* is any formula $g \Vdash p$.
Let Ω be the collection of the positive observations induced by **P**.
1. *any disjunction of positive observations is still a positive observation*:
every subset X of Ω has a join (or *sup*), $\bigvee X$
2. *any finite conjunctions of positive observations is still a positive observation*:
every finite subset X of Ω has a meet (or *inf*), $\bigwedge X$
3. *to affirm O and at least one of the $O_i \in Y$, we must affirm at least O and one of the O_i* (*frame distributivity*): binary meets distribute over joins: $O \wedge (\bigvee Y) = \bigvee \{O \wedge O_i : O_i \in Y\}$.
A structure $\langle \Omega, \wedge, \vee \rangle$ with the above properties is called a *frame*.
If, moreover, we define:
$a \Longrightarrow b = \bigvee \{x : x \wedge a \leq b\}$ and $\neg a = a \Longrightarrow 0$, where $0 = \bigvee \emptyset$,
then a frame becomes a *complete Heyting algebra* (see Part II).
Frames and complete Heyting algebras are the same objects; differences arise when we consider morphisms (transformations) between objects, since in the case of frames, morphisms are not required to preserve $\Longrightarrow$.

Window 4.1. **The logic of finite positive observations**

The logic modelled by frames is called "Geometric Logic" or "Observation Logic" (see Frame 4.4.2).

An elementary example of frame is given by the following binary-tree observation: let G be a set of machines whose behaviours are checked at some intervals of time. Any machine may generate, asynchronously, a binary string by concatenating a 0 or a 1 to its preceding output. The set of all possible binary strings is denoted by M. If a machine m outputs a string s, we write $m \Vdash s$.

Let us stop the observations of the system at point in time t and record all the observed output relations $\Vdash$ between machines in G and strings in M. This means that the evolution of $\Vdash$ for each machine is recorded and we obtain a *P-system* $\langle G, M, \Vdash \rangle$. Let us denote the substring relation with the symbol $\ll$ (hence $01 \ll 011$ and $01 \ll 010$, and so on). In this *P-system* the set of possible future stories of a machine x such that at point in time t, $x \Vdash s$ is given by the principal order filter $\uparrow_{\ll} s$ which is represented by the output s originating this order filter. Otherwise stated, the possible future evolutions of an observation $x \Vdash s$ (that is, any observation of the form $x \Vdash s'$ for $s \ll s'$) are represented

by their potential $x \Vdash s$ (because if s is 0110, then it represents, at point in time t, also 01100, 01101, 011001, 011000, etc.). This means that if at point in t the last observation for a subset G' of machines is s (for instance 0110), then we can momentarily equalize all the members of G'. In fact our best evidence is that all those machines output string s, so that at point in time t, they can be considered to have the same behavior, regardless of their future development. Obviously, at point in time $t+1$ some of the machines in G' could output s' and some others s'' such that $s' \neq s''$ (for instance $s' = 01101$ and $s'' = 01100$). However they cannot be distinguished at point in time t: we have no information for performing this distinction, yet.

However, we must distinguish a machine m whose output at point in time t is string s from a machine m' whose output at t is a string $s' \ll s$. Indeed, the latter's next output could be a string s'' incompatible with s, or it could definitely stop or loop at position s'. In both cases, we cannot legitimately equalize the behaviours of m and m' at time t.

Now we shall apply the approach suggested by Pointless Topology.

Consider the set of machines $G = \{m1, m2, m3, m4, m5\}$ and the set of observations $M = \{\perp, \langle 0 \rangle, \langle 1 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle, ...\}$. Imagine that we make our observations in the interval of time $[t_0, t]$ collecting the following output relations ($\perp$ denotes the void string):
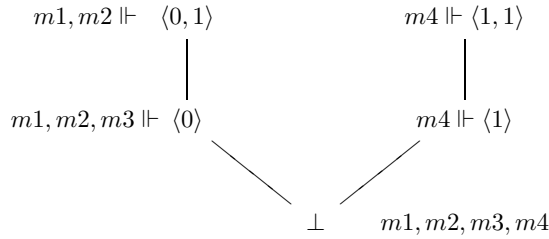
We obtain the following *P-system* $\langle G, M, \Vdash \rangle$:

$\Vdash$	$\langle 0 \rangle$	$\langle 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 1, 1 \rangle$
$m1$	1	0	1	0
$m2$	1	0	1	0
$m3$	1	0	0	0
$m4$	0	1	0	1

Therefore the set $\mathcal{SUB}(M)$ is given by:

$sub(\perp) = \{m1, m2, m3, m4\}$, $sub(\langle 0 \rangle) = \{m1, m2, m3\}$, $sub(\langle 1 \rangle) = \{m4\}$, $sub(\langle 0, 1 \rangle) = \{m1, m2\}$, $sub(\langle 1, 1 \rangle) = \{m4\}$.

So, let us define on G a topological space $\langle G, \Omega_{\mathcal{SUB}}(G)\rangle$ by tacking $\mathcal{SUB}(M)$ as a sub-base (or by tacking $\{\emptyset\} \cup \mathcal{SUB}(M)$ as a base). $\Omega_{\mathcal{SUB}}(G)$ is the following distributive lattice:

$$\{m1, m2, m3, m4\}$$

$$\{m1, m2, m3\} \qquad\qquad \{m1, m2, m4\}$$

$$\{m1, m2\} \qquad\qquad \{m4\}$$

$$\emptyset$$
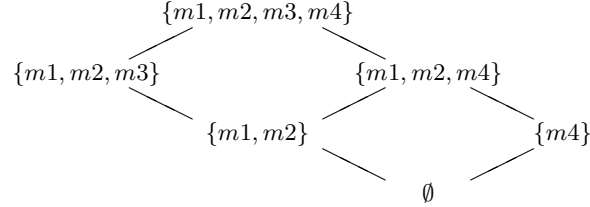
The element $\{m1, m2\}$ corresponds to the property $\bot \wedge \langle 0 \rangle \wedge \langle 0, 1 \rangle$, $\{m4\}$ corresponds to $\bot \wedge \langle 1 \rangle \wedge \langle 1, 1 \rangle$, while $\{m1, m2, m4\}$ corresponds to $\bot \wedge ((\langle 0 \rangle \wedge \langle 0, 1 \rangle) \vee (\langle 1 \rangle \wedge \langle 1, 1 \rangle))$ and so on. Let us consider the set of co-prime (join-irreducible) elements of $\Omega_{\mathcal{SUB}}(G)$,

$$\mathcal{J}(\Omega_{\mathcal{SUB}}(G)) = \{\{m1, m2\}, \{m4\}, \{m1, m2, m3\}\}.$$

Indeed, given two elements s and s' of M, either they are incompatible, meaning that they have sub-strings of the same length that differ at least in one position, or one is a substring of the other. Clearly, if $s \ll s'$, then $sub(s') \subseteq sub(s)$, because in order to output s' a machine must output also substring s. Therefore, if $s \ll s'$, then $sub(s) \cup sub(s') = sub(s)$. If s and s' are incompatible $sub(s) \cup sub(s') \neq sub(s'')$ for any $s'' \in M$. It follows that for any $s \in M$, $sub(s)$ is a co-prime element. Since the other elements of $\Omega_{\mathcal{SUB}}(G)$ are unions of elements of $\mathcal{SUB}(M)$, the latter is the set of all co-prime elements of the distributive lattice $\Omega_{\mathcal{SUB}}(G)$.

Exercise 4.1. *Show the* P-system *which results by recording just the most updated output for each machine. Show that in this case* $\mathcal{J}(\Omega_{\mathcal{SUB}}(G))$ *is a partition.*

Let us now define on G the specialization preorder $\preceq$ induced by $\Omega_{\mathcal{SUB}}(G)$. Since in order to compute the specialization preorder co-prime open sets are sufficient, we shall have for any $x, x' \in G$, $x \preceq x'$ iff for all $s \in M$, if $x \Vdash s$ then $x' \Vdash s$, iff for all $s \in M$, if $x \Vdash s$ then there exists $s' \gg s$ such that $x' \Vdash s'$), iff for all $s \in M$, if $x \in sub(s)$ then $x' \in sub(s)$.

Here are some examples: $m1 \not\preceq m3$ and $m2 \not\preceq m3$, because $m1, m2 \in \{m1, m2, m3\}$ but $m3 \notin \{m1, m2\}$; on the contrary, $m3 \preceq m1$ and $m3 \preceq m2$.

The specialization preorder $\langle G, \preceq \rangle$, the poset of co-prime elements $\langle \mathcal{J}(\Omega_{\mathcal{SUB}}(G)), \preceq \rangle$, function ψ and the partial order $\langle G/_{\equiv}, \sqsubseteq \rangle$ are depicted in the following diagrams:

$$\langle G, \preceq \rangle$$

$$m1, m2$$

$$m3 \quad m4$$

$$\langle G/_{\equiv}, \sqsubseteq \rangle \qquad\qquad \langle \mathcal{J}(\Omega_{\mathcal{SUB}}(G)), \preceq \rangle \qquad\qquad G$$

$$\text{nat}$$

$$[m1]_{\equiv} \xleftarrow{\ \psi^{-1}\ } \{m1, m2\} \xleftarrow{\ \psi\ } m1$$

$$\xleftarrow{\ \psi\ } m2$$

$$[m3]_{\equiv} \qquad\qquad [m4]_{\equiv} \qquad\qquad\qquad m3$$

$$\psi^{-1} \qquad\qquad \psi^{-1} \qquad\qquad \psi$$

$$\{m1, m2, m3\} \xleftarrow{} \{m4\} \xleftarrow{\ \psi\ } m4$$
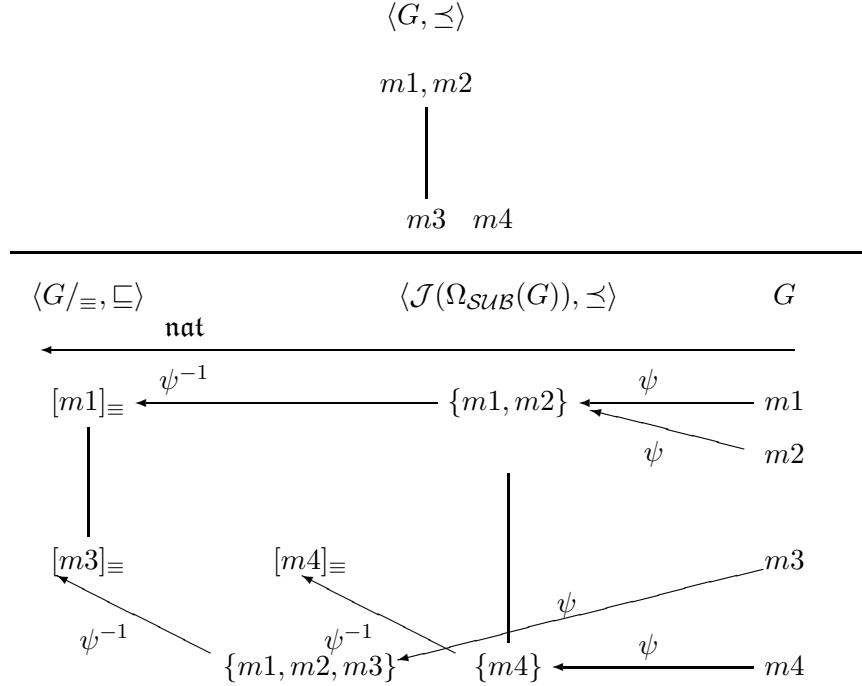
Indeed, machines $m1$ and $m2$ always output the same string. In other terms their behaviours are indistinguishable at point in time t, that is the time we stopped the observations. Thus they must be equalized and in fact they collapse into the same abstract point $\{1, 2\}$ via function ψ or via function $\text{nat} = \psi \circ \psi^{-1}$.

Clearly $\langle G, \Omega_{\mathcal{SUB}}(G) \rangle$ and $\langle \mathcal{J}(\Omega_{\mathcal{SUB}}(G)), \Omega_{\preceq}(\mathcal{J}(\Omega_{\mathcal{SUB}}(G))) \rangle$ (or $\langle G/_{\equiv}, \Omega_{\sqsubseteq}(G/_{\equiv}) \rangle$), fail to be homeomorphic because we have collected less outputs than machines and this happens exactly when at least two machines always output the same strings. Through the above techniques we can equalise these machines.

4.6   Frame – Quanta of Information

4.6.1   Quanta at a Location and Ortholattices

If we assume the "same-up-to-some-tolerated-difference" principle, then objects will be perceived to be connected by a tolerance, or coherence, relation, R, that is a symmetric and reflexive binary relation on U.

The resulting structure $\langle U, R \rangle$ is called a *Proximity Space*, an important mathematical object since it has strict connections with ortholattices and Quantum Logics (cf. Bell's quoted work and see later Frame 15.10 of Part 3).

Moreover the dual relation $-R$, called *orthogonality relation* determines the (better known) dual space $\langle U, -R \rangle$, called *Orthogonality Space*.

A *Proximity Space* $\mathbf{T}$ is a set X equipped with a *coherence relation* T (that is a symmetric and reflexive relation).

For any $x \in X$, the *quantum at location* x, QL_x, is defined by:

$QL_x = \{x' \in X : \langle x, x' \rangle \in T\} = T^{\smile}(x) = T(x) = \langle T^{\smile} \rangle(x) = \langle T \rangle(x) = \downarrow_T x = \uparrow_T x$.

$Part(X)$, the family of all unions of quanta, is called a *complete quantum assemblage*. It is possible to prove that $\mathbf{Sat}_{QL}(\mathbf{T}) = (Part(X), \wedge, \vee, {}^*, X, \emptyset)$

is a complete ortholattice when the operations are defined by:

$\bigvee \{U_i\}_{i \in I} = \bigcup \{U_i\}_{i \in I}$;

$\bigwedge \{U_i\}_{i \in I} = \bigcup \{QL_x : QL_x \subseteq \bigcap \{U_i\}_{i \in I}\}$;

$U^* = \bigcup \{QL_x : x \notin U\}$.

Window 4.2. **Proximity Spaces and Complete Quantum Assemblages**

A complete quantum assemblage of a Proximity Space is a complete ortholattice and any complete ortholattice is isomorphic to a complete quantum assemblage of a Proximity Space.

Consider the following proximity space $\mathbf{T}$ (clearly a Proximity Space $\mathbf{T} = \langle X, T \rangle$ is a *P-system* $\langle X, X, T \rangle$):

T	a	b	c	d
a	1	1	1	0
b	1	1	0	0
c	1	0	1	1
d	0	0	1	1

It induces the following quantum assemblage $\mathbf{Sat}_{QL}(\mathbf{T})$ that we compare with $\mathbf{Sat}_Q(\mathbf{T})$:
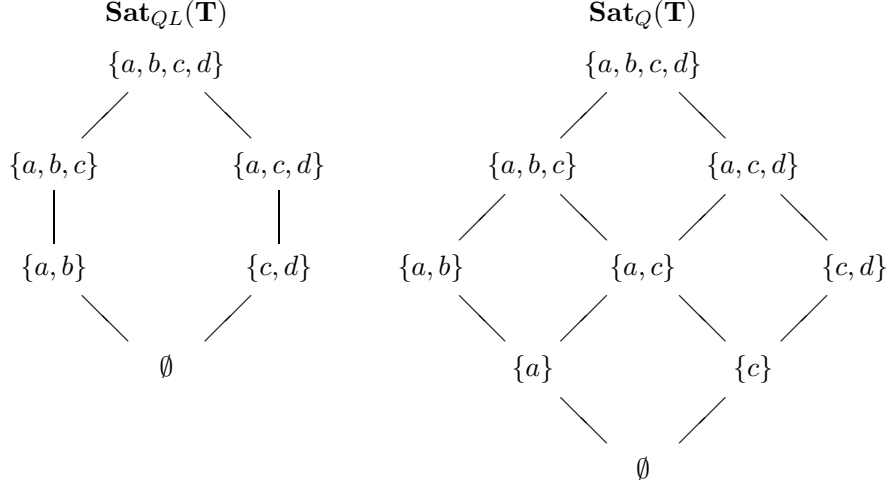
In $\Omega_{QL}(\mathbf{T})$, $\{a,b,c\} \wedge \{a,c,d\}$ is computed as follows: $\{a,b,c\} \cap \{a,c,d\} = \{a,c\}$ but for all $X \in \Omega_{QL}(\mathbf{T})$, $X \nsubseteq \{a,c\}$. Hence $\{a,b,c\} \wedge \{a,c,d\} = \emptyset$.

As to orthogonality, since $\{a,b\}^* = QL_c \cup QL_d = \{a,c,d\} \cup \{c,d\} = \{a,c,d\}$ and $\{a,c,d\}^* = QL_b = \{a,b\}$.

There is no way to recover quantum assemblages from quanta of information. In fact by means of the induced IQRS, $\mathbf{Q}(\mathbf{T})$, quanta of information can just reproduce the behaviour of $-\mathbf{T}$, or direct or inverse images of $R_{\mathbf{T}}$ (by means of the equivalences of *Proposition* 3.1.3). But they cannot recover all and only the direct (or, equivalently, inverse) images of T, because T is not a preorder. For instance, in $\mathbf{Sat}_Q(\mathbf{T})$ we have $\{a,b\} = Q_b^{\mathbf{T}}$ but $Q_c^{\mathbf{T}} = \{c\}$ is not an element of $\mathbf{Sat}_{QL}(\mathbf{T})$. Moreover, $\mathbf{Sat}_Q(\mathbf{Q}(\mathbf{T}))$ is isomorphic to $\mathbf{Sat}_Q(\mathbf{T})$ for T is a tolerance relation.

On the contrary, if we are given an Information System $\mathbf{S}$, $R_{\mathbf{S}}$ is a preorder so that in view of *Proposition* 3.1.3 we have $Q_x^{\mathbf{Q}(\mathbf{S})} = R_{\mathbf{S}}^{\smile}(x) = QL_x$. Moreover, if $\mathbf{S}$ is an *A-system* (or it is dichotomic or functional) then $R_{\mathbf{S}}$ is also symmetric, so it is a tolerance relation too. It follows that $\Omega_{QL}(\mathbf{S}) = \Omega_Q(\mathbf{S})$, so that $\mathbf{Sat}_{QL}(\mathbf{S})$ is a distributive ortholattice, hence a Boolean algebra, because all distributive ortholattices are Boolean algebras (the additional Boolean features are indeed given by transitivity of $R_{\mathbf{S}}$). But from *Corollary* 3.1.4, $\Omega_Q(\mathbf{S}) = \Omega_Q(\mathbf{Q}(\mathbf{S}))$. So we have proved *Proposition* 3.3.6 in another way.

In the case of *P-systems*, since symmetry is not granted, Ω_Q obtains distributive lattices but not ortholattices, hence not Boolean algebras, in general.

Notice that any transitive Proximity Space induces a Boolean algebra, but the converse implication is not valid. Indeed, we can have non-transitive Proximity Spaces such that the induced quantum assemblage is a Boolean algebra. Here an example:

R	1	2	3	4
1	1	1	1	0
2	1	1	0	0
3	1	0	1	0
4	0	0	0	1

The family of quanta at a point is $\{QL_1 = \{1,2,3\}, QL_2 = \{1,2\}, QL_3 = \{1,3\}, QL_4 = \{4\}\}$. It is easy to see that the resulting quantum assemblage coincides with the Boolean algebra whose atoms are QL_2, QL_3 and QL_4. It is interesting to notice that this happens because $QL_1 = QL_2 \cup QL_3$. If, on the contrary, this coincidence fails, we no longer obtain a Boolean algebra. For instance if in the above *P-system* in addition we had $\langle 3, 4 \rangle \in R$, then $QL_3 = \{1,3,4\}$ so that the element QL_1 would be neither an atom nor join-accessible from atoms.

4.6.2 A Topo-Algebraic Reading of IQRSs

We have already proved that for any Information System $\mathbf{S}$ over a set G, $\Omega_Q(\mathbf{S})$ can be made into a lattice of open subsets of G. Now we shall give a brief deeper insight into this perspective.

Any IQRS is basically a preorder $\mathbf{O}$. We can then consider the family of order filters $\mathscr{F}(\mathbf{O})$. In view of the results collected so far, it is immediate to verify that if $\mathbf{O} = \mathbf{Q}(\mathbf{S})$ then $\mathscr{F}(\mathbf{O}) = \{R(X) : X \subseteq G\} = \Omega_Q(\mathbf{S})$. Indeed this family determines the so-called *Alexandrov topology* over $\mathbf{O}$, whose lattice of open subsets is, thus, $\mathbf{Sat}_Q(\mathbf{O}^{\smile})$.

Moreover, $\mathscr{F}(\mathbf{O}) \subseteq \wp(G)$. So let $in : \mathscr{F}(\mathbf{O}) \longmapsto \wp(G)$ be the inclusion map. This inclusion function is a lattice homomorphism between $(\mathscr{F}(\mathbf{O}), \subseteq)$ and $(\wp(G), \subseteq)$. Then it is both a $\vee$-preserving and a $\wedge$-preserving map, so that in view of *Lemma* 1.4.3 we can ask what its upper and lower adjoints are.

Lemma 4.6.1. *Let $\mathbf{O} = \langle G, R \rangle$ be a preordered set. Let $\mathscr{F}(\mathbf{O})$ be the family of its order filters and $in : \mathscr{F} \longmapsto \wp(G)$ an inclusion function.*

Then the upper adjoint "in$^+$" of the inclusion map "in", coincides with the interior operator $\mathbb{I}$ induced by $\mathscr{F}(\mathbf{O})$ qua Alexandrov topology on $\mathbf{O}$.

Proof. From *Proposition* 1.4.7.(3), for any $X \in \wp(G)$, $in^+(X) = \bigvee(in^{-1}(\downarrow_\subseteq X)) = \bigcup\{Y \in \mathscr{F}(\mathbf{O}) : in(Y) \subseteq X\}$. Since $in(Y) = Y$, $in^+(X)$ is the greatest element of $\mathscr{F}(\mathbf{O})$ contained in X.   QED

Lemma 4.6.2. *For any set $X \subseteq G$ and any $g \in G$, $g \in [R](X)$, iff $g \in \mathbb{I}(X)$.*

Proof. For any $X \subseteq G$, $g \in \mathbb{I}(X)$ if and only if $g \in \bigcup\{Y \in \mathscr{F}(\mathbf{O}) : Y \subseteq X\}$ if and only if $g \in \{z \in G : \forall z' \in G(\langle z, z'\rangle \in R \Rightarrow z' \in X)\} = [R](X)$.   QED

Lemma 4.6.3. *The lower adjoint "in$^-$" of the inclusion map "in", coincides with the closure operator $\mathbb{C}^{op}$ induced by $\mathscr{F}(\mathbf{O}^\smile)$ qua Alexandrov topology on $\mathbf{O}^\smile$.*

Proof. $in^-(X) = \bigwedge(in^{-1}(\uparrow_\subseteq X)) = \bigcap\{Y \in \mathscr{F}(\mathbf{O}) : Y \supseteq X\} = \{z \in G : \exists z'(\langle z, z'\rangle \in R^\smile \ \& \ z' \in X)\} = \mathbb{C}^{op}(X)$.   QED

Corollary 4.6.1. *For any set $X \subseteq G$, for any $g \in G$, $g \in \langle R^\smile\rangle(X)$ if and only if $g \in \mathbb{C}^{op}(X)$.*

Proof. Indeed $\mathbb{C}^{op}(X) = R(X)$, any X.   QED

This reasoning can be dualized by starting from the family of order ideals $\mathscr{I}(\mathbf{O}) = \{R^\smile(X) : X \subseteq G\}$, thus obtaining the operators $\langle R\rangle$ and $[R^\smile]$. Notice that $\mathbf{O}^\smile = \mathbf{Q}(\mathbf{O})$ so that $\langle R^\smile\rangle = Q_{(\ldots)}^{\mathbf{Q}(\mathbf{O})}$ and $\langle R\rangle = Q_{(\ldots)}^{\mathbf{O}}$. Therefore, if $\mathbf{O}$ is the IQRS induced by $\mathbf{S}$, $\mathbf{Q}(\mathbf{S})$, then $Q_X^{\mathbf{S}} = Q_X^{\mathbf{Q}(\mathbf{Q}(\mathbf{S}))} = Q_X^{\mathbf{Q}(\mathbf{O})} = \langle R^\smile\rangle(X) = R(X) = \langle R_{\widetilde{\mathbf{S}}}^\smile\rangle(X) = R_{\mathbf{S}}(X)$.

Finally, since *in* is a lattice homomorphism between $(\mathscr{F}(\mathbf{O}), \subseteq)$ and $(\wp(O), \subseteq)$, we can recover the adjointness properties of these operators:

$$(1) \ \langle R^\smile\rangle \dashv [R]; \qquad (2) \ \langle R\rangle \dashv [R^\smile].$$

TERMINOLOGY AND NOTATION. Given an ordered set $\mathbf{O}$, in general we shall denote the structure $(\mathscr{F}(\mathbf{O}), \subseteq)$ with $\mathbf{F}(\mathbf{O})$.

4.6.3   Duality Between P-Systems and Preorders

A more conceptual proof of *Proposition* 3.3.4 Let $\mathscr{F}(\mathbf{O})$ be the set of order filters of $\mathbf{O}$. Thus $\mathscr{F}(\mathbf{O}) = \Omega_Q(\mathbf{Q}(\mathbf{O}))$ (i.e. $\Omega_Q(\mathbf{O}^\smile)$), so

that we know that $\mathscr{F}(\mathbf{O})$ can be made into the distributive lattice $\mathbf{Sat}_Q(\mathbf{Q}(\mathbf{O}))$. Then let $\mathcal{J}(\mathbf{Sat}_Q(\mathbf{Q}(\mathbf{O})))$ be the set of co-prime elements of $\mathbf{Sat}_Q(\mathbf{Q}(\mathbf{O}))$. We have seen in the Introduction that co-prime elements have the form $\uparrow_R x$, i.e. $R(x)$, for some element $x \in G$ and that they may be understood as properties fulfilled by those elements g of G such that $g \in R(x)$. Thus, let us set $g \Vdash x$ if and only if $g \in\uparrow x$. Then we can define an Information System $\mathbf{I}(\mathbf{O})$ as $\langle G, \mathcal{J}(\mathbf{Sat}_Q(\mathbf{Q}(\mathbf{O}))), \Vdash\rangle$. Thus, $\langle g, g'\rangle \in R_{\mathbf{I}(\mathbf{O})}$ iff $g' \in Q_g^{\mathbf{I}(\mathbf{O})}$, iff $\Vdash (g) \subseteq \Vdash (g')$, iff $g \in R(x) \Rightarrow g' \in R(x)$ for all $R(x) \in \mathcal{J}(\mathbf{Sat}_Q(\mathbf{Q}(\mathbf{O})))$. In particular, since R is reflexive, $g \in R(g)$ so that $g' \in R(g)$ holds, i.e. $\langle g, g'\rangle \in R$. Conversely, if $\langle g, g'\rangle \in R$ and $\langle x, g\rangle \in R$, for transitivity $\langle x, g'\rangle \in R$ too. It follows that $g \in R(x) \Rightarrow g' \in R(x)$, all $x \in G$.

Consider the following preorder $\mathbf{O} = \langle G, R\rangle$:

R	1	2	3	4	5	$R_{\mathbf{O}}$	1	2	3	4	5
1	1	1	1	1	1	1	1	0	0	0	0
2	0	1	0	1	1	2	1	1	0	0	0
3	0	0	1	0	0	3	1	0	1	0	0
4	0	0	0	1	1	4	1	1	0	1	1
5	0	0	0	1	1	5	1	1	0	1	1

It is immediate to verify that $R_{\mathbf{O}} = R^{\smile}$ so that the family $\mathscr{F}(\mathbf{O}) = \{R(X) : X \subseteq G\}$ coincides with $\Omega_Q(\mathbf{Q}(\mathbf{O}))$ (that is, $\Omega_Q(\mathbf{O}^{\smile})$), which in turn, once equipped with the set-theoretical operations, gives the distributive lattice depicted on the left with set of co-prime elements depicted on the right:
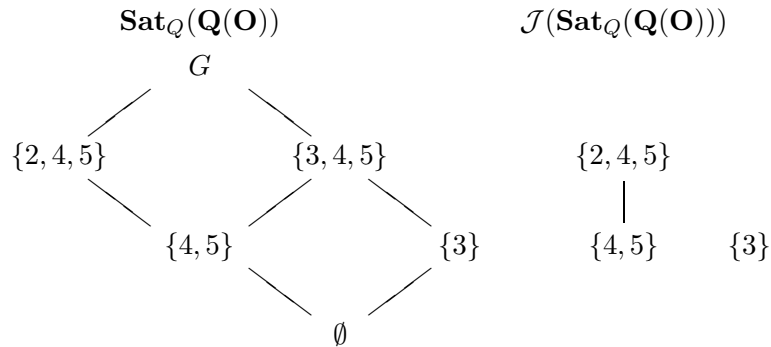
Now we verify that the required Information System $\mathbf{I}(\mathbf{O})$ is $\langle G, \mathcal{J}(\mathbf{Sat}_Q(\mathbf{Q}(\mathbf{O}))), \Vdash\rangle$ where $g \Vdash X$ if and only if $g \in X$, all $g \in G$ and $X \in \mathcal{J}(\mathbf{Sat}_Q(\mathbf{Q}(\mathbf{O})))$.

⊩	{3}	{4,5}	{2,4,5}
1	0	0	0
2	0	0	1
3	1	0	0
4	0	1	1
5	0	1	1

One can easily verify that for all $X \subseteq G, Q_X^{\mathbf{I(O)}} = Q_X^{\mathbf{O}}$.

Hence $\mathbf{Q(I(O))} \equiv_I \mathbf{O}$.

Dually, let $\mathbf{S}$ be the above Information System $\mathbf{I(O)}$. Since $R_{\mathbf{S}} = R$ we have that $\mathbf{Q(S)} \cong_I \mathbf{O}$ and $\mathbf{I(Q(S))} \cong_I \mathbf{I(O)} = \mathbf{S}$.

4.7 Frame – Information Systems

Information Systems as related to Rough Set Theory were introduced by Z. Pawlak in Pawlak [1981]. Immediately, a large amount of researches and studies followed together with the applications of Rough Set Theory in a number of fields among which we mention:

- Decision support systems

- Knowledge discovery in databases

- Pattern recognition

- Mereology

It is practically impossible to cite even the major contributions to the theory, so we shall just sketch some developments towards generalisations.

4.7.1 Generalising Information Relations

As we know from *Definition* 1.2.4, we can consider *A-systems* $\langle G, \{V_a\}_{a \in At}, At \rangle$ in which for each $a \in At$, $a : G \longmapsto \wp(V_a)$. That is, an object g can be associated with more than one value for any attribute a. We recall that this kind of *A-systems* is called *non-deterministic*. Non-deterministic *A-systems* have been originally studied both by Z. Pawlak and, especially, by E. Orłowska.

On this basis, a number of *information relations* among objects have been defined, given $A \subseteq At$, in Konrad et al. [1981], Orłowska [1985] and Vakarelov [1991]:

	Indistiguishability Relations
Strong indiscernibility	$\langle x, y \rangle \in ind_A$ iff $a(x) = a(y)$, for all $a \in A$.
Strong similarity	$\langle x, y \rangle \in sim_A$ iff $a(x) \cap a(y) \neq \emptyset$, for all $a \in A$.
Strong forward inclusion	$\langle x, y \rangle \in fin_A$ iff $a(x) \subseteq a(y)$, for all $a \in A$.
Strong backward inclusion	$\langle x, y \rangle \in bin_A$ iff $a(x) \supseteq a(y)$, for all $a \in A$.
Strong negative similarity	$\langle x, y \rangle \in nim_A$ iff $-a(x) \cap -a(y) \neq \emptyset$, for all $a \in A$.
Strong incomplementarity	$\langle x, y \rangle \in icom_A$ iff $a(x) \neq -a(y)$, for all $a \in A$.

Moreover, we have the weak version of the above notions when the defining properties are required for some $a \in A$ instead of all $a \in A$. Therefore we have weak indiscernibility, $wind_A$, weak similarity, $wsim_A$, weak forward inclusion $wfin_A$, weak backward inclusion, $wbin_A$, weak negative similarity, $wnim$, weak incomplementarity $wicom_A$.

Moreover, by taking the complement of either $a(x)$ or $a(y)$ in the above definitions, one obtains *distinguishability relations*. So, for instance, the relation *lort* defined by $\langle x, y \rangle \in lort_A$ iff $-a(x) \subseteq a(y)$, for all $a \in A$, is called *strong left orthogonality*, the relation $wrnim : \langle x, y \rangle \in wrnim_A$ iff $a(x) \cap -a(y) \neq \emptyset$ is called *weak right negative similarity* and so on (for details see Orłowska [1989] and Demri & Orłowska [2002]).

4.7.2 Generalizing Indiscernibility Relation

Other early generalisations suggested weakening the properties of an equivalence relation, thus using tolerance relations (by dropping transitivity) or even generic relations (see Frame 4.10).

A further generalisation is obtained by exploiting Fuzzy Set Theory. In Fuzzy Set Theory instead of a classical characteristic function $\chi_X(x) : X \longmapsto \{0, 1\}$, which evaluates the positive or negative membership of x in a set X, we have a membership function $\mu_F(x) : X \longmapsto L$, where L is an ordered set of values (possibly a complete lattice and usually the real interval $[0, 1]$). This function gives the degree of membership of x in X.

In Dubois & Prade [1992] two approaches are suggested to get a fuzzy extension of the upper and lower approximations. First, instead of a classical set one can approximate a fuzzy set by means of an equivalence relation E on G. To define this notion, given any element X_i of

G/E we use a map $\omega(X_i) = \{y : \langle x, y \rangle \in X_i\}$ – this way we distinguish between X_i as a member of the quotient set G/E and $\omega(X_i)$ as a subset of G.

Then, the upper and lower approximation of a fuzzy subset F of G is defined by, respectively:

$\mu_{(uE)(F)}(X_i) = sup\{\mu_F(x) : \omega(X_i) = [x]_E\}$ and $\mu_{(lE)(F)}(X_i) = inf\{\mu_F(x) : \omega(X_i) = [x]_E\}$, where $\mu_{(uE)(F)}(X_i)$, resp. $\mu_{(lE)(F)}(X_i)$, is the degree of membership of X_i in $(uE)(F)$, resp. in $(lE)(F)$.

In the second approach one defines a fuzzy similarity relation on G. A fuzzy similarity relation R is a fuzzy set on $G \times G$. The usual properties of a fuzzy similarity relation R are: (i) reflexivity ($\mu_R(x, x) = 1$), (ii) symmetry ($\mu_R(x, y) = \mu_R(y, x)$), and (iii) $*$-transitivity ($\mu_R(x, z) \geq \mu_R(x, y) * \mu_R(y, z)$, where $*$ is any operation satisfying $a * b \leq min(a, b)$ – a sort of logical "and").

Using a fuzzy similarity relation, R, the fuzzy equivalence class $[x]_R$ for objects close to x can be defined as: $\mu_{[x]_R} = \mu_R(x, y)$, that is, the extent to which two elements x and y are similar.

Then, the fuzzy lower and upper approximations are defined as:

$\mu_{(lR)(F_i)}(X) = inf_{x \in G}(max\{1 - \mu_{F_i}(x), \mu_X(x)\}), \forall i;$
$\mu_{(uR)(F_i)}(X) = sup_{x \in G}(min\{\mu_{F_i}(x), \mu_X(x)\}), \forall i,$

where F_i denotes a fuzzy equivalence class belonging to G/R.

In the same year the notion of a *fuzzy rough set* was independently introduced in Lin [1992] and in Hadjimichael & Wong [1993] where, further, some links are established between this notion and neighborhood systems (see Frame 4.10).

4.7.3   Generalising from Sets to Relations

A further generalisation is approximating relations instead of sets. This idea was introduced in Skowron & Stepaniuk [1993] and discussed in Pagliani [1996] from the point of view of Relation Algebra (see Frame 15.18.4 of Part III).

In the following quick presentation we shall not deal with the inclusion vagueness function ν and the structural function P, for which we refer to the quoted work.

Thus let $\{\mathbf{C}_i = \langle U_i, E_i \rangle\}_{1 \leq i \leq k}$ be a family of Approximation Systems.

By $\mathcal{R} = \mathbf{C}_1 \times, ..., \times \mathbf{C}_k = \langle U, E \rangle$ we denote an approximation space such that $U = U_1 \times, ..., \times U_k$, and, for $x = \langle x_1, ..., x_k \rangle \in U$, we set $E(x) = E_1(x_1) \times, ..., \times E_k(x_k)$. Then the following are the definitions for the lower and, respectively, upper approximations of a relation $R \subseteq U$:

Definition 4.7.1. *Let β be a real number within the range $0 \leq \beta < 0.5$. Let $card(X)$ denote the cardinality of the set X. Then,*

$$(i) \; \mathcal{L}(\mathcal{R}, R) = \{x \in U : \frac{card(R \cap E(x))}{card(E(x))} \geq 1 - \beta\};$$

$$(ii) \; \mathcal{U}(\mathcal{R}, R) = \{x \in U : \frac{card(R \cap E(x))}{card(E(x))} > \beta\}.$$

where $card(X)$ denotes the cardinality of the set X.

4.8 Frame – Dichotomic, Complementary and Functional Systems

4.8.1 Dichotomic Systems

First notice that the reverse of *Proposition* 3.1.2 does not hold. For instance, in a *P-system* $\mathbf{P}'$ such that $G = \{1, 2, 3, 4\}, M = \{A, B, C\}$ and $\Vdash (1) = \{A, B\}, \Vdash (2) = \{A, B\}, \Vdash (3) = \{B, C\}, \Vdash (4) = \{B, C\}$, Q_g is an equivalence class, any $g \in G$, though $\mathbf{P}'$ is neither dichotomic nor functional.

Let us now define the notion of a *complementary system* of a *P-system*:

Definition 4.8.1. *Let $\mathbf{P} = \langle G, M, \Vdash \rangle$ be a P-system. A system $\overline{\mathbf{P}} = \langle G, \overline{M}, \overline{\Vdash} \rangle$ is called a complementary system of $\mathbf{P}$ if:*

(i) $\overline{M} = \{\overline{m} : m \in M\}$,   (ii) $g \overline{\Vdash} \overline{m}$ *iff* $g \nVdash m$.

If we are given a *P-system* $\mathbf{P}$, then a quanta of information at location X with respect to a complementary system $\overline{\mathbf{P}}$ will be denoted by $\overline{Q_X}$ (i.e. $\overline{Q_X} = Q_X^{\overline{P}}$).

To be precise, we have to discharge any m such that $\Vdash^{\smile}(m) = G$ before forming $\overline{\mathbf{P}}$, otherwise we do not obtain a *P-system* because $\overline{m}$ would be such that $\overline{\Vdash}^{\smile}(\overline{m}) = \emptyset$ so that $\overline{\Vdash}$ would not be onto.

As we know, quanta of information in *P-systems* and in complementary systems are dual to each other, in the sense that $g \in Q_{g'}^P$ if and only if $g' \in Q_g^{\overline{P}}$.

Let us now introduce the notion of a sum of *P-systems* over the same domain:

Definition 4.8.2 (Sum of *P-systems*). *Let* $\mathbf{P} = \langle G, M, \Vdash \rangle$ *and* $\mathbf{P}' = \langle G, M', \Vdash' \rangle$ *be two P-systems over the same domain G. The* sum *of* $\mathbf{P}$ *and* $\mathbf{P}'$ *is defined as:* $\mathbf{P} \oplus \mathbf{P}' = \langle G, M \cup M', \Vdash \cup \Vdash' \rangle$.

Clearly, for any *P-system* $\mathcal{P} = \langle G, M, \Vdash \rangle$, the system $\mathcal{D}(\mathbf{P}) = \mathbf{P} \oplus \overline{\mathbf{P}}$ is dichotomic, so that in view of *Lemma* 3.1.1 $R_{\mathcal{D}(\mathbf{P})}$ is an equivalence relation.

But we can obtain the same result from the fact that if $g' \in Q_g$ then surely $g' \in Q_g \restriction M$ and $g' \in Q_g \restriction \overline{M}$, so that in view of *Proposition* 3.1.3 $g \in Q_{g'} \restriction M$ and $g \in Q_{g'} \restriction \overline{M}$ too. If we build the dichotomic system $\mathcal{D}(\mathbf{P})$ we obtain the same information as $\mathcal{N}(\mathbf{P})$. In fact the column of m coincides with that of m_1 while the column of $\overline{m}$ coincides with that of m_0, all $m \in P$. Hence $\mathcal{D}(\mathbf{P}) \cong_I \mathcal{N}(\mathbf{P})$.

We can readily verify that $\mathcal{D}(\mathcal{N}(\mathbf{A})) \cong_I \mathcal{N}(\mathbf{A}) \cong_I \mathbf{A}$, for any *A-system* $\mathbf{A}$.

4.8.2 Complementary Systems I

Any dichotomic *P-system* $\mathbf{P}$ has at least an i-equivalent system of the form $\mathcal{D}(\mathbf{P}')$ for some $\mathbf{P}'$. In fact if $\mathbf{P}'$ is a dichotomic *P-systems* then starting from any singleton $X_1 = \{m_1\}$ we can proceed in the following manner: $m \in X_{n+1}$ if for no $m_i \in X_n$ either $\Vdash^\smile (m_i) = \Vdash^\smile (m)$ or m_i is a complementary property of m. Let us set $X = X_n$ if $X_{n+1} = X_n$. Clearly $X \subseteq M$ so that we can define a relation $\Vdash'$ as the restriction of $\Vdash$ to X. Let us set $\mathbf{P}' = \langle G, X, \Vdash' \rangle$. Since $\mathbf{P}$ is dichotomic, $\overline{X} \subseteq M \cap -X$ so that $\mathcal{D}(\mathbf{P}')$ is a subsystem of $\mathbf{P}$ such that for any $p \notin X \cup \overline{X}$ there is m' either in X or in $\overline{X}$ such that $\Vdash^\smile (m) = \Vdash^\smile (m')$. Thus $\mathcal{D}(\mathbf{P}') \cong_I \mathbf{P}$.

4.8.3 Complementary Systems II

In general, if $\mathbf{A}$ is an *A-system*, then $\mathcal{N}(\mathbf{A})$ is not necessarily a dichotomic system. However $\mathcal{N}(\mathbf{A}) \cong_I \mathcal{N}(\mathcal{N}(\mathbf{A}))$ which is dichotomic (see *Corollary* 3.4.1). Notice that $\mathcal{N}(\mathcal{N}(\mathbf{A}))$ is informationally equivalent to the system defined as follows:

1. For each a_v in $\mathcal{N}(\mathbf{A})$, if V_a is not a singleton set $\neg a_v = \{a_{v'}\}_{v' \neq v}$, while if $V_a = \{v\}$ then set $\neg a_v = \{a_{v'}\}$. We set $P = \{a_v\}_{v \in V_a} \cup \{\neg a_v\}_{v \in V_a}$.

2. For each $g \in G$ set $g \Vdash^* \neg a_v$ if and only if $g \not\Vdash a_v$ and $g \Vdash^* a_v$ if and only if $g \Vdash a_v$.

   Clearly $\neg a_v$ is the complementary copy of a_v. Thus,

3. set $\mathbf{S} = \langle G, P, \Vdash^* \rangle$. By easy inspection we can verify that $\mathbf{S}$ is a dichotomic system and that $\mathbf{S} \cong_I \mathcal{N}(\mathbf{A})$.

Conversely, since given any *P-system* $\mathbf{P}$, $\mathcal{N}(\mathbf{P})$ induces an equivalence relation, we can ask whether it "is", in some form, an *A-system*. But we know that it is trivially an *A-system* where the set of attributes values is $V = \{0, 1\}$ and $m_1(g) = 1$ iff $g \Vdash m_1$ iff $g \not\Vdash m_0$ iff $m_0(g) = 0$, $m_1(g) = 0$ iff $g \Vdash m_0$ iff $g \not\Vdash m_1$ iff $m_0(g) = 1$.

We have just to show that $\langle G, \mathcal{N}(M), \{0, 1\} \rangle$ is informationally equivalent to $\mathcal{N}(\mathbf{P})$. But this can be verified by trivial inspection.

4.8.4   Complementary Systems III

Complementary systems reflect the "exactly-the-same" point of view advocated by classical Rough Set Theory. It is worth noticing that this point of view is based on an implicit "closed world assumption", stating that if an object g does not manifest a property p then it implicitly manifests the opposite property $\overline{p}$.

The close world assumption reflects, to some extent, the idea that the information system we are dealing with describes a complete state of affairs. Indeed, we have mathematically proved that adding the opposite properties makes any quantum of information Q_g into an equivalence class modulo "fulfilling exactly the same properties."

4.8.5   Dichotomic Systems and Functional Systems

We know that if $\mathbf{S}$ is a dichotomic system or a functional system then $R_{\mathbf{S}}$ is an equivalence relation (see *Proposition* 3.1.2). Thus a question naturally arises as how to define a functional system $F(\mathbf{S})$ informationally equivalent to $\mathbf{S}$. The answer is simple and general.

Consider any Information System $\mathbf{S}$. If $\mathbf{S}$ is a *P-system* consider it as an *A-system*. Any tuple $t \in \prod_{a \in At} V_a$ is a combination of attribute-values and has the form $\langle a_{1_m}, \ldots, a_{j_n} \rangle$. We set $g \Vdash^* t$ only if $a_i(g) = a_{i_m}$ for any $a_i \in At$ and $a_{i_m} \in t$. We say that the resulting system $\langle G, \prod_{a \in At} V_a, \Vdash^* \rangle$ is the required $F(\mathbf{S})$. Clearly $\Vdash^*$ is a map because no $g \in G$ can satisfy different tuples (otherwise g would take different

attributes values for some attribute). Thus $R_{F(\mathbf{S})}$ is an equivalence relation such that $\langle g, g' \rangle \in R_{F(\mathbf{S})}$ only if $a(g) = a(g')$ for all $a \in At$ (or in M). It follows that $\mathcal{N}(\mathbf{S}) \cong_I F(\mathbf{S})$ so that if $\mathbf{S}$ is dichotomic or it is an A-system then $R_{\mathbf{S}} = R_{F(\mathbf{S})}$ and $\mathbf{S} \cong_I F(\mathbf{S})$.

4.8.6 Functional Systems and Approximations

Consider the following FP-system $\mathbf{F}' = \langle \{1, 2, 3, 4\}, \{a, b\}, \Vdash \rangle$ and its kernel $\Vdash \otimes \Vdash^{\smile}$ (alias $\kappa_{\hat{|\Vdash}}$):

$\Vdash$	a	b		$\Vdash \otimes \Vdash^{\smile}$	1	2	3	4
1	1	0		1	1	0	1	0
2	0	1		2	0	1	0	1
3	1	0		3	1	0	1	0
4	0	1		4	0	1	0	1

First of all, notice, for instance, that $[e]\langle i\rangle(\{3\}) = [e](\{a\}) = \{1, 3\} = \langle e\rangle\langle i\rangle(\{3\})$. Indeed in FP-systems $[e] = \langle e\rangle$. Moreover, we can trivially verify that $R_{\mathbf{F}'} = \Vdash \otimes \Vdash^{\smile}$, which induces the equivalence classes: $\{1, 3\}$ and $\{2, 4\}$. Now let us compute some example of interior and closure: we have already seen that $cl(\{3\}) = \{1, 3\}$ but $\{1, 3\} = (uR_{\mathbf{F}'})(\{3\})$. As for the interior, $int(\{1, 2, 4\}) = \langle e\rangle[i](\{1, 2, 4\}) = \langle e\rangle(\{b\}) = \{2, 4\} = (lR_{\mathbf{F}'})(\{1, 2, 4\})$.

4.8.7 Dichotomic Systems and Approximations

Here is a dichotomic system $\mathbf{D} = \langle \{1, 2, 3, 4\}, \{a, \overline{a}, b, \overline{b}, c, \overline{c}\}, \Vdash \rangle$ such that cl is not a classical upper approximation and int is not a classical lower approximation:

$\Vdash$	a	$\overline{a}$	b	$\overline{b}$	c	$\overline{c}$
1	1	0	0	1	0	1
2	1	0	0	1	0	1
3	0	1	0	1	0	1
4	0	1	0	1	1	0

The equivalence classes of $R_{\mathbf{D}}$ are $\{1, 2\}, \{3\}, \{4\}$. Easily, we compute $int(\{2, 3\}) = \emptyset$ (because $[i](\{2, 3\}) = \emptyset$), while $(lR_{\mathbf{D}})(\{2, 3\}) = \{3\}$. Also $cl(\{1, 4\}) = [e](\{a, \overline{a}, \overline{b}, c, \overline{c}\}) = \{1, 2, 3, 4\} \neq \{1, 2, 4\} = (uR_{\mathbf{D}})(\{1, 4\})$.

4.9 Frame – Concept Lattices

4.9.1 Formal Contexts and Formal Concept Analysis

Formal Concept Analysis, or FCA, was introduced by R. Wille in Wille [1982]. The starting points of FCA are *Formal Contexts* which coincide with our *P-systems*. Indeed, we used the symbols "G" and "M" after the German terms "*Gegenstände*" ("objects") and, respectively, "*Merkmale*" ("properties") used by Wille in the quoted paper.

Wille's original purpose was to answer some psychologists' requirements about the introduction of non numerical data analysis techniques.

Wille's proposal was the manipulation of data by means of the polarity $\langle est, \mathcal{ITS} \rangle$.[2] Given a Formal Context (i.e. a *P-systems*) $\mathbf{P} = \langle G, M, \Vdash \rangle$, any pair of the form $\langle A, B \rangle$ where $A \subseteq G, B \subseteq M$ and $A = [[e]](B)$ and $B = [[i]](A)$, is called a *Formal Concept*. If the above equalities hold, then in view of *Corollary* 2.3.1, we know that $A = est(B)$ and $B = \mathcal{ITS}(A)$, so that A is the extent of B and B is the intent of A in that all of the properties in B are fulfilled by all of the objects in A.

The family of all formal concepts induced by $\mathbf{P}$ is denoted as $\mathcal{B}(\mathbf{P})$. Given two formal concepts $\langle A, B \rangle$ and $\langle A', B' \rangle$, if $A \subseteq A'$ (or, equivalently, $B \supseteq B'$, because the right and left ordered structures of the polarity $\langle est, \mathcal{ITS} \rangle$ are opposite to each other), then the former is said to be a *subconcept* of the latter (or the latter a *superconcept* of the former), in symbols $\langle A, B \rangle \preceq \langle A', B' \rangle$.

Also, we know that $\langle \mathcal{B}(\mathbf{P}), \preceq \rangle$ is a complete lattice, because $\preceq$ defines the following *sup* and *inf* operations:

$$\bigvee_{i \in I} \langle A_i, B_i \rangle = \langle est(\bigcup_{i \in I} A_i), \bigcap_{i \in I} B_i \rangle; \bigwedge_{i \in I} \langle A_i, B_i \rangle = \langle \bigcap_{i \in I} A_i, \mathcal{ITS}(\bigcup_{i \in I} B_i) \rangle$$

This can be directly derived from *Proposition* 2.3.2 after considering that intersections on $\mathbf{G}$ turn into unions on $\mathbf{M^{op}}$ and the other way around.

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[2]We use our notation.

Let us consider the following formal context (from Wille [1982]):

$\Vdash$	Size			Distance from sun		Moon	
	Small	Medium	Large	Near	Far	Yes	No
Mercury	x			x			x
Venus	x			x			x
Earth	x			x		x	
Mars	x			x		x	
Jupiter			x		x	x	
Saturn			x		x	x	
Uranus		x			x	x	
Neptune		x			x	x	
Pluto	x				x	x	

Let us call it the "planet context", denoted by **C**. Using some abbreviations, we can see that $G = \{Me, V, E, Ma, J, S, U, N, P\}$ and $M = \{Ss, Sm, Sl, Dn, Df, My, Mn\}$. It induces the following lattices of extents and, on the right, intents:
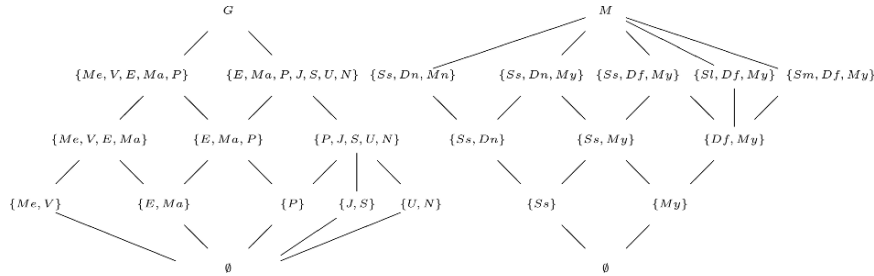
Any extent is obtained by an application of est. For instance $est(\{J, S, P\}) = [[e]][[i]](\{J, S, P\}) = [[e]](\{Df, My\}) = \{J, S, P, U, N\}$. Dually, any intent is an application of $\mathcal{ITS}$. For example, $[[i]][[e]](\{Dn, My\}) = [[i]](\{E, Ma\}) = \{Dn, My, Ss\}$.

If we combine isomorphic elements of the two lattices into pairs we obtain formal concepts. For instance the isomorphic image of $\{Me, V\}$ is $[[i]](\{Me, V\}) = \{Ss, Dn, Mn\}$, while the isomorphic image of $\{E, Ma, P\}$ is $[[i]](\{E, Ma, P\}) = \{Ss, My\}$, and so on. Thus the formal concepts induced by the planet contexts are $\langle\{Me, V\}, \{Ss, Dn, Mn\}\rangle, \langle\{E, Ma, P\}, \{Ss, My\}\rangle$, and so on.

It is straightforward to verify that formal concept formation is not additive. Indeed, $est(\{J\}) \cup est(\{P\}) = \{J, S\} \cup \{P\} = \{J, S, P\} \subsetneq$

$\{P, J, S, U, N\} = est(\{J, S\} \cup \{P\})$. By dual-symmetry, the same happens of intents. For instance $\mathcal{ITS}(\{Ss, Dn\} \cup \{Df, My\}) = M \supsetneq \{Ss, Dn, Df, My\} = \mathcal{ITS}(\{Ss, Dn\}) \cup \mathcal{ITS}(\{Df, My\})$.

Also it is immediate to verify an instance of the fact that $[[i]]$ is an intisomorphism: $[[i]](\{Me, V, E, Ma\}) \cap [[i]](\{E, Ma, P\}) = \{Ss, Dn\} \cap \{Ss, My\} = \{Ss\} = [[i]](\{Me, V, E, Ma, P\} = [[i]](\{Me, V, E, Ma\} \cup \{E, Ma, P\})$.

Similar examples verify the same property of $[[e]]$ (easy exercise left to the reader).

4.9.2   Formal Concepts and Approximation Operators

Notice that since $\mathbf{C}$ is a nominalised system, in view of *Corollary* 3.4.1 and *Proposition* 3.1.2, $R_\mathbf{C}$ is an equivalence relation, so that $\mathbf{E} = \langle G, G, R_\mathbf{C} \rangle$ is an Indiscernibility Space and $\langle G, (lR_\mathbf{C}), (uR_\mathbf{C}) \rangle$ is a Pawlak Approximation System (we recall that $(lR_\mathbf{C})$ and $(uR_\mathbf{C})$ coincide with $int^\mathbf{E}$ and, respectively, $cl^\mathbf{E}$).

We can easily verify that formal concepts in general differ from either (generalised) lower or upper approximations or both. Here are some examples:

Set	est	int	cl	$(lR_\mathbf{C})$	$(uR_\mathbf{C})$
$\{E, Ma, J, S\}$	$\{E, Ma, J, S, U, N, P\}$	$\{J, S\}$	$\{E, Ma, J, S, P\}$	$\{E, Ma, J, S\}$	$\{E, Ma, J, S\}$
$\{V, P\}$	$\{Me, V, E, Ma, P\}$	$\emptyset$	$\{Me, V, E, Ma, P\}$	$\{P\}$	$\{Me, V, P\}$

Intuitively, $est(X)$ collects all the elements of G which are glued together by means of the properties which are shared by all the elements of X. In fact the operator $[[i]](X)$ in $est(X)$ looks for those properties whose extensions include X. To put it another way, $est(X)$ has a somewhat "structural" flavor, since their elements are gathered together by means of what we can term the "intensional backbone" of X. On the contrary $(lR_\mathbf{C})(X)$ (and $(uR_\mathbf{C})(X)$, respectively) adds together all the equivalence classes modulo E in order to get the maximal (minimal, respectively) set which is included in X (which includes X, respectively). For that reason, additivity of equivalence classes is a nice property (meaning that if we add two equivalence classes modulo E we obtain an equivalence class modulo an equivalence relation coarser than E).

In a sense, the two generalised approximation operators, int and cl represent an intermediate approach, in that the operator $[i](X)$ in

$int(X)$ looks for those properties whose extensions are included in X, hence something similar to a sort of "intensional core" of X. Similarly, $[e]$ in $cl(X)$ looks for those objects whose intension is included in the union of all the properties fulfilled by the elements of X, hence for "subconcepts" of $\langle i \rangle(X)$.

However, see below an interesting question:

Exercise 4.2. *One can verify that for each $g \in G$, $cl(\{g\}) = est(\{g\}) = (uR_{\mathbf{C}})(\{g\})$. Why?*

From the above exercise it follows that in the planet context, extents induced by singletons coincide with equivalence classes modulo the equivalence relation induced by the context (that is, for any $g \in G$, $est(\{g\}) = [g]_{R_{\mathbf{C}}}$).

The above considerations apply to nominalised contexts. Now we show a more general relationship between extents and upper approximations:

Lemma 4.9.1. *Let $\mathbf{P} = \langle G, M, \Vdash \rangle$ be a P-system and let E be an equivalence relation on G. Then for all $X \subseteq G$, $est(X) \subseteq \bigcup\limits_{x \in est(X)} [x]_E = (uE)(est(X))$.*

Proof. Trivially, because E is reflexive.   QED

Proposition 4.9.1. *Let $\mathbf{P} = \langle G, M, \Vdash \rangle$ be a P-system and let $E_{\Vdash}$ be the equivalence relation defined by $\langle g, g' \rangle \in E_{\Vdash}$ if and only if $\langle i \rangle(g) = \langle i \rangle(g')$, for $g, g' \in G$. Then for all $X \subseteq G$, $est(X) = \bigcup\limits_{x \in est(X)} [x]_{E_{\vDash}} = (uE_{\Vdash})(est(X))$.*

Proof. From the above Lemma $est(X) \subseteq \bigcup\limits_{x \in est(X)} [x]_{E_{\Vdash}}$. Vice-versa, if $g \in \bigcup\limits_{x \in est(X)} [x]_{E_{\Vdash}}$ then there is $g' \in est(X)$ such that $\langle i \rangle(g) = \langle i \rangle(g')$. It immediately follows that $g \in est(X)$.   QED

Thus any extent of a formal concept is a definable set modulo $E_{\Vdash}$. However, since $(uE_{\Vdash})$ is additive on $\wp(G)$ while est is not, the family of extents $\{est(X) : X \subseteq G\}$ is a subset of the family of definable sets $\{(uE_{\vDash})(X) : X \subseteq G\}$.

Corollary 4.9.1. *For any P-system $\mathbf{P}$ and Pawlak Approximation System $\mathbf{E}$ on the same set of objects, G, for any $X \subseteq G$, $est^{\mathbf{P}}(X) \subseteq cl^{\mathbf{E}}(est^{\mathbf{P}}(X))$. If for any X, $cl^{\mathbf{E}}(X) = E_{\Vdash}(X)$, then $est^{\mathbf{P}} = cl^{\mathbf{E}}est^{\mathbf{P}}$.*

4.9.3 Combining Classical Approximation Systems and Concept Lattices

Stated above are some basic relationships between Pawlak's Approximation Systems and Formal Concepts. Moreover, researches have been developed to understand general relationships between Rough Set Theory and FCA and to combine the two approaches (see [Pagliani 1993a, 1996, 1998c], [Kent 1993, 1996], [Yao & Chen 2004]).

Particularly E. R. Kent's approach is a fundamental background for any attempt to merge the two data analysis techniques. We shall give a taste of this approach. Using our notation and concepts, we can recast Kent's approach in a Multi-agent pre-topological Perception System $\langle G, M, int^{\mathbf{E}}, cl^{\mathbf{E}}, \langle e \rangle^{\mathbf{C}}, \langle i \rangle^{\mathbf{C}}, [[e]]^{\mathbf{C}}, [[i]]^{\mathbf{C}} \rangle$, where the operators decorated with "$\mathbf{E}$" are induced by an Indiscernibility Space $\langle G, G, E \rangle$ (i.e. $\mathbf{E} = \langle G, int^{\mathbf{E}}, cl^{\mathbf{E}} \rangle$ is a Pawlak Approximation System) and the operators decorated with "$\mathbf{C}$" are induced by a *P-system* (formal context) $\mathbf{C} = \langle G, M, \Vdash \rangle$. Then Kent defines the *upper approximation of the context* $\mathbf{C}$ with respect to $\mathbf{E}$, $cl^{\mathbf{E}}(\mathbf{C})$, as follows:

Definition 4.9.1. $cl^{\mathbf{E}}(\mathbf{C}) = \langle G, M, \Vdash^{cl^{\mathbf{E}}} \rangle$, *where* $\Vdash^{cl^{\mathbf{E}}}$ *is defined pointwise by setting for each* $m \in M$, $\langle e \rangle^{cl^{\mathbf{E}}(\mathbf{C})}(m) =_{def} cl^{\mathbf{E}} \langle e \rangle^{\mathbf{C}}(m)$.

In other words, for all $g \in G$ and $m \in M$, $g \Vdash^{cl^{\mathbf{E}}} m$ if and only if there is $g' \in G$ such that $g' \in cl^{\mathbf{E}}(g)$ and $g' \Vdash m$.

Then formal concepts are formed according to the transformed formal context, that is, a formal concept in $cl^{\mathbf{E}}(\mathbf{C})$ is a pair of the form $\langle est^{cl^{\mathbf{E}}(\mathbf{C})}(X), [[i]]^{cl^{\mathbf{E}}(\mathbf{C})}(X) \rangle$ for some $X \subseteq G$ (or, equivalently, of the form $\langle [[e]]^{cl^{\mathbf{E}}(\mathbf{C})}(Y), \mathcal{ITS}^{cl^{\mathbf{E}}(\mathbf{C})}(Y) \rangle$ for some $Y \subseteq M$).

Now two considerations are in order.

First of all, one must take a great care while applying Kent's approach, because in the case of dichotomic context it may lead to contradictions, easily. For instance, if we are to transform the planet-context by means of a Pawlak Approximation System $\mathbf{N}$ induced by an equivalence relation N, classifying the planets inside the asteroid belt ($\{Me, V, E\}$), or outside of it ($\{Ma, J, S, U, N, P\}$), then we have $cl^{\mathbf{N}}(\langle e \rangle^{\mathbf{C}}(\{My\})) = cl^{\mathbf{N}}(\{E, Ma, J, S, U, N, P\}) = G$ and, simultaneously, $cl^{\mathbf{N}}(\langle e \rangle^{\mathbf{C}}(\{Mn\})) = cl^{\mathbf{N}}(\{Me, V\}) = \{Me, V, E\}$, so that in the upper approximation of the planet-context *Mercury*, *Earth* and *Venus* would fulfill both *Moon-yes* and *Moon-no*.

Second, we wonder whether when we apply $cl^{\mathbf{E}}$ to $est^{\mathbf{C}}$ we obtain the same result as when we apply $est^{cl^{\mathbf{E}}(\mathbf{C})}$, that is, if for any $X \subseteq G$, $cl^{\mathbf{E}}(est^{\mathbf{C}}(X)) = est^{cl^{\mathbf{E}}(\mathbf{C})}(X)$.

The answer is negative:

Lemma 4.9.2. *Let* $\mathbf{C} = \langle G, M, \Vdash \rangle$ *be a formal context and* $\mathbf{E}$ *a PAS on* G. *Then for all* $m \in M$, $\langle e \rangle^{cl^{\mathbf{E}}(\mathbf{C})}(m) \supseteq \langle e \rangle^{\mathbf{C}}(m)$.

Proof. Indeed, $\langle e \rangle^{cl^{\mathbf{E}}(\mathbf{C})}(m) = cl^{\mathbf{E}}(\langle e \rangle^{\mathbf{C}}(m))$ and since $cl^{\mathbf{E}}$ is increasing we have the result.   QED

Corollary 4.9.2. *Let* $\mathbf{C} = \langle G, M, \Vdash \rangle$ *be a formal context and* $\mathbf{E}$ *a PAS on* G. *Then* $[[i]]^{cl^{\mathbf{E}}(\mathbf{C})}(X) \supseteq [[i]]^{\mathbf{C}}(X)$, *any* $X \subseteq G$.

Proof. From the above Lemma, $\bigcap\limits_{x \in X} \langle i \rangle^{cl^{\mathbf{E}}(\mathbf{C})}(x) \supseteq \bigcap\limits_{x \in X} \langle i \rangle^{\mathbf{C}}(x)$ and we obtain the result from definition.   QED

Proposition 4.9.2. *Let* $\mathbf{C} = \langle G, M, \Vdash \rangle$ *be a formal context and* $\mathbf{E}$ *a PAS on* G. *Then for all* $X \subseteq G, est^{cl^{\mathbf{E}}(\mathbf{C})}(X) \subseteq cl^{\mathbf{E}}(est^{\mathbf{C}}(X))$.

Proof. Clearly for all $g \in G$, $g \in cl^{\mathbf{E}}(est^{\mathbf{C}}(X))$ if and only if for all $m \in [[i]]^{\mathbf{C}}(X)$, $g \in cl^{\mathbf{E}}\langle e \rangle^{\mathbf{C}}(m)$. Thus suppose $g \notin cl^{\mathbf{E}}(est^{\mathbf{C}}(X))$. Then there is $m \in [[i]]^{\mathbf{C}}$ such that $g \notin cl^{\mathbf{E}}\langle e \rangle^{\mathbf{C}}(m)$. That is $g \notin \langle e \rangle^{cl^{\mathbf{E}}(\mathbf{C})}(m)$. Now, from the above Corollary we have $m \in [[i]]^{cl^{\mathbf{E}}(\mathbf{C})}$, too. But, obviously, $g \in est^{cl^{\mathbf{E}}(\mathbf{C})}(X)$ if and only if for all $m \in [[i]]^{cl^{\mathbf{E}}(\mathbf{C})}(X)$, $g \in \langle e \rangle^{cl^{\mathbf{E}}(\mathbf{C})}(m)$. Thus $g \notin est^{cl^{\mathbf{E}}(\mathbf{C})}(X)$. Hence by contraposition we obtain the thesis.   QED

Trivial counterexamples show that the reverse inclusion is false (see below).

Third, we can ask about $est^{\mathbf{C}}cl^{\mathbf{E}}$. Actually, this operator has an unpredictable behaviour with respect to both $cl^{\mathbf{N}}(\mathbf{C})$ and $cl^{\mathbf{E}}est^{\mathbf{C}}$.

EXAMPLE

Let $\mathbf{P}$ be the planet context of the above examples, let $\mathbf{N}$ be the above Pawlak Approximation System, and let $\mathbf{X}$ be a Pawlak Approximation System induced by an equivalence relation gathering together $\{Me, P\}$ (the two extremes of the Solar system) and $\{V, E, Ma, J, S, U, N\}$ (the intermediate planets). Then:

(*i*) $cl^{\mathbf{N}}(est^{\mathbf{P}}(\{E\})) = cl^{\mathbf{N}}(\{E, Ma\}) = G$.
(*ii*) $est^{cl^{\mathbf{N}}(\mathbf{P})}(\{E\}) = \{Me, V, E\}$.[3]

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[3]Because in the system $cl^{\mathbf{N}}(\mathbf{P})$, $\langle i \rangle(\{E\}) = \langle i \rangle(\{Me\}) = \langle i \rangle(V) = \{Ss, Dn, My, Mn\}$.

(iii) $est^{\mathbf{P}}(cl^{\mathbf{N}}(\{E\})) = est^{\mathbf{P}}(\{Me, V, E\}) = \{Me, V, E, Ma\}$.

(iv) $cl^{\mathbf{N}}(est^{\mathbf{P}}(\{J\})) = cl^{\mathbf{N}}(\{J, S\}) = \{Ma, J, S, U, N, P\}$.

(v) $est^{\mathbf{P}}(cl^{\mathbf{N}}(\{J\})) = est^{\mathbf{P}}(\{Ma, J, S, U, N, P\}) = \{E, Ma, J, S, U, N, P\}$.

(vi) $est^{\mathbf{P}}(cl^{\mathbf{X}}(\{Me\})) = \{Me, V, E, Ma, P\}$.[4]

(vii) $est^{cl^{\mathbf{X}}(\mathbf{P})}(\{Me\}) = G$.[5]

According to (i) and (ii) the reverse inclusion of *Proposition* 4.9.2 does not hold. From (i), (ii) and (iii), we have $est^{cl^{\mathbf{N}}(\mathbf{P})} \subsetneqq est^{\mathbf{P}}cl^{\mathbf{N}} \subsetneqq cl^{\mathbf{N}}est^{\mathbf{P}}$. According to (iv) and (v), $cl^{\mathbf{N}}est^{\mathbf{P}} \subsetneqq est^{\mathbf{P}}cl^{\mathbf{N}}$. Finally, from (vi) and (vii) $est^{\mathbf{P}}cl^{\mathbf{X}} \subsetneqq est^{cl^{\mathbf{X}}(\mathbf{P})}$.

Dually, we can define the *lower approximation* $int^{\mathbf{E}}(\mathbf{C}) = \langle G, M, \Vdash^{int^{\mathbf{E}}} \rangle$ of a formal context $\mathbf{C}$ with respect to a PAS $\mathbf{E}$ as follows:

Definition 4.9.2. $int^{\mathbf{E}}(\mathbf{C}) = \langle G, M, \Vdash^{int^{\mathbf{E}}} \rangle$, *where* $\Vdash^{int^{\mathbf{E}}}$ *is defined point-wise by setting for each* $m \in M$, $\langle e \rangle^{int^{\mathbf{E}}(\mathbf{C})}(m) = int^{\mathbf{E}}\langle e \rangle^{\mathbf{C}}(m)$.

From the decreasing properties of int we immediately have that for all $X \subseteq G$, $[[i]]^{int^{\mathbf{E}}(\mathbf{C})}(X) \subseteq [[i]]^{\mathbf{C}}(X)$, so that with a similar reasoning as for the previous Proposition we obtain:

Proposition 4.9.3. *Let* $\mathbf{C} = \langle G, M, \Vdash \rangle$ *be a formal context and* $\mathbf{E}$ *a PAS on* G. *Then for all* $X \subseteq G, est^{int^{\mathbf{E}}(\mathbf{C})}(X) \supseteq int^{\mathbf{E}}(est^{\mathbf{C}}(X))$.

To sum up, for all $X \subseteq G$:

$$int^{\mathbf{E}}(est^{\mathbf{C}}(X)) \subseteq est^{int^{\mathbf{E}}(\mathbf{C})}(X) \subseteq est^{cl^{\mathbf{E}}(\mathbf{C})}(X) \subseteq cl^{\mathbf{E}}(est^{\mathbf{C}}(X))$$

The two transforms $cl^{\mathbf{E}}(\mathbf{C})$ and $int^{\mathbf{E}}(\mathbf{C})$ are described by Kent in an elegant manner which can be better explained within Relation Algebra (that is what will be done in Frame 15.18.5 of Part III).

Notice that instead of $cl^{\mathbf{E}}$ we can apply $\langle i \rangle^{\mathbf{E}}$ to $est^{\mathbf{C}}$ and obtain the same result because E is reflexive (so that $x \in \langle i \rangle^{\mathbf{E}}(x)$, all $x \in G$). Moreover, since E is symmetric we can use $\langle e \rangle^{\mathbf{E}}$, as well. But if E were not symmetric and reflexive, then we should obtain truly different operators $\langle i \rangle^{\mathbf{E}}est^{\mathbf{C}}$, $\langle e \rangle^{\mathbf{E}}est^{\mathbf{C}}$. In addition, we can generalise Kent's approach by applying after $est^{\mathbf{C}}$ any applicable operator from any other Information System or I-Quantum Relation System $\mathbf{S}$ on G. Clearly, the relationships between these operators and $est^{cl^{\mathbf{S}}(\mathbf{C})}$ must be studied

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[4]Because the only property shared by Me and P is Ss.

[5]Because for all $g \in \{V, E, Ma, J, S, U, N\}$, $\langle i \rangle^{cl^{\mathbf{X}}(\mathbf{P})}(\{x\}) = M$.

case by case, as well as their application meanings. Further, one should analyse what happens in case of particular relationships between $\mathbf{C}$ and E (for instance, when E is the equivalence relation induced by $\mathbf{C}$ or when $E = R_{\mathbf{C}}$). Finally, one should analyse what happens when $\mathbf{C}$ is a nominal or dichotomic system.

4.9.4 Combining Non-Classical Approximation Systems and Concept Lattices

Variations of concept lattices have been introduced by means of the above operators, namely "object oriented concepts" of the form $\langle int(X), [i](X) \rangle$ (see Yao & Chen [2004]) and "property oriented concepts" of the form $\langle (cl(X), \langle i \rangle(X) \rangle$ (see Düntsch & Gegida [2002]) (we use our own terminology and notation).

These constructions are intuitively explained in the following way: if we are given an object oriented concept $\langle X, Y \rangle$, then if an object possesses a property in Y then it belongs to X. Symmetrically, if $\langle X, Y \rangle$ is a property oriented formal concept, and a property is possessed by an object in X then it must be in Y.

Thus these facts are exploited, for instance, in order to predict the membership of an object by looking at its properties. Indeed, we have $\bigcup\limits_{y \in [i](X)} \langle e \rangle(y) \subseteq X$.

However, the reverse implication holds in a weaker form: $int(X) \subseteq \bigcup\limits_{y \in [i](X)} \langle e \rangle(y)$. Since $\bigcup\limits_{y \in [i](X)} \langle e \rangle(y) \subseteq int(X)$ we obtain the equation $\bigcup\limits_{y \in [i](X)} \langle e \rangle(y) = int(X)$, so that in the case of an object oriented formal concept, $\langle X, Y \rangle$ we obtain the features which characterizes formal concepts, namely that (generalised) intents and (generalised) extents uniquely determine each other: $\bigcup\limits_{y \in Y} \langle e \rangle(y) = X$.

Dually, one can prove: (a) $X \subseteq \bigcap\limits_{y \in [[i]](X)} \langle e \rangle(y)$, (b) $\bigcap\limits_{y \in [[i]](X)} \langle e \rangle(y) \subseteq est(X)$, (c) $est(X) = \bigcap\limits_{y \in [[i]](X)} \langle e \rangle(y)$ so that for any (usual) formal concept $\langle X, Y \rangle$, $X = \bigcap\limits_{y \in Y} \langle e \rangle(y)$.

4.9.5 Nominal Systems and Conceptual Scaling

The transformation of *A-systems* into *P-systems* has been studied in Wille [1982], Vakarelov [1997] and Pagliani [2005].

In Formal Concept Analysis nominalisation is an instance of a more comprehensive techniques called *Conceptual scaling* (see Ganter & Wille [1989]). Indeed, there is no unique way to transform an Attribute System $\mathbf{S} = \langle G, \{V_a\}_{a \in At}, At \rangle$ (or a "Multivalued context" in FCA terminology) into a *P-system*, because any transformation is, in fact, a matter of interpretation. Roughly speaking, conceptual scaling is a method to give a meaning to each set of attribute-values V_a (for $a \in At$) by turning it from a flat set to a hierarchical domain. In other terms, conceptual scaling gives a conceptual interpretation of the attributes and their values. For instance, if an attribute a takes numerical values $V_a = \{0, 1, 2, ...n\}$ giving a "smaller-higher" order to V_a may be important to a conceptual interpretation of data. It follows that we have to link $\mathbf{S}$ with the *P-system* $\mathbf{A}^{\geq} = \langle V_a, V_a, \geq \rangle$ (or $\mathbf{A}^{\leq} = \langle V_a, V_a, \leq \rangle$), called a *conceptual scale*. Once we have defined conceptual scales for each attribute we combine them together in a common scale $\mathbf{Sc}$ (see the quoted paper for a list of combining techniques).

In a wider setting a *scale* is a *P-system* $\mathbf{Sc} = \langle G_{\mathbf{Sc}}, M_{\mathbf{Sc}}, \Vdash_{\mathbf{Sc}} \rangle$ and the connection between $\mathbf{S}$ and $\mathbf{Sc}$ is given by a mapping $\sigma : G \longmapsto G_{\mathbf{Sc}}$ such that if X is an extent in $\mathbf{Sc}$ (that is, $X = est^{\mathbf{Sc}}(X)$) then the pre-image $\sigma^{-1}(X)$ is an extent in $\mathbf{S}$ (that is, $est^{\mathbf{S}}(\sigma^{-1}(X)) = \sigma^{-1}(X)$).

Nominalisation is obtained when one applies the scale $\mathbf{Sc} = \langle \bigcup \{V_a\}_{a \in At}, \bigcup \{V_a\}_{a \in At}, = \rangle$ (called, indeed, *nominal scale*), "*which, at least, would conceptually separate different values*" (Wille [1992]).

In Vakarelov [1997] the transformation from (and to) *P-systems* to (and from) *A-systems* is framed within a wider study about dependency and consequence relations. Vakarelov's approach is slightly different from Wille's, because he defines a *P-system* as a triple $\mathbf{S} = \langle G, M, obs \rangle$ (we are using our own terminology and notation). Moreover, Vakarelov takes into account the more general case of non-deterministic information systems (see Frame 4.7.1).

Given an Information System, for each $a \in At$ one can define a *P-system* $\mathbf{A}_a = \langle G, V_a, obs_a \rangle$, where $obs_a(g) = a(g)$. In the *A-system* $\mathbf{A}$ of Example 1.2.1, $obs_A(a) = \{1\}$, $obs_{A'}(a) = \{b\}$, $obs_{A''}(a) = \{\alpha\}$, which amounts to saying that $\langle a, A_1 \rangle, \langle a, A_b' \rangle$ and $\langle a, A_\alpha'' \rangle$ belong to $\Vdash_{\mathbf{A}}^{\mathcal{N}}$ (see Example 3.4.1).

4.10   Frame – Neighborhood Systems

We introduce a topic which will be largely discussed in Part III.

Topological concepts have been widely applied in data analysis since its very beginning. However within a framework close to our approach we must quote the pioneering work by T. Y Lin: [Lin 1988, 1989 and 1998] (see also Yao & Lin [1997], Yao [1999] and Yao & Chen [2004]). According to Lin's approach, Rough Set Theory is generalised by using arbitrary binary relations instead of equivalence relations.

It is worthwhile to briefly discuss some intuitions which led to this generalisation. Consider an *A-system* $\mathbf{A} = \langle G, At, \{V_a\}_{a \in At} \rangle$. We can group the values of each single set V_a by means of binary relations, thus obtaining a family of binary relations on the sets of values, denoted by $\{R_a^j\}_{a \in At, j \in J}$, for some index family J. For instance, in the *A-system* $\mathbf{A}$ of Example 1.2.1 one may classify the values b and c of attribute A' as "close", while b and f may be classified as "very close" (suppose b, c and f are locations). Thus we have two binary relations: $Close_{A'}$ and $Very-close_{A'}$. This may be helpful, for instance, in order to answer a query asking for an item with value 1 for attribute A and value f for attribute A'. Indeed there are no items with such an evaluation, but we can approximate a correct answer by returning the answer "a'''", because it has value 1 for A and its actual value b for A' is "very-close" to the required value f.

The reader will find further details in the Frame section of Part III.

As a matter of fact, the family $\{R_a^j\}_{a \in At, j \in J}$ is a sort of conceptual scaling (see Frame 4.9.5).

Generally speaking, in this way we associate each object $g \in G$ with a family of neighborhoods, each one defined as $\langle R_a^{j \smile} \rangle(g)$.

One can frame this approach in an abstract setting.

Let U, U' be sets. Let us consider the elements of U as connected (classified, characterised, labeled, perceived, ...) by means of the characteristic features of the elements of another set U', By means of a map n, any element u of U will be associated with one or more elements of $\wp(U')$, obtaining a family of subsets of U', such that each $N \in n(u)$ represents a particular relationship between the elements of U'. Therefore, the fact that the image of u along n is a set of sets and not just a set, reflects the fact that any element of $n(u)$ represents, in principle, a different classification of the elements of U *via* the elements of U'.

Moreover, Yao and Lin have generalised this approach to fuzzy relations (see Frame 4.7.2).

In Pagliani [2003] this approach, as to usual relations, has been further generalised and the pre-topological and topological properties of the resulting systems have been systematically studied in Pagliani [2002], following the approach of Point-free Topology (see below Frame 4.11).

The first section of Part III is devoted to neighborhood systems.

4.11   Frame – Basic Pairs and Point-Free Topology

P-systems appear in different mathematical contexts and under different names, such as *Basic pairs*, *Formal contexts* (see Frame 4.9) and *Chu spaces* (see Frame 4.12).

The material in the present Chapter is based on the "Basic pairs" approach and on original researches. So, it is worthwhile to give a preview of the "Padua" approach on point-free topology through basic pairs.

Concrete pre-topological spaces are fundamentally describable by means of a map n linking elements from a set A with a family of subsets of A, called "neighborhoods of a". Given $a \in A$ and $X \subseteq A$, we record the fact that $X \in n(a)$ by $a \Vdash X$. Usually we shall denote the set $n(a)$ with $\mathcal{N}_a$, so that $\mathcal{N}_a = \{X : a \Vdash X\}$. Mathematically, $\Vdash$ is a subset of the Cartesian product $A \times \wp(A)$. The family $\mathcal{N}(A) = \{\mathcal{N}_a\}_{a \in A}$ is called a (*concrete*) *neighborhood system* and the pair $\langle A, \{\mathcal{N}_a\}_{a \in A} \rangle$ will be called a *concrete neighborhood space*. As we have seen, a neighborhood of a collects the elements that are close to a relative to a specific criterion (the neighbors of a).

Lifting the level of abstraction we can get rid of the elements of the neighborhoods and refer to each of them as to elementary entities. In other words, we can consider any element of $\wp(A)$ as an element of a set B. Therefore, we can consider two sets, A and B and think of the elements of A as pure "indices" of *sets* of observables, as illustrated above.

On this basis, a new approach to pre-topological and topological formal spaces has been introduced in Sambin & Gebellato [1998] (see also Sambin [2001]), which subsumes the original theory introduced in [Sambin 1987 and 1989] after suggestions from P. Martin-Löf's Intuitionistic Type Theory. This is the approach known as "point-free topology".

Given a basic pair $\langle A, B, \Vdash \rangle$, the operator $\mathcal{A}$ makes it possible to synthesise the formal properties of formal pre-topologies in logical (or, better, type theoretical) terms.

This goal is achieved by defining a relation, denoted by the symbol ◀, which connects (subsets of) formal neighborhoods with subsets of formal neighborhoods to be interpreted as a formal pre-covering relation, because it is the formal counterpart of the "concrete" concept of an "adherence":

Definition 4.11.1. *Let $\langle A, B, \Vdash \rangle$ be a basic pair. Then for any $b \in B$ and $Y, Y' \subseteq B$, the following relation is called a* formal pre-cover, *or shortly a* semi-cover:

(**basis**): $b \blacktriangleleft Y$ *iff* $b \in \mathcal{A}(Y)$, (**step**): $Y \blacktriangleleft Y'$ *iff* $\forall y \in Y, y \blacktriangleleft Y'$.

The relationships between different kinds of formal and concrete neighborhood systems have been analysed in Pagliani [2002], where the adjointness properties of basic operators were presented.

The formal (pre)topology approach will be widely investigated in Part III. Here, for the convenience of the interested reader, we shall give a table of translation between the terminology and symbols used in this book and the usual notation of formal topology.

Present notation	$\langle e \rangle$	$[i]$	$\langle i \rangle$	$[e]$
Formal topology	ext	$\square$	$\diamondsuit$	$rest$

4.12   Frame – Chu Spaces

A *Chu space* is, in itself, a *P-system*. The difference lies in the fact that actually we have to speak of Chu spaces in the plural mode, linked together by morphisms (cf. Barr [1979], Pratt [1999]). More precisely, given two Chu spaces $\mathbf{C} = \langle G, M, \vDash \rangle$ and $\mathbf{C}' = \langle G', M' \vDash' \rangle$ a morphism from $\mathbf{C}$ to $\mathbf{C}'$ is a pair of functions $\langle f_M, f_G \rangle$ with $f_M : M \longmapsto M'$ and $f_G : G' \longmapsto G$ such that for any $m \in M, g' \in G', f_G(g') \vDash m$ if and only if $g' \vDash' f_M(m)$. That is, the following diagram commutes in Chu spaces:

$$M \xrightarrow{\ f_M\ } M'$$

(diagram: $M \xrightarrow{f_M} M'$, with $\Vdash$ on left side $M \to G$, $\Vdash'$ on right side $M' \to G'$, and $G \xleftarrow{f_G} G'$ on bottom)

For example, consider a *P-system* $\mathbf{P} = \langle G, M, \vDash \rangle$ and add to G a new object along with the observed properties, which remain unchanged, thus obtaining $\mathbf{P}' = \langle G', M, \vDash' \rangle$. Then the pair of functions $\langle I_M, I_G \rangle$, where I_M is the identity function on M and I_G is the injection of objects, is a Chu morphism.

Another example is the following:

Let $\mathbf{P} = \langle A, M, \vDash \rangle$ and $\mathbf{Q} = \langle A, M, \vDash' \rangle$, where

$\vDash$	a	b
$g1$	1	1
$g2$	1	0

$\vDash'$	a	b
$g1$	1	1
$g2$	1	1

$f_M(m) = m,\ f_G(x) = g1,$
any $m \in M, x \in G$

One can easily check that the pair $\langle f_M, f_G \rangle$ gives a Chu-mapping between $\mathbf{P}$ and $\mathbf{Q}$.

This simple example is used in Zhang [2004] to show that Chu-mappings do not necessarily preserve formal concepts. Indeed, in our notation, $\langle \{g1, g2\}, \{a\} \rangle$ is a formal concept in $\mathbf{P}$ while $\langle \{g1, g2\}, f_M(\{a\}) \rangle = \langle \{g1, g2\}, \{a\} \rangle$ is not a formal concept of $\mathbf{Q}$. In fact, $\mathbf{Q}$ induces the only formal concept $\langle \{g1, g2\}, \{a, b\} \rangle$. We note, incidentally, that, on the contrary, f_G preserves the latter formal concept: $\langle f_G(\{g1, g2\}), \{a, b\} \rangle = \langle \{g1\}, \{a, b\} \rangle$ which is a formal concept in $\mathbf{P}$.

4.13   Frame – Intuitionism, Modalities and Relational Semantics

In early 1960s, Saul Kripke introduced models for some modal logics and for Intuitionistic logic ([Kripke 1963a, b]). In a sense, Kripke's proposal collected, rationalized and simplified previous modelling techniques but it created a sensation within the logic community and indeed turned into a standard because of its intuitive flavour and straight relationships with other mathematical fields such as Relation algebra, Topology and Sheaf Theory.

A *Kripke frame* is a pair $\langle W, R \rangle$, where W is a non-empty set, and R is a binary relation on W. Elements of W are usually called *possible worlds*, and R is known as the *accessibility relation*.

A propositional *Kripke model* is a triple $\langle W, R, \vDash \rangle$, where $\langle W, R \rangle$ is a Kripke frame, and $\vDash$ is a relation between possible worlds and formulas from a propositional language $\mathcal{L}$, called *forcing*.

The forcing discipline between possible worlds and atomic formulas of $\mathcal{L}$, is called a *set-up*.

Relations R and $\vDash$ fulfill particular properties depending on the logical system we are to model. For instance, if Intuitionistic Logic is to be modeled, R must be a preorder and the following set-up discipline is applied, for any $w, w' \in W$ and atomic p:

$$if\ w \vDash p\ and\ \langle w, w' \rangle \in R,\ then\ w' \vDash p \quad (monotonicity)$$

so that any atomic formula (and, by induction, any formula – see below) is mapped on an order filter (i.e. an R-neighborhood) of the frame.

On the contrary, if we are dealing with a Modal Logic, R does have to fulfill specific properties depending on the logic to be modeled (cf. Part III). The forcing relation between worlds and atomic formulas is arbitrary and does not depend on the relation R, which plays a role just in the forcing clauses for modal formulae (see below), so that any atomic formula is mapped onto some generic subset of W.

4.13.1 Necessity and Possibility

As to well formed formulas, again the forcing discipline changes along with the logical system. We give, as examples, the forcing discipline for Intuitionistic Logic and standard (Diodorean) modal logics:

Forcing	Intuitionistic logic	Modal logic
$w \vDash \neg\alpha$	$\forall w'((\langle w, w' \rangle \in R) \Rightarrow (w' \nvDash \alpha))$	$w \nvDash \alpha$
$w \vDash \alpha \vee \beta$	$w \vDash \alpha$ or $w \vDash \beta$	$w \vDash \alpha$ or $w \vDash \beta$
$w \vDash \alpha \wedge \beta$	$w \vDash \alpha$ & $w \vDash \beta$	$w \vDash \alpha$ & $w \vDash \beta$
$w \vDash \alpha \Longrightarrow \beta$	$\forall w'((\langle w, w' \rangle \in R$ & $w \vDash \alpha) \Rightarrow (w' \vDash \beta))$	$w \vDash \neg\alpha$ or $w \vDash \beta$
$w \vDash \Box\alpha$		$\forall w'((\langle w, w' \rangle \in R) \Rightarrow (w' \vDash \alpha))$
$w \vDash \Diamond\alpha$		$\exists w'(\langle w, w' \rangle \in R$ & $w' \vDash \alpha)$

We shall set, for any formula α, $[\![\alpha]\!] = \{w \in W : w \vDash \alpha\}$. From the above definition, it is clear that $[\![\alpha]\!] \implies [\![\beta]\!]$ holds if and only if $[\![\alpha]\!] \subseteq [\![\beta]\!]$.

The properties of the modal operators defined by means of a relation depend on the properties of the relation itself. This is not surprising. For instance we have seen in *Corollary* 3.3.4 that $[R_{\mathbf{S}}]$ and $[R_{\mathbf{S}}^{\smile}]$ are interior operators. Hence, for instance, $[R_{\mathbf{S}}](X) \subseteq X$ and $[R_{\mathbf{S}}][R_{\mathbf{S}}](X) = [R_{\mathbf{S}}](X)$. These properties depend on *Proposition* 3.3.1 which in turn depends on reflexivity and, respectively, transitivity of the relation $R_{\mathbf{S}}$ (cf. *Proposition* 3.1.3).

Indeed, in general we have the following correspondences, given a Kripke frame $\langle W, R \rangle$:

1. $[R]([\![\alpha]\!]) \implies [\![\alpha]\!]$, for all formula α, is fulfilled in the class of reflexive frames.

2. $[R][R]([\![\alpha]\!]) \Longleftrightarrow [R]([\![\alpha]\!])$, for all formula α, is fulfilled in the class of transitive frames.

We shall see in Frame 4.13.4 that, however, some important properties of relations cannot be expressed in pure necessity and possibility terms.

4.13.2   Basic Operators as Modal Operators

In view of the above forcing rules, if we interpret the condition $w \vDash \alpha$ in a Kripke model with the condition $w \in [\![\alpha]\!]$, where w belongs to a set W representing the validity domain of a formula α, then it is not difficult to understand why we used the symbols $[i], [e], \langle i \rangle, \langle e \rangle, [R], \langle R \rangle, [R^{\smile}]$ and $\langle R^{\smile} \rangle$ to denote our operators. Indeed we leave as an easy verification that, for instance, $[R](X)$ meets the forcing clause for $\Box(X)$ if we consider *P-system* as generalised Kripke frames, generalised in the sense that the accessibility relation, $\Vdash$ in this case, is between the elements of two possibly distinct sets.

On the contrary we can consider a Quantum Relational System as a normal Kripke frame, for Intuitionistic Logic or an S4 modal logic, because any i-quantum relation is a preorder. Eventually, if $R_{\mathbf{S}}$ is an equivalence relation, then $\mathbf{Q}(\mathbf{S})$ is a Kripke frame for $\mathbf{S}$5-modal logic. The reader will find every details in Part III.

4.13.3 Ramified Tense Logic

Another kind of modal interpretation can be of some help to recall some concepts introduced in this Chapter. In fact we can consider an IQRS as a Kripke model $\mathbf{O} = \langle W, R, \vDash \rangle$ for a ramified tense logic like K_t (see Prior [1957]).

For tense formulas α we have the following clauses:

1. $w \models G(\alpha)$ iff $\forall w'((\langle w, w' \rangle \in R) \Rightarrow (w' \models \alpha))$ ("always in the future").

2. $w \models P(\alpha)$ iff $\exists w'(\langle w, w' \rangle \in R \ \& \ w' \models \alpha)$ ("sometimes in the past").

3. $w \models H(\alpha)$ iff $\forall w'((\langle w', w \rangle \in R) \Rightarrow (w' \models \alpha))$ ("always in the past").

4. $w \models F(\alpha)$ iff $\exists w'(\langle w', w \rangle \in R \ \& \ w' \models \alpha)$ ("sometimes in the future").

From the discussion of Section 3.3 of this Chapter, we can conclude that the following correspondence holds, with respect to any IQRSs:

Basic operator	$[R]$	$\langle R^{\smile} \rangle$	$[R^{\smile}]$	$\langle R \rangle$
Modeled tense operator	G	P	H	F

4.13.4 Necessity and Sufficiency Operators

In Humberstone [1983] I. Lloyd Humberstone considered relational logics in which modal operators are defined in terms of a notion of "inaccessibility". More precisely, given a Kripke model $\mathbf{K} = \langle W, R, \vDash \rangle$ we can consider a model on the complementary frame $-\mathbf{K} = \langle W, -R \rangle$. Obviously, if $\langle w, w' \rangle \in R$ means "world w' is accessible from world w" then $\langle w, w' \rangle \in -R$ means "world w' is inaccessible from world w".

On $-\mathbf{K}$ one can define the same *sufficiency* operator $[[R]]$ which was introduced in this Chapter. Indeed, we know that the equality $[[R]](X) = [-R](-X)$ holds (see Exercise 2.5), so that the notion of "inaccessibility" lays at the core of the notion of a "sufficiency operator".

Indeed, Humberstone's original operator is defined as $\blacksquare(X) = [-R](X)$, so that $[[R]]$ is an immaterial departure from the original definition.

Technically the operation $[[R]]$, together with its dual $\langle\langle R\rangle\rangle$ and the usual modalities, is able to express properties of relational frames which are not expressible in the customary modal language. For instance:

1. The class of irreflexive frames, expressed by: $[[R]](\alpha) \implies [\![\alpha]\!]$, all formula α,

2. The class of asymmetric frames, expressed by: $[\![\alpha]\!] \implies \langle\langle R\rangle\rangle([\![\alpha]\!])$, all formula α,

3. The class of intransitive frames, expressed by: $[[R]]([\![\alpha]\!]) \implies [[R]][[R]]([\![\alpha]\!])$, all formula α,

and other properties.

But, as Humberstone points out:

"Apart from any purely technical interest such examples may have, some of them are of obvious relevance when the unenriched language is thought of in tense-logical terms (thinking of $\square$ as Prior's G, – see Frame 4.13.3 (authors' note)).

Further applications are suggested by the fact that we may sometimes wish to consider an operator O with the reading: $O(\alpha)$ is true iff α is true only at accessible (sc. R-related) worlds, since $O(\alpha)$ may be defined as $\blacksquare(\sim\alpha)$ (i.e. $[[R]](\alpha)$ – authors' note). For example, in deontic logic, where $\square$ and $\lozenge$ are thought of as expressing obligatoriness and permissibility, respectively, one may wish to consider the unorthodox notion of strong permissibility, P, defined thus: $P(\alpha) =_{df} [[R]](\alpha)$, since all frames will then validate the equivalence $(P(\alpha) \wedge P(\beta)) \longleftrightarrow P((\alpha) \vee (\beta))$ much loved by the proponents of 'free-choice' permission." (we used our own symbols).

Notice that the latter equivalence stated in the quotation is exactly the anti-additivity of $[[R]]$ and $[[R^{\smile}]]$.

4.13.5   Modal Operators and Information Systems

The fact that derivative operators in Concept Lattices are sufficiency operators was probably first noted in Pagliani [1996] were the formula $[[E]][E](R)$ (there written $[E](R)[E]$) was exploited in order to approximate a relation R by means of an equivalence relation E, in the framework of Relation Algebra (see also Pagliani [1998b] for further generalisations and applications).

Within the same framework Ewa Orłowska noted that the so-called "weakest pre-specification" in program semantics is indeed an application of a sufficiency operator (see also Frame 4.14.1).

More details about the above results are in Frame 15.18.2 of Part III.

Here it must be noted that sufficiency operators have been systematically studied in the field of Information Systems and Spatial Reasoning by E. Orłowska and I. Düntsch (cf. [Düentsch Orlowska 1999 and 2001]).

An interesting application is given by Qualitative Geometry (cf. the latter paper). Indeed, in spatial reasoning one have to deal with *contact relations*.

A relation C on a set U is a contact relation if the following properties are satisfied:

(R) C is reflexive; (S) C is symmetric; (E) $C(x) = C(y)$ implies $x = y$.

In order to express contact relations one needs both necessity and sufficiency operators. In fact reflexivity can be expressed by necessity (cf. Frame 4.13.1). But in view of Frame 4.13.4 symmetry is likely to need sufficiency in order to be expressed. Further the extensionality principle (E) happens to need both of them.

Indeed one has to use modal structures of type $\langle \wp(U), \langle R \rangle, [[R]] \rangle$ from a Kripke frame $\langle U, R \rangle$. Such structures (or better their abstract counterparts – see the quoted papers) are called *Mixed modal-sufficiency algebras* or MIAs.

Let us define $e : \wp(U) \times \wp(U) \longmapsto \wp(U); e(X,Y) = [R](X) \cap [[R]](Y)$ and set for all $X \subseteq U$, $m(X) = e(X, -X)$. Then R is a contact relation if and only if the following properties hold:

(R') $[R](X) \subseteq X$; (S') $X \subseteq [[R]][[R]](X)$;
(E') $m(-(e(X,X) \cap -Y)) \cup m(-(e(X,X) \cap Y)) = U$.

Another interesting application of the sufficiency operator is given in Gegida & Düntsch [2002]. Suppose we are given a set Q of *problems* and a set S of *skills*. Let $R \subseteq Q \times S$ a binary relation. There are variants with respect to the interpretation of relation R: a) $\langle q, s \rangle \in R$ if with skill s it is possible to solve q; b) $\langle q, s \rangle \in R$ if skill s is necessary to solve q and $R(q)$ is the minimal set of skills which are sufficient to solve q. We can then define the usual operators $[R] : \wp(S) \longmapsto \wp(Q)$, $\langle R^\smile \rangle : \wp(Q) \longmapsto \wp(S)$, $[[R]]$ and $[[R^\smile]]$ (we are using our notation). Therefore, for any $P \subseteq Q$, if $q \in [R]\langle R^\smile \rangle(P)$ then with the skills which

are needed to solve all the problems in P it is possible to solve q:

$$q \in [R]\langle R^{\smile}\rangle(P) \text{ iff } \bigcap\{R(p) : p \in P\} \subseteq R(q).$$

Sufficiency operators give a symmetric and dual reading: if $q \in [[R]][[R^{\smile}]](P)$ then the minimal set of skills sufficient to solve q is included in the minimal set of skills which is sufficient to solve all the problems in P:

$$q \in [[R]][[R^{\smile}]](P) \text{ iff } R(q) \subseteq \bigcup\{R(p) : p \in P\}.$$

Finally, it is worth mentioning that the operator $[[R]]$ has ben used in Cattaneo et al. [1993] to model an intuitionistic-like orthocomplementation in Quantum Logic (for this logic see also Frame 4.6.1).

4.14   Frame – Galois Adjunctions

The notion of a "Galois Adjunction" is derived from Évariste Galois' investigations of necessary and sufficient conditions for a polynomial equation to be solvable by radicals. In this analysis Galois developed the notion of a correspondence between groups and fields (Galois' results were however published fourteen years after his early death in the *Journal de mathématiques pures et appliquées*, 1846).

Roughly speaking, given a field K and an extension K' of K, let F be the set of subfields of K' that contain K. Consider the ordered set $\langle F, \subseteq \rangle$. Let $e \in F$ and let us denote by $G(K'/e)$ the group of field automorphisms f of K' such that $f(e) = e$. Now, let S be the set of subgroups of $G(K'/e)$. Consider $\langle S, \subseteq \rangle$. Given a subgroup $g \in S$, let us define $Fix(g)$ to be the field of all elements of K' that are fixedpoints of all elements of g. Then the maps $e \longmapsto G(K'/e)$ and $g \longmapsto Fix(g)$ form an axiality.

This technique was generalised by Oystein Ore in his classic work, where the notion of a Galois connection was in fact introduced. Moreover, in Galois connections Saunders Mac Lane recognised the structure of an adjoint pair of functors (see Frame 4.14.4). Later on, the topic was rearranged by Melvin Janowitz and Thomas Blyth in Blyth & Janowitz [1972] within the general framework of residuated lattice.

The centrality of the notion of a Galois adjunction is witnessed by the number of examples in mathematics and logic, as we see below.

4.14.1   Galois Adjunctions in Computer Science

The notion of a Galois adjunction has been largely exploited in Logic and Computer Science. As an example we cite (Gierz et al. [1980]) and the references quoted thereby. A seminal and interesting application is given by Rohit Parikh in Parikh [1982]. Let W be a set of program states and L a language. Suppose $\vDash$ is a satisfaction relation between the elements of W and L. Given $\Gamma \subseteq L$ we can define the "model" of Γ as the set $Mod(A) = \{s \in W : \forall A \in \Gamma, s \vDash A\}$. Dually, given $X \subseteq W$ we define the "theory" of X as the set $Th(X) = \{A \in L : \forall s \in X, s \vDash A\}$. It is clear that $\langle Th, Mod \rangle$ forms a Galois connection between $\wp(W)$ and $\wp(L)$. Moreover, one can prove that the closure operator $J(X) = Mod(Th(X))$ on W is topological if the following conditions are satisfied: (*falsehood*): there is an element $\bot \in L$ such that $Mod(\bot) = \emptyset$, (*disjunction*): for every $A, B \in L$ there is a C such that $Mod(C) = Mod(A) \cup Mod(B)$.

   If the above applications aim at approximating states in programs which fulfill some required property, Galois connections are also used, in Computer Science, to approximate computations on a given domain by means of computations on a simpler domain.

   For instance consider $L = \langle \wp(\mathbb{Z}). \subseteq \rangle$, where $\mathbb{Z}$ is the set of integers. Any $X \subseteq \mathbb{Z}$ can be approximated by its minimum and maximum elements, so that we can define a function $\alpha(X) = [min(X), max(X)]$. We set a new domain $L' = \{\bot\} \cup \{[l, u] : l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{+\infty\}, l \leq u\}$. L' is ordered as follows: a) $\bot \sqsubseteq [l, u]$, any $[l, u] \in L'$; b) $[l_1, u_1] \sqsubseteq [l_2, u_2]$ iff $l_2 \leq l_1 \leq u_1 \leq u_2$. For instance $[3, 9] \sqsubseteq [3, 11]$ and $\alpha\{2, 5, 3\} = [2, 5]$. We further set $\alpha(\bot) = \emptyset$.

   Now we have to define a function δ from L' to L such that $L' \dashv^{\alpha, \delta} L$. But we know how to do it: $\delta([l, u]) = sup(\alpha^{-1}(\downarrow [l, u])) = \bigcup \{X : l \leq min(X) \leq max(X) \leq u\} = \{x : l \leq x \leq u\}$. For instance $\delta([2, 5]) = \bigcup\{\{2, 5\}, \{3, 4\}, \{2, 3, 5\}, ...\} = \{2, 3, 4, 5\}$. We further set $\delta(\bot) = \emptyset$.

   Let us verify an example of the connection properties: $\alpha(\{2, 3, 5\}) = [2, 5] \sqsubseteq [1, 5]$ and $\{2, 3, 5\} \subseteq \{1, 2, 3, 4, 5\} = \delta([1, 5])$. As to the other way around, $\{1, 3\} \subseteq \{1, 2, 3, 4\} = \delta([1, 4])$ and $\alpha(\{1, 3\}) = [1, 3] \sqsubseteq [1, 4]$.

   Now we can associate any $X \subseteq \mathbb{Z}$ with the assertion $X(\Pi(x))$ stating that the value of a variable x during execution of program Π can vary just on X. Thus if $X \subseteq X'$ the assertion $X(\Pi(x))$ is more precise than $X'(\Pi(x))$, because in the former the variability of x is more restricted. In this sense "$\subseteq$" is said to be a "concrete approximation relation".

Moving to the side of abstraction L' we have that $\alpha(\{1,2,5\}) = [1,5] \sqsubseteq [1,8] = \alpha(\{1,2,5,8\})$ (as the upper adjoint α is monotone). We have also, for instance, $[1,5] \sqsubseteq [-3,5]$ and $[1,9] \sqsubseteq [-3,5]$ but $[1,9] \not\sqsubseteq [-3,5]$ because $9 \not\leq 5$ and $[-3,5] \not\sqsubseteq [1,9]$ for $1 \not\leq -3$. In fact, $[1,5](\Pi(x))$ is the best abstract approximation of the concrete assertion $\{1,2,5\}(\Pi(x))$. In other terms, $\alpha(X)$ is the *abstraction* of X in the sense that it is the most precise approximation of X in L'. In turn, $\delta(y)$ is the *concretisation* of y, that is, the most imprecise element of L which can be approximated by y. Obviously, in moving from concrete to abstract approximations we can lose interesting information.

A comparison between this approach and other techniques with the same aim can be found in Cousot & Cousot [1992].

Another prominent application in Computer Science, namely the calculus of *weakest pre (post) specifications*, is presented in Hoare & He [1986], and briefly discussed in Part III, as mentioned in Frame 4.13.5.

Finally we must mention the *inverse limit construction* for recursive domain. This technique is based on the concept of a *projection pair*, $\langle \epsilon, \pi \rangle$, where ϵ and π are continuous functions such that $\mathbf{D} \dashv^{\epsilon,\pi} \mathbf{E}$, $\mathbf{D}$ and $\mathbf{E}$ are *Domains* (that is, roughly speaking, sets equipped with a partial order $\sqsubseteq$ to be intended as "(information) approximation" – see Frame 4.4.2), and ϵ is injective. Thus, as we know from Section 1.3 and *Proposition* 1.4.10, π is onto and is a retraction of ϵ so that $\epsilon \circ \pi = 1_D$ and $\pi \circ \epsilon \sqsubseteq 1_E$.

As is explained in Paulson [1987], "If $\mathbf{D}$ can be embedded in $\mathbf{E}$, this means that $\mathbf{E}$ has a richer structure, and can represent the structure $\mathbf{D}$. If elements of $\mathbf{D}$ are mapped into $\mathbf{E}$ and back again, no information is lost. The mapping in the reverse direction, from $\mathbf{E}$ to $\mathbf{D}$ and back, may lose information. [...] A projection pair is something like a pair of isomorphism functions, but isomorphisms preserve information in both directions".[6]

Since ϵ determines π we can indicate that there is a projection pair between $\mathbf{D}$ and $\mathbf{E}$ by writing $\mathbf{D} \lhd_\epsilon \mathbf{E}$. If we have a chain of embeddings $\mathbf{D}_0 \lhd_{\epsilon_0} \mathbf{D}_1 \lhd_{\epsilon_1} \cdots \lhd_{\epsilon_{n-1}} \mathbf{D}_n \cdots$ the inverse limit construction makes it possible to construct a domain $\mathbf{D}_\infty$ that contains all the $\mathbf{D}_n$ and such that it includes least upper bounds of increasing chains (a sort of continuity property).

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[6] Pay attention that in the quoted book $\epsilon \circ \pi$ is reversed in the functional notation $\pi \circ \epsilon$.

Exercise 4.3. *Let $D = \langle\{\bot, 0, 1, \top\}, \sqcap\rangle$ with the operation $\sqcap$ given by the ordering $\bot \leq x$, any x in D, $0 \leq \top, 0 \leq 0$, $1 \leq \top, 1 \leq 1$, $\top \leq \top$. Let $\alpha : L \longmapsto D$ defined as:*

$$\alpha(X) = \begin{cases} \bot & \text{if } X = \emptyset \\ 1 & \text{if } \forall n \in X(n > 0) \\ 0 & \text{if } \forall n \in X(n < 0) \\ \top & \text{otherwise} \end{cases}$$

Compute the lower adjunction δ of α.

4.14.2   Galois Adjunctions and Dedekind Cuts

A fundamental application of the notion of a Galois adjunction is given by Dedekind's definition of real numbers by means of a series of encapsulated rational intervals (cf. R. Dedekind, "*Stetigkeit und irrationale Zahlen*", Vieweg, 1872). A typical example of a Dedekind cut is given by the pair

$$\langle\{a \in \mathbb{Q} : (a^2 < 2) \vee (a \leq 0)\}, \{b \in \mathbb{Q} : (b^2 \geq 2) \wedge (b > 0)\}\rangle,$$

which represents the real number $\sqrt{2}$.

Dedekind's approach can be straightforwardly generalised as follows. Let $\langle X, \leq\rangle$ be an ordered set then $\mathbf{X} = \langle X, X, \leq\rangle$ is a *P-system*. Let us call an $A \subseteq X$ such that $A \in \Gamma_{est}(\mathbf{X})$ a "cut". Thus $p \in A$ if p is less than or equal to all the elements which are greater than or equal to all the elements of A.

It follows that cuts coincide with formal concepts when $\mathbf{X}$ is regarded as a Formal Context (see Frame 4.9). The set of cuts on X is called the *Dedekind-MacNeille completion* (or *completion by cuts* or *normal completion*) of X and denoted as $\mathcal{DM}(\mathbf{X})$ (see H. K. MacNeille, "Partially ordered sets". *Trans. Amer. Math. Soc.*, 42, 1937, pp. 90–96). From *Proposition* 2.3.2.(5) we immediately have that the Dedekind-MacNeille completion of any ordered set X is a complete lattice. Moreover, the map $\phi : X \longmapsto \mathcal{DM}(\mathbf{X}); \phi(x) = \langle e \rangle(x) = \downarrow x$ is an order embedding. That is, ϕ preserves all existing infs and sups of X.

EXAMPLE
Consider the preordered set $\mathbf{Q}(\mathbf{P}) = \langle G, R_{\mathbf{P}}\rangle$ where $R_{\mathbf{P}}$ is the i-quantum relation depicted in EXAMPLE 3.1.2. Then $\mathcal{DM}(\mathbf{Q}(\mathbf{P})) = \mathbf{Sat}_{est}(\mathbf{Q}(\mathbf{P}))$ is:

$$G$$

$$\{a', a'', a'''\} \qquad\qquad \{a, a', a'''\}$$

$$\{a', a'''\}$$

$$\{a'''\} \qquad\qquad\qquad \{a\}$$

$$\emptyset$$
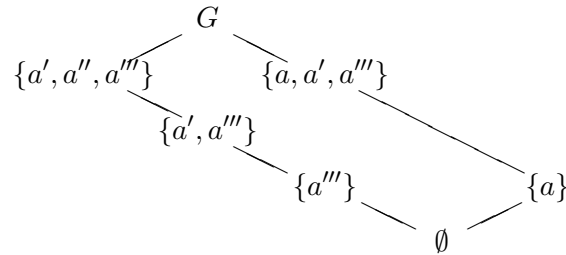
In Sambin & Gebellato [1998] one can find a connection between construction of points in Formal Topology (cf. Frame 4.11) and Dedekind's construction of real numbers.

4.14.3   Galois Adjunctions at Large

Galois adjunctions appear in a number of fields and under different forms. In general one can say that when two statements are connected by an "if and only if" operator, then the two statements are likely to form a Galois adjunctions. Let us see some example:

Logic	Set Theory	Number Theory
$\neg a \Rightarrow b \equiv \neg b \Rightarrow a$	$-X \subseteq Y \equiv -Y \subseteq X$	$-n \leq m \equiv -m \leq n$
$a \wedge b \Rightarrow c \equiv b \Rightarrow (a \Rightarrow c)$	$X \cap Y \subseteq Z \equiv X \subseteq -Y \cup Z$	$n + m \leq z \equiv n \leq z - m$

Indeed the implication operator $\Rightarrow$ defines an order among propositions (i.e. $a \leq b$ iff $a \Rightarrow b$ is true). Thus the first logical statement reads "$\neg(a) \leq b$ iff $a \geq \neg(b)$" so that it is an example of a Galois connection, because the order is reversed by switching the application of function $\neg$. The second logical statement reads "$\wedge_a(b) \leq c$ iff $b \leq \Rightarrow_a (c)$, so that we are in the presence of a Galois adjunction $\wedge \dashv \Rightarrow$ (note that we have parameterized the logical constants $\wedge$ and $\Rightarrow$ by means of formulas). Since the lower adjunction $\wedge$ must preserve co-limits, i.e. *sups*, it follows that for the second statement to hold $\wedge$ must distribute over $\vee$. The second logical statement is also called "Currying property" (we refer the reader to Part II for a detailed discussion on these topics).

Strictly connected with the latter example is the following one. Let X be a set and $A \subseteq X$. Let $F_A(B) = A \cap B$ and $G_A(B) = -A \cup B$, for any $B \subseteq X$. Then $F \dashv G$ on $\langle \wp(X), \subseteq \rangle$. Indeed, this is an instance (in a Boolean algebra of sets) of the adjunction between meet and implication.

To conclude, another very general example of Galois adjunction is indeed a generalisation of Galois' original application. Let X be any mathematical object with an underlying set, such as a group, a ring or a vector space or others. For any subset A of X, let $F(A)$ be the smallest subobject of X which contains A (for instance the subgroup, subring or subspace generated by A). On the other side, for any subobject B of X, let $G(B)$ be the underling set generating B. Then we have $F \dashv G$ (G is also called a "forgetful" functor because it forgets the structure of its arguments).

Exercise 4.4. *in Corollary 1.4.1 we have proved that the adjunction relationship* $\langle \wp(A), \subseteq \rangle \dashv^{f^{\rightarrow}, f^{\leftarrow}} \langle \wp(B), \subseteq \rangle$ *holds. Find the upper adjoint of* $f^{\leftarrow}$ *(if any).*

4.14.4 Galois Connections and Representation Theorems

The duality between I-quantum-relational structures and *P-systems* which has been proved in *Proposition* 3.3.4, discussed in Frame 4.6.3 and applied in Frames 4.5 and 4.6.2 is an example of more general techniques aimed at establishing duality connections between some kinds of lattices and some kinds of (topological) ordered structures (or at *representing* one structure in terms of the other).

The origin of these results must be credited to Marshall Stone (see Stone [1936]).

As reported in Johnstone [1982], "[...] Stone was neither an algebraist nor a logician. It was his work in [linear operators in Hilbert space] which led him to the consideration of *algebras of commuting projections* in Hilbert space; it was known that these could be given the structure of Boolean algebras, but they had no natural representation as algebras of subsets."

Stone realized that a Boolean algebra is a ring in which $a \cdot a = a$, for all a. Thus, in analogy with rings he understood the importance of (prime) ideals to set the carrier of the representation construction. Moreover, he realized that the set of prime ideals could be made into a topological space in which principal ideals correspond to *clopen* (that is, both open and closed) sets. Since there is a 1-1 correspondence between principal ideals and elements of the algebra (namely $f(\uparrow x) = x$) we can recover an isomorphic copy of the original algebra.

Independently Garrett Birkhoff arrived at an equivalent duality result between distributive lattices and partially ordered sets (Birkhoff

[1933]). Indeed the results in the present Part and the discussion about pointless topology in the Introduction refer to Birkhoff's duality.

Of late Priestley was able to "topologize" Birkhoff's construction and represent bounded distributive lattices by means of compact *totally ordered disconnected topological spaces* (Priestley [1984]).

In Frame 4.14.2 we have seen that Galois connections make ordered sets into complete lattices (Dedekind-McNeille completion).

Thus, in view of the duality theorems and the (pre)topological properties of Galois operators we can wonder whether Galois connections may be used in order to make particular ordered sets into distributive lattices.

The answer, for the finite case, is positive.

For a formal context $\mathbf{C} = \langle X, X \neq \rangle$, the Concept Lattice $\mathcal{B}(\mathbf{C})$ is isomorphic to the Boolean lattice of all subsets of X. In fact, for any $A \subseteq X$, $[[\neq]](A) = \{x : \forall a(a \in A \Rightarrow a \neq x)\} = -A$. Therefore, $est(A) = A$ and $\mathcal{ITS}(A) = -A$.

For a formal context $\mathbf{C} = \langle X, X \not\geq \rangle$, the Concept Lattice $\mathcal{B}(\mathbf{C})$ is isomorphic to the distributive lattice $\mathbf{I}(\mathbf{C})$ of order ideals of $\mathbf{C}$. In fact for any $A \subseteq X$, $[[\not\geq]](A) = - \downarrow A = \uparrow A'$, for some $A' \subseteq X$. It follows that the elements of $\mathcal{B}(\mathbf{C})$ are exactly the pairs $\langle A, -A \rangle$, for $A \in \mathbf{I}(\mathbf{C})$ (see Wille [1982] and Hartung [1993] for other interesting examples and a thorough study).

Notice that $\mathbf{F}(\mathbf{Sat}_Q(\mathbf{P}))$ (see the example of Frame 4.14.2) is given by closing $\mathbf{Sat}_{est}(\mathbf{Q}(\mathbf{P}))$ under unions and is anti-isomorphic to $\mathbf{Sat}_Q(\mathbf{P})$ (cf. Example 3.3.1).

4.14.5   Galois Adjunctions, Isomorphisms and Approximation: A Note

Notice that a Galois adjunction establishes weaker links between the involved structures than an isomorphism does. In fact, as we have seen, a Galois adjunction gives rise to an isomorphism between certain substructures.

This is why it acts so properly in defining a "natural" understanding of how the elements of a structure may approximate the elements of another (or, more precisely, they can approximate each other).

However these remarks must be complemented with another notable feature of a Galois adjunction (and adjoint functors at large, see below): it picks a particular "optimal" approximation (minimal or maximal) from the set of possible approximations.

4.15 Frame – Categories and Adjoint Functors

Actually, the notions of an "adjoint pair" and of a "duality" are fully defined, in a general setting, within *Category Theory*, where $\leq$ and $\leq'$ are replaced by the morphisms that define the categories we are dealing with. Roughly speaking, categorical morphisms are associative and transitive correspondences between the objects of the given category. Moreover any category is equipped with a null-morphism ν such that for any object o, $\nu(o) = o$. In a category, morphisms are denoted by arrows.

The idea of an adjoint functor was formulated by Daniel Kahn in Kahn [1958] and elaborated by distinguished category theoreticians such as Alexander Grothendieck (see Grothendieck [1957–1962]) and William Lawvere (see Lawvere [1969]).

Eventually, Saunders Mac Lane interpreted Galois connections as adjoint functors (with the right category reversed – see Mac Lane [1971]).

Here we just show the formal definition of adjoint functors with some additional comments related to the topics discussed in this Part.

In the Introduction we have informally used a category-like terminology when speaking of the set of $(0,1)$-homomorphisms in the construction of the abstract points from a frame.

We give now the formal definition of this basic categorial machinery.

Definition 4.15.1. *A category* **C** *consists of a class* $\mathcal{OB}_{\mathbf{C}}$ *of objects, a class* $\mathcal{HOM}_{\mathbf{C}}$ *of morphisms between members of* $\mathcal{OB}_{\mathbf{C}}$. *A morphism is the abstraction of a function or mapping and is denoted by an arrow between objects. The class of morphisms between two objects* $X, Y \in \mathcal{OB}_{\mathbf{C}}$ *is denoted* $\mathcal{HOM}_{\mathbf{C}}(X, Y)$. *Morphisms must fulfill the following properties:*

- *(Composition) If* $f \in \mathcal{HOM}_{\mathbf{C}}(X, Y)$ *and* $g \in \mathcal{HOM}_{\mathbf{C}}(Y, Z)$ *then the composition* $f \circ g \in \mathcal{HOM}_{\mathbf{C}}(X, Z)$ *is always defined (some authors write "$g \circ f$" or "gf"). Such a composition is often described by saying than the following diagram commutes (that is, the two paths along the arrows coincide):*

$$X \xrightarrow{\;f\;} Y$$

with $f \circ g$ and g leading to Z.

- *(Identity) For every object X there exists a morphism $1_X : X \longrightarrow X$ such that for every morphisms $f : A \longrightarrow B$ we have $f \circ 1_B = 1_A \circ f$.*

- *(Associativity) For every $f, g, h \in \mathcal{HOM}_{\mathbf{C}}$ the following equation holds, whenever the compositions are defined: $h \circ (f \circ g) = (h \circ f) \circ g$.*

The interpretation of morphisms depends on the category. So, for instance, if the objects of $\mathbf{C}$ are sets, morphisms are functions. If the objects of $\mathbf{C}$ are groups then morphisms are group homomorphisms, if they are topological spaces then morphisms are continuous functions.

Particularly, if the objects of $\mathbf{C}$ are lattices then morphisms are lattice homomorphisms and if the objects are ordered sets then morphisms are order preserving maps.

Finally, we can represent every partially ordered set as a category in which the objects are the elements of the set and there is at most one morphism between two objects x and y representing the order relation (that is, $x \longrightarrow y$ if and only if $x \leq y$).

One can extend the notion of a morphism from objects of a category to categories themselves. These extensions are called *functors*.

It follows that a functor from a category $\mathbf{A}$ to a category $\mathbf{B}$ will map elements of $\mathcal{OB}_{\mathbf{A}}$ to elements of $\mathcal{OB}_{\mathbf{B}}$ as well as elements of $\mathcal{HOM}_{\mathbf{A}}$ to elements of $\mathcal{HOM}_{\mathbf{B}}$:

Definition 4.15.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be categories. A functor F from $\mathbf{A}$ to $\mathbf{B}$ is a mapping that associates to each object X in $\mathbf{A}$ an object $F(X)$ in $\mathbf{B}$, associates to each morphism $f : X \longrightarrow Y$ in $\mathbf{A}$ a morphism $F(f) : F(X) \longrightarrow F(Y)$ in $\mathbf{B}$, such that the following two properties hold:*

- $F(id_X) = id_{F(X)}$ *for every object $X \in \mathbf{A}$.*

- $F(f \circ g) = F(f) \circ F(g)$, *for all morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$.*

At this point we can introduce the notion of *adjoint functors*:

Definition 4.15.3. *A pair of adjoint functors between two categories* $\mathbf{C}$ *and* $\mathbf{D}$ *consists of two functors* $F : \mathbf{C} \longmapsto \mathbf{D}$ *and* $G : \mathbf{D} \longmapsto \mathbf{C}$ *and a natural isomorphism* $\phi : \mathcal{HOM}_{\mathbf{D}}(F-,-) \longmapsto \mathcal{HOM}_{\mathbf{C}}(-,G-)$ *consisting of bijections* $\phi_{X,Y} : \mathcal{HOM}_{\mathbf{D}}(F(X),Y) \longmapsto \mathcal{HOM}_{\mathbf{C}}(X,G(Y))$, *for all objects* X *in* $\mathbf{C}$ *and* Y *in* $\mathbf{D}$ *and such that the following diagram commutes for all* $f : X' \longrightarrow X$ *in* $\mathbf{C}$ *and* $g : Y \longrightarrow Y'$ *in* $\mathbf{D}$ *(naturality of* ϕ*):*

$$
\begin{array}{ccc}
\mathcal{HOM}_{\mathbf{C}}(X,G(Y)) & \xrightarrow{\;\;\mathcal{HOM}(f,G(g))\;\;} & \mathcal{HOM}_{\mathbf{C}}(X',G(Y')) \\
\Big\| \phi_{X,Y} & & \Big\| \phi'_{X',Y'} \\
\mathcal{HOM}_{\mathbf{D}}(F(X),Y) & \xrightarrow{\;\;\mathcal{HOM}(F(f),g)\;\;} & \mathcal{HOM}_{\mathbf{D}}(F(X'),Y')
\end{array}
$$

As we can easily realize, an adjointness relation does not hold in a void, but it holds with respect to a pair of fixed structures: the domain of F, that will be called the *left category*, and the domain of G that will be called the *right category*.

Every adjoint pair fulfills some interesting properties that we have proved from the abstraction level of Galois adjunction in *Proposition* 1.4.8.(9).

Namely, adjoint pair of functors defines a *unit* η from the functor $1_{\mathbf{C}}$ to $GF(X)$, for all $X \in \mathbf{C}$:

$$\eta_X : X \longrightarrow GF(X); \eta_X = \phi_{X,F(X)}(1_{F(X)})$$

Analogously, one may define a *co-unit* ϵ from FG to $1_{\mathbf{D}}$, for all $Y \in \mathbf{D}$:

$$\epsilon_Y : FG(Y) \longrightarrow Y; \epsilon_Y = \phi_{G(Y),Y}^{-1}(1_{G(Y)})$$

Moreover, units and co-units have the following properties:

$$1_F = (F\eta) \circ (\epsilon G) : F \longrightarrow FGF \longrightarrow F$$

$$1_G = (\eta G) \circ (G\epsilon) : G \longrightarrow GFG \longrightarrow G$$

As we have seen in *Lemma* 1.4.2 (as to Galois adjunction) the most important property of adjoints is "continuity", which we now describe

in a more abstract setting: every functor that has a left adjoint (therefore, a right adjoint) preserves limits (or it "commutes" with limits, or it is *continuous*) and every functor which has a right adjoint (therefore, a left adjoint) preserves colimits (or it "commutes" with colimits, or it is *cocontinuous*).

Further, we can restate *Corollary* 2.3.1.(5) and (6) at a higher level of abstraction by saying that adjoint pairs extend equivalences in the sense that if $\mathbf{C}_1$ is the full subcategory of $\mathbf{C}$ consisting of those objects X of $\mathbf{C}$ for which η_X is an isomorphism and, dually, if $\mathbf{D}_1$ is the full subcategory of $\mathbf{D}$ consisting of those objects Y of $\mathbf{D}$ for which ϵ_Y is an isomorphism, then F and G restricted to $\mathbf{C}_1$ and $\mathbf{D}_1$ yield inverse equivalences of these subcategories. In a sense adjoints are "generalised" inverses. However a right inverse of F (i.e. a functor G such that FG is naturally isomorphic to $1_{\mathbf{D}}$) needs not be a right (or left) adjoint of F.

Finally, the conditions stated in *Proposition* 1.4.7 and *Lemma* 1.4.3 are subsumed by the *Freyd Adjoint Functor Theorem* which states that G has a left adjoint if and only if it is continuous and for every object X of $\mathbf{C}$ there exists a family of morphisms $f_i : X \longrightarrow G(Y_i), i \in I$ such that any other morphism $h : X \longrightarrow G(Y)$ uniquely factorises with some f_i, meaning that the following equation holds: $h = f_i \circ G(t)$, for some $i \in I$ and some $t : Y_i \longrightarrow Y$.

An analogous existence result holds of right adjoints.

4.16   Solutions

- Exercise 1.1

  - (A) A section chooses an element from $f^{\leftarrow}(b)$, for any element b of B. Hence the number of possible sections is given by $\prod_{b \in B} card(f^{\leftarrow}(b))$, where $card(X)$ denotes the cardinality of the a set X. It follows that if there is a $b \in B$ such that for no $a \in A, b = f(a)$, then $card(f^{\leftarrow}(b)) = 0$ and $\prod_{b \in B} card(f^{\leftarrow}(b)) = 0$.

  - (B) A retraction r maps any $f(a)$ to a itself. Then surely in any retraction, for each b such that $b = f(a)$ for some $a \in A$, $r(b) = a$ is required. As to the remaining elements of B, any combination is admissible. It follows that the number of retractions is given by $card(A)^{(card(B)-card(A))}$.

- (C) Let s be a section of f. Then for all $b \in B, f(s(b)) = b$. Moreover, for any $g : C \longmapsto B$, the composition $g \circ s$ is a map from C to A such that $(g \circ s) \circ f = g \circ (s \circ f) = g \circ 1_B = g$. Let $C = B$ and $g = 1_B$. Then $1_B \circ s \circ f = 1_B$. This means that any $b \in B$ must be the $f - image$ of some $a \in A$.

- (D) Let r be a retraction of f. Then for all $a \in A, r(f(a)) = a$. Moreover, for any two maps $g : C \longmapsto A$, $h : C \longmapsto A$, if $g \circ f = h \circ f$ then $g = h$. In fact, $g = g \circ 1_A = g \circ (f \circ r) = (g \circ f) \circ r = (h \circ f) \circ r = h \circ (f \circ r) = h \circ 1_A = h$. In particular, let $C = A$ and $h = g = 1_A$. Then if $f(1_A(a)) = f(1_A(a'))$, we have $1_A(a) = 1_A(a')$, i.e. $a = a'$, so that f is injective.

- Exercise 1.2 The map f' cannot have an upper adjoint because it is not $\vee$-preserving. In fact $f'(c \vee d) = f'(1) = y \neq x = x \vee x = f'(c) \vee f'(d)$. Similarly, f'' cannot have a lower adjoint because it does not preserve meets ($f''(a \wedge b) = f''(c) = x \neq y = y \wedge y = f''(a) \wedge f''(b)$). Notice, though, that both f' and f'' are monotonic.

- Exercise 1.3 (**1**) From *Proposition* 1.4.8.(5) we have $\iota\sigma(x \vee y) \geq \iota\sigma(x) \vee \iota\sigma(y) = x \vee y$. But from *Corollary* 1.4.2.(1) $x \vee y \geq \iota\sigma(x \vee y)$. Thus $\iota\sigma(x \vee y) = x \vee y$ whenever x and y are fixpoints of $\iota\sigma$. (**2**) By duality from (**1**). (**3**) and (**4**). From *Proposition* 1.4.8.(7) we have $\varepsilon\varsigma(x \wedge y) \leq \varepsilon\varsigma(x) \wedge \varepsilon\varsigma(y) = x \wedge y$ because $x, y \in Im_{\iota\sigma}$. But from *Corollary* 1.4.2.(2) $x \wedge y \leq \varepsilon\varsigma(x \wedge y)$. Thus $\varepsilon\varsigma(x \wedge y) = x \wedge y$ whenever x and y are fixpoints of $\varepsilon\varsigma$. Analogously for $\varsigma\varepsilon$.

- Exercise 2.1 $-(R(X)) = -\{y : \exists x(x \in X \ \& \ \langle x, y \rangle \in R)\} = \{y : \forall x \neg (x \in X \ \& \ \langle x, y \rangle \in R)\} = \{y : \forall x(x \notin X \vee \langle x, y \rangle \notin R)\} = \{y : \forall x(x \in X \Rightarrow \langle x, y \rangle \notin R)\} = -\{y : \forall x(\langle x, y \rangle \in R \Rightarrow x \notin X)\}$.

- Exercise 2.2

  - (A) (1) $\langle\langle i \rangle\rangle(\{a, a'\}) = \{b, b'\}$; (2) $[[i]](\{a, a''\}) = \{b''\}$; (3) $[i](\{a, a'\}) = \{b, b'\}$. Notice that the first and the third results coincide just by chance.

  - (B) $\langle\langle i \rangle\rangle(X) = -\{b : \forall a(a \in X \vee \langle b, a \rangle \in \Vdash^{\smile})\} = -\{b : \forall a \neg (a \notin X \ \& \ \langle b, a \rangle \notin \Vdash^{\smile})\} = -\{b : \neg \exists a(a \notin X \ \& \ \langle b, a \rangle \notin \Vdash^{\smile})\} = \{b : \exists a(a \in -X \ \& \ \langle b, a \rangle \notin \Vdash^{\smile})\}$.

But the last condition can be read: $\{b : \exists a(\langle a, b\rangle \in - \Vdash$ & $a \in -X)\}$ which describes the set $\{b : \exists a(a \in -X$ & $\langle a, b\rangle \in - \Vdash)\}$, that is, $(- \Vdash)(-X)$.

- Exercise 2.3

  - (A) $\langle \alpha \rangle$ distributes over unions. Directly from definitions, in view of the fact that $\exists x(x \in A \cup B)$ is equivalent to $\exists x(x \in A) \vee \exists x(x \in B)$.

  - (B) $[\alpha]$ distributes over intersections. Directly from definitions, in view of the fact that $\forall x(x \in A \cap B)$ is equivalent to $\forall x(x \in A) \wedge \forall x(x \in B)$.

  - (C) (i) $[[i]](X \cup X') = [[i]](X) \cap [[i]](X')$, any $X, X' \subseteq G$; (ii) $[[e]](Y \cup Y') = [[e]](Y) \cap [[e]](Y')$, any $Y, Y' \subseteq M$. Transform the defining formulas as follows, where $\alpha(x, y) \equiv y \in obs(x)$ or $\alpha(x, y) \equiv x \in sub(y)$: (a) $\forall x(x \in A \cup B \Rightarrow \alpha(x, y))$, (b) $\forall x(\neg(x \in A \cup B) \vee \alpha(x, y))$, (c) $\forall x((\neg(x \in A) \wedge \neg(x \in B)) \vee \neg\alpha(x, y))$, (d) $\forall x((\neg(x \in A) \vee \alpha(x, y)) \wedge (\neg(x \in B) \vee \alpha(x, y)))$, (e) $\forall x(\neg(x \in A) \vee \alpha(x, y)) \wedge \forall x(\neg(x \in B) \vee \alpha(x, y))$, (f) $\forall x(x \in A \Rightarrow \alpha(x, y)) \wedge \forall x(x \in B \Rightarrow \alpha(x, y))$. Now, by forming the related sets with respect to the variable y, we obtain the result.

  - (D) Since $X \cap X' \subseteq X, X \cap X' \subseteq X'$ and $[[i]]$ is anti-tonic, $[[i]](X \cap X') \supseteq [[i]](X), [[i]](X \cap X') \supseteq [[i]](X')$. Hence $[[i]](X \cap X') \supseteq [[i]](X) \cup [[i]](X')$, any $X, X' \subseteq G$. Similarly we can prove that $[[e]](Y \cap Y') \supseteq [[e]](Y) \cup [[e]](Y')$, any $Y, Y' \subseteq M$.

- Exercise 2.4

  - (A) $int(X) = \langle e\rangle[i](X)$, but $[i](X) = \{m : \langle e\rangle(m) \subseteq X\}$. Hence $int(X) = \langle e\rangle(\{m : \langle e\rangle(m) \subseteq X\})$. Moreover, $\langle e\rangle(\{m : \langle e\rangle(m) \subseteq X\}) = \langle e\rangle(\bigcup\{\{m\} : \langle e\rangle(m) \subseteq X\})$ and since $\langle e\rangle$ is additive the latter expression turns into $\bigcup\{\langle e\rangle(m) : \langle e\rangle(m) \subseteq X\}$.

  - (B) $cl(X) = -int(-X)$. Thus $cl(X) = -\langle e\rangle(\bigcup\{\{m\} : \langle e\rangle(m) \subseteq X\}) = [e](-\bigcup\{\{m\} : \langle e\rangle(m) \subseteq X\}) = [e](\bigcap\{-\{m\} : \langle e\rangle \subseteq -X\})$. But $\langle e\rangle(m) \subseteq -X$ & $\langle e\rangle(m') \subseteq -X$ implies $\langle e\rangle(m) \cup \langle e\rangle(m') \subseteq -X$ which is equivalent, by

additivity of $\langle e \rangle$, to $\langle e \rangle(\{m\} \cup \{m'\}) \subseteq -X$. Thus, by applying a De Morgan law we obtain $[e](\bigcap\{-\{m\} : \langle e \rangle(m) \subseteq -X\}) = [e](\bigcup\{\{m\} : \neg(\langle e \rangle(m) \subseteq -X)\}) = [e](\{m : \langle e \rangle(m) \cap X \neq \emptyset\})$.

- (C) From the above proof, $cl(X) = [e](\bigcap\{-\{m\} : \langle e \rangle(m) \subseteq -X\}) = [e](\bigcap\{-\{m\} : X \subseteq -\langle e \rangle(m) \subseteq\})$.

- (D) From the above proof, $cl(X) = [e](\bigcap\{-\{m\} : X \subseteq -\langle e \rangle(m) \subseteq\})$. But m is arbitrary; so set $-\{m\} = Z$ for some $Z \subseteq G$. Moreover, $-\langle e \rangle(m) = [e](-\{m\})\}$ so that the above equation turns into $cl(X) = [e](\bigcap\{Z : X \subseteq [e](Z)\})$.

- Exercise 2.5

  - (A) We know that $Y = [i](X)$ and $Y' = [i](X')$ for some $X, X' \subseteq G$ (see *Lemma* 2.3.1.(4)). Hence, from hypothesis, $\langle e \rangle[i](X) = \langle e \rangle[i](X')$ and, therefore, $[i]\langle e \rangle[i](X) = [i]\langle e \rangle[i](X')$. But $[i]\langle e \rangle[i] = [i]$, whence $[i](X) = [i](X')$ so that $Y = Y'$.

  - (B) Let us notice that $[[i]](A)$ has the logical form $\forall x(A(x) \Rightarrow C(x))$. Therefore: $[[i]](A \cap B)$ has the logical form $\forall x((A(x) \& B(x)) \Rightarrow C(x))$ which is equivalent to $\forall x(\neg A(x) \vee \neg B(x) \vee C(x))$ which is implied by $\forall x(\neg A(x)) \vee C(x)) \vee \forall x(\neg B(x) \vee C(x))$. The last formula is the logical companion of $[[i]](A) \cup [[i]](B)$.
    Similarly, $[[i]](A) \cap [[i]](B)$ has the logical form $\forall x(\neg A(x)) \vee C(x)) \& \forall x(\neg B(x) \vee C(x))$ which is equivalent to $\forall x((\neg A(x) \vee C(x)) \& (\neg B(x) \vee C(x)))$, so to $\forall x(C(x) \vee \neg(A(x) \vee B(x)))$ and finally to $\forall x((A(x) \vee B(x)) \Rightarrow C(x)))$. The last formula is the logical counterpart of $[[i]](A \cup B)$.

  - (C) The answer is "No". Indeed, we can have $X \subseteq \langle e \rangle(Y)$ but for all $b \in Y$, $X \nsubseteq \langle e \rangle(b)$.

  - (D) $\langle e \rangle(m) \supseteq X$ if and only if for all $g \in X$, $g \Vdash m$, if and only if $m \in [[i]](X)$.

  - (E) $\neg\langle e \rangle(-Y) = \{a \in G : \neg\exists b(b \in -Y \& a \in sub(b))\} = \{a \in G : \forall b(b \notin -Y \vee a \notin sub(b))\} = \{a \in G : \forall b(a \in sub(b) \Rightarrow b \in Y))\} = [e](Y)$.

  - (F) $\langle\langle i \rangle\rangle(X) = -[[i]](-X) = -\{b : \forall a(a \notin X \Rightarrow a \Vdash b)\} = \{b : \neg\forall a(a \in X \vee a \Vdash b)\} = \{b : \exists a(a \notin X \& a \nVdash b)\}$.

- (G) Trivially, since $a \Rightarrow b \equiv \neg b \Rightarrow \neg a$.

- (H) $\mathcal{A}(\{b\}) = \{b\}$, $\mathcal{A}(\{b'\}) = \mathcal{A}(\{b,b'\}) = \mathcal{A}(\{b',b''\}) = \mathcal{A}(\{b,b',b''\}) = \{b,b',b''\}$, $\mathcal{A}(\{b''\}) = \{b''\}$, $\mathcal{A}(\{b'''\}) = \mathcal{A}(\{b'',b'''\}) = \{b'',b'''\}$, $\mathcal{A}(\{b,b'\}) = \mathcal{A}(\{b',b''\}) = \mathcal{A}(\{b,b',b''\}) = \{b,b',b''\}$, $\mathcal{A}(\{b,b'''\}) = \mathcal{A}(\{b',b'''\}) = \mathcal{A}(\{b,b',b'''\}) = \mathcal{A}(\{b',b'',b'''\}) = \mathcal{A}(\{b,b'',b'''\}) = \mathcal{A}(M) = M$, $\mathcal{A}(\emptyset) = \emptyset$.

  $\mathcal{C}(\emptyset) = \mathcal{C}(\{b\}) = \mathcal{C}(\{b'\}) = \mathcal{C}(\{b''\}) = \mathcal{C}(\{b,b''\}) = \mathcal{C}(\{b',b''\}) = \emptyset$, $\mathcal{C}(\{b'''\}) = \mathcal{C}(\{b,b'''\}) = \mathcal{C}(\{b'',b'''\}) = \mathcal{C}(\{b,b'',b'''\}) = \{b'''\}$, $\mathcal{C}(\{b,b'\}) = \mathcal{C}(\{b,b',b''\}) = \{b,b'\}$, $\mathcal{C}(\{b',b'''\}) = \{b',b'''\}$, $\mathcal{C}(\{b,b',b'''\}) = \{b,b',b'''\}$, $\mathcal{C}(\{b',b'',b'''\}) = \{b',b'',b'''\}$, $\mathcal{C}(M) = M$.

  $int(\emptyset) = int(\{a'\}) = int(\{a'''\}) = int(\{a',a'''\}) = \emptyset$, $int(\{a\}) = int(\{a,a'\}) = int(\{a,a'''\}) = int(\{a,a',a'''\}) = \{a\}$, $int(\{a''\}) = int(\{a',a''\}) = int(\{a'',a'''\}) = \{a''\}$, $int(\{a,a''\}) = \{a,a''\}$, $int(\{a,a',a''\}) = \{a,a',a''\}$, $int(\{a',a'',a'''\}) = \{a',a'',a'''\}$, $int(G) = G$.

  $cl(\{a\}) = \{a\}$, $cl(\{a'\}) = cl(\{a',a'''\}) = \{a',a'''\}$, $cl(\{a''\}) = cl(\{a',a''\}) = cl(\{a'',a'''\}) = cl(\{a',a'',a'''\}) = \{a',a'',a'''\}$, $cl(\{a'''\}) = \{a'''\}$, $cl(\{a,a'\}) = cl(\{a,a'''\}) = cl(\{a,a',a'''\}) = \{a,a',a'''\}$, $cl(\{a,a''\}) = cl(\{a,a',a''\}) = cl(\{a,a'',a'''\}) = cl(G) = G$, $cl(\emptyset) = \emptyset$.

- (I) $cl(X) = \{g : \forall b(g \Vdash b \Rightarrow b \in \{m : \exists a(a \in X \,\&\, a \Vdash m)\})\} = \{g : \forall b(g \Vdash b \Rightarrow \exists a(a \in X \,\&\, a \Vdash b))\}$. So $-cl(-X) = \{g : \neg \forall b(g \Vdash b \Rightarrow \exists a(a \notin X \,\&\, a \Vdash b))\} = \{g : \exists b(g \Vdash b \,\&\, \neg \exists a(a \notin X \,\&\, a \Vdash b))\} = \{g : \exists b(g \Vdash b \,\&\, \forall a \neg(a \notin X \,\&\, a \Vdash b))\} = \{g : \exists b(g \Vdash b \,\&\, \forall a(a \in X \vee a \nVdash b))\} = \{g : \exists b(g \Vdash b \,\&\, \forall a(a \Vdash b \Rightarrow a \in X))\} = \{g : \exists b(g \Vdash b \,\&\, b \in \{m : \forall a(a \Vdash m \Rightarrow a \in X)\})\} = int(X)$.

- Exercise 3.1 From *Proposition* 3.1.3 $x \in \langle R \rangle(x')$ iff $x \in Q_{x'}^{\mathbf{R}}$. It follows immediately that for any $X \subseteq A, \langle R \rangle(X) = Q_X^\cup$. Since $Q_{X \cup Y}^\oplus = Q_X^\oplus \cap Q_Y^\oplus \subseteq Q_X^\cup \cup Q_Y^\cup = Q_{X \cup Y}^\cup$ we immediately have $Q_X^\cup \supseteq Q_X^\oplus$. Finally, $[[-R^\smile]](X) = \{a : \forall x(x \in X \Rightarrow \langle a, x \rangle \notin R)\} = \{a : \nexists x(x \in X \,\&\, \langle a, x \rangle \in R)\} = -\langle R \rangle(X)$. Similarly

we obtain $[[-R]](X) = -\langle R^\smile \rangle(X)$. Hence $[[-R]][[-R^\smile]](X) = -\langle R^\smile \rangle - \langle R \rangle(X) = [R^\smile]\langle R \rangle(X) = \langle R \rangle(X)$.   QED

- Exercise 3.2 Suppose $X = Q_A \cap Q_B$ and $z \in Q_X$. Then $\exists z' \in Q_A \cap Q_B$ such that $z \in Q_{z'}$. Clearly $z' \in Q_A$ and $z' \in Q_B$. By q-transitivity we have $z \in Q_A$ and $z \in Q_B$ so that $z \in X$. Thus, $Q_X \subseteq X$. But $X \subseteq Q_X$, by reflexivity, so that $X = Q_X$.   QED

- Exercise 4.1 In this case we have that $\Vdash$ is the following map:

$\Vdash$	$\langle 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 1, 1 \rangle$
$m1$	0	1	0
$m2$	0	1	0
$m3$	1	0	0
$m4$	0	0	1

Therefore, $\mathcal{J}(\Omega_{\mathcal{SUB}}(G)) = \{\{m1, m2\}, \{m3\}, \{m4\}\}$.

- Exercise 4.2 Because the planet context is nominalised.

- Exercise 4.3
  $\delta(x) = \bigcup\{\alpha^\leftarrow(\downarrow x)\}$. Hence: $\delta(\bot) = \bigcup\{\emptyset\} = \emptyset$; $\delta(1) = \bigcup\{X : \forall n \in X, n > 0\} = \{n : n > 0\}$.
  $\delta(0) = \bigcup\{X : \forall n \in X, n > 0\} = \{n : n > 0\}$; $\delta(\top) = \delta(\bot) \cup \delta(1) \cup \delta(0) \cup -(\delta(1) \cup \delta(0)) = \mathbb{Z}$.

- Exercise 4.4
  The upper adjoint of $f^\leftarrow$ is the map $h : A \longmapsto B; h(X) = \{y : f^\leftarrow(\{y\}) \subseteq X\}$.

Part II

The Logico-Algebraic
Theory of Rough Sets

Chapter 5

Logic and Rough Sets:
An Overview

"Any specific object has a specific logic" K. Marx.

Since the present Part has a certain complexity, it is worth introducing, with some details, the intuitive motivations of the entire picture and their connections with the mathematical machinery which will be used.

5.1 Foreword

Thus, let us sum up what we have discussed and discovered as far as now.

In Rough Set Theory, the starting point is a collection of observations which are stored in an Information System $\mathbf{I}$ and which induces an indiscernibility space $\langle U, E \rangle$. We denote the family of all basic categories by $IND(\mathbf{I})$.

We have seen that from any Information System $\mathbf{I}$ one can compute the extension $[\![\mathcal{D}]\!]$ on the *universe U* of a *basic property* $\mathcal{D}$ which we call a $\mathbf{I}$-*basic property*, because it can be formulated using the linguistic material from $\mathbf{I}$.[1]

$\mathbf{I}$-*basic properties* make it possible to classify the objects from U into different disjoint equivalence classes which are to be intended as

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[1]For instance, if $\mathbf{I}$ is an Attribute Systems, a deterministic property is a conjunction of sentences of the form "$a_i = v_j$", where a_i ranges on the *set of parameters At*, v_j ranges on the *set of values V*.

the gnoseological co-ordinates interpreting the universe U, the basis of the organization of U from a conceptual point of view. In a literal mathematical sense, $IND(\mathbf{I})$ is a basis for a topological space that provides the "gnoseological geometry" of our world. Objects belonging to the same class are *indiscernible* by means of our system of information $\mathbf{I}$. Moreover, from the hypotheses about *P-systems* and *A-systems*, stated in Part I, every object from U will belong to the extension of some $\mathbf{I}$-*basic property*. So we obtain the first important characteristic for our analysis:

Axiom 5.1.1. *The set $IND(\mathbf{I}) = \{[\![\mathcal{D}]\!] : \mathcal{D} \text{ is a } \mathbf{I}-\text{basic property}\}$ is a partition of U.*

These classes, or blocks, are the atoms of more complex conceptual constructions. In Rough Set Theory, they are called "elementary" (or "basic") "classes" (or "categories") and we adopt this use.

As noted at the very beginning of the Introduction, this construction is fundamental. Call it "*grouping*", "*association*", "*categorization*", we hardly can find an analysis of human knowledge leaving it out of consideration: "*When you learn a concept, you learn how to treat different things as instances of the same category. Without this classification procedure, thinking would be impossible because each event or entity would be unique*" (Johnson-Laird [1988], page 132).

Because $IND(\mathbf{I})$ coincides with the family of the equivalence classes modulo the equivalence relation induced by the Information System $\mathbf{I}$, on a more abstract level we can start from any generic Indiscernibility Space $\langle U, E \rangle$, where $E \subseteq U \times U$ is an arbitrary equivalence relation.

The topological space for which the family U/E forms a basis, is called an *Approximation Space*. Nonetheless, in the present book we also use this term to denote the *frame* (complete distributive lattice) of its open subsets, denoted by $\mathbf{AS}(U/E)$ (the context shall avoid any confusion). Any open set is the union of basic sets. Therefore, they are extensions of *disjunctions* of basic properties, called thereafter "$\mathbf{I}$-*properties*". So, given an Information System $\mathbf{I}$, the *Approximation Space* $\mathbf{AS}(U/E)$ induced by $\mathbf{I}$, represents in fact this kind of linguistic description of concepts. This intuitively explains why an Approximation Space is defined as the set of all the unions of elementary classes plus the empty-set $\emptyset$ (an arbitrary $\mathbf{I}$-property could have an empty extension).

Thus, from an algebraic point of view, we have:

Axiom 5.1.2. *For any Indiscernibility Space $\langle U, E \rangle$, the Approximation Space* $\mathbf{AS}(U/E)$ *is the Boolean algebra of sets for which U/E is the set of atoms.*

Now, as we know, the second basic maneuver is to contrast $\mathbf{AS}(U/E)$ with the result of another categorization, say the Indiscernibility Space $\langle U, E' \rangle$. However, for the sake of maximal generalization, we shall assume that any arbitrary subset of U can be brought in contrast with $\mathbf{AS}(U/E)$, so that the second categorization will be assumed to be the discrete one ($\langle x, y \rangle \in E'$ if and only if $x = y$), if not otherwise stated. In other terms, any subset of U can play the role of a pre-figure. Thus we assume that the foreground Approximation Space will always be the powerset $\wp(U)$.[2]

For this reason the structure of $\mathbf{AS}(U/E')$ does not count and we shall reserve by now the name "Approximation Space" to the background space, and the term "datum" to the elements of the foreground space.

With respect to this structure we have the following fact:

Axiom 5.1.3. *For any Indiscernibility Space $\langle U, E \rangle$, the Approximation Space* $\mathbf{AS}(U/E)$ *is a subalgebra of the Boolean algebra of sets* $\mathbf{B}(U)$ *defined on the powerset $\wp(U)$ of the universe U.*

In accordance with these assumptions, Approximation Spaces are given a more general interpretation. In fact, if we assume that a generic subset of elements of U is the extension of a generic "datum" virtually definable on U, then the fact that an Approximation Space $\mathbf{AS}(U/E)$ is generally a strict subalgebra of $\wp(U)$, displays the popular observation that usually we do not have a complete information about any situation we face with, in a pretty "concrete" and "tangible" manner. In other words, the granularity of the knowledge represented by our properties, generally does not allow the exact representation of arbitrary concepts,

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[2]For the sake of generalisation, but also because we are working within the monological approach. From this point of view, the foreground space is always subordinated to the background. Otherwise stated, the foreground space is inert, so that any subset of this space may be conceived of as a "crude" datum to be analysed, but not to be questioned. In the dialogical approach we do not have "crude data" any longer, but interacting categories.

but just an approximation depending on the gnoseological material at our disposal. Hence the term "Approximation Space".

Let us then consider an arbitrary set $X \subseteq U$. Obviously, either $X \in$ $\mathbf{AS}(U/E)$ or not. In the first case X can be exactly described by means of an $\mathbf{I}$-property, which can be named a *background property*, in view of the discussion in the Introduction. In the second case we cannot use $\mathbf{I}$-properties for a direct description of X, which can be approximated by means of an *upper approximation*, $(uE)(X)$, and a *lower approximation*, $(lE)(X)$. If $\langle U, \mathbf{AS}(U/E) \rangle$ is intended as a topological space, we know that $(uE)(X)$ is the closure $\mathbb{C}(X)$ and $(lE)(X)$ is the interior $\mathbb{I}(X)$.

However, in general in between $(lE)(X)$ and $(uE)(X)$ we have the topological boundary of X: $\mathbb{B}(X) = \mathbb{C}(X) \cap -\mathbb{I}(X) = (uE)(X) \cap -(lE)(X)$. Notice that the boundary of X is the set of points which are neither in the lower approximation, nor in the complement of the upper approximation of X: $\mathbb{B}(X) = \mathbb{C}(X) \cap -\mathbb{I}(X) = -(-\mathbb{C}(X) \cup \mathbb{I}(X)) = -(-(uE)(X) \cup (lE)(X))$. From the point of view of Approximation Spaces, two sets that have exactly the same upper and lower approximations can be considered equivalent, and one obtains:

Definition 5.1.1. *A* rough set *of $\langle U, E \rangle$ is an equivalence class of subsets of U modulo the equality of their upper and lower approximations. Such an equivalence relation is called a* rough equality*.

The family of all rough sets induced by an Approximation Space $\mathbf{AS}(U/E)$ is called a *Rough Set System* and is denoted by $RS(U/E)$.

5.2 Rough Set Systems and Three-Valued Logics

As we have mentioned in the Introduction, one can give a logical interpretation to this machinery. The upper approximation $(uE)(X)$ is the set of elements that *possibly* belong to X since they share the same $\mathbf{I}$-*properties* with some element actually in X. In other words, if $x \in (uE)(X)$, then we *can* associate it to X, even if it does not actually belong to this set, since some "twin" of x belongs to X already. On the other hand $(lE)(X)$ is the set of elements of X that *necessarily* belong to X since there are not elements outside of X which are describable by means of the same $\mathbf{I}$-*properties*. In negative terms: if $x \in X$ but there is an x' belonging to $-X$ which is indiscernible from x, then in

order to obtain the lower approximation of X, we discard x too, since its membership is *accidental*, according to the conceptual background represented by $\mathbf{AS}(U/E)$.

Example 5.2.1. Possibility and necessity in an information system

Consider the information system of Example 2.4 in the Introduction. Let X be the set $\{d, e, f, g\}$, which in $\mathbf{AS}(U/A')$ is characterized by the property "*Comfort = medium*". We can notice that in $\mathbf{AS}(U/A)$ the patient c may be associated to X since patient d belongs to X and is indiscernible from c which is as like as any patient having *Temperature = normal*, *Hemoglobin = good*, *Blood Pressure = low* and *Oxygen Saturation = good*. So we can assume that it is not impossible for these patients to have *Comfort = medium* because we have examples of patients with the same attribute-values that have this rate. In fact $(uE_A)(X) = \{c, d, e, f, g\}$. On the other hand, all the patients with *Temperature = low*, *Hemoglobin = good*, *Blood Pressure = normal* and *Oxygen Saturation = fair*, have *Comfort = medium*. This means, for instance, that e necessarily belongs to X, since we do not have counterexamples of patients with the same characteristics but with *Comfort ≠ medium*. This fact is reflected by the equation $(lE_A)(X) = \{e, f, g\}$.

From this point of view, for any set X we have two *definite* or *certain* states: the lower approximation (interior of X, necessary part of X), which means "definitely yes", and the complement of the upper approximation, $-(uE)(X) = -\mathbb{C}(X)$, (exterior of X, impossible part of X) which means "definitely no". Since $-\mathbb{C}(X) = \mathbb{I}(-X)$, this element coincides with the complementary figure $\neg X$. Everything that is neither in $(lE)(X)$ nor in the complementary figure, $\neg X$, is in the boundary of X, $\mathbb{B}(X)$. Indeed a boundary is a region of doubt: if $x \in \mathbb{B}(X)$, then we can say nothing certain about the membership of x in X. We cannot say either that x is certainly (or necessarily) in X, or that x has certainly nothing to do with X: in fact it *could* belong to X, since it is indiscernible from some element of X; but it could belong to $-X$, too, because it is indiscernible from some element of $-X$.

It follows that, generally, between $(lE)(X)$ and its complementary figure $\neg X$, there is not an empty region and that the union of $(lE)(X)$ and $\neg X$ does not give the unit universe U. In other words, the law of Excluded Middle is not uniformly valid for rough sets.

So we begin to see that the classical two-valued characteristic function must be generalized by a three-valued one, if we want to grasp this situation. It follows that in general Rough Set Systems are likely to have strict relationships with some three-valued logico-algebraic system. Actually, more than one of these systems are related to this

information analysis and the reason depends on the deeper meaning of our construction.

In fact, the topological space $\langle U, \mathbf{AS}(U/E) \rangle$ may fulfill different separation properties depending on the granularity of our knowledge. In turn, this depends on the level of accuracy of the attributes. We have the best separation properties when $\mathbf{AS}(U/E) = \wp(U)$. In this case our topology is the *discrete* topology (which is Hausdorff and completely disconnected) and one can single out each element of U. Otherwise stated, in a sense we have enough properties for "*naming*" any single element of U. On the contrary, when no object can be discerned from the others, we have the *trivial* topology: $\mathbf{AS}(U/E) = \{\emptyset, U\}$. Using the famous sentence of the German philosopher G. W. F. Hegel, this is like the "*night in which all cows are black*". Indeed in this case we have the weakest separation property.

However, usually we shall have intermediate cases in which some elements can be singularly "*named*", while others cannot be individualised by the information at our disposal: in general in $\langle U, E \rangle$ some equivalence classes are *singletons* while others are not.

5.3   Exact and Inexact Local Behaviours in Rough Set Systems

Let us denote by B^* the family of the equivalence classes that are singletons, and by P^* the family of the equivalence classes that have cardinality strictly greater than 1. As mentioned in the introduction. B^* and P^* do not have the same logical role in the construction of a Rough Set System. In fact the elements in B^* are *exact* in nature, because they do not have any boundary, any region of doubt, so that they should enjoy the principle that in Classical Logic reflects completeness and exactness: Excluded Middle.[3] Indeed, given a set X and an open singleton $\{s\}$, either $\{s\}$ is included in $(lE)(X)$ or it is included in $-(uE)(X)$.[4] On the contrary any basic open set with at least two elements may be included in the boundaries of at least two different sets. This means that if there is no singleton in $\mathbf{AS}(U/E)$, then there are at least two sets X such that $(uE)(X) = U$ and $(lE)(X) = \emptyset$. In other words, there are at least two *undefinable* sets.

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[3]If $\{x\}$ is a singleton, then x is an isolated point, in topological terms.

[4]In topological terms: an isolated element cannot be a member of any boundary.

Therefore, it is not difficult to understand why the class of the undefinable sets can play the role of intermediate value: this class represents situations in which everything could be true, or everything could be false. Thus, the rough set of all the undefinable sets is another "night in which all cows are black".

Example 5.3.1. Exact and inexact information

In the Approximation Space induced by the set A of attributes in the Information System of Example 2.4 discussed in the Introduction, we have two non singleton atoms, $\{c, d\}$ and $\{e, f\}$, and five singletons, $\{a\}, \{b\}, \{g\}, \{h\}, \{i\}$. The singleton $\{a\}$, for instance, is uniquely defined by the property "*Temperature = low, Hemoglobine = fair, Blood Pressure = low, Oxygen Saturation = fair*". This property applies only to the element a so that we have complete and unique information about a, because the attribute-values we are dealing with make it possible to distinguish a from all the other elements of U.

On the contrary, the element c fulfills the same property as the element d, so that we do not have enough information in At to distinguish c (or d).

Clearly, as far as we deal with the set of attributes A, we do not have subsets of U that are undefinable, because for instance if $\{a\}$ is included in the upper approximation of a subset of U, then it is included in its lower approximation, too. It follows that there are not sets X such that $(uE_A)(X) = U$ and $(lE_A) = \emptyset$.

Now consider, instead, the following sub-table of the same Information System, where $U^* = \{a, b, c, d, e, f\}$ and $A^* = \{Temperature, Hemoglobin\}$:

v	$Temperature$	$Hemoglobin$
a	low	fair
b	low	fair
c	normal	good
d	normal	good
e	low	good
f	low	good

Clearly $E/_{A^*} = \{\{a, b\}, \{c, d\}, \{e, f\}\}$. So the induced Approximation Space has three atoms and none of them is a singleton. If we contrast the set $X = \{a, c, e\}$ against $E/_{A^*}$ then we find $(uE_{A^*})(X) = U^*$ and $(lE_{A^*})(X) = \emptyset$. In fact it is impossible to find a disjunction of basic properties exactly describing some member of X but not all the members of U. Hence X is an undefinable set. Indeed, the process of peaking up an element out of each (non singleton) equivalence class gives us a combination of eight undefinable sets:

$$\{\{a, c, e\}, \{a, c, f\}, \{a, d, e\}, \{a, d, f\}, \{b, c, e\}, \{b, c, f\}, \{b, d, e\}, \{b, d, f\}\}$$

This collection is therefore the rough set of all sets X such that $(uE_{A^*})(X) = U^*$ and $(lE_{A^*}) = \emptyset$. Therefore, it represents all the undefinable sets.

Therefore, we may suppose that for any rough set there are two distinct *local* logical behaviours: one is classical and localized on $B = \bigcup B^*$, whereas the other one, localized on $P = \bigcup P^*$, is purely three-valued. It is the combination of these local behaviours that defines the overall logical features of rough sets. It follows that the construction of $RS(U/E)$ will depend on the parameter B (or, equivalently, P).

Moreover, in $RS(U/E)$, any rough set induced by an element of the Approximation Space $\mathbf{AS}(U/E)$ has a particular logical behaviour too: such an element corresponds to an exactly definable subset of U, hence, again, it should fulfill Classical Logic, but within the logical environment determined by the overall Rough Set System. And, as just seen, this environment might be three-valued.

Thus we have two levels of local logical behaviours: one is related to the internal definition of rough sets, the other deals with the global logical properties of Rough Set Systems.

The first level completely depends on the parameter B (or P). These parameters cannot be recovered from the "*geometrical*" shape of the Approximation Space $\mathbf{AS}(U/E)$, except for trivial cases. It follows that in general an inspection of the atoms is unavoidable in order to define $RS(U/E)$. Because the information provided by this inspection does not have any theoretic content, we call B and P *external parameters* or *empirical parameters* and we say that they are able to distinguish the classical local behaviour within an Approximation Space.

On the contrary, we can analyse the lattice structure of $RS(U/E)$ from a pure abstract point of view. In fact, also in this case we have to use a particular parameter, but curiously enough, though it is induced by the empirical parameter B, nevertheless it is definable in $RS(U/E)$ by means of a mere lattice-theoretic definition. For this reason we call it an *internal parameter* or *theoretical parameter* and we shall see that it distinguishes a classical local behaviour within a Rough Set System.

It follows that Rough Set Systems should be analysed using some notion able to manage the concept of "**it is locally the case that**".

For this purpose we shall exploit the mathematical notions of a "*Grothendieck topology*" and a "*Lawvere-Tierney operator*" which have been introduced to deal with local properties.

5.4 Representing Rough Sets

A *rough set* is an equivalence class of sets modulo the equality of their approximations. Thus a rough set from U belongs to $\wp(\wp(U))$.

However a rough set is naturally and more comfortably representable by a pair $\langle X_1, X_2 \rangle$ of elements of $\mathbf{AS}(U/E)$, where X_1 and X_2, are the two approximations.

So, consider the (by now informal) family

Definition 5.4.1. $RS(U/E) = \{\langle X_1, X_2 \rangle \in \mathbf{AS}(U/E) \times \mathbf{AS}(U/E)$: $\langle X_1, X_2 \rangle$ *is a Rough Set in* $\mathbf{AS}(U/E)\}$.

We immediately have the problem of the formal and abstract characterization of the sentence "is a Rough Set in $\mathbf{AS}(U/E)$". A first sub-problem is:

Problem 5.4.1. *For any Approximation Space* $\mathbf{AS}(U/E)$, *determine the* internal formal characteristics *that must be satisfied by an ordered pair to represent a rough set.*

The answer depends on the intuitive motivations that drive our reading of the nature of rough sets. A first, and in a sense the most immediate and "naive", solution is considering pairs of the form

$$\langle (uE)(X), (lE)(X) \rangle \tag{5.4.1}$$

This ordered pair uniquely describes the equivalence class in question.

From this point of view, the "internal property" to be fulfilled by a pair $\langle X_1, X_2 \rangle$ in order to belong to $RS(U/E)$ is necessarily:

$$X_2 \subseteq X_1 \tag{5.4.2}$$

because the first element X_1 stands for the upper approximation of a set X and X_2 stands for its lower approximation. Thereafter we call such a representation the *decreasing representation* of a rough set.

A second reading, probably less "naive" but still intuitive, is suggested by the application of Rough Set Theory to some semantics for Logics of Knowledge and Learning (see the Frame section of Part III and is connected with the following intuition: any rough set represents what definitely is known to satisfy a concept and what definitely is known not to satisfy it. Between the two areas, eventually, there is a *doubtful region* which is due to the incompleteness of our knowledge.

Thereafter, from this point of view the "internal property" of a pair $\langle X_1, X_2 \rangle$ is necessarily:

$$X_1 \cap X_2 = \emptyset \tag{5.4.3}$$

and we call it the *disjoint representation* of a rough set.

We have already seen that in a more logical setting, the upper approximation $(uE)(X)$ corresponds to the modal application $M(X)$ – "what *possibly* belongs to X" – and the lower approximation $(lE)(X)$ corresponds to the modal application $L(X)$ – "what *necessarily* belongs to X". According to this reading, the decreasing representation of a rough set is of the type

$$\langle M(X), L(X) \rangle \tag{5.4.4}$$

However, consider $-M(X)$. Since $-M(X)$ means "it is *impossible* to belong to X", we have that $L(X)$ and $-M(X)$ are the only statements expressing "certainty". Thus a definite knowledge about a specific phenomenon will be expressed by a pair

$$\langle L(X), -M(X) \rangle \tag{5.4.5}$$

that is to say, $\langle$*maximal internal body, complementary body*$\rangle$. In order to make rough sets reflect the above intuition, one must represent them as a pair

$$\langle (lE)(X), -(uE)(X) \rangle \tag{5.4.6}$$

that is exactly the *disjoint representation* of a rough set.

So we shall set:

Definition 5.4.2 (Decreasing representation of rough sets). *For any Approximation Spaces* $\mathbf{AS}(U/E)$ *and* $X \subseteq U$*:* $rs(X) = \langle (uE)(X), (lE)(X) \rangle$*.*

Definition 5.4.3 (Disjoint representation of rough sets). *For any Approximation Spaces* $\mathbf{AS}(U/E)$ *and* $X \subseteq U$*:* $rs'(X) = \langle (lE)(X), -(uE)(X) \rangle$*.*

The application rs will be called a "rough set map".

From the involution property of "$-$", one easily shows that the two representations are interchangeable.

Although a choice between the two representations is somewhat arbitrary, since we prefer to deal with the two standard modalities (Necessity and Possibility) we adopt the decreasing representation.[5]

Therefore, we assume by default the decreasing representation until the disjoint representation is explicitly considered. Moreover, when the context is clear, with the term "rough set" we shall denote the decreasing (disjoint) representations of a rough set (which, actually, is an equivalence class).

Example 5.4.1. Representing rough sets

Let us represent some rough sets induced by the Approximation Space $X \in \mathbf{AS}(U/E_A)$ of Example 1.2.3 (cf. also Example 1.2.5).

Disjoint representation:

$$\{\{a\},\{c\}\} \quad \longrightarrow \quad \langle\emptyset,\{b,d\}\rangle; \quad \{\{b,a\},\{b,c\}\} \quad \longrightarrow \quad \langle\{b\},\{d\}\rangle;$$
$$\{\{d,a\},\{d,c\}\} \quad \longrightarrow \quad \langle\{d\},\{b\}\rangle; \quad \{\{b,d,a\},\{b,d,c\}\} \quad \longrightarrow \quad \langle\{b,d\},\emptyset\rangle.$$

and any $X \in \mathbf{AS}(U/E_A)$ is represented by $\langle X, -X \rangle$.

Decreasing representation:

$$\{\{a\},\{c\}\} \quad \longrightarrow \quad \langle\{a,c\},\emptyset\rangle; \quad \{\{b,a\},\{b,c\}\} \quad \longrightarrow \quad \langle\{a,b,c\},\{b\}\rangle;$$
$$\{\{d,a\},\{d,c\}\} \quad \longrightarrow \quad \langle\{a,c,d\},\{d\}\rangle; \quad \{\{b,d,a\},\{b,d,c\}\} \quad \longrightarrow \quad \langle U,\{d,b\}\rangle.$$

and any $X \in \mathbf{AS}(U/E_A t)$ is represented by $\langle X, X \rangle$.

A second sub-problem is:

Problem 5.4.2. *For any Approximation Space* $\mathbf{AS}(U/E)$*, determine the* internal empirical characteristics *of the ordered pairs representing a rough set.*

More precisely, this problem is related to the previous discussion about singleton and non singleton basic categories. If we assume the decreasing representation we have to notice that not every pairs of elements fulfilling the formal property (5.4.2) are legal. In other words, (5.4.2) is a necessary but not sufficient condition for a pair to represent a rough set of an Information System **I**.

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[5]Other reasons supporting this choice can be found in the Frame section of Part II. However, from a strictly mathematical point of view the disjoint representation is to be preferred because it has more general applications – see Example 9.6.1 of Section 9.6.

In fact, as we already know, if an elementary class S is a singleton then for any $X \subseteq U$, S is included either in $(lE)(X)$ or in $-(uE)(X)$. Thus S belongs to $(lE)(X)$ whenever S is included in $(uE)(X)$. This is the required general characteristic. It follows, for instance, that the pair $\langle S, \emptyset \rangle$ it is not a legal one although it fulfills property 5.4.2, while, for example, $\langle S, S \rangle$ is.

In the same case, if we assume the disjoint representation, we have to discard, for instance, the pair $\langle \emptyset, \emptyset \rangle$: indeed it enjoys property 5.4.3 but it is clear that the singleton S must necessarily be included either in X_1 or in X_2, for any pair $\langle X_1, X_2 \rangle$. Again 5.4.3 is only a necessary condition.

The problem becomes thereafter:

Problem 5.4.3. *For any Approximation Space $\mathbf{AS}(U/E)$ characterize the set $RS(U/E)$ within the family of elements of the Cartesian product $\mathbf{AS}(U/E) \times \mathbf{AS}(U/E)$ which fulfill property (5.4.2) (or (5.4.3) if we prefer the disjoint representation).*

The solution of this problem will be given within the following more general:

Problem 5.4.4. *Determine if there is some logico-algebraic structure behind Rough Sets Systems.*

Example 5.4.2. Local validity in Rough Set Systems – 1

Let us consider the information system of Example 2.3. According to it, the pair $\langle \{a, b, c, d\}, \{a, c, d\} \rangle$ is not a legal rough set (in decreasing representation). In fact if $b \in (uE_A)(X)$, for some $X \subseteq U$, then for some $x \in X$, $\langle x, b \rangle \in E_A$ But since $\{b\}$ is a singleton we have $\langle b, x \rangle \in E_A$ if and only if $x = b$, so that $b \in X$ too. It follows that $\{b\} \subseteq X$. Hence, $\{b\} \subseteq (lE_{At})(X)$ and $b \in (lE_{At})(X)$.

The union B of all the singletons is $\{b, d\}$ and we have that for any $x \in B$, either $x \in (lE_{At})(X)$ or $x \in -(uE_{At})(X)$. Otherwise stated, $x \in (uE_{At})(X)$ if and only if $x \in (lE_{At})(X)$. This means that for any rough set $\langle X_1, X_2 \rangle$, $X_1 \cap B = X_2 \cap B$.

In disjoint representation, the above considerations lead to the fact that for any $X \subseteq U$, for any rough set $\langle X_1, X_2 \rangle$, $X_1 \cup X_2 \supseteq B$, i.e. $(lE_{At})(X) \cup -(uE_{At})(X) \supseteq \{b, d\}$.

Let us depict the situation in Figures 5.1 and 5.2 below.

Given the universe U, a usual set X has a complement $-X$ such that $X \cup -X = U$ (Figure 5.1 left). In an Approximation Space, on the contrary we have $(lE)(X) \cup -(uE)(X) \subseteq U$ (Figure 5.1 right). The intermediate area is the boundary of X.

But if the union B of all the singletons is not void (Figure 5.2 left), we have a different situation: any subset B' of B is a sub-body with it own complement $-B'$ as complementary body. Indeed, $(lE)(B') \cup -(uE)(B') = B \cup B' = B$ (Figure 5.2 right).
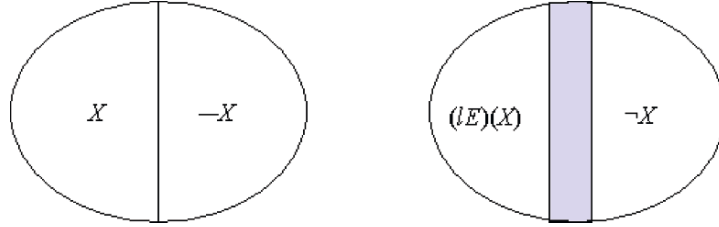
Figure 5.1: An empty union B of singletons
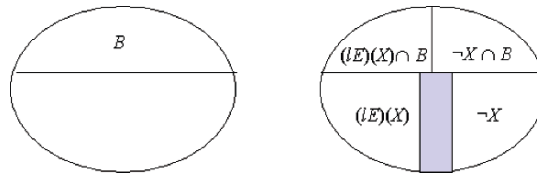
Figure 5.2: A non-empty union B of singletons – subsets of B behave as usual

5.5 Rough Set Systems, Local Validity, and Logico-Algebraic Structures

Notwithstanding its "practical" motivations, Rough Set Theory happens to be able to model a number of logical systems.

Indeed, Rough Set Systems have many connections with Heyting and bi-Heyting algebras, Łukasiewicz algebras, Post algebras, Stone algebras, Chain Based Lattices and P-algebras. In what follows we provide the overall picture of these connections.

First of all we have to show that the language-oriented operations provided by Logic are meaningful in Rough Set Systems. As a matter of fact, this is partially true on the part of the operations "and" and "or".

Indeed, let X, Y, Z be subsets of U. We have:

- Interpretation of the operation $\wedge$: if $rs(X) \wedge rs(Y) = rs(Z)$, then Z is a maximal set in the class $\{X' \cap Y' : rs(X') = rs(X) \; and \; rs(Y') = rs(Y)\}$.

- Interpretation of the operation $\vee$: if $rs(X) \vee rs(Y) = rs(Z)$, then Z is a minimal set in the class $\{X' \cup Y' : rs(X') = rs(X) \; and \; rs(Y') = rs(Y)\}$.

It follows that rs distributes over $\vee$ just with respect to the upper approximations and, dually, it distributes over $\wedge$ just with respect to the lower approximations (details later in the text). Hence, the two binary connectives $\vee$ and $\wedge$ make sense in defining a Rough Set Logic, under the limitations of the above proviso.

Now, it is well known that the set $\mathbf{B}^{[n]} = \{\langle a_1, \ldots, a_{n-1}\rangle \in \mathbf{B}^{n-1} : a_i \geq a_j$ for $i \leq j\}$, where $\mathbf{B}$ is a Boolean algebra, is an example of n-valued Łukasiewicz algebra (see Boicescu et al. [1991]).

Thus $\mathbf{AS}(U/E)^{[3]}$ is a three-valued Łukasiewicz algebra.

From this consideration it follows that $RS(U/E)$ is a *substructure* of $\mathbf{AS}(U/E)^{[3]}$ if we assume the decreasing representation.

On the side of the disjoint representation, if $\mathbf{D}$ is a finite distributive lattice with least element $\perp$, then the set $K(\mathbf{D}) = \{\langle a_1, a_2\rangle \in \mathbf{D}^2 : a_1 \wedge a_2 = \perp\}$ is an example of De Morgan algebra. In particular if $\mathbf{D}$ is a finite Boolean algebra, then $K(\mathbf{D})$ is a Post algebra of order three. Since $\mathbf{AS}(U/E)$ is a Boolean algebra, from the above considerations it follows that if we assume the disjoint representation, then $RS(U/E)$ is a *substructure* of the Post algebra of order three, $K(\mathbf{AS}(U/E))$.

Our last problem can now be restated in the following way:

Problem 5.5.1. *For any Approximation Space* $\mathbf{AS}(U/E)$, *characterize within* $\mathbf{AS}(U/E)^{[3]}$ *and within* $K(\mathbf{AS}(U/E))$ *the logical status of the substructure* $RS(U/E)$ *using only information-oriented parameters depending on* $\mathbf{AS}(U/E)$.

In this Part we shall start answering these questions by representing $RS(U/E)$ as a *semi-simple Nelson algebra*. We decide to start with this interpretation for a couple of reasons. First, although David Nelson introduced his systems in order to circumvent some non constructivistic features of intuitionistic negation (in connection with Kleene's notion of "Recursive Realizability"), Nelson's deep intuitive motivations can be completely framed in our context:

"*In general, an experimental verification of a statement consists of an operation followed by an observation of a property.* [...] *However, if we have not observed the property, there remains an ambiguous situation insofar as the truth of a statement is concerned. The failure to observe the property may be significant of the falsity of the statement or may merely be an indication of some deficiency on the part of the observer.* [...] *In view of this ambiguity, it might be maintained that*

every significant observation must be an observation of some property and further that the absence of a property P if it may be established empirically at all, must be established by the observation of (another) property N which is taken as a token for the absence of P." (Nelson [1959], page 208).

On the basis of these intuitions, in the quoted paper David Nelson introduced a logical system named **S**, which makes it possible to distinguish concepts such as "from A a contradiction is provable" and "the negation of A is established", which are usually unified.

We call this difference the issue of "separation of concepts", and we record it by saying that in the former case the proof ends with a weak form of negation, $\dashv\alpha$, and in the latter with a strong form of negation, $\sim\alpha$. System **S** is strictly connected to semi-simple Nelson algebras, that constitute a subvariety of the class of Nelson algebras, which in its turn is connected with the system **N** introduced in Nelson [1949].[6]

The second reason to start with semi-simple Nelson algebras is the fact that the duality theory of these algebras provides us with the mathematical machinery that is needed in order to exhibit a rigorous characterization of $RS(U/E)$. The main result of this approach is that for any Approximation Space $\mathbf{AS}(U/E)$ the Rough Set System $RS(U/E)$ can be made into a finite semi-simple Nelson algebra, which is precisely definable by means of the parameter B (viz. the union of all the singleton elementary classes) that was discussed in the previous subsections. We shall use B for filtering $RS(U/E)$ out of $\mathbf{AS}(U/E)^{[3]}$ and $K(\mathbf{AS}(U/E))$.

This use of B will be completely framed within the theory of Grothendieck Topologies, because it will be based on the notion of "local validity", as has been anticipated.

These Nelson algebras will be proved to be equipped with a pseudo-complementation, $\neg$, which, in fact, can be defined by means of the weak negation $\dashv$ and the strong negation $\sim$.

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[6]One of the principal differences between **N** and **S** is that in **S** we have just a restricted form of *thinning*, namely $\frac{\alpha,\alpha,\alpha\vdash\beta}{\alpha,\alpha\vdash\beta}$. As always happens, restrictions on structural rules make formerly unified logical meanings split. The above restricted form of thinning is shared also by three-valued Łukasiewicz logic which may be defined by consistently extending **S** by means of the axiom $\alpha\equiv\sim\alpha$, for a suitable formula α (cf. the discussion below about the connection between semi-simple Nelson algebras and three-valued Łukasiewicz algebras, and about central elements).

What is the rough set interpretation of these negations?

- Strong negation "$\sim$": we have $\sim rs(X) = rs(-X)$, so that the strong negation of a rough set equals the rough set of its set-theoretical complement. In other words, the strong negation faithfully represents the set-theoretical complement at the rough set level.

- Pseudo-complementation "$\neg$": if $\neg rs(X) = rs(Y)$, then Y is the greatest definable set disjoint from $(uE)(X)$.

- Weak negation "$\lnot$": if $\lnot rs(X) = rs(Y)$, then Y is the greatest definable set disjoint from $(lE)(X)$.

Thus negations have a straightforward meaning in Rough Set Systems.

Moreover, the above algebraic structures may also be regarded as bi-Heyting algebras. More precisely, on can show that the operator $\lnot$ is the pseudo-complementation in the co-Heyting algebra $RS(U/E)^{op}$ that is obtained by reversing the order, thus swapping $\wedge$ and $\vee$, 1 and 0 (and defining a dual relative pseudo-complementation). Therefore in $RS(U/E)$, if we set $1 = \langle U, U \rangle$ and $0 = \langle \emptyset, \emptyset \rangle$ we have, for any a:

$$a \vee \neg a \leq 1, a \wedge \neg a = 0;$$
$$a \vee \lnot a = 1, a \wedge \lnot a \geq 0;$$
$$a \vee \sim a \leq 1, a \wedge \sim a \geq 0.$$

These failures of the laws of contradiction and excluded middle, have an immediate informational interpretation, displayed by the following symmetries, for $a = \langle (uE)(X), (lE)(X) \rangle$:

$$a \vee \neg a = \langle U, -\mathbb{B}(X) \rangle = a \vee \sim a$$
$$a \wedge \lnot a = \langle \mathbb{B}(X), \emptyset \rangle = a \wedge \sim a.$$

So, it is absolutely evident that the lack of the classical principles is due to the presence of the doubtful boundary region: the excluded middle and the law of contradiction are valid up to the presence of a non-empty boundary. Indeed, if $\mathbb{B}(X) = \emptyset$, then $\langle U, -\mathbb{B}(X) \rangle = \langle U, U \rangle$, which is the top element, while $\langle \mathbb{B}(X), \emptyset \rangle = \langle \emptyset, \emptyset \rangle$, which is the bottom element.

Particular attention is given to the logico-algebraic characterization of definable sets. It is possible to define, by means of the weak negation and the pseudo-complementation, two operators $\lnot\lnot$ and $\neg\neg$. These operators project any rough set X onto particular exact elements, that

is elements $\langle X_1, X_2 \rangle$ such that $X_1 = X_2$ (assuming X to be in decreasing representation). More precisely, $\neg\neg$ is a possibility operator , while $\lrcorner\lrcorner$ is a necessity modalizator.

The interpretation in Rough Set Theory of these modalities is:

- "Possibility" operator: if $\neg\neg rs(X) = rs(Y)$, then Y is the least definable set containing X. That is, $\neg\neg rs(X) = rs((uE)(X))$;

- "Necessity" operator: if $\lrcorner\lrcorner rs(X) = rs(Y)$, then Y is the greatest definable set included in X. That is, $\lrcorner\lrcorner rs(X) = rs((lE)(X))$.

As it will more detailed in Frame 10.12.4, it is worth mentioning that the operation $\lrcorner$ was exploited in Lawvere [1982] to give a logical account for the notions of a "boundary", "essential core of a body" and "subbody" or "body", in the context of Continuum Physics. If we compare our terminology with Lawvere's, we can observe that the notion of "essential core of a body" corresponds to our "maximal internal body". In our terminology, however, a "body" is so if it coincides with its own essential core, that is to say if it is a regular element.[7]

It is clear that, because of their atomicity, singleton elementary classes are sub-bodies that either belong to X or to its complementary figure $\neg X$, for *any* given subspace X of the universe of discourse. Otherwise stated, $B \subseteq X \cup \neg X$. There is no notion of a boundary involving B: any point which can be isolated by an elementary class, cannot belong to any boundary. It follows that for all $a \in RS(U/E)$ we have $a \vee \neg a = \langle U, -\mathbb{B}(X) \rangle \geq \langle U, B \rangle$ and $a \wedge \lrcorner a = \langle \mathbb{B}(X), \emptyset \rangle \leq \langle -B, \emptyset \rangle$, so that the law of excluded middle and the law of contradiction are valid with respect to the subspace B.

At this point, Grothendieck topologies display their power, as we shall see in Chapter 7. Indeed, our use of Grothendieck topologies has the objective to formally render, from a mathematical point of view, that in a part of our universe we have to apply Classical Logic, while in the remaining part we have to apply a three-valued Logic. Roughly speaking, given the family $\mathbf{G}$ of open sets of a Grothendieck topology over a universe U, we say that a property $\mathcal{P}$ is *locally valid* on a set $X \subseteq U$, if its domain of validity, $[\![\mathcal{P}]\!]$, has a *large enough intersection* with X, where the meaning of "large enough" is given by the topology $\mathbf{G}$;

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[7]Therefore, we are not able to distinguish between a body and its essential core, while we can distinguish the maximal internal body within a generic set (or prefigure). As a matter of fact, our topology is coarser than Lawvere's.

namely, if $[\![\mathcal{P}]\!] \cap X \in \mathbf{G}$. So we shall define a suitable Grothendieck topology $\mathbf{G}_B$ on $\mathbf{AS}(U/E)^{[3]}$, depending on the parameter B, such that the disjunction $a \vee \lrcorner a$ is absolutely valid while $a \vee \neg a$ is greater than or equal to the *local top element* $\langle U, B \rangle$ (i.e. the transformation *via* $\mathbf{G}_B$ of the absolute top element $\langle U, U \rangle$) and the conjunction $a \wedge \neg a$ is absolutely invalid, while $a \wedge \lrcorner a$ is less than or equal to the *local bottom element* $\langle -B, \emptyset \rangle$ (i.e. the transformation *via* $\mathbf{G}_B$ of the absolute bottom element $\langle \emptyset, \emptyset \rangle$).

Hence Grothendieck topologies will code the fact that both excluded middle and the law of contradiction are locally valid with respect to the sub-universe B, according to the picture of Figure 5.3.
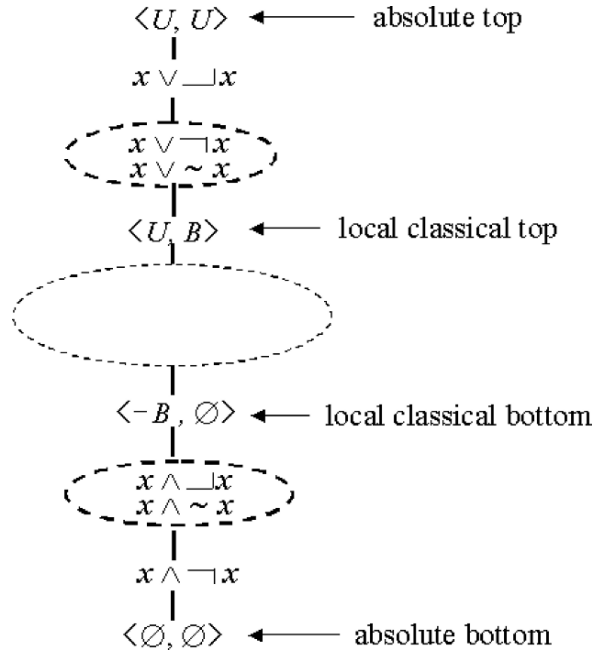
Figure 5.3: Local and global elements

Example 5.5.1. Local validity in Rough Set Systems – 2

Negations and boundaries:

Given a rough set $x = \langle X_1, X_2 \rangle$ (in decr. repr.), $\sim x = \langle -X_2, -X_1 \rangle = \langle -(lE)(X), -(uE)(X) \rangle$ that is the rough set of $-X$. $\lrcorner x = \langle -X_2, -X_2 \rangle = \langle -(uE)(X), -(uE)(X) \rangle$ that is the rough set of $(uE)(-X)$ (of $-(lE)(X)$). $\neg x = \langle -X_1, -X_1 \rangle = \langle -(lE)(X), -(lE)(X) \rangle$, that is the rough set of $(lE)(-X)$ (of $-(uE)(X)$). If $y =$

$\langle Y_1, Y_2 \rangle$, we define $x \wedge y$ and $x \vee y$ point-wise:

$$x \wedge y = \langle X_1 \cap Y_1, X_2 \cap Y_2 \rangle,\ x \vee y = \langle X_1 \cup Y_1, X_2 \cup Y_2 \rangle$$

(for all these operations see Window 7.1).

In the information system of Example 2.3, we have for instance that if $a = \langle \{a, c, b\},$
$\{b\} \rangle$ then $a \wedge \sim a = \langle \{a, c, b\}, \{b\} \rangle \wedge \langle \{a, c, d\}, \{d\} \rangle = \langle \{a, c\}, \emptyset \rangle$. Now, $\langle \{a, c, b\}, \{b\} \rangle$ represents the rough set $\{\{a, b\}, \{c, b\}\}$. Let us consider, for instance, $\{a, b\}$. The boundary $\mathbb{B}(\{a, b\})$ is $\{a, b, c\} \cap -\{b\} = \{a, c\}$. It follows that $a \wedge \sim a = \langle \mathbb{B}(\{a, b\}), \emptyset \rangle$. Hence $\neg(a \wedge \sim a) = \neg(a \wedge \neg a) = \langle -\mathbb{B}(\{a, b\}), -\mathbb{B}(\{a, b\}) \rangle = \langle \{a, c\}, \{a, c\} \rangle$.

On the contrary, $a \vee \sim a = \langle \{a, b, c, d\}, \{b, d\} \rangle = \langle U, -\mathbb{B}(\{a, b\}) \rangle$, so that $\neg(a \vee \sim a) = \neg(a \vee \neg a) = \langle -- \mathbb{B}(\{a, b\}), -- \mathbb{B}(\{a, b\}) \rangle = \langle \mathbb{B}(\{a, b\}), \mathbb{B}(\{a, b\}) \rangle$.

Local Validity:
Let us consider again the rough set $a = \langle \{a, c, b\}, \{b\} \rangle$. Then $a \vee \neg a = \langle \{a, c, b\}, \{b\} \rangle \vee \langle \{d\}, \{d\} \rangle = \langle U, \{b, d\} \rangle = \langle U, B \rangle$. However, if we take the illegal pair $a' = \langle \{a, c, b\}, \emptyset \rangle$, then $a' \vee \neg a' = \langle U, \{d\} \rangle \lneqq \langle U, B \rangle$. So the property $x \vee \neg x \geqslant \langle U, B \rangle$ reflects our constraint on the admissible forms of rough sets with decreasing representation.

On the other side, $a \wedge \neg a = a \wedge \langle \{c, d\}, \{c, d\} \rangle = \langle \{c\}, \emptyset \rangle \leq \langle -B, \emptyset \rangle$. Again, $a' \wedge \neg a' = a' \wedge \langle U, U \rangle = a' \gneqq \langle -B, \emptyset \rangle$. Henceforth, also the property $x \wedge \neg x \leqslant \langle -B, \emptyset \rangle$ testifies for the legality of x.

Once we have accomplished this logico-algebraic interpretation of Rough Set Systems, we can exploit well-known relationships between the class of semi-simple Nelson algebras and the class of *three-valued Łukasiewicz algebras* in order to move from Nelson's philosophical issues concerning the separation of concepts to the standpoint of Multi-Valued Logics. It will be proved that for any Approximation Space $\mathbf{AS}(U/E)$ the Rough Set System $RS(U/E)$ is a finite three-valued Łukasiewicz algebra. In this framework the projection operators correspond to the two endomorphisms provided by these algebras.

The logical status of the intermediate value in these algebras is worth being discussed.

Generally, three-valued Łukasiewicz algebras lack the presence of a central element. An element x is called *central* if $\sim x = x$. One can prove that in $RS(U/E)$, *qua* three-valued Łukasiewicz algebra, there is at most one central element. Now we show that there is a central element *only if there are not singleton elementary categories*. In fact, we know that in this case we have at least two undefinable sets whose corresponding rough set is $\langle U, \emptyset \rangle$ (by definition of "undefinable set", the closure of these sets is the entire universe, while their interior is empty). It happens that $\sim \langle U, \emptyset \rangle = \langle U, \emptyset \rangle$.

Moreover, in this specific case $RS(U/E)$ can be made into a *Post algebra of order three*, characterized by the three-element chain

$$\langle \emptyset, \emptyset \rangle \leq \langle U, \emptyset \rangle \leq \langle U, U \rangle.$$

However, in general we do not have such a central element because $B \neq \emptyset$. In this case is it impossible to define an algebraic structure exhibiting a three-element chain of values, in full generality? It is possible, if we conceive, once more, the concept of an intermediate value in a *relative* manner, not in an *absolute* one. This means that the property "to be an intermediate element" must be given a local, or relative, meaning exactly as the notion of a "rough set" was given, exploiting the Grothendieck topology $\mathbf{G}_B$, a meaning relative to the sub-universe B of the exact pieces of information. In this way we enter the realm of the generalizations of Post algebras called *Chain Based Lattices* investigated by Epstein and Horn.

Particularly, we shall see that for any Approximation Space $\mathbf{AS}(U/E)$, the Rough Set System $RS(U/E)$ is a $P_2 - lattice$ of order three characterized by means of the parameter B. Under this interpretation, the above local top element $\langle U, B \rangle$ and local bottom element $\langle -B, \emptyset \rangle$ play the role of intermediate and, respectively, co-intermediate elements.

If we compare the fact that $\langle U, \emptyset \rangle$ means "totally undefinable" with the local top and bottom elements, we find a meaningful reading for the intermediate value of a Rough Set Systems *qua* P_2-lattices: the worst informational situation is $\langle U, B \rangle$ which means "totally undefined up to B".

So one can pass from an extreme situation when $B = U$, to an opposite extreme situation when $B = \emptyset$, through an intermediate one when $U \neq B \neq \emptyset$. In the first case $\langle U, B \rangle = \langle U, U \rangle = \sim \langle \emptyset, \emptyset \rangle = \langle -B, \emptyset \rangle$. In the second $\langle U, U \rangle \neq \langle U, B \rangle = \langle -B, \emptyset \rangle \neq \langle \emptyset, \emptyset \rangle$. In the intermediate case $\langle U, U \rangle \neq \langle U, B \rangle \neq \langle -B, \emptyset \rangle \neq \langle \emptyset, \emptyset \rangle$.

We illustrate these situations in Figure 5.4 below.

Moreover we shall show that the pseudo-supplementation and the dual pseudo-supplementation which are definable in P_2-lattices play the same roles as the projection operators in semi-simple Nelson algebras and the two endomorphisms in Łukasiewicz three valued algebras.

It will also be proved that any finite semi-simple Nelson algebra, three-valued Łukasiewicz algebra, Post algebra of order-three or $P_2 -$lattice of order three, is isomorphic to the rough sets system induced
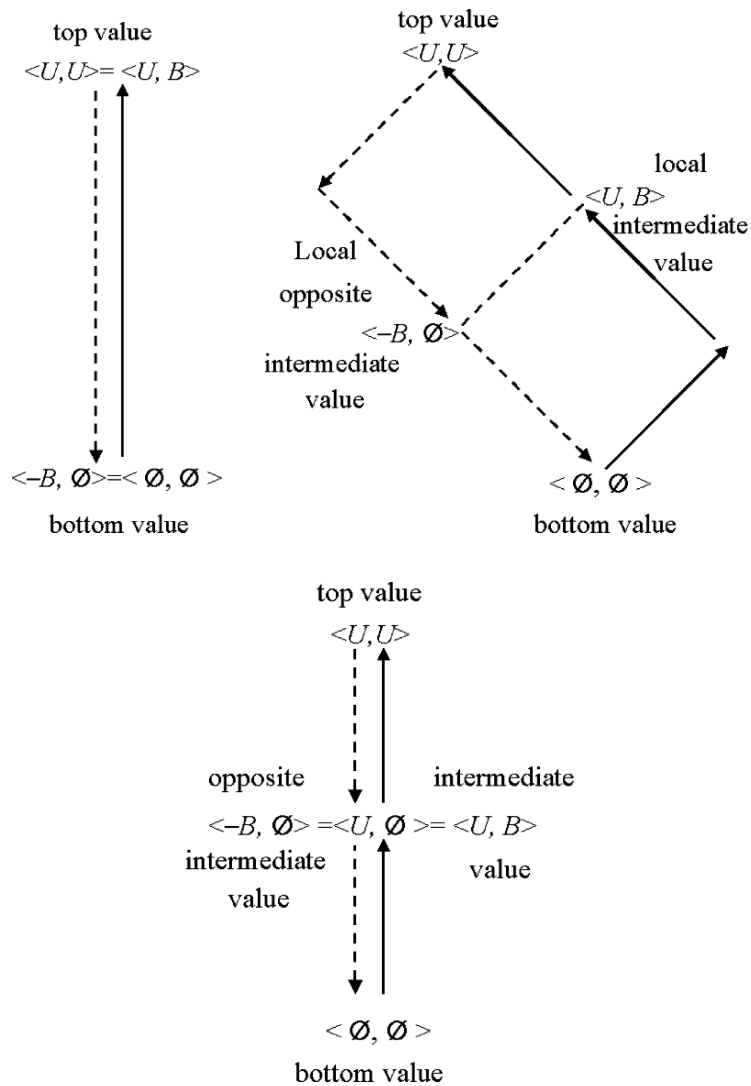
Figure 5.4: Three-valued lattices connected with Rough Set Systems

by some Approximation Space $\mathbf{AS}(U/E)$. More importantly, we shall exhibit a logico-algebraic decomposition of the structure of Rough Sets Systems (hence of semi-simple Nelson algebras, three-valued Łukasiewicz algebras, Post algebra or P_2-lattices of order three), based on the distinction between *locally exact* (or Boolean) part and *locally inexact* (or Postean) part of an Information System.

The first section of the Part will run as follows.

- We formally define the sets B and P and explain why they induce local logical behaviours in an Approximation Space.

- We introduce the mathematical notions of a "Grothendieck Topology" and a "Lawvere-Tierney operator", underlining their suitability for managing the notion "to be locally valid".

- The set B and its dual P will be exploited as information-dependent logico-topological parameters in order to define a Grothendieck topology for identifying $RS(U/E)$ within the set of all the ordered pairs of decreasing elements of $\mathbf{AS}(U/E)$.

- Via two Lawvere-Tierney operators, defined by means of B and P and inherited from the previous step, we shall define two modal operators M and L in $RS(U/E)$ that parallel the upper and the lower approximations, respectively. We shall see that M is an example of a closure operator induced by a well-known Grothendieck topology, namely the dense topology on the dual space of $RS(U/E)$ *qua* Heyting algebra while L is the closure operator induced by the dense topology on the dual space of the opposite Heyting algebra $RS(U/E)^{op}$.

- Using the above machinery we shall be able to show when and how a Rough Set System can be made into a Boolean algebra, a Łukasiewicz algebra, a Post algebra, a P_2-lattice, a P-algebra or a Nelson algebra. We shall see the roles played in these constructions by the notions of a "central element" and an "intermediate value", and the knowledge-oriented content that they are given in our setting.

- Finally by means of two additional Lawvere-Tierney operators based on the parameters P and B, we define a couple of new Grothendieck closure operators which make it possible to discover the double local logical nature of the above algebraic structures: the Post-like one (related to the inexact information of a knowledge system) and the Boolean one (related to its exact information).

In the second section of the Part the above results will be linked with an analysis of the notion of a "constructive logical system", by discussing the following points:

- The difference between the "truth-oriented" and the "knowledge-oriented" approaches in Logic.

- Why a knowledge-oriented approach leads us to the rejection of some classical principles and the assumption of new principles such as explicit definability (any derivable existential sentence must be explicitly instantiated by a closed term) and the disjunction principle (a disjunction is provable if at least one disjoint sentence is explicitly derivable). These principles define what are usually accepted as "constructive systems".

- The limits of this understanding of a "constructive system" and their relations with the classical definition of the concept of "knowledge".

- What is hidden in the knowledge-oriented approach. More precisely the difference between the logical status of atomic and non-atomic sentences.

- As a consequence the need to make classical and constructive systems coexist either by endowing constructive systems with well-suited classical principles or by adopting "context operators" which are able to identify the logical environment of a sentence, either classical or constructive, thus making the logical understanding of a sentence explicit.

To conclude, we shall record two notable conclusions:

R1 The "context operators" are the starting points of an approach to study maximal constructive logics, that is, constructive logics embedding a maximal amount of classical principles, in the sense that they cannot be augmented with any new principle without making them collapse into a non constructive system.

R2 The "context operators" are tightly connected with the Lawvere-Tierney operators which we use to formalise the notion of "local validity" and to define Rough Set Systems, both from a philosophical and a technical point of view.

Chapter 6

Basic Logico-Algebraic Structures

In order to appreciate the polymorphism of Rough Set Systems the essential ideas and notions of the logico-algebraic structures we shall deal with will be introduced.

In Mathematical toolkit 16.3 the reader will find the basic principles of bounded lattices. Moreover, all the algebraic structures needed are not only bounded lattices, but *finite distributive* bounded lattices, that is, finite structures $\mathbf{D} = \langle A, \vee, \wedge, 0, 1 \rangle$ such that the following distributive properties hold:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{6.0.1}$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{6.0.2}$$

REMARKS. The restriction to finite structures is not a limitation when we have to deal with practically given Rough Set Systems. This consideration lies behind our choice to focus on finite algebras. However, in general the results which will follow do not require finiteness. Anyway, we shall indicate when the finiteness assumption is essential to prove a result.

Among bounded distributive lattices Heyting algebras play a prominent role.

6.1 Heyting Algebras

Heyting algebras aim at modeling Intuitionistic Logic. In contrast to Classical Logic which maintains that given a formula α the negation $\sim\alpha$ is conceptually the complement of α, so that $\alpha\vee\sim\alpha$ holds true, Intuitionism maintains that the negation of α holds whenever any attempt to prove α *implies* a contradiction. In turn, a formula α implies a formula β if one can *transform* any proof of α into a proof of β.

With a questionable abuse, in Intuitionistic Logic the set of proofs for a formula α is usually identified with α itself.

Indeed this shift can been contested in various degrees. At the very beginning of constructive researches, the Russian mathematician A. Kolmogorov intended a formula as a problem and Yu. T. Medvedev made this idea explicit by claiming that a formula has a meaning only if coupled with its set of admissible solutions. In the last decade of the XX century Linear logicians advocated that the original Intuitionistic spirit leads to a "proof semantics" instead of a "formula semantics". Synthesizing, in a sense, the two approaches, P. A. Miglioli introduced the Evaluation Form Semantics, in which each formula α is interpreted by means of the set of proofs ending with α. These issues will be discussed in the second section of the Part, where, moreover, we shall see that they lead to a logical setting which is fully shared by Rough Set Theory.

However, in what follows we start with the "formula semantics" approach.

The provability of a formula β from α, $\alpha\vdash\beta$, is read, *via* the "deduction theorem" as $\vdash\alpha\longrightarrow\beta$. In lattices, $\alpha\vdash\beta$ is modeled by the relation $[\![\alpha]\!]\leq[\![\beta]\!]$, where $[\![x]\!]$ is the interpretation of formula x in the given lattice. Moreover, for any element w of the lattice, $w\leq[\![\beta]\!]$ means that β is valid at w. Thus we have that $[\![\alpha\longrightarrow\beta]\!]$ is the largest element w of the lattice such that $w\wedge[\![\alpha]\!]\leq[\![\beta]\!]$. Indeed, if this inequality holds, either both α and β are valid at w, or β is already valid *without the assistance* of α, so that β keeps holding true if we add α to the premises. Of course, one may question whether this addition is a faithful interpretation of the intended meaning of intuitionistic implication, even within the formula semantics approach.[1] However, this is the interpretation embedded in Heyting algebras in which, therefore, for any $a, b \in A$ there must be an element $a\Longrightarrow b$ such that the following

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[1]Clearly this immaterial transformation of a superfluous proof of α into an already given proof of β is not acceptable if we think that any "transformation" must involve some *relevant* connection between the two sides of the move. For instance, Relevance Logics do not admit unlimited instances of the *weakening rule* $\frac{\Gamma\vdash\beta}{\alpha,\Gamma\vdash\beta}$, which corresponds to the questionable maneuvers we are discussing.

relation holds, for any $c \in A$:

$$c \wedge a \leq b \ \textit{iff}\ c \leq a \Longrightarrow b \tag{6.1.3}$$

which states, indeed, that $a \Longrightarrow b$ is the largest element w such that $w \wedge a \leq b$.

The element $a \Longrightarrow b$ is called the *pseudo-complement* of a *relative to* b. It follows that if we model any contradiction with the bottom element of the algebra, 0, then the negation will be defined as

$$\neg a = a \Longrightarrow 0 \tag{6.1.4}$$

The element $\neg a$ is called the *pseudo-complement* of a.

Thus we eventually arrive at the following definition:

Definition 6.1.1 (Heyting algebras). $\mathbf{H} = \langle A, \wedge, \vee, \Longrightarrow, \neg, 0, 1 \rangle$ *is called a* Heyting algebra *if it is a bounded lattice such that the operation* $\Longrightarrow$ *fulfills the relation* (6.1.3).

Notice that we are not required to explicitly assume that $\mathbf{H}$ is distributive. Indeed let us make the binary operations $\wedge$ and $\Longrightarrow$ into two families of unary operations $\{\wedge_x\}_{x \in A}$ and $\{\Longrightarrow_x\}_{x \in A}$, respectively (in other words, we parameterize $\wedge$ and $\Longrightarrow$ with the elements of the algebra). Then, from (6.1.3) we have, for any $a, b, c \in A$:

$$\wedge_a(c) \leq b \ \textit{iff}\ c \leq \Longrightarrow_a (b) \tag{6.1.5}$$

At this point the reader has already recognized that (6.1.5) defines a Galois adjunction $\wedge \dashv \Longrightarrow$ on $\mathbf{H}$. In fact, $\Longrightarrow$ is upper adjoint to $\wedge$. Therefore, $\wedge$ preserves joints, *qua* lower adjoint, so that $\mathbf{H}$ must be a distributive lattice (cf. *Lemma* 1.4.2) of Chapter 1.

REMARKS. The parameterizations above are instances of *Currying* – after the logician H. Curry – that is, the technique to transform a binary function $f(x, y)$ into a unary function f' which takes as input the first argument, x, and returns a new function f'' which takes as input the second argument, y, and returns the required result.

Any finite distributive lattice is a Heyting algebra, in fact we can set:

$$a \Longrightarrow b = \bigvee \{x : x \wedge a \leq b\} \tag{6.1.6}$$

Heyting algebras feature important properties. For instance the pseudo-complement of a is not necessarily the complement of a. Indeed we have that $a \vee \neg a \leq 1$, so that the *Excluded Middle* does not hold in Heyting algebras (just as it does not hold in Intuitionistic Logic). It follows that, in general, $a \vee b = 1$ only if $a = 1$ or $b = 1$, which models the so-called *disjunction property* (see below Chapter 9). However $a \wedge \neg a = 0$, so that the *Law of Contradiction* still holds. Moreover, in any Heyting algebra we have:

$$a \leq \neg\neg a \qquad\qquad (6.1.7)$$

Hence $\neg\neg a \implies a \leq 1$, so that deducing a contradiction from the assumption that a formula α is false, does not amount to a proof of α itself (although it may be a useful information[2]). Anyway, $\neg 0 = 1$ and $\neg 1 = 0$ so that if applied to the top and bottom elements $\neg$ is involutive (i.e. $\neg\neg x = x$).

In Heyting algebras the first De Morgan law holds:

$$\neg(a \vee b) = \neg a \wedge \neg b \qquad\qquad (6.1.8)$$

However one can prove that the second De Morgan rule

$$\neg(a \wedge b) = \neg a \vee \neg b \qquad\qquad (6.1.9)$$

does not hold (in fact, if 6.1.9 held, then since $a \wedge \neg a = 0$, we would obtain a weak form of Excluded Middle, namely $\neg a \vee \neg\neg a = 1$). Actually in Heyting algebras the second De Morgan law is weakened to

$$\neg(a \wedge b) \geq \neg a \vee \neg b \qquad\qquad (6.1.10)$$

Definition 6.1.2. *A* Boolean algebra *is a Heyting* **H** *algebra s.t.* $\forall a \in A,$

$$a \vee \neg a = 1 \qquad\qquad (6.1.11)$$

In Heyting algebras there are two kinds of notable elements, which will play important roles in our analysis:

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[2]It says that trying to prove α is not a desperate attempt.

Definition 6.1.3. *Let x be an element of a Heyting algebra* **H**, *then*

    1. x is said to be dense *iff $\neg x = 0$ (iff $\neg\neg x = 1$).*

    2. x is said to be regular *iff $\neg\neg x = x$.*

If $\mathbf{H}^{op}$ is a Heyting algebra, too (which always happens if **H** is finite) then we can define a dual relative pseudo-complementation, as the relative pseudo-complementation in $\mathbf{H}^{op}$:

Definition 6.1.4 (Lower adjoint to join)**.** *Given a lattice* $\mathbf{L} = \langle A, \wedge, \vee, 0, 1 \rangle$, *let us set* $\forall a, b, x \in A$,

    (i) *$a \vee x \geq b$ if and only if $x \geq a \Longleftarrow b$; (ii) $\lrcorner a = a \Longleftarrow 1$.*

Then $a \Longleftarrow b$, if it exists, is the smallest element x of A such that $a \vee x$ is greater than or equal to b, while $\lrcorner a$ is called the *dual pseudo-complement* (or *co-intuitionistic negation*) of a and it is the smallest element x of A such that $a \vee x = 1$.

    The operator $\lrcorner$ will be a pillar of our algebraic construction of Rough Set Systems.

Definition 6.1.5 (Co-Heyting algebras)**.** *If for any $a, b \in A$, $a \Longleftarrow b$ is defined, then* $\langle A, \vee, \wedge, \Longleftarrow, \lrcorner, 0, 1 \rangle$ *is called a* co-Heyting algebra.

Definition 6.1.6 (Bi-Heyting algebras)**.** *A* bi-Heyting algebra *is a bounded distributive lattice which is both a Heyting and a co-Heyting algebra.*

Example 6.1.1. Heyting algebras
Below we depict a Heyting algebra **A**. We shall use this algebra in a number of subsequent examples.
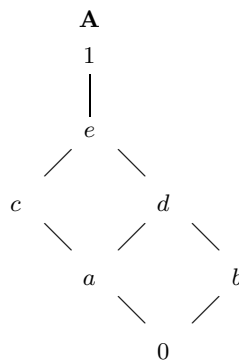
In the Heyting algebra **A** the pseudo-complement of b relative to a is c, that is, $b \Longrightarrow a = c$. Indeed, the set of elements x such that $x \wedge b \leq a$ is $\{0, a, c\}$, and c is

the largest in such set. Obviously, if $x \leq y$ then $x \implies y = 1$. The element c happens to be the pseudo-complement $\neg b$, too. In fact it is the largest element whose meet with b is 0. One can verify that $\neg a = b$ so that $a \leq \neg\neg a = \neg b = c$. **A** is also a co-Heyting algebra. Suffice it to reverse the lattice upside-down. We can verify that in **A**, $c \Longleftarrow d = b$, which is $c \implies d$ in $\mathbf{A}^{op}$. However, notice that for each element of $\mathbf{A}^{op}$ distinct from 1, the pseudo complement is 1. Thus for each element $x \neq 1$ of **A**, $\neg\!\!\!\neg x = 1$. On the contrary, $\neg\!\!\!\neg 1 = 0$.

6.2 Nelson Algebras

The failure of (6.1.9) implies that we can prove that a conjunction is false without being able to exhibit which formula is actually false. To circumvent this draw-back Nelson algebras were introduced – after David Nelson's logical work (see previous Chapter for Nelson's logical account). Nelson algebras feature a strong form of negation, $\sim$, such that the first De Morgan rule holds as well as the involution property:

$$\sim\sim a = a \qquad (6.2.12)$$

From (6.1.8) and (6.2.12) we immediately obtain the second De Morgan rule:

$$\sim a \lor \sim b = \sim\sim (\sim a \lor \sim b) = \sim (\sim\sim a \land \sim\sim b) = \sim (a \land b).$$

The properties about negation so far collected lead to an intermediate kind of lattices:

Definition 6.2.1 (De Morgan lattices). *A De Morgan lattice is a bounded distributive lattice* $\mathbf{D} = \langle A, \land, \lor, \sim, 0, 1 \rangle$ *in which (6.1.8), (6.1.9) and (6.2.12) hold (putting "$\sim$" instead of "$\neg$" in the first two equations).*

From the two De Morgan rules it follows that $\sim 1 = 0$ and $\sim 0 = 1$ (for instance, $a \lor \sim 0 = \sim (\sim a \land 0) = \sim 0$, any a, so that ~ 0 is the top element).

However, Excluded Middle as well as the Law of Contradiction fail even for strong negated formulas. That is, $a \lor \sim a \leq 1$ and $a \land \sim a \geq 0$.

For some elements, however, both principles hold. The family of such elements is called the *center*, $\mathcal{CTR}(\mathbf{D})$ of the lattice **D**. Thus a

center is the site where $\sim$ behaves as a Boolean complementation and it is always not empty because at least 1 and 0 belong to it.[3]

Moreover, we also obtain the contraposition rule:

$$a \leq b \ \textit{iff} \ \sim b \leq \sim a \tag{6.2.13}$$

because if $a \leq b$ then $a \vee b = b$, so that $\sim b = \sim (a \vee b) = \sim a \wedge \sim b$, i.e. $\sim b \leq \sim a$.

Now we state another principle required to define Nelson algebras. For all $a, b \in A$:

$$a \wedge \sim a \leq b \vee \sim b \tag{6.2.14}$$

This principle tells us that we can have at most one element a such that $\sim a = a$. Such an element is called, if any, *central element* (not to be confused with the center of the algebra). If a lattice has a central element, them it is called *centered*.

All this material makes it possible to define another intermediate class of lattices:

Definition 6.2.2 (Kleene algebras). *A De Morgan lattice* $\mathbf{K} = \langle A, \wedge, \vee, \sim, 0, 1 \rangle$ *such that* (6.2.14) *holds, is called a* Kleene algebra.

Moreover, in Nelson algebras, we have an implication which fulfills the following weaker adjointness property with conjunction. For any $a, b, c \in A$:

$$a \wedge c \leq \sim a \vee b \ \textit{iff} \ c \leq a \longrightarrow b \tag{6.2.15}$$

So notice, first of all, that $\sim a$ is not defined as $a \longrightarrow 0$ and, secondly, that $\longrightarrow$ is a weaker form of implication than the intuitionistic $\Longrightarrow$, and we call it *weak relative pseudo-complementation* or, shortly, *weak implication*. In fact we have that $a \longrightarrow b$ does not imply $\sim b \longrightarrow \sim a$. Therefore, if we set $a \preceq b$ iff $a \longrightarrow b = 1$, then $\preceq$ happens to be a preorder so that it does not coincide with the partial order $\leq$ of the lattice.

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[3]More precisely, one should speak of a *center with respect to a negation,* "*'*", because the center of a lattice $\mathbf{L}$ is defined as the set $\{x : \exists x'(x \wedge x' = 0 \ \& \ x \vee x' = 1)\}$. However we shall see that any additional qualification is superfluous in the lattices we are dealing with.

Of course, in any De Morgan lattice we can define an implication by cases $a \twoheadrightarrow b = \sim a \vee b$. This implication has nothing to do with the weak implication. However, we can restate (6.2.15) as follows:

$$a \longrightarrow b = a \Longrightarrow (a \twoheadrightarrow b) \qquad (6.2.16)$$

where, $\Longrightarrow$ is a relative pseudo-complementation.

It follows that in those Kleene algebras in which the relative pseudo-complementation of a relative to $a \twoheadrightarrow b$ is uniformly definable (surely all finite algebras) we can also define a weak implication $\longrightarrow$. But in order to obtain Nelson algebras from this kind of Kleene algebras we have to provide $\longrightarrow$ with the additional property:

$$(a \wedge b) \longrightarrow c = a \longrightarrow (b \longrightarrow c), \forall a, b, c \in A \qquad (6.2.17)$$

which, incidentally, is the logical form of the transformation of a function with two arguments, a and b, into two functions with one argument each, i.e. *Currying*.

Definition 6.2.3 (Nelson algebras)**.** *A Kleene algebra* $\mathbf{N} = \langle A, \wedge, \vee, \sim, \longrightarrow, 0, 1 \rangle$ *such that* (6.2.15) *and* (6.2.17) *hold, is called a* Nelson *algebra.*

Relation (6.2.15) allows sufficient room to define another type of negation in Nelson algebras:

$$\lrcorner a = a \longrightarrow 0 \qquad (6.2.18)$$

For reasons that will be clear in a while, we call "$\lrcorner$" *weak negation*. In view of (6.2.15) $\lrcorner a$ is the largest element x such that $x \wedge a \leq \sim a$.

Definition 6.2.4 (Semi-simple Nelson algebras)**.** *A Nelson algebra* $\mathbf{N}$ *is called semi-simple if and only if $a \vee \lrcorner a = 1$, for any $a \in A$.*

In view of *Definition* 6.1.4, $\lrcorner$ is a dual pseudo-complementation, in semi-simple Nelson algebras (in generic Nelson algebras this does not hold). Therefore in these algebraic structures we shall adopt the symbol $\urcorner$ instead of $\lrcorner$. But, whenever required to avoid confusion, we shall eventually distinguish $\urcorner$ and $\lrcorner$, also in semi-simple Nelson algebras.

Clearly, if a Nelson algebra is a Heyting algebra we have the relative pseudo-complementation $\Longrightarrow$. Moreover, one can set an extensional implication "$\rightarrowtail$" as follows:

$$a \rightarrowtail b = (a \longrightarrow b) \wedge (\sim b \longrightarrow \sim a) \qquad (6.2.19)$$

Specifically, in Nelson algebras in which a pseudo-complementation relative to 0 is defined for any element, we can set a pseudo-complementation "$\neg$".[4]

Let us now set in a Nelson algebra $\mathbf{N}$, for all a, b:

$$a \supset b = \mathord{\sim} \mathbin{\raise1pt\hbox{$\lrcorner$}} \mathord{\sim} a \vee b \vee (\mathbin{\raise1pt\hbox{$\lrcorner$}} a \wedge \mathbin{\raise1pt\hbox{$\lrcorner$}} \mathord{\sim} b) \tag{6.2.20}$$

$$\neg a = a \supset 0 = \mathord{\sim} \mathbin{\raise1pt\hbox{$\lrcorner$}} \mathord{\sim} a \tag{6.2.21}$$

Later on we shall prove that in semi-simple Nelson algebras $\supset$ is a relative pseudo-complementation and $\bar{\neg}$ a pseudo-complementation. Therefore, as above, in semi-simple Nelson algebras we shall denote $\bar{\neg}$ with $\neg$, if not otherwise stated to avoid risk of confusion. Nonetheless, in order to keep track of the general distinction between these operations, we shall not denote $\supset$ with $\Longrightarrow$. Thus we have the following correspondence table:

Heyting/co-Heyting algebras	Nelson algebras	Semi-simple Nelson algebras
$\Longrightarrow$	$\Longrightarrow$	$\supset$
	$\supset$	$\supset$
	$\longrightarrow$	$\longrightarrow$
	$\rightarrowtail$	$\rightarrowtail$
$\neg$	$\neg$	$\neg$ or $\bar{\neg}$
	$\bar{\neg}$	$\neg$ or $\bar{\neg}$
$\lrcorner$	$\lrcorner$	$\lrcorner$ or $\dot{\lrcorner}$
	$\dot{\lrcorner}$	$\lrcorner$ or $\dot{\lrcorner}$

We shall prove that in semi-simple Nelson algebras:

$$a \wedge \neg a = 0 \leq a \wedge \mathord{\sim} a = a \wedge \mathbin{\raise1pt\hbox{$\lrcorner$}} a \tag{6.2.22}$$

$$a \vee \mathbin{\raise1pt\hbox{$\lrcorner$}} a = 1 \geq a \vee \mathord{\sim} a = a \vee \neg a \tag{6.2.23}$$

Hence the following holds:

$$\neg a \leq \mathord{\sim} a \leq \mathbin{\raise1pt\hbox{$\lrcorner$}} a \tag{6.2.24}$$

But for some elements the three negations collapse. It can be shown that if two of them coincide, then all three coincide and the site of the elements a such that $\mathord{\sim} a = \neg a = \mathbin{\raise1pt\hbox{$\lrcorner$}} a$ is the *center*, $\mathcal{CTR}(\mathbf{N})$, of $\mathbf{N}$.

---

[4] Actually this is the case of any finite Nelson algebra, because they are Heyting algebras too. But there are infinite Nelson algebras which are not Heyting algebras (see Frame 10.9.3). Nevertheless, the relative pseudo-complementation of a with respect to $a \twoheadrightarrow b$ is defined for all elements a and b in a Nelson algebra. Obviously.

Example 6.2.1. De Morgan, Kleene and Nelson algebras
We depict a De Morgan lattice **M**, a Kleene algebra **C** and a Nelson algebra **D**.
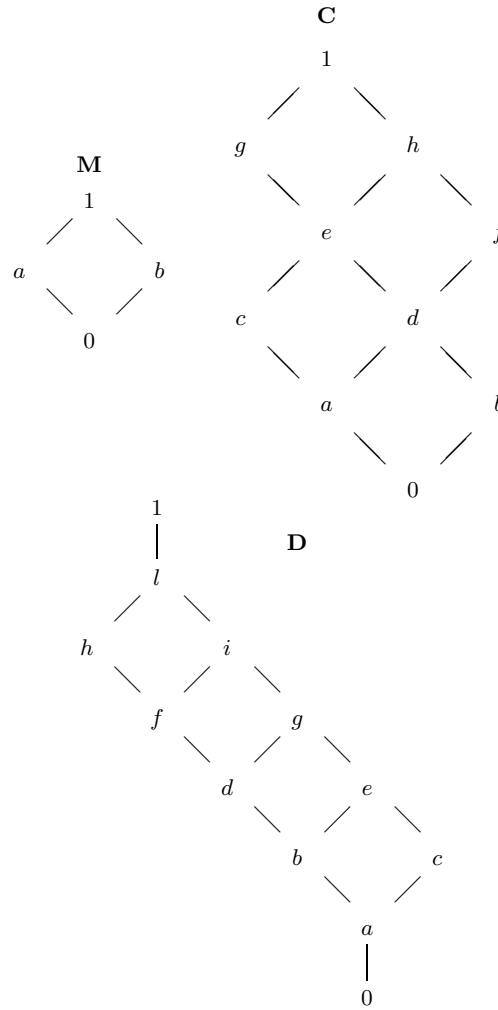
In the De Morgan lattice **M** we have $\sim a = a$, $\sim b = b$, $\sim 1 = 0$ and $\sim 0 = 1$. So **M** has two central elements, a and b. For no definition of $\sim$, **M** could have just one central element because, otherwise $\sim$ would not be involutive because for at least one non-central element x we would have $\sim\sim x \neq x$. Clearly, $a \wedge \sim a = a$ and $b \vee \sim b = b$ are incomparable in **M** so that property (6.2.14) does not hold.

In the Kleene lattice **C** the weak negation $\sim x$ is the element which is symmetric to x: $\sim 1 = 0$, $\sim g = b$, $\sim e = d$, $\sim c = f$ and $\sim a = h$. There is no central element here.

C is a Heyting algebra (trivially, because it is finite), so that we can define a relative pseudo-complementation $\Longrightarrow$. We can then adopt definition (6.2.16) to set a

weak implication "$\longrightarrow$". However, this weak implication fails to fulfill the Currying property (6.2.17). In fact, for instance, $g \longrightarrow (f \longrightarrow c) = g \longrightarrow c = h$, whereas $(g \wedge f) \longrightarrow c = 1$. We leave the computation to the reader. Notice that $g \longrightarrow c = h$ while $g \Longrightarrow c = c$.

In the Nelson lattice $\mathbf{D}$ the different negations are given by the following table:

	1	a	b	d	c	f
$\sim$	0	l	i	g	h	e
$\lrcorner$	0	1	1	1	h	e
$\lrcorner\,\lrcorner$	1	0	0	0	c	h
$\urcorner$	0	l	i	c	h	c
$\urcorner\,\urcorner$	1	0	0	h	c	h

From the above table we can compute the negations of any element. For instance, from $\lrcorner f = e$ and $\lrcorner\,\lrcorner f = h$, we deduce that $\lrcorner e = h$. With the same method we obtain $\urcorner e = f$; in fact, using $\sim\sim x = x$, we have $\urcorner e =\sim \lrcorner \sim e =\sim \lrcorner f =\sim e = f$, and so on (eventually using $\sim \urcorner \sim = \lrcorner$).

Let us compute an instance of weak relative pseudo-complementation: $e \longrightarrow d$, on the basis of (6.2.15). We have that $\{x : x \wedge e \leq (\sim e \vee d)\} = \{x : x \wedge e \leq (f \vee d)\} = \{x : x \wedge e \leq f\} = \{h, f, d, b, a, 0\}$. The largest element of this set is h. Therefore $e \longrightarrow d = h$.

Further, we can verify an instance of the Currying property of $\longrightarrow$:
$i \longrightarrow (e \longrightarrow d) = i \longrightarrow h = h = e \longrightarrow d = (i \wedge e) \longrightarrow d$.

Since $\mathbf{D}$ is a finite distributive lattice, it is also a Heyting algebra. However, the operation $\supset$ defined in (6.2.20) does not coincide with the relative pseudo-complementation $\Longrightarrow$ of $\mathbf{D}$ *qua* Heyting algebra. For instance, $b \supset 0 = i$ while $b \Longrightarrow 0 = 0$ (of course, the weak relative pseudo-complementation $\longrightarrow$ does not coincides either with $\Longrightarrow$ or with $\supset$: $b \longrightarrow 0 = 1$).

Finally, notice that $c \vee \lrcorner c = c \vee h = l \lneqq 1$. It follows that $\mathbf{D}$ is not semi-simple.

6.3 N-Valued Łukasiewicz Algebras

Semi-simple Nelson algebras are strongly linked to three-valued Łukasiewicz algebras. Hence this kind of multi-valued logico-algebraic systems deserves to be introduced.

Definition 6.3.1. *An n-valued Łukasiewicz algebra, for $n \geq 2$, is a de Morgan lattice $\langle A, \wedge, \vee, \sim, 0, 1 \rangle$ with $n-1$ unary operators $\phi_1, \phi_2, ..., \phi_{n-1}$ satisfying the following identities:*

1. $\phi_i(x \vee y) = \phi_i(x) \vee \phi_i(y)$; $\phi_i(x \wedge y) = \phi_i(x) \wedge \phi_i(y)$, $1 \leq i \leq n-1$.

2. $\phi_i(x) \wedge \phi_j(x) = \phi_j(x)$, $1 \leq i \leq j \leq n-1$.

3. $\phi_i(x) \vee \sim \phi_i(x) = 1$; $\phi_i(x) \wedge \sim \phi_i(x) = 0$, $1 \leq i \leq n-1$.

4. $\phi_i(\sim x) =\sim \phi_{n-i}(x)$, $1 \leq i \leq n-1$.

5. $\phi_i(\phi_j(x)) = \phi_j(x)$, $1 \leq i, j \leq n-1$.

6. $x \vee \phi_1(x) = \phi_1(x)$; $x \wedge \phi_{n-1}(x) = \phi_{n-1}(x)$.

7. $\phi_i(0) = 0$; $\phi_i(1) = 1$, $1 \leq i \leq n-1$.

8. $\sim x \wedge \phi_{n-1}(x) = 0$; $\sim x \vee \phi_1(x) = 1$.

9. $y \wedge (x \vee \sim \phi_i(x) \vee \phi_{i+1}(y)) = y$, $1 \leq i \leq n-2$.

In Łukasiewicz algebras one can define an upper adjoint to $\wedge$, $\sqsupset$, called a *Moisil residuation*. In particular, if the algebra is three-valued the definition is:

$$a \sqsupset b = b \vee \sim \phi_1(a) \vee (\sim\phi_2(a) \wedge \phi_1(b)) \qquad (6.3.25)$$

As to the relations between three-valued Łukasiewicz algebras and semi-simple Nelson algebras, we shall see that everything is based on the fact that the operators ϕ_i are definable as double dual pseudo-complementations or double pseudo-complementations.

6.4 Chain-Based Lattices

It is not difficult to exhibit Nelson algebras where some element cannot be defined in terms of polynomial of other elements (think of a Nelson algebra where 1 is co-prime).

But there are distributive lattices such that each element is a combination of elements of the center and of another sublattice forming a chain $\{0 = e_0 \leq e_1 \leq ... \leq e_{n-1} = 1\}$, called a *chain base*.

Prototypes of such lattices are *Post algebras* but they can be generalised in different ways and these generalisations are given the collective name *Chain-based lattices*. Post algebras and some Chain-based lattices play a real important role in our discussion in that their fundamental ingredients, a chain base and a centre, make it possible to represent the polymorphism of Rough Set Systems.

Let us then introduce their features. A few of them will be left unproven and the reader is addressed to the literature quoted in the Frame section for further details.

Here we mention some possible conditions on a bounded distributive lattice $\mathbf{L}$. In what follows, by $\mathcal{CB}(\mathbf{L})$ we denote the chain base of $\mathbf{L}$, if it exists.

Basically, the principles below regard the capability to generate all the elements of a lattice by means of the chain and the center, the shape of the chain and its position within the lattice, and the possibility to project the elements of the lattice (specifically the elements of the chain) onto the center.

(cbl-1) **Generation by chain and center**. A first condition is about the aforementioned possibility to recover any element of $\mathbf{L}$ by means of elements forming a chain in $\mathbf{L}$ and elements of the center. Namely:

$\mathbf{L}$ *is generated by* $\mathbf{B} \cup C$ *where* $\mathbf{B}$ *is a Boolean subalgebra of* $\mathcal{CTR}(\mathbf{L})$ *and* $C \subseteq \mathcal{CB}(\mathbf{L})$.

Intuitively, any element x is identified by means of two co-ordinates, $\mathcal{CTR}(\mathbf{L})$ and $\mathcal{CB}(\mathbf{L})$:
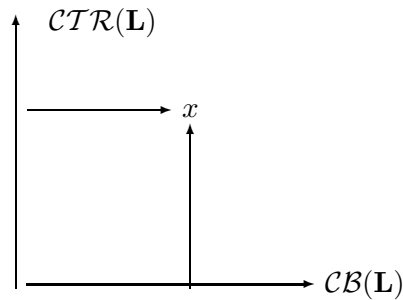
(cbl-2) **Projection onto the center**. The second principle deals with the existence of operations which projects each element onto the center of the lattice. Namely:

For all $x, y \in \mathbf{L}$, *there is a greatest element* $e \in \mathcal{CTR}(\mathbf{L})$ *such that* $e \wedge x \leq y$. *Such* e *is denoted by* $x \stackrel{c}{\Longrightarrow} y$ *and it is called the pseudo-supplement of* x *relative to* y.

In particular $!x = 1 \stackrel{c}{\Longrightarrow} x$ is called the *pseudo-supplement of* x and by definition it is the largest element of the center, below x.

(cbl-3) **Chaining with respect to the relative pseudo-complementation**. This principle states that the chain behaves as chain also

with respect to the order embedded in the relative pseudo-complementation. Namely:

$e_{i+1} \implies e_i = e_i$ *for all sequences* $\langle e_{i+1}, e_i \rangle$ *of elements of the chain base.*

(cbl-4) **Existence of the pseudo-supplement for any element of** $\mathcal{CB}(\mathbf{L})$. $!e_i$ *exists for any element* e_i *of the chain base.*
(cbl-5) **Linearity with respect to the relative pseudo-complementation**. *For any* $x, y \in \mathbf{L}, (x \implies y) \vee (y \implies x) = 1$.
(cbl-6) **Linearity with respect to the relative pseudosupplementation**. *For any* $x, y \in \mathbf{L}, (x \overset{c}{\implies} y) \vee (y \overset{c}{\implies} x) = 1$.
(cbl-7) **Position of the chain base in the lattice**. This principle is about the position of the elements of the chain base with respect to the bottom element: $!e_{n-2} = 0$.

Now we list some algebraic structures that can be defined on the basis of the above properties.[5]

Definition 6.4.1. *If* $\mathbf{L}$ *satisfies* (cbl-1), *then it is called a* $P_0 - lattice$.

Definition 6.4.2. *A* $P_0 - lattice$ *which satisfies* (cbl-3) *and* (cbl-4), *is called a* $P_2 - lattice$.

In a $P_2 - lattices$ $\mathbf{L}$ any element x has a representation $x = \bigvee_{i=1}^{n-1} (b_i \wedge e_i)$ where $e_i \in \mathcal{CB}(\mathbf{L})$ and $b_i \in \mathcal{CTR}(\mathbf{L})$. If $b_i \geq b_{i+1}$ for all i, then the representation is called *monotone*. If $b_i \wedge b_j = 0$, for $i \neq j$, then it is called *disjoint*.[6] Using these representations one can define a relative pseudo-complementation $\ni$. It follows that any $P_2 - lattice$ is a Heyting algebra.[7]

It can be shown that for a lattice $\mathbf{L}$ to be a $P_2 - lattice$ it is sufficient to satisfy (cbl-1), to be pseudo-supplemented and to fulfill the following additive property: $!(x \vee y) = !x \vee !y$.

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[5] Actually, we list only the lattices we are interest in. For a more general analysis we address the reader to the quoted literature.

[6] Hence, in this context the meaning of the terms "disjoint" and "monotone" do not coincide with the terms "disjoint" and "decreasing" in the context of representation of rough sets.

[7] For the interested reader, here is the definition: $a \ni b = b \vee \bigvee_{i=0}^{n-1} (b_i \wedge c_i)$, where $a = \bigvee_{i=1}^{n-1} (b_i \wedge e_i)$ is a disjoint representation of a, $b = \bigvee_{i=1}^{n-1} (c_i \wedge e_i)$ is a monotone representation of b, $c_0 = 1$ and, finally, $b_0 = \bigwedge_{i=1}^{n-1} -b_i$, where "$-$" is the Boolean complement in $\mathcal{CTR}(L)$.

Definition 6.4.3. *If* **L** *satisfies* (cbl-2) *and* (cbl-6) *then both* **L** *and its dual* **L**^{op} *are Heyting algebras satisfying* (cbl-2)*,* (cbl-5) *and* (cbl-6)*. Such a lattice is called a* P-algebra*.*

In *P-algebras, qua* bi-Heyting algebras, we have both $\Longrightarrow$ and $\Longleftarrow$. In addition, we find the dual operations of $\stackrel{c}{\Longrightarrow}$ and !. They are denoted by $\stackrel{c}{\Longleftarrow}$ and ¡, respectively, and called the *dual relative pseudo-supplementation* and the *dual pseudo-supplementation*, respectively. Since $¡x = 0 \stackrel{c}{\Longleftarrow} x$, we have that $¡x$ is the smallest element of the center, above x.

It can be shown that any $P_2 - lattice$ is a $P - algebra$. Hence a $P_2 - lattice$ can be equipped with the operator ¡. Vice-versa any $P - algebra$ fulfilling (cbl-1) is a $P_2 - lattice$.

Definition 6.4.4. *If* **L** *is a* P_2*-lattice satisfying* (cbl-7)*, then it is called a* Post-algebra*.*

From the above definitions, it follows that a Post algebra of order n, for $n \geq 2$, is a Heyting algebra with n nullary operations $0 = e_0, e_1, ..., e_{n-1} = 1$, forming a chain base, and $n - 1$ unary operators $D_i(x) = e_i \stackrel{c}{\Longrightarrow} x$, for $1 \leq i \leq n - 1$, satisfying the identities for P_2-lattices and (cbl-7).

The unary operations are analogous to those in Łukasiewicz algebras.

Notice that these projection operators will be massively used to model the lower and the upper approximations of rough sets.

6.5 Relationships, Analogies and Differences Between Structures

A number of relationships among the above structures are known in logic and algebraic literature (cf. the Frame section). These relationships while revealing certain analogies among the above systems, require some deeper explanation. Actually, we have to notice the following "strange facts":

1. Any Post algebra of order n exhibits an n-element chain of values.

2. Any P_2-lattice of order n is a principal ideal of a Post algebra of order n.

3. Any P_2-lattice of order n exhibits an n-element chain of values. This sentence seems in contrast with the two previous statements.

Indeed, if **P** is a Post algebra of order n, then its chain of values is aligned upon the bottom element so that some of this values will be surely excluded by any principal ideal of **P** (except the non proper ideal $\downarrow 1$).

4. In general a three-valued Łukasiewicz algebra does not have a three-element chain of values. We have a chain if and only if the algebra is centered. In this case, the central element is the intermediate value.

5. A finite P_0-lattice of order three can be made into a P_2-lattice of order three and as such it can exhibit a three-element chain of values, even if it is not centered. Since any finite P_0-lattice of order three is also a three-valued Łukasiewicz algebra, at a first sight we have a contradiction with the previous statement: what about the intermediate value?

6. Any finite P_0-lattice of order three is a semi-simple Nelson algebra. But it is also a $P-$algebra and as such it has projection operators onto the center (as like as 3-valued Łukasiewicz algebras), though Nelson algebras do not have projection operators.

In the next Example we shall see how these "mysteries" are represented from a formal point of view. After that we shall explain them in terms of "information granules".

Example 6.5.1. Semi-simple Nelson, Łukasieicz and Chain-based lattices

Here we show an example of semi-simple Nelson algebra, Chain-based lattices and three-valued Łukasiewicz algebras. Pay attention to the fact that the depicted lattices serve as example of different kind of algebras, for a reason that will be crystal clear in a few lines.
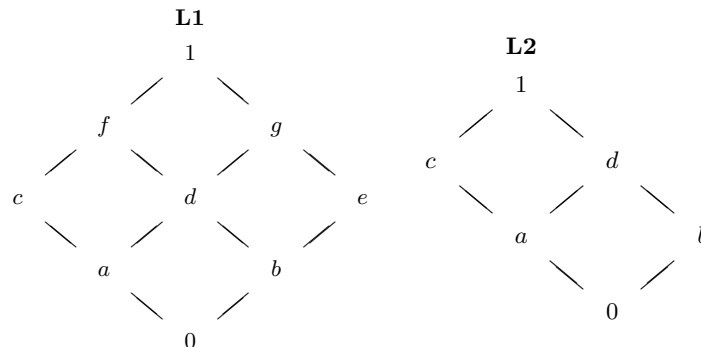
This is the table of the three negations in **L1** and **L2**:

L1	0	a	b	c	d	e	f	g	1
$\sim$	1	g	f	e	d	c	b	a	0
$\dashv$	1	1	1	e	1	c	e	c	0
$\neg$	1	e	c	e	0	c	0	0	0

L2	0	a	b	c	d	1
$\sim$	1	d	c	b	a	0
$\dashv$	1	1	c	b	c	0
$\neg$	1	b	c	b	0	0

1. We leave as an easy exercise to verify that both **L1** and **L2** are Nelson algebras. Moreover, by easy inspection of the tables, one can verify that for any x, $x \vee \dashv x = 1$. It follows that both **L1** and **L2** are semi-simple. For this reason we shall adopt the notation $\dashv$ and $\neg$ instead of $\dashv$ and, respectively, $\neg$.

   Since they are finite distributive lattices they can also be made into Heyting algebras. In this case the implication $\supset$ defined in (6.2.20) coincides with the relative pseudo-complementation $\Longrightarrow$, differently from the Nelson algebra **D** of Example 6.2.1 which is not semi-simple. For instance, $a \supset 0 = \sim \dashv \sim a \vee 0 \vee (\dashv a \wedge \dashv \sim 0) = \sim \dashv g \vee (1 \wedge 0) = \sim c = e = a \Longrightarrow 0$. Anyway, $\longrightarrow$ does not correspond with $\supset$ and $\Longrightarrow$. In fact, $a \longrightarrow 0 = a \Longrightarrow (\sim a \vee 0) = a \Longrightarrow g = 1$. The center of **L1**, $\mathcal{CTR}(\mathbf{L1})$, is $\{0, c, e, 1\}$, while $\mathcal{CTR}(\mathbf{L2}) = \{0, b, c, 1\}$. For all these elements, $x \wedge \sim x = 0$ and $x \vee \sim x = 1$ (easy inspection). Finally, $\sim d = d$, so that **L1** is a centered algebra and d its central element. On the contrary, there is no central element in **L2**.

2. **L1** has a 3- elements chain of values $\langle e_0, e_1, e_2 \rangle$ provided by $\langle 0, d, 1 \rangle$. Further, one can verify that $!e_{n-2} = 0$. In fact, $e_{n-2} = d$, so that $!e_{n-2} = !d = 1 \overset{c}{\Longrightarrow} d = \bigvee \{x : x \wedge 1 \leq d \ \& \ x \in \mathcal{CTR}(\mathbf{L1})\} = \bigvee \{0\} = 0$.

   Finally, $D_1(x) = e_1 \overset{c}{\Longrightarrow} x = d \overset{c}{\Longrightarrow} x$ so that one obtains the following table:

	0	a	b	c	d	e	f	g	1
D_1	0	c	e	c	1	e	1	1	1
D_2	0	0	0	c	0	e	c	e	1

   Since it is also a Heyting algebra, we have verified that **L1** can be made into a Post algebra of order three.

3. Both **L1** and **L2** can be made into three-valued Łukasiewicz algebras by setting $\phi_1 = D_1$ and $\phi_2 = D_2$ in **L1**. As to **L2**:

	0	a	b	c	d	1
ϕ_1	0	c	b	c	1	1
ϕ_2	0	0	b	c	b	1

4. **L1** and **L2** are also P_0-lattices. This is obvious for **L1**, because it is a Post algebra. As for **L2** consider that $\mathbf{C} = \langle 0, a, 1 \rangle$ is a chain base. Let us see a decreasing representation of the only element which is neither in the center nor in the chain: $d = (1 \wedge a) \vee (b \wedge 1)$ (notice that the elements of the center, 1 and b, are in decreasing order). A disjoint representation of d is given by $(c \wedge a) \vee (b \wedge 1)$.

5. However, the position of the chain base **C** prevents **L2** to be a $P2-$lattice. Indeed, although (cbl-4) of Section 6.4 is satisfied because $!e_i$ exists for any element e_i of the chain base (easy verification), nonetheless (cbl-3) does not hold. In fact, we have $a \Longrightarrow 0 = b$, so that $e_1 \Longrightarrow e_0 \neq e_0$.

However it can be made into a P_2-lattice by setting $e_1 = d$. In fact $d \Longrightarrow 0 = 0$. With this new chain, the element a (which now is neither in the chain nor in the center) is represented as $c \wedge d$. The new intermediate value is the least dense element of the lattice, because $\neg\neg d = \neg 0 = 1$ and the other dense element is 1 (this move is justified in Frame 10.1).

Further, with this new chain the projection operators ! and ¡ coincide with ϕ_2 and, respectively, ϕ_1. Moreover, the reader will easily notice that the operators D_1 and ϕ_1 coincide with the operator $\dashv\dashv$, while D_2 and ϕ_2 coincide with $\neg\neg$. This does not happen by chance, as we shall prove in the text.

Here below we depict the application of the modal operators $\dashv\dashv = \phi_2 = D_2 =!$ and $\neg\neg = \phi_1 = D_1 =$¡:
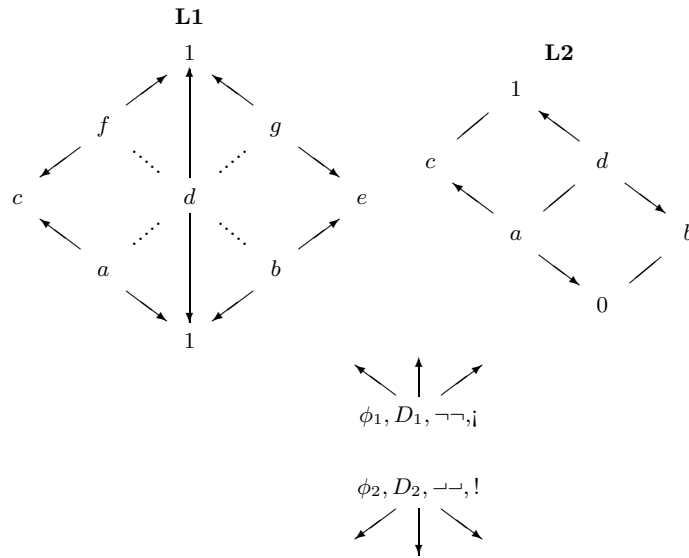
Chapter 7

Local Validity, Grothendieck Topologies and Rough Sets

7.1 Representing Rough Sets

The first step is to represent rough sets. Thus, we now give the formal definition of a rough set and the formal definition of the decreasing representation of rough sets which was adopted in the Introduction.

Definition 7.1.1. *Given an Indiscernibility Space $\langle U, E \rangle$,*

1. *Two sets $X, Y \in \wp(U)$ are called* rough top equal, *$X \simeq Y$, iff* $(uE)(X) = (uE)(Y)$.

2. *Two sets $X, Y \in \wp(U)$ are called* rough bottom equal, *$X \eqsim Y$, iff* $(lE)(X) = (lE)(Y)$.

3. *Two sets $X, Y \in \wp(U)$ are called* rough equal, *$X \approx Y$, iff $X \simeq Y$ and $X \eqsim Y$.*

4. *A set $X \in \wp(U)$ is called* definable *iff $X = (lE)(X) = (uE)(X)$.*

5. *A set $X \in \wp(U)$ is called* undefinable *iff $(lE)(X) = \emptyset$ and $(uE)(X) = U$.*

6. *Any equivalence class of subsets of U modulo the relation $\approx$ is called a* rough set.

We represent rough sets, with respect to an Indiscernibility Space $\langle U, E \rangle$ by means of ordered pairs of the form $\langle (uE)(X), (lE)(X) \rangle$, and if there is no risk of confusion we shall call them rough sets *tout-court*. Therefore, we define the following *rough-set map*:

Definition 7.1.2. *Let $\langle U, E \rangle$ be an Indiscernibility Space and $\mathbf{AS}(U/E)$ the Approximation Space induced by it. Then*

1. *$rs : \wp(U) \longmapsto \mathbf{AS}(U/E) \times \mathbf{AS}(U/E); \quad rs(X) = \langle (uE)(X), (lE)(X) \rangle$, is called a* rough-set map.

2. *The image Im_{rs} will be called the* Rough Set System *of the Indiscernibility Space $\langle U, E \rangle$ and denoted by $RS(U/E)$.*

By extension we say that two rough sets $\langle A_1, A_2 \rangle, \langle B_1, B_2 \rangle$ are top equal, if and only if $A_1 = B_1$, bottom equal if and only if $A_2 = B_2$. Since the context will make the meaning clear, we shall use for these relations the same symbols "$\simeq$" and, respectively, "$\eqsim$".

From the above definitions we have that any ordered pair $\langle (uE)(X), (lE)(X) \rangle$ such that $(uE)(X) = (lE)(X)$ is the image of an equivalence class modulo E with exactly one element: X. In this case X is an exactly definable set and $rs(X) = \langle X, X \rangle$.

Definition 7.1.3. *Given an Indiscernibility Space $\langle U, E \rangle$, any rough set of type $\langle X, X \rangle$, for $X \subseteq U$, is called an* exact rough set.

Although this term may appear contradictory, it is meaningful and it can be found elsewhere in literature (see, for instance, [Cleave, 1974]).

Obviously, for any $X \in \mathbf{AS}(U/E)$, $rs(X)$ is an exact rough set.

7.1.1 Local Logical Behaviours in Rough Set Systems

TERMINOLOGY AND NOTATION
From now on we set $a = \langle a_1, a_2 \rangle, b = \langle b_1, b_2 \rangle$ and, in general, $x = \langle x_1, x_2 \rangle$. We shall use this notation also for generic rough sets, reserving capital Latin letters, for instance $X = \langle X_1, X_2 \rangle$, for specific subsets of the universe U or when we want to stress the fact that the elements of a rough set are sets. The context will clarify the level of abstraction we are referring to. Moreover, since the equivalence relation E will be always understood, we shall avoid its reference in the notation of Approximation Spaces and Rough Set Systems and we shall use $\mathbf{AS}(U)$ and $RS(U)$, respectively. In any case, we do not change the notation of upper and lower approximations.

Since an Approximation Space is a Boolean algebra and Rough Sets are ordered pairs of decreasing elements of this Boolean algebra, we have to consider the logical operations which can be performed on generic ordered pairs of decreasing elements of a generic Boolean algebra $\mathbf{A} = \langle A, \vee, \wedge, \Longrightarrow, \neg, 1, 0 \rangle$, where for any $a, b \in A, a \Longrightarrow b = \neg a \vee b$:

---

1. $1 = \langle 1, 1 \rangle$ (top element).
2. $0 = \langle 0, 0 \rangle$ (bottom element).
3. $a \vee b = \langle a_1 \vee b_1, a_2 \vee b_2 \rangle$ (meet).
4. $a \wedge b = \langle a_1 \wedge b_1, a_2 \wedge b_2 \rangle$ (join).
5. $a \longrightarrow b = \langle a_2 \Longrightarrow b_1, a_2 \Longrightarrow b_2 \rangle$ (weak implication).
6. $\sim a = \langle \neg a_2, \neg a_1 \rangle$ (strong negation).
7. $\lrcorner a = \langle \neg a_2, \neg a_2 \rangle$ (weak negation).
8. $a \supset b = \sim \lrcorner \sim a \vee b \vee (\lrcorner a \wedge \lrcorner \sim b)$ (intuitionistic implication).
9. $\neg a = \sim \lrcorner \sim a = \langle \neg a_1, \neg a_1 \rangle$ (intuitionistic negation),
where $1, 0, \wedge, \vee, \Longrightarrow$ and $\neg$ applied inside the ordered pairs are the operations of the underlying Boolean algebra.
Definition: $a \leq b$ iff $a \vee b = b$.
Facts (to be proved in Frame 10.5):
(a) $a \wedge c \leq b$ iff $c \leq a \supset b$.
(b) $a \leq b$ iff $a \wedge b = a$, iff $a \longrightarrow b = \sim b \longrightarrow \sim a = 1$, iff $a \supset b = 1$,
    iff $a_1 \leq b_1$ and $a_2 \leq b_2$.
(c) $\neg \neg a = \lrcorner \sim a$; $\lrcorner \lrcorner a = \sim \lrcorner a$; (d) $\neg a = a \supset 0$; $\lrcorner a = a \longrightarrow 0$.
(e) $\sim \neg a = \lrcorner \sim a$; $\sim \lrcorner a = \neg \sim a$.

Window 7.1. **Operations on ordered pairs of decreasing elements of a Boolean algebra**

It is not difficult to verify that any Rough Set System $RS(U/E)$ is closed under the operations listed in *Window* 7.1. Therefore, it is a complete distributive lattice with top element $\langle U, U \rangle$ and bottom element $\langle \emptyset, \emptyset \rangle$. In any Rough Set System, $\langle X, Y \rangle \leq \langle X', Y' \rangle$ if and only if $X \subseteq X'$ and $Y \subseteq Y'$.

However, before trying to understand what happens when we apply the above operations to a Rough Set System $RS(U)$, this system must be exactly defined.

More precisely, we know that $RS(U)$ is a subset of $\mathbf{AS}(U) \times \mathbf{AS}(U)$. Then the first question is: how can we identify $RS(U)$ within the above

direct product (also denoted by $\mathbf{A}^2$)? An initial answer is given by the following

Lemma 7.1.1. *Given any Approximation Space* $\mathbf{AS}(U)$*, if* $\langle X, Y \rangle$ *is a rough set then* $X \supseteq Y$*.*

Proof. Trivially, since for any X, $(uE)(X) \supseteq (lE)(X)$.   QED

It follows that $RS(U)$ will be within two extreme systems that we describe in general:

Definition 7.1.4. *for any Boolean algebra* $\mathbf{A}$*,*
1. $\mathcal{P}(\mathbf{A}) = \{\langle x_1, x_2 \rangle \in \mathbf{A}^2 : x_1 \geq x_2\}$*.*
2. $\mathcal{B}(\mathbf{A}) = \{\langle x_1, x_2 \rangle \in \mathbf{A}^2 : x_1 = x_2\}$*.*

Indeed we have:

$$\mathcal{B}(\mathbf{AS}(U)) \subseteq RS(U) \subseteq \mathcal{P}(\mathbf{AS}(U)) \tag{7.1.1}$$

(in particular, from *Definition* 7.1.3, $rs(\mathbf{AS}(U)) = \mathcal{B}(\mathbf{AS}(U))$. Moreover it is easy to show that any Boolean algebra $\mathbf{A}$ is representable in the following isomorphic form:

$$\mathbf{B}(\mathbf{A}) = \langle \mathcal{B}(\mathbf{A}), \vee, \wedge, \dashv, \langle \emptyset, \emptyset \rangle, \langle U, U \rangle \rangle. \tag{7.1.2}$$

On the contrary, as we shall see, $\mathcal{P}(\mathbf{AS}(U))$ is a Post algebra of order three. Hence we have the following information: first, exact rough sets (that is, of type $\langle X, X \rangle$) form a Boolean algebra; second, this Boolean algebra must be embedded in $RS(U)$; third $RS(U)$ is in between a Post algebra of order three and a Boolean algebra.

In order to proceed we must consider closely the philosophy of the theory. In fact, if we take it seriously, we have to distinguish two sorts of elementary classes.

Definition 7.1.5. *Let* $\langle U, E \rangle$ *be an Indiscernibility Space. We set:*

(1) $B^* = \{X \in U/E : card(X) = 1\}$;
(2) $P^* = \{X \in U/E : card(X) \geq 2\}$;
(3) $B = \bigcup B^*$;
(4) $P = \bigcup P^*$.

where $card(X)$ *denotes the cardinality of the set* X*.*

The distinction between these sets is essential: indeed, given an elementary class X if $X = \{x\}$, for $x \in U$, then our information about its

unique element is perfect: X is a sort of fixed point in the knowledge extraction process, in the sense that more information cannot be transformed into a narrower analysis of X: its unique element x is completely determined already.

On the other hand, if $card(X) \geq 2$, then our information about its elements is not complete. Therefore we can have some $A \subseteq U$ such that some but not all the elements of X are in A, so that neither $X \subseteq (lE)(A)$ nor $X \subseteq -(uE)(A)$. So X is included in $(uE)(A) \cap -(lE)(A)$, i.e. X is included in the *doubtful* region (or *boundary* in topological terms) of the description of A via the information at our disposal.

This difference is immediately reflected in the behaviour of the following generalized characteristic function χ:

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in -(uE)(A) \\ \delta & \text{if } x \in (uE)(A) \cap -(lE)(A) \\ 1 & \text{if } x \in (lE)(A) \end{cases}$$

If $X = \{x\} \in U/E$, then for all $A \subseteq U$, $\chi_A(x)$ takes value 0 or 1: $\{x\}$ cannot be part of any boundary and hence, for any singleton X, we have:

$$X \subseteq (lE)(A) \; if \; and \; only \; if \; X \subseteq (uE)(A). \tag{7.1.3}$$

Thus χ has a **local Boolean behaviour** on B and a **local three-valued behaviour** on P.

Indeed, condition (7.1.3) is tantamount to

$$(lE)(A) \cap B = B \cap (uE)(A), \; any \; A \subseteq U \tag{7.1.4}$$

That is, the upper and lower approximations must locally (that is, on B) coincide. But in view of *Definition* 7.1.3, this means that any rough set is a classic rough set as to its B-part. So we start seeing that describing rough sets uniformly by means of a generalized three-valued characteristic function is only half of the job. If we forget the second half we miss the target.

Therefore, in order to exhibit a well-founded mathematical explanation of the algebraic mechanism we are required to describe, we need some subtle notion able to define the concept

$$\boxed{\text{IT IS LOCALLY THE CASE THAT}}$$

in a formal setting.

Since U/E is the set of atoms of the Boolean algebra $\mathbf{AS}(U)$, we shall denote it with $atoms(\mathbf{AS}(U))$, too.

If $\mathbf{S}$ is a mathematical structure with carrier X, for instance an ordered set $\mathbf{P} = \langle X, \leq \rangle$, whenever required we shall distinguish $\mathbf{S}$ from X. Thus we shall write, for example, $p \in X$ to denote an element p of $\mathbf{S}$. However, if there is no need for this distinction, we shall use the notation $p \in \mathbf{S}$ as an equivalent expression.

7.2   Some Duality of Distributive Lattices

Before looking for a suitable mathematical tool, we must go through a short excursus on duality theory of distributive lattices and Heyting algebras.

Duality theorems state a "dialectic" relationships between two different classes of structures, in the sense that the members of one class can be recovered from members of the dual class, and vice-versa, in a "continuous way" that we are going to specify.

Typically the elements of one class are structurally simpler than the other, and, for this reason, they receive also the name "spectral spaces".

The general schema of a duality is the following:

$$
\begin{array}{ccc}
\mathbf{A}' & \xrightarrow{\ \phi\ } & \mathbf{B}' \\
\big\uparrow{\scriptstyle h} & & \big\downarrow{\scriptstyle g} \\
\mathbf{A} & \xleftarrow[\ \eta\ ]{} & \mathbf{B}
\end{array}
$$

which reads: Suppose $\mathbf{A}$ and $\mathbf{A}'$ belong to a class $\mathcal{A}$ and $\mathbf{B}$ and $\mathbf{B}'$ belong to a class $\mathcal{B}$. Then, if $\mathbf{A}'$ is dual to $\mathbf{B}'$ and $\mathbf{B}'$ is *transformed* into $\mathbf{B}$ *via* a mapping g, then the dual of $\mathbf{B}$ is transformed, *via* h, into $\mathbf{A}'$. Otherwise stated, duality reflects the transformations which are typical of a given classe in the other class.

In our framework, distributive lattices are dual to partially ordered sets (eventually equipped with some additional features), transformations of distributive lattices are lattice $\{0, 1\}$-homomorphisms, and

transformations of partially ordered sets are order preserving maps (and any additional dual feature).

As a matter of fact the definition of "abstracts points" as "bundles of properties", described in the Introduction at Subsection 5.1, is such a dual construction. Indeed, the partial order between abstract points of a frame **A** of observed properties stocks sufficient information to recover **A** itself.

Here, we sketch the fundamental points of the duality relations of distributive lattices, in the <u>finite case</u> (which will be understood by default). Notice, that they can be extended to the infinite case by improving, in a sense, the topological component of the results we are going to describe.

Any finite distributive lattice $\mathbf{L} = \langle L, \leq \rangle$ is dual to the partial ordered set of its own co-prime elements $\mathbf{J}(\mathbf{L}) = \langle \mathcal{J}(\mathbf{L}), \leqslant \rangle$, where $p \leqslant p'$ only if $p \geq p'$. Any partially ordered set $\mathbf{P} = \langle P, \leq \rangle$ is dual to the lattice of its filters $\mathbf{F}(\mathbf{P}) = \langle \mathscr{F}(\mathbf{P}), \leqslant \rangle$, where for all $\uparrow A, \uparrow B \in \mathscr{F}(\mathbf{P})$, $\uparrow A \leqslant \uparrow B$ if $\uparrow A \subseteq \uparrow B$, so that $\uparrow p \leqslant \uparrow p'$ whenever $p' \leq p$.

One can prove that if **L** is a finite distributive lattice then

$$\mathbf{L} \cong_d \mathbf{F}(\mathbf{J}(\mathbf{L})) \tag{7.2.5}$$

where $\cong_d$ denotes the isomorphism between finite distributive lattices. Conversely, if **P** is a partially ordered set, then

$$\mathbf{P} \cong_p \mathbf{J}(\mathbf{F}(\mathbf{P})) \tag{7.2.6}$$

where $\cong_p$ denotes the isomorphisms between partially ordered sets.

The isomorphism (7.2.5) is given by the function ϕ described in Section 5.1 of the Introduction which, we recall, for any $a \in \mathbf{L}$ is defined as:

$$\phi(a) = \{p \in \mathcal{J}(\mathbf{L}) : p \leq a\} \tag{7.2.7}$$

(notice that $\leq$ is the partial order of **L**). This isomorphism will be understood for all duality results which will be stated from now on.

The direction of the ordering on $\mathbf{J}(\mathbf{L})$ or $\mathbf{F}(\mathbf{P})$ is by no means essential; we assume the above ordering to follow the usual preference of logicians for thinking in terms of filters (truth) instead of ideals (falsity).

If we consider $\mathbf{J}(\mathbf{L})$ as a Kripke frame for Intuitionistic Logic, then the monotonicity condition of the forcing clauses (cf. condition (*monotonicity*) of Frame 4.13 of Part I) states that for any formula α its

interpretation $[\![\alpha]\!]$ in $\mathbf{J(L)}$ is a filter. Therefore $\mathbf{F(J(L))}$ is the family of possible validity domains of formulas.

Example 7.2.1. Duality for finite distributive lattices

Consider the Heyting algebra $\mathbf{A}$ of Example 6.1.1. For this algebra the duality construction runs as follows.

$$\{1,a,b,c\}$$
$$\mathbf{F(J(A))} \quad |$$
$$a \quad \mathbf{J(A)} \qquad\qquad \{a,b,c\} \qquad\qquad \{a\} \quad \mathbf{J(F(J(A)))}$$
$$| \qquad\qquad\qquad \diagup \quad \diagdown \qquad\qquad |$$
$$c \qquad\quad b \quad\; \{a,c\} \qquad \{a,b\} \qquad \{a,c\} \qquad\qquad \{b\}$$
$$\diagdown \quad \diagup \qquad \diagdown \quad \diagup \quad \diagdown \qquad \diagdown \quad \diagup$$
$$1 \qquad\qquad\qquad \{a\} \qquad \{b\} \qquad \{1,a,b,c\}$$
$$\diagdown \quad \diagup$$
$$\emptyset$$
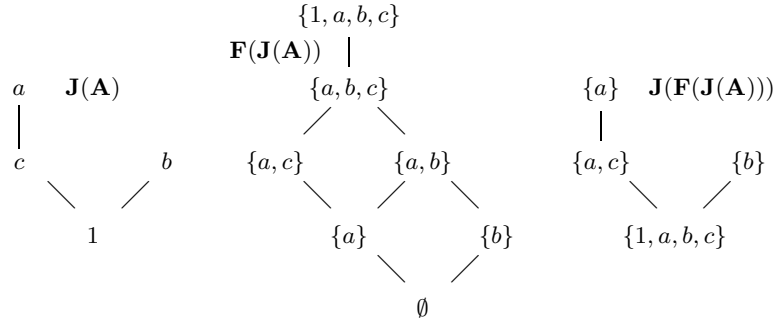
Thus, for instance $\phi(d) = \{a,b\}$ because a and b are the elements of $\mathcal{J}(\mathbf{A})$ below d in $\mathbf{A}$.

7.2.1   Duality for Heyting Algebras

Given a distributive lattice $\mathbf{L}$ of subsets of a universe U, we have seen that $\mathbf{L} = \mathbf{F(S(L))}$, where $\mathbf{S(L)}$ is U preordered by the specialization preorder $\preceq$ induced by $\mathbf{L}$. Thus $\mathbf{L}$ is the *Alexandrov Topology* on U induced by $\preceq$ and can be equipped with the following operations, besides $\cap$, $\cup$ and the complementation $-$:

(a) $A \implies B = \mathbb{I}(-A \cup B)$ (where $\mathbb{I}$ is the *interior operator*).
(b) $\neg A = \mathbb{I}(-A) = -\mathbb{C}(A)$ (where $\mathbb{C}$ is the *closure operator*).

Indeed, $-A \cup B$ is the largest subset X of U such that $X \cap A \subseteq B$, so that its interior is the largest element of $\mathbf{L}$ with such property.

As we have shall see in Subsection 7.3.1, the definitions above explain some terminology: for instance in the lattice $\mathbf{L}'$ of the example of subframe 10.3.1 below, the element $\{d,c\}$ is *dense*, in topological terms, since $\mathbb{I}\mathbb{C}(\{d,c\}) = \{a,b,c,d\} = 1$. But $\{d,c\}$ is *dense* also in logico-algebraic terms (see *Definition* 6.1.3). In fact from (a) and (b), for any $X \in L'$, $\mathbb{I}\mathbb{C}(X) = -\mathbb{C} - \mathbb{C}(X) = \neg\neg X$.

Similarly we have that $\mathbb{I}\mathbb{C}(\{d,c\}) \neq \{d,c\}$. Thus $\{d,c\}$ is *regular* neither in topological nor in logico-algebraic terms. On the contrary $\{c\}$ is regular.

Practically, in order to find the closure $\mathbb{C}(X)$ of a subset $X \subseteq U$ we have just to compute $\downarrow_{\preceq} X$ in $\mathbf{S}(\mathbf{L})$. In turns, the interior $\mathbb{I}(X)$ is the largest element of $\mathbf{L}$ included in X, and we can compute it by exploiting either of the following relations:

$$x \in \mathbb{I}(X) \ \textit{iff} \ \forall x'(x \preceq x' \Rightarrow x' \in X); \qquad (7.2.8)$$

$$\mathbb{I}(X) = -\mathbb{C}(-X) = - \downarrow_{\preceq} (-X). \qquad (7.2.9)$$

In $\mathbf{L}'$, for example, $\mathbb{I}(\{b,d,a\}) = - \downarrow_{\preceq} (-\{b,d,a\}) = - \downarrow_{\preceq} \{b',c\} = - \{b,b',a,c\} = \{d\}$; $\mathbb{C}(\{c\}) = \downarrow_{\preceq} \{c\} = \{c,a\}$. Thus $\neg\{c\} = - \{c,a\} = \{b,b',d\}$.

Exercise 7.1. *Prove that $A \Longrightarrow B = \neg(A \cap -B)$.*

This way we arrive at the following definition:

Definition 7.2.1. *A finite Heyting space is a poset $\mathbf{X} = \langle X, \leq \rangle$. Given a Heyting space $\mathbf{X}$ we can define its dual Heyting algebra $\mathbf{H}(\mathbf{X})$ in the following way: $\mathbf{H}(\mathbf{X}) = \langle \mathbf{F}(\mathbf{X}), \wedge, \vee, \Longrightarrow, \neg, 0, 1 \rangle$, where for any $A, B \in \mathbf{F}(\mathbf{X})$: (i) $A \Longrightarrow B = - \downarrow (A \cap -B)$; (ii) $\neg A = A \Longrightarrow \emptyset = - \downarrow A$, (iii) $A \wedge B = A \cap B$, (iv) $A \vee B = A \cup B$, (v) $1 = X$, (vi) $0 = \emptyset$.*

Conversely, given a finite Heyting algebra $\mathbf{A} = \langle A, \wedge, \vee, \Longrightarrow, \neg, 0, 1 \rangle$ we can obtain its dual Heyting space $\mathbf{HS}(\mathbf{A})$ by setting $\mathbf{HS}(\mathbf{A}) = \mathbf{J}(\mathbf{A})$.

We have that if $\mathbf{A}$ is a finite Heyting algebra and $\mathbf{X}$ is a finite Heyting space, then $\mathbf{A} \cong_h \mathbf{H}(\mathbf{HS}(\mathbf{A}))$ and $\mathbf{X} \cong_p \mathbf{HS}(\mathbf{H}(\mathbf{X}))$, where $\cong_h$ is the Heyting algebras isomorphism and $\cong_p$ is the isomorphism between posets.

7.3   Grothendieck Topologies

We ended Subsection 7.1.1 with a question: "Is there any mathematical tool which is able to deal with the concept *is locally the case that* ..."?

Fortunately, we can find such a notion in a powerful mathematical field: Grothendieck Topologies and their logico-algebraic interpretation introduced by Lawvere and Tierney.

Let $\mathbf{O} = \langle O, \leq \rangle$ be a preorder. We recall that for any $X \subseteq O$, the set $\uparrow X = \{p \in O : \exists x \in X \wedge x \leq p\}$, is called the *order filter* or *sieve* generated by X.

In particular, for any $p \in O$, $\uparrow p = \{p' \in O : p \leq p'\}$ is called the *principal filter* generated by p.

Let $\mathbf{F}(\mathbf{O})$ be the set of order filters over $\mathbf{O}$, ordered by $\subseteq$, i.e. $\mathbf{F}(\mathbf{O}) = \langle \mathscr{F}(\mathbf{O}), \subseteq \rangle$.

A Grothendieck Topology maps any element $p \in O$ to a particular subset $J_{[p]}$ of the family $\mathscr{F}(\uparrow p)$ of order filters over the subpreorder $\langle \uparrow p, \leq_p \rangle$, where $\leq_p$ is the preorder induced by $\leq$ on $\uparrow p$. More precisely, a Grothendieck Topology is a map $j : O \longmapsto \wp(\mathscr{F}(\mathbf{O})); j(p) = J_{[p]} \subseteq \mathscr{F}(\uparrow p)$ such that:

GT1. $\uparrow p \in J_{[p]}, \forall p \in O$;
GT2. $\uparrow p' \cap S \in J_{[p']}, \forall p' \geq p, \forall S \in J_{[p]}$;
GT3. $\forall p \in O, S \in J_{[p]}, S' \subseteq \uparrow p$, for $S' \in \mathscr{F}(\mathbf{O})$, if $\forall p' \in S, S' \cap \uparrow p' \in J_{[p']}$ then $S' \in J_{[p]}$.

If a filter S belongs to $J_{[p]}$, then we say that "*S covers p*".
By extension we shall call the set
$\mathbf{G} = \{J_{[p]} : p \in O\}$ a "Grothendieck Topology", too;
The structure $\langle O, \leq, \mathbf{G} \rangle$ is called an *ordered site*.

Window 7.2. **Grothendieck topologies (over preorders)**

Axiom GT1 (identity) states that the principal filter $\uparrow p$ covers p.
Axiom GT2 (stability) states that the restriction of a cover of an object p to a subfilter of p generated by p', is a cover of p'.
Axiom GT3 (transitivity) states that subcovers of covers are again covers.

Like any other topology, a Grothendieck topology $\mathbf{G}$ induces a closure operator. In particular, given an ordered site, one can uniquely define a Grothendieck closure operator, J, in the following way:

$$J : \mathscr{F}(\mathbf{O}) \longmapsto \mathscr{F}(\mathbf{O}); J(S) = \{p : S \cap \uparrow p \in J_{[p]}\}, \; for \; J_{[p]} \in \mathbf{G}.$$
(7.3.10)

The formal properties of J are the following:

$$(i) \; J(X) \supseteq X; \;\; (ii) \; J(J(X)) = J(X); \qquad (7.3.11)$$
$$(iii) \; J(X \cap Y) = J(X) \cap J(Y)$$

Notice that J is multiplicative and not additive, differently from the usual topological closure operators.

We address the reader's attention to the fact that $\mathbf{F}(\mathbf{O})$ is a topology in itself. Indeed it is the *Alexandrov topology* on $\mathbf{O}$. Thus an element of $\mathbf{G}$ is a family of open sets of a (usual) topology.

We want to emphasize the fact that Grothendieck topologies have actually been introduced in order to grasp the notion "to be locally valid". Intuitively: given a universe U and a Grothendieck topology on it, one can say that a property α is locally valid on a subset A of U, if the domain of validity of α, $[\![\alpha]\!]$, has a large enough intersection with A, where the meaning of "large enough", as we are going to explain, is provided by the Grothendieck topology we are working with.

7.3.1   A Fundamental Example: The Dense Topology

A well-known example of a Grothendieck topology is provided by the *dense topology* over an intuitionistic Kripke model. We remind from Frame 4.13 of Part I that a model of this type is a pair $\mathbf{M} = \langle \mathbf{W}, \models \rangle$, where $\mathbf{W} = \langle W, \leq \rangle$. Remember that the intuitionistic forcing discipline requires that the interpretation of any well formed formula α of an intuitionistic language $\mathcal{L}$ is an order filter $[\![\alpha]\!]$ of $\mathbf{W}$, hence a member of $\mathscr{F}(\mathbf{W})$. If $w \in [\![p]\!]$, then we say that w *forces* the validity of p, $w \models p$:

$$\forall w \in W, \forall \alpha \in \mathcal{L}, w \models \alpha \ \textit{iff} \ w \in [\![\alpha]\!]. \tag{7.3.12}$$

Moreover, recall that:

$$w \models \neg\alpha \ \textit{iff} \ \forall w' \geq w, w' \not\models \alpha. \tag{7.3.13}$$

Now consider for any $w \in W$ the set $J^{den}_{[w]}$ of filters S of $\mathscr{F}(\uparrow w)$ that are *dense* in $\uparrow w$, that is to say, such that $\forall w' \geq w, \exists w'' \geq w'$ such that $w'' \in S$. It is possible to prove that the family $\mathbf{G} = \{J^{den}_{[w]} : w \in W\}$ is a Grothendieck topology. The domain of validity of a formula α, $[\![\alpha]\!]$, is "large enough" with respect to a filter $\uparrow w$, if $[\![\alpha]\!] \cap \uparrow w$ covers w in the dense topology, that is, if $\{w' \geq w : w' \models \alpha\} \in J^{den}_{[w]}$. In this case we say that α is *locally valid* at level w, denoted by $w \models \langle l \rangle(\alpha)$, with respect to the dense topology. From (7.3.13) it is not difficult to obtain:

Proposition 7.3.1. *For any formula α, for any intuitionistic Kripke model $\mathbf{M}$, for any $w \in W$, $w \models \langle l \rangle(\alpha)$ if and only if $w \models \neg\neg\alpha$.*

The modal operator $\langle l \rangle$ is called a *Lawvere's local operator*. As is well-known, these operators algebraically correspond to particular closure operators over Heyting algebras. In our case following (7.3.10) one can define a Grothendieck closure operator J^{den} on $\mathbf{F}(\mathbf{W})$ and prove that for any order filter S, $J^{den}(S) = \mathbb{I}\mathbb{C}(S)$, where $\mathbb{I}$ and $\mathbb{C}$ are the interior and, respectively, closure operators induced by $\mathbf{F}(\mathbf{W})$ *qua* Alexandrov topology. In fact if $\exists w'' \geq w'$ such that $w'' \in S$, then $w' \in \mathbb{C}(S)$. If this happens for all $w' \geq w$, since by transitivity $w \in \mathbb{C}(S)$ we obtain, moreover, $w \in \mathbb{I}\mathbb{C}(S)$ (otherwise stated, for no $w' \geq w, w' \in \mathbb{I}(-S)$).

Since from duality theory the set of order filters over any preorder is a Heyting algebra, we can wonder whether J^{den} is an example of some particular kind of operators on Heyting algebras. Indeed in the notion of a *Lawvere-Tierney operator* we find the proper generalization, that will be useful throughout this study:

Definition 7.3.1. *Given a Heyting algebra* $\mathbf{H} = \langle H, \vee, \wedge, \Longrightarrow, \neg, 0, 1 \rangle$, *a Lawvere-Tierney operator* J *on* $\mathbf{H}$ *is a map* $\mathbf{H} \longmapsto \mathbf{H}$ *such that for all* $a, b \in \mathbf{H}$:

(1) $a \leq J(a)$;   *(2)* $J(J(a)) = J(a)$;   *(3)* $J(a \wedge b) = J(a) \wedge J(b)$.

Any Lawvere-Tierney operator J induces an equivalence relation $\equiv_J$ on H in the following way: $a \equiv_J b$ iff $J(a) = J(b)$. By $[a]_{\equiv_J}$, we denote the equivalence class of a modulo $\equiv_J$. One can prove:

Proposition 7.3.2. *For any Lawvere-Tierney operator* J *on a Heyting algebra* $\mathbf{H}$,

(1) the relation $\equiv_J$ *is a congruence on* $\mathbf{H}$;
(2) For any $a \in \mathbf{H}$, $J(a) = \bigvee [a]_{\equiv_J}$.

The set of fixed points of J is then

$$J(\mathbf{H}) = \{J(a) : a \in \mathbf{H}\} = \{\bigvee [a]_{\equiv_J} : a \in \mathbf{H}\}.$$

Thus from this more abstract point of view, we have:

Proposition 7.3.3. *For any Heyting algebra* $\mathbf{H}$, $\neg\neg : \mathbf{H} \longmapsto \mathbf{H}$ *is a Lawvere-Tierney operator.*

Proposition 7.3.4. *Given a partially ordered set* $\mathbf{O}$, *the Lawvere-Tierney operator* "$\neg\neg$" *on the Heyting algebra* $\mathbf{F}(\mathbf{O})$ *coincides with the Grothendieck closure operator induced by the dense topology.*

Proof. Consider $\mathbf{F}(\mathbf{O})$ as a frame of open subsets of a topological space. Thus $\mathbf{F}(\mathbf{O})$ can be made into a Heyting algebra if we define

$x \implies y = \mathbb{I}(-x \cup y)$, for any element $x, y \in \mathbf{F(O)}$ (see Subsection 7.2.1). Hence for any x, $\neg x = x \implies 0 = \mathbb{I} - x$. It follows that $\neg\neg x = \mathbb{I} - \mathbb{I} - x = \mathbb{I}\mathbb{C} - -x = \mathbb{I}\mathbb{C}(X)$. From this and the results quoted after *Proposition* 7.3.1, we obtain the proof. $\qquad$ QED

A fixed point of the operator $\neg\neg$ is an element p of $\mathbf{H}$ such that $\neg\neg p = p$. These elements are called $\neg\neg-$ *saturated* or *regular* (after the fact that in a topological space a set X is called regular if $X = \mathbb{I}\mathbb{C}(X)$).

The family $Reg(\mathbf{H}) = \{x \in \mathbf{H} : \neg\neg x = x\}$ of the regular elements of a Heyting algebra $\mathbf{H}$ can be made into a Boolean algebra $\mathbf{REG(H)} = \langle Reg(\mathbf{H}), \vee_{\neg\neg}, \wedge, \implies, \neg, 0, 1 \rangle$, where $a \vee_{\neg\neg} b = \neg\neg(a \vee b)$. In fact $\mathbf{REG(H)}$ cannot inherit $\vee$ from $\mathbf{H}$ because a Lawvere-Tierney operator is in general just multiplicative (for other Lawvere-Tierney operators, J, things could be a little bit more complicated, since in general $J(0) \neq 0$; hence $J(\neg a) \neq \neg a$). Thus $\mathbf{REG(H)}$ is not, generally, a subalgebra of $\mathbf{H}$.

On the contrary, if $\neg\neg$ happens to be an endomorphism in $\mathbf{H}$, then $\mathbf{REG(H)}$ is the *centre* $\mathcal{CTR}(\mathbf{H})$, that is the subalgebra of all the elements of $\mathbf{H}$ that have a Boolean behaviour. We now only anticipate that any Rough Set System is a Heyting algebra with this property.

Note that a comprehensive example of a Grothendieck topology will be given in a few pages.

7.4 Lawvere-Tierney Operators and Rough Set Systems

We shall use several Lawvere-Tierney operators in order to add stronger logical properties to different basic logico-algebraic structures. Therefore, the notion of *filtering* will be pervasive throughout this analysis. In fact, "filtering" is the algebraic way for adding logical properties.

Given the system of all possible rough sets, $\mathcal{P}(\mathbf{AS}(U))$ (see *Definition* 7.1.4), we shall add more structure by filtering it by means of the following constraint:

$$-x_1 \cup x_2 \supseteq B, \ for \ all \ \langle x_1, x_2 \rangle \in \mathcal{P}(\mathbf{AS}(U)). \qquad (7.4.14)$$

We recall that B is the union of all singleton elementary classes. Since $x_1 \supseteq x_2$, this is tantamount to $x_1 \cap B = x_2 \cap B$, i.e. condition (7.1.4). In logico-algebraic terms, since $x_1 \geq x_2$ we have $x_2 \implies x_1 = U$, while

(7.4.14) gives $x_1 \implies x_2 \supseteq B$. Therefore $x_1 \iff x_2 \supseteq B$. Thus B acts as a *local truth*, a local top element that validates classical tautologies. Indeed "to be valid" for a property α means that the domain of validity $[\![\alpha]\!]$ is (greater than or) equal to the top element 1. Hence "to be locally valid" for a property α means that $[\![\alpha]\!]$ is greater than or equal to the local top element, in our case B.

REMARKS. This is not a usual notion in logic. Indeed when in Part III we shall restate everything in a modal language, we shall realize that we are actually requiring that some (otherwise invalid) modalised formulas – in our case $\Box(A) \vee \neg \Diamond(A)$ – must be valid with respect to a specific part of the universe U.

Example 7.4.1. Semi-simple Nelson algebras from Boolean algebras
Consider the Boolean algebra $\mathbf{B}$, and the derived lattices of ordered pairs of decreasing elements of $\mathbf{B}$.
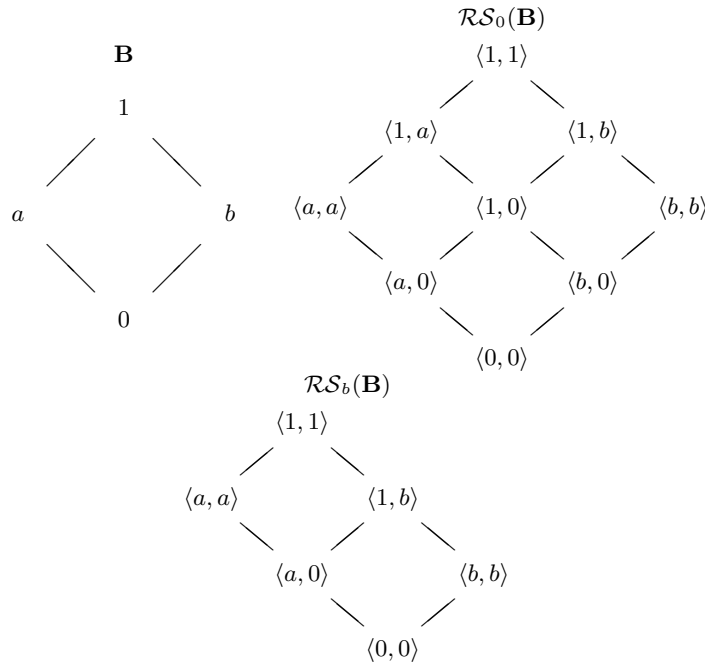
Notice that $\mathcal{RS}_0(\mathbf{B})$ and $\mathcal{RS}_b(\mathbf{B})$ are isomorphic to the lattices $\mathbf{L1}$ and, respectively, $\mathbf{L2}$ of Example 6.5.1. Let us abstract from sets. In the first lattice the filtering parameter is the element 0 and in the second the element b.

The lattice $\mathcal{RS}_0(\mathbf{B})$ consists of all the ordered pairs of decreasing elements of $\mathbf{B}$, because for all $x, y \in \mathbf{B}, x \implies y \geq 0$, so that the filtration clause is immaterial, and only the constraint $x_1 \geq x_2$ applies.

On the contrary, in $\mathcal{RS}_b(\mathbf{B})$ the ordered pair $\langle 1, 0 \rangle$ is not accepted, because $1 \geq 0$ but $1 \implies 0 = 0 \not\geq b$. For the same reason we discard $\langle 1, a \rangle$ and $\langle b, 0 \rangle$, too. On the contrary, $\langle a, 0 \rangle$ is a valid pair because $a \implies 0 = b \geq b$. Similarly for the other pairs of $\mathcal{RS}_b(\mathbf{B})$.

Given the operations listed in *Window* 7.1 it is not difficult to compute the operations in these lattices. For instance, $\dashv \langle 1, b \rangle = \langle \neg b, \neg b \rangle = \langle a, a \rangle$, $\sim \langle 1, b \rangle = \langle \neg b, \neg 1 \rangle = \langle a, 0 \rangle$, $\neg \langle 1, b \rangle = \langle \neg 1, \neg 1 \rangle = \langle 0, 0 \rangle$, $\langle 1, 0 \rangle \vee \langle b, b \rangle = \langle 1, b \rangle$, $\langle 1, 0 \rangle \wedge \langle b, b \rangle = \langle b, 0 \rangle$, and so on.

The center $\mathcal{CTR}(\mathcal{RS}_0(\mathbf{B}))$ is given by the ordered pairs such that $x_1 = x_2$, which crown the lattice. For any x, $\dashv\dashv x$ is the largest element of the center below x, while $\neg\neg x$ is the smallest element of the center above x. For instance $\dashv\dashv \langle 1, a \rangle = \dashv \langle \neg a, \neg a \rangle = \langle \neg\neg a, \neg\neg a \rangle = \langle a, a \rangle$ and $\neg\neg \langle 1, a \rangle = \langle 1, 1 \rangle$. Notice that $\langle \neg\neg x, \neg\neg y \rangle = \langle x, y \rangle$ only because $\neg$ is a Boolean complement in $\mathbf{B}$.

The same considerations apply to $\mathcal{RS}_b(\mathbf{B})$ and $\mathcal{RS}_b(\mathbf{B})^{op}$.

Since in any Boolean algebra $-x_1 \cup x_2$ is equivalent to $x_1 \implies x_2$, let us start examining the Lawvere-Tierney operators by considering the following operator J^a:

Definition 7.4.1. *Let* $\mathbf{H}$ *be a Heyting algebra. For any* $a, x \in \mathbf{H}$, $J^a(x) = a \implies x$. *By extension we set* $J^a(\mathbf{H}) = \{J^a(x) : x \in \mathbf{H}\}$.

Then we obtain an equivalence relation $\equiv_{J^a}$:

Definition 7.4.2. *For any* $a, x, y \in \mathbf{H}$, $x \equiv_{J^a} y$ *iff* $J^a(x) = J^a(y)$ *iff* $(a \implies x) = (a \implies y)$.

Proposition 7.4.1. *For any Heyting algebra* $\mathbf{H}$ *and any element* $a \in \mathbf{H}$, J^a *is a Lawvere-Tierney operator.*

Exercise 7.2. *Prove statement 7.4.1.*

Corollary 7.4.1. *Let* $\mathbf{H}$ *be a Heyting algebra and* $a \in \mathbf{H}$. *Then:*

1. $\equiv_{J^a}$ *is a congruence relation.*

2. *For any* $x \in \mathbf{H}$, $J^a(x) = \bigvee [x]_{\equiv_{J^a}}$.

3. $J^a(\mathbf{H})$ *is lattice-isomorphic to the sublattice* $\Omega_{J^a} = \{p : p \leq a\}$.

The last point of *Corollary* 7.4.1 means that a becomes really a local top element in $J^a(\mathbf{H})$.

We leave the proof as an exercise.

Now we prove a condition which is equivalent to $\equiv_{J^a}$ and which will be extremely helpful.

Proposition 7.4.2. *For any* $a, b, c \in \mathbf{H}$, $b \equiv_{J^a} c$ *iff* $a \wedge b = a \wedge c$.

Proof. Suppose $(a \implies b) = (a \implies c)$. Then, *a fortiori*, $(a \implies c) \wedge a \leq b$ and $(a \implies b) \wedge a \leq c$. But for any $x, y \in \mathbf{H}$, $x \implies y \geq y$. Therefore $c \wedge a \leq b$ and $b \wedge a \leq c$, so that $c \wedge a \wedge b = c \wedge a$ and $b \wedge a \wedge c = b \wedge a$. Hence $c \wedge a = b \wedge a$. Vice-versa, suppose $c \wedge a = b \wedge a$. Clearly $(a \implies b) \wedge a \leq b$, but also $(a \implies b) \wedge a \leq a$, so that $(a \implies b) \wedge a \leq a \wedge b$. From this and the hypothesis we obtain $(a \implies b) \wedge a \leq a \wedge c$ and, *a fortiori*, $(a \implies b) \wedge a \leq c$. It follows that $a \implies b \leq a \implies c$. Symmetrically we obtain $a \implies c \leq a \implies b$. Thus $a \implies c = a \implies b$.   QED

The above Proposition provides us with an alternative definition of the equivalence relation $\equiv_{J^a}$:

$$b \equiv_{J^a} c \quad iff \quad \exists z \in \uparrow a \text{ such that } b \wedge z = c \wedge z. \qquad (7.4.15)$$

Then we shall also say that $\equiv_{J^a}$ *is induced by the filter* $\uparrow a$.

Exercise 7.3. *Prove that the above condition* $(7.4.15)$ *is equivalent to that of Proposition 7.4.2 (hints: prove that if $s \wedge b = s \wedge c$ and $s' \leq s$ then $s' \wedge b = s' \wedge c$).*

Example 7.4.2. Grothendieck topologies

Since $J(X) = \{p : X \cap \uparrow p \in J_p\}$ we obtain the following equivalent definitions which make it possible to link Grothendieck topologies to Lawvere-Tierney operators:

$$J_p = \{\uparrow p \cap X : p \in J(X)\} \qquad (7.4.16)$$

$$J_p = \{X : X \subseteq \uparrow p \ \& \ p \in J(X)\} \qquad (7.4.17)$$

$$J_p = \{X : X \subseteq \uparrow p \ \& \ X \equiv_J \uparrow p\} \qquad (7.4.18)$$

Exercise 7.4. *Prove that the above definitions are equivalent.*

Consider the partially ordered set $\mathbf{J}(\mathbf{L})$ and its dual Heyting algebra $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ of Example 7.2.1. Let us construct the dense topology on $\mathbf{J}(\mathbf{L})$. The minimal dense element of $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ is $\{a, b\}$. We know that $x \in J^{\{a,b\}}(X)$ if and only if $x \in \neg\neg X$, that is, $- \downarrow - \downarrow X \supseteq \uparrow x$. Thus, for instance, $c \in J^{\{a,b\}}(\{a\}) = \{a, c\} = J^{\{a,b\}}(\{a, c\})$, $c \in J^{\{a,b\}}(\{a, b\}) = J^{\{a,b\}}(\{a, b, c\}) = J^{\{a,b\}}(\mathcal{J}(\mathbf{L})) = \mathcal{J}(\mathbf{L})$, and we have that the cover $J_{[c]}^{\{a,b\}}$ of the element c is $\{\uparrow c \cap X : c \in J^{\{a,b\}}(X)\} = \{\{a, c\} \cap \{a\}, \{a, c\} \cap \{a, c\}, \{a, c\} \cap \{a, b\}, \{a, c\} \cap \{a, b, c\}, \{a, c\} \cap \mathcal{J}(\mathbf{L})\} = \{\{a\}, \{a, c\}\}$.

Here we have the entire topology:

x	a	b	c	d
$J_{[x]}^{\{a,b\}}$	$\{\{a, b\}, \{a, b, c\}, \mathcal{J}(\mathbf{L})\}$	$\{\{a\}\}$	$\{\{b\}\}$	$\{\{a\}, \{a, c\}\}$

- Let us verify the defining conditions of the dense topology (cf. Subsection 7.3.1). Consider for instance the element $\{a\}$. First of all let us check the *inclusion condition*: $\{a\} \in \mathbf{F}(\uparrow c) = \{\{a\}, \{a, c\}\}$, because it is an order

filter included in $\uparrow c$. We have now to verify the *cofinality condition*: $\forall w \in \{a, c\}, \exists w' \in \{a\}$ such that $w' \geq w$. But a is such a required element w', trivially. Hence $\{a\}$ is an element of $J_{[c]}^{\{a,b\}}$. On the contrary, $\{a, b\}$ satisfies just the cofinality condition but not the inclusion condition with respect to $\uparrow c$. Thus $\{a, b\} \notin J_{[c]}^{\{a,b\}}$. Finally $\{a\} \notin J_{[1]}^{\{a,b\}}$, because it satisfies just the inclusion condition but not the cofinality condition with respect to $\uparrow 1$.

- Indeed, we also can verify that for any $X \in \mathbf{F}(\mathbf{J}(\mathbf{L}))$, $J^{\{a,b\}}(X) = \bigcup[X]_{\equiv_{J\{a,b\}}}$. For instance, we have $\{a\} \cap \{a, b\} = \{a, c\} \cap \{a, b\}$. Hence $\{a, c\} \equiv_{J\{a,b\}} \{a\}$ and it is immediate to see that $[\{a\}]_{\equiv_{J\{a,b\}}} = \{\{a\}, \{a, c\}\}$. But $\bigcup\{\{a\}, \{a, c\}\} = \{a, c\}$. Thus $J^{\{a,b\}}(\{a\}) = \{a, c\} = \neg\neg\{a\}$.

- Let us verify that the Grothendieck topology and the Lawvere-Tierney operator $J^{\{a,b\}}$ go together well. Since, for instance, $J^{\{a,b\}}(\{a\}) = \{p : \uparrow p \cap \{a\} \in J_{[p]}^{\{a,b\}}\}$, we have $\uparrow c \cap \{a\} = \{a\} \in J_{[c]}^{\{a,b\}}$, $\uparrow a \cap \{a\} = \{a\} \in J_{[c]}^{\{a,b\}}$, $\uparrow b \cap \{a\} = \emptyset \notin J_{[c]}^{\{a,b\}}$, and so on, obtaining at the end $J^{\{a,b\}}(\{a\}) = \{a, c\}$.

- Finally, let us verify the axioms for Grothendieck topologies (cf. *Window* 7.2):

    (GT1): $\uparrow 1 = U \in J_{[1]}^{\{a,b\}}$, $\uparrow a = \{a\} \in J_{[a]}^{\{a,b\}}$, $\uparrow b = \{b\} \in J_{[c]}^{\{a,b\}}$, $\uparrow c = \{a, c\} \in J_{[c]}^{\{a,b\}}$;

    (GT2): $\uparrow a \cap \{a, c\} = \{a\} \in J_{[a]}^{\{a,b\}}$, $\uparrow c \cap \{a, c\} = \{a, c\} \in J_{[c]}^{\{a,b\}}$. Thus (GT2) holds of every element of $J_{[c]}^{\{a,b\}}$ and all elements of $\uparrow c$.

    $\uparrow 1 \cap \{a, b\} = \{a, b\} \in J_{[1]}^{\{a,b\}}$, $\uparrow 1 \cap \{a, b, c\} = \{a, b, c\} \in J_{[1]}^{\{a,b\}}$, $\uparrow 1 \cap \mathcal{J}(\mathbf{L}) = \mathcal{J}(\mathbf{L}) \in J_{[1]}^{\{a,b\}}$, $\uparrow a \cap \{a, b\} = \{a\} \in J_{[a]}^{\{a,b\}}$, $\uparrow a \cap \{a, b, c\} = \{a\} \in J_{[a]}^{\{a,b\}}$, $\uparrow a \cap \mathcal{J}(\mathbf{L}) = \{a\} \in J_{[a]}^{\{a,b\}}$, $\uparrow c \cap \{a, b\} = \{a\} \in J_{[c]}^{\{a,b\}}$, $\uparrow c \cap \{a, b, c\} = \{a, c\} \in J_{[c]}^{\{a,b\}}$, $\uparrow c \cap \mathcal{J}(\mathbf{L}) = \{a, c\} \in J_{[c]}^{\{a,b\}}$, $\uparrow b \cap \{a, b\} = \{b\} \in J_{[b]}^{\{a,b\}}$, $\uparrow b \cap \{a, b, c\} = \{b\} \in J_{[b]}^{\{a,b\}}$, $\uparrow b \cap \mathcal{J}(\mathbf{L}) = \{b\} \in J_{[b]}^{\{a,b\}}$. Thus (GT2) holds of any element of $J_{[1]}^{\{a,b\}}$ and all elements of $\uparrow 1$. And so on, more trivially, for b and a.

    (GT3): Let us take an arbitrary element of $\mathcal{J}(\mathbf{L})$, say c (which instantiates the variable p of the axiom). Let us take all the subfilters of $\uparrow c$ that belong to $J_{[c]}^{\{a,b\}}$; so the conclusion of the axiom, (i.e. $S' \in J_{[c]}^{\{a,b\}}$) holds independently of any premise. Let us now instantiate p with b. But $\{b\}$ is the only element of $J_{[b]}^{\{a,b\}}$, and the only non-empty subset of $\{b\}$. Finally b is the only element of $\{b\}$. Thus the conclusion is a tautological "$X \in Y$ iff $X \in Y$". Same when we set $p = b$. Now let us consider a. Set $S = \{a, b\}$ and $S' = \{a, c\}$ (which is possible because $\uparrow 1 = \mathcal{J}(\mathbf{L})$). Set now $p' = b$, for $a \in S$. We have $\uparrow a \cap \{a, c\} = \{a\} \in J_{[a]}^{\{a,b\}}$, but $\{a, c\} \notin J_{[1]}^{\{a,b\}}$. Indeed, when we set $p' = b$ we have $\uparrow b \cap \{a, c\} = \emptyset \notin J_{[b]}^{\{a,b\}}$. Thus the precondition of the axiom does not hold for all $p' \in \{a, b\}$. We leave as an exercise to check the axiom for the other cases.

Now some further considerations are in order. We can verify, for instance, that $J^{\{a,b\}}(\emptyset) = \emptyset$. This means that for no $x \in \mathcal{J}(\mathbf{L})$, $\emptyset \in J_{[x]}^{\{a,b\}}$. On the contrary, consider the Grothendieck topology corresponding to the LT- operator $J^{\{b\}}$. We

have in this case that $\{a\} \cap \{b\} = \emptyset \cap \{b\} = \{a, c\} \cap \{b\} = \emptyset$, so that $\{a\} \equiv_{J\{b\}}$ $\{a, c\} \equiv_{J\{b\}} \emptyset$. Hence $J^{\{b\}}_{[a]} = \{\emptyset, \{a\}, \{a, c\}\}$ and, thus, $J^{\{b\}}(\emptyset) = \{a, c\}$. It follows that $J^{\{b\}}(\neg\{a, c\}) = J^{\{b\}}(\{b\}) = \mathcal{J}(\mathbf{L})$ (indeed, $J^{\{b\}}(\neg\{a, c\}) = J^{\{b\}}(\neg\{a\}) = J^{\{b\}}(\neg\emptyset)$). But $\mathcal{J}(\mathbf{L}) \neq \{b\} = \neg\{a, c\}$.

From the above discussion we have that in order to define a Grothendieck topology on a poset $\mathbf{X} = \langle U, \leq \rangle$ we can select a filter F out of $\mathscr{F}(\mathbf{X})$, then for any $p \in U$ and $S \in \mathscr{F}(\mathbf{X})$ we have that $S \in J^{F}_{[p]}$ if $S \subseteq\, \uparrow p$ and $\uparrow p \equiv_{JF} S$. We say that the topology is *induced by* F.

Exercise 7.5.
(A) Prove that in a Grothendieck topology induced on a poset $\mathbf{X} = \langle U, \leq \rangle$ by a filter $\uparrow X$ on $\mathbf{F}(\mathbf{X})$, $\emptyset \in J^{X}_{[a]}$ only if $a \in \neg X$.
(B) Prove that there is an element a such that $\emptyset \in J^{X}_{[a]}$ only if X is strictly less than the least dense element of $\mathbf{F}(\mathbf{X})$.

In the following Example 7.4.3 we shall see that the Boolean algebra $\mathbf{REG}(\mathbf{H})$ is isomorphic to $\mathbf{H}/_{\equiv_{J^d}}$ where $\equiv_{J^d}$ is the equivalence relation induced by the filter of all and only the *dense elements* of $\mathbf{H}$, that is, elements a such that $\neg\neg a = 1$ (again, the terminology has a topological origin, since a set X is dense in a topological space $\langle U, \Omega(U) \rangle$ if and only if $\mathbb{IC}(X) = U$, if and only if $\neg\neg X = U$ in $\Omega(U)$ *qua* Heyting algebra). In other terms, the dense topology provides us with the geometrical counterpart of the well-known Gödel-Glivenko result about the double negation validity of classical tautologies in intuitionistic propositional logic (see Section 9.6).

Example 7.4.3. Center of a Heyting algebra
Consider the Heyting algebra $\mathbf{A}$ of Example 6.1.1. The set $Reg(\mathbf{A})$ of all regular elements of $\mathbf{A}$ is $\{1, 0, b, c\}$ ($\neg\neg c = c$, and so on) and forms a Boolean algebra $\mathbf{REG}(\mathbf{A})$ because $\neg$ is involutive on the elements of $Reg(\mathbf{A})$.

But $\mathbf{REG}(\mathbf{A})$ is not a subalgebra of $\mathbf{A}$. In fact $b \vee c = e$ but e is not regular. Indeed the joint in $\mathbf{REG}(\mathbf{A})$ is given by $b \vee_{\neg\neg} c =_{def} \neg\neg(b \vee c) = \neg\neg e = 1$. This fact is connected with the issue of *non-standard constructive systems* that will be discussed in Section 9.4.

Finally, notice that d, e and 1 are *dense* elements ($\neg d = \neg e = \neg 1 = 0$) and that d is the least dense element of $\mathbf{A}$.

Now, if we take the filter $\uparrow d$ of all dense elements, we can define a congruence $\equiv_{J^d}$ as follows (see *Definition* 7.4.2 and *Proposition* 7.4.2):

$$a \equiv_{J^d} b \text{ iff } \exists z \in\, \uparrow d(a \wedge z = z \wedge b) \tag{7.4.19}$$

One can prove that the quotient algebra $\mathbf{A}/_{\equiv_{Jd}}$ is a Boolean algebra isomorphic to the algebra of regular elements $\mathbf{REG}(\mathbf{A})$. The isomorphism is obtained by mapping a congruence class onto its largest element and, in the opposite direction, by mapping a regular element onto its congruence class:
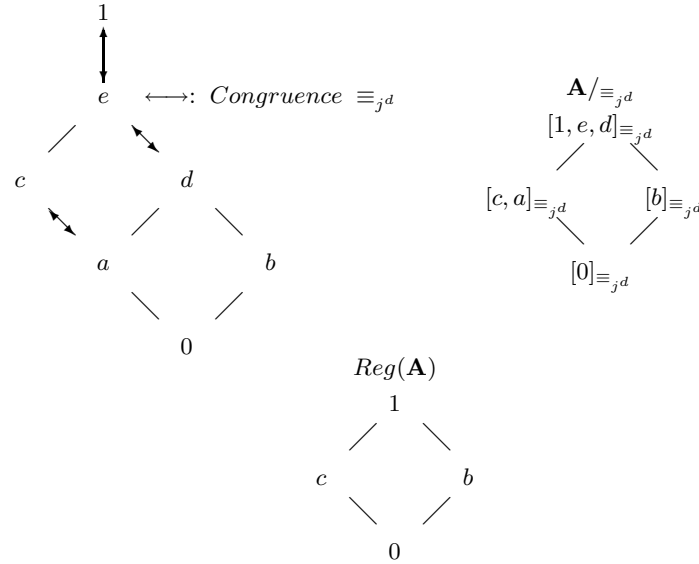
Now let us come back to Approximation Spaces and Rough Set Systems.

From now on B will be a generic element of $\mathbf{AS}(U)$ and $P = -B$. Thus B and P will be considered as formal parameters. The dependence of B and P on $\mathbf{AS}(U)$ is understood and has no influence in the development of the reasoning, because we shall treat Approximation Spaces as generic finite Boolean algebras.

The operator J^B makes it possible to filter out $\mathbf{AS}(U)$ by means of the filter $\uparrow B$, via the congruence relation $\equiv_{J^B}$, producing the following set:

$$
\begin{aligned}
\mathcal{RS}_B(\mathbf{AS}(U)) &= \{\langle a_1, a_2 \rangle \in \mathbf{AS}(U)^2 : a_2 \Longrightarrow a_1 \\
&= U \ \& \ a_1 \Longrightarrow a_2 \equiv_{J^B} U\}. \qquad (7.4.20)
\end{aligned}
$$

Proposition 7.4.3. *For any Approximation Space* $\mathbf{AS}(U)$, $\mathcal{RS}_B$ $(\mathbf{AS}(U)) = RS(U)$.

Proof. For any $a \in \mathcal{RS}_B(\mathbf{AS}(U))$, $a_2 \Longrightarrow a_1 = U$ if and only if $-a_2 \cup a_1 = U$, if and only if $a_1 \supseteq a_2$, as required by *Lemma* 7.1.1.

Moreover $a_1 \implies a_2 \equiv_{J^B} U$ if and only if $-a_1 \cup a_2 \equiv_{J^B} U$. Therefore, since $B \implies U = U$, from *Definition* 7.4.2 the equivalence holds if and only if $U = -B \cup -a_1 \cup a_2$, if and only if $(-a_1 \cup a_2) \supseteq B$, as required by (7.4.14). Alternatively we can exploit *Proposition* 7.4.2 and obtain directly condition (7.1.4).   QED

(In view of the above Proposition, when we do not need any reference to the parameters B or P, we can denote $\mathcal{RS}_B(\mathbf{AS}(U))$ with $RS(U)$).

It follows that for any rough set a, a singleton elementary class X must be included in the lower approximation a_2 or in the complement $-a_1$ of the upper approximation. Thus X cannot be included in $a_2 \cap -a_1$, that is, X will not be included in the boundary of any $A \in rs^{-1}(\langle a_1, a_2 \rangle)$, as required. This is the role of the above filtering.

Corollary 7.4.2. *(1)* $\mathcal{P}(\mathbf{AS}(U)) = \mathcal{RS}_\emptyset(\mathbf{AS}(U))$; *(2)* $\mathcal{B}(\mathbf{AS}(U)) = \mathcal{RS}_U(\mathbf{AS}(U))$.

Thus $\mathcal{P}$ and $\mathcal{B}$ represent two well determined extreme situations.

We have seen that our second clause in the definition of $\mathcal{RS}_B$ $(\mathbf{AS}(U))$ warrants what is required: within any rough set the equality between its lower and upper approximations must be locally valid on B. Of course a set such that its lower and upper approximations coincide, is a usual set. But rough sets must behave as usual on B: there is no reason be rough there to.

TERMINOLOGY AND NOTATION. As usual, from now on, let us denote $\mathbf{AS}(G/R_\mathbf{A})$ with the shorter notation $\mathbf{AS}(G)$.

Example 7.4.4. Algebraic constructions of Rough Set Systems
Consider the *A-system* $\mathbf{A} = \langle G, At, \{V_a\}_{a \in At} \rangle$ of Example 1.2.1 of Chapter 1. We remind that the i-quantum relation $R_\mathbf{A}$ is an equivalence relation so that $\mathbf{Q}(\mathbf{A}) = \langle G, R_\mathbf{A} \rangle$ is an Indiscernibility Space. The induced Approximation Space $\mathbf{AS}(G/R_\mathbf{A})$ is depicted in Example 3.3.1 of Chapter 3 and coincide with $\mathbf{Sat}_Q(\mathbf{A})$. In this space, the external parameter is the local Boolean universe B, that is, the union of all singleton elementary classes. Hence $B = \bigcup\{\{a'\}, \{a'''\}\} = \{a', a'''\}$ and $\neg B = -B = \{a, a''\}$ (because $\mathbf{AS}(G)$ is a Boolean algebra of sets). We remind the definition of the operator J^B: $J^B(X) = B \implies X$. Hence $J^B(X) = \neg B \vee X = -B \cup X$. Thus, for $X, Y \in \mathbf{AS}(G)$, $X \cong_{J^B} Y$ if and only if $-B \cup X = -B \cup Y$ if and only if $B \cap X = B \cap Y$ (in fact, if $x \in B \cap X$ but $x \notin B \cap Y$, then $x \in X$ and $x \notin Y$. Thus, $x \in -B \cup X$ but $x \notin -B \cup Y$). For instance, $\{a, a', a''\} \cong_{J^B} \{a'\}$ because $-\{a', a'''\} \cup \{a, a', a''\} = \{a, a', a''\} = -\{a', a'''\} \cup \{a'\}$.

X is a fixed point of the Lawvere-Tierney operator J^B if and only if $B \implies X = X$, if and only if $-B \cup X = X$, if and only if $-B \subseteq X$. It follows that the family of fixed points of J^B is $\uparrow \neg B$, which is isomorphic to $\downarrow B$ *via* the complementation $\neg$.

The geometry of the equivalence classes of $\mathbf{AS}(G)$ modulo $\equiv_{JB}$ and its fixed points is shown in the diagram below, where the double arrow $\longleftrightarrow$ show the equivalence $\equiv_{JB}$:
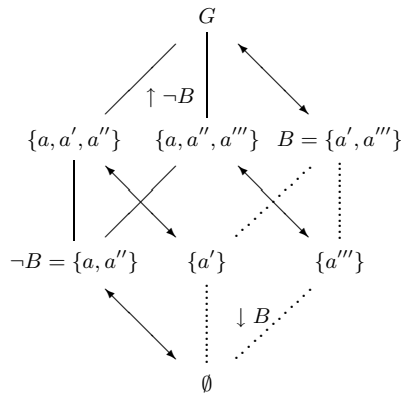
Each fixpoint is the largest element of an equivalence class modulo $\equiv_{JB}$ and for each $X \in \mathbf{AS}(G)$, $J^B(X) = \bigvee[\{X\}]_{\equiv_{JB}}$. For instance, $\{a, a', a''\} = \bigvee[\{a'\}]_{\equiv_{JB}} = \bigcup\{\{a'\}, \{a, a', a''\}\}$ and $J^B(\{a'\}) = B \implies \{a'\} = \{a, a', a'''\}$.

Here, the Rough Set System induced by $\mathbf{AS}(G)$ is depicted in both disjoint and decreasing representation, along with the Rough Set System represented as a lattice of equivalence classes modulo rough equivalence.

$$\{[X]_{\approx} : X \subseteq G\}$$

$$\langle G, G \rangle$$

$$\langle G, \{a', a'''\} \rangle$$

$$\langle \{a, a', a''\}, \{a, a', a''\} \rangle \qquad \langle \{a, a'', a'''\}, \{a, a'', a'''\} \rangle$$

$$\langle \{a', a'''\}, \{a', a'''\} \rangle$$

$$\langle \{a, a', a''\}, \{a'\} \rangle \qquad \langle \{a, a'', a'''\}, \{a'''\} \rangle$$

$$\langle \{a, a''\}, \{a, a''\} \rangle$$

$$\langle \{a'\}, \{a'\} \rangle \qquad \langle \{a'''\}, \{a'''\} \rangle$$

$$\langle \{a, a''\}, \emptyset \rangle$$

$$\langle \emptyset, \emptyset \rangle$$

$$\mathcal{RS}_B(\mathbf{AS}(G))$$

$$\langle G, \emptyset \rangle$$

$$\langle \{a', a'''\}, \emptyset \rangle$$

$$\langle \{a, a', a''\}, \{a'''\} \rangle \qquad \langle \{a, a''', a''\}, \{a'\} \rangle$$

$$\langle \{a', a'''\}, \{a, a''\} \rangle$$

$$\langle \{a'\}, \{a'''\} \rangle \qquad \langle \{a'''\}, \{a'\} \rangle$$

$$\langle \{a, a''\}, \{a', a'''\} \rangle$$

$$\langle \{a'\}, \{a, a'', a'''\} \rangle \qquad \langle \{a'''\}, \{a, a', a''\} \rangle$$

$$\langle \emptyset, \{a', a'''\} \rangle$$

$$\langle \emptyset, G \rangle$$
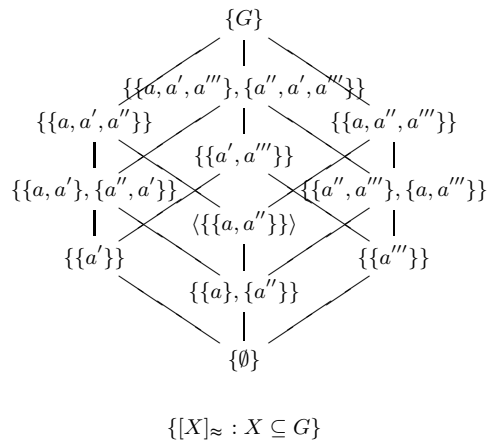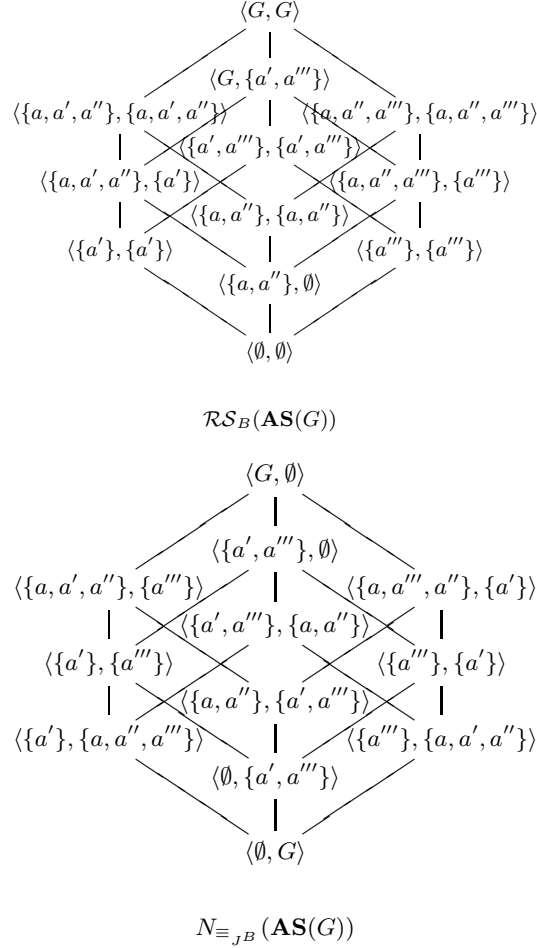
$$N_{\equiv_{JB}}(\mathbf{AS}(G))$$

The notation $N_{\equiv_{JB}}(\mathbf{AS}(G))$ will be justified later in this Part. If not otherwise stated, from now on we shall refer to the lattice $\mathcal{RS}_B(\mathbf{AS}(G))$.

One can notice that for all $X \in \mathbf{AS}(G)$, $X \equiv_{JB} G$ if and only if $B \subseteq X$. In fact, $X \equiv_{JB} G$ iff $B \implies X = B \implies G = G$, iff $B \subseteq X$. It follows that $\langle X_1, X_2 \rangle \in N_{\equiv_{JB}}(\mathbf{AS}(G))$ only if B is distributed between X_1 and X_2, because we must have $X_1 \cup X_2 \equiv_{JB} G$, hence $X_1 \cup X_2 \supseteq B$. For instance, $\langle \{a, a''\}, \{a'\} \rangle$ belongs to $\mathcal{P}(\mathbf{AS}(G))$ but not to $N_{\equiv_{JB}}(\mathbf{AS}(G))$ because $\{a, a''\} \cup \{a'\} \not\supseteq \{a', a'''\}$. On the contrary, $\langle \{a'\}, \{a'''\} \rangle$ belongs to $N_{\equiv_{JB}}(\mathbf{AS}(G))$ because $\{a'\} \cup \{a'''\} \supseteq \{a', a'''\}$. In the decreasing representation, the same example reads as $\langle \{a, a'', a'''\}, \{a, a''\} \rangle \notin \mathcal{RS}_B(\mathbf{AS}(G))$ because $-\{a, a'', a'''\} \cup \{a, a''\} = \{a, a', a''\} \not\supseteq \{a', a'''\}$ and $\langle \{a, a', a''\}, \{a'\} \rangle$ belongs to $\mathcal{RS}_B(\mathbf{AS}(G))$ because $-\{a, a', a''\} \cup \{a'\} \supseteq \{a', a'''\}$.

About the latter examples, notice that $\langle \{a, a', a''\}, \{a'\} \rangle$ is such that $\{a, a', a''\} \cap B = \{a'\} = B \cap \{a'\}$, while $\langle \{a, a'', a'''\}, \{a, a''\} \rangle$ is such that $\{a, a'', a'''\} \cap B = \{a'''\} \neq \emptyset = B \cap \{a, a''\}$.

In the Hasse diagrams below, notice how the elements of the centre are placed on the vertexes of the edges forming a Boolean algebra:

$$\langle G, G \rangle$$

$$\langle \{a, a', a''\}, \{a, a', a''\}\rangle \qquad \langle \{a, a'', a'''\}, \{a, a'', a'''\}\rangle$$

$$\langle \{a', a'''\}, \{a', a'''\}\rangle$$

$$\langle \{a, a''\}, \{a, a''\}\rangle$$

$$\langle \{a'\}, \{a'\}\rangle \qquad \langle \{a'''\}, \{a'''\}\rangle$$

$$\langle \emptyset, \emptyset \rangle$$

$$\mathcal{CTR}(\mathcal{RS}_B(\mathbf{AS}(G)))$$

$$\langle G, \emptyset \rangle$$
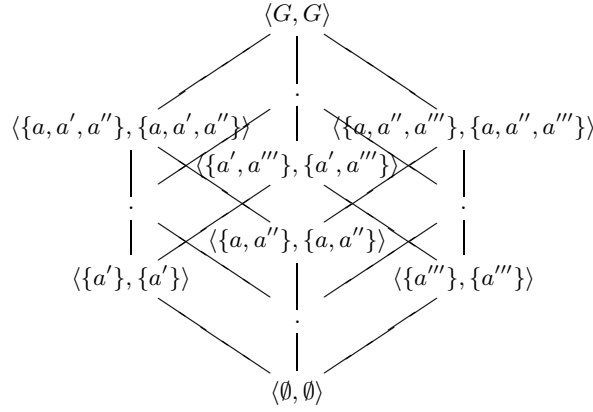
$$\langle \{a, a', a''\}, \{a'''\}\rangle \qquad \langle \{a, a''', a''\}, \{a'\}\rangle$$

$$\langle \{a', a'''\}, \{a, a''\}\rangle$$

$$\langle \{a, a''\}, \{a', a'''\}\rangle$$

$$\langle \{a'\}, \{a, a'', a'''\}\rangle \qquad \langle \{a'''\}, \{a, a', a''\}\rangle$$

$$\langle \emptyset, G \rangle$$
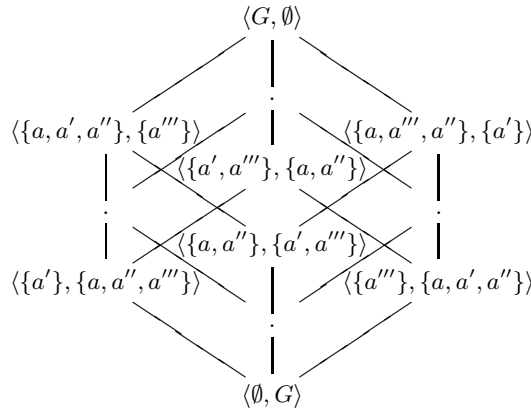
$$\mathcal{CTR}(N_{\equiv_{JB}}(\mathbf{AS}(G)))$$

Both $\neg\langle\{a, a'', a'''\}, \{a'''\}\rangle$ (i.e $\langle\{a, a', a''\}, \{a, a', a''\}\rangle$) and $\neg\langle\{a, a'', a'''\}, \{a'''\}\rangle$ (i.e. $\langle\{a'\}, \{a'\}\rangle$) are elements of type $\langle X_1, X_2 \rangle$ such that $X_1 = X_2$. The three negations coincide when applied to these elements because $\neg X_2 = \neg X_1$. It follows that all these elements X are complemented because $X \vee \neg X = X \vee \sim X = X \vee \neg X = 1$ and $X \wedge \neg X = X \wedge \sim X = X \wedge \neg X = 0$. Thus $\mathcal{CTR}(\mathcal{RS}_B(\mathbf{AS}(G))) = \{\langle X_1, X_2 \rangle : X_1 = X_2\}$.

We can now notice that there is another algebraic interpretation of the above filtering.

In fact, $\mathcal{RS}_B(\mathbf{AS}(U))$ can be recovered by filtering $\mathcal{P}(\mathbf{AS}(U))$ modulo the filter $\uparrow \langle U, P \rangle$. Actually, by applying the related Lawvere-Tierney operator, we can prove:

Lemma 7.4.1. *For any $a, b \in \mathcal{P}(\mathbf{AS}(U))$, $a \equiv_{J\langle U,P \rangle} b$ if and only if a and b are top equal (that is, their first elements, a_1 and b_1, coincide) and the P-parts of a_2 and b_2 are equal.*

Proof. $a \wedge \langle U, P \rangle = b \wedge \langle U, P \rangle$ if and only if $a_1 \cap U = b_1 \cap U$ and $a_2 \cap P = b_2 \cap P$, if and only if $a_1 = b_1$ and $a_2 \cap P = b_2 \cap P$.   QED

Lemma 7.4.2. $\langle x_1, x_2 \rangle = J^{\langle U,P \rangle}(a)$ *if and only if $x_1 = a_1$ and $x_2 = a_2 \cup (B \cap x_1)$.*

Proof. Since any a differs from all the other elements of $[a]_{\equiv_{J\langle U,P \rangle}}$ just in the B-part of a_2 and $a_1 \supseteq a_2$, when we take $\bigvee [a]_{\equiv_{J\langle U,P \rangle}}$ we add to a_2 the B-part of a_1 (i.e. x_1).   QED

It follows that if $\langle x_1, x_2 \rangle = J^{\langle U,P \rangle}(a)$, then $x_1 \cap B = x_2 \cap B$ and hence, in view of condition (7.1.3), it is a (legal) rough set. On the other side, if a is a rough set then $a = J^{\langle U,P \rangle}(x)$ for some $x \in \mathcal{P}(\mathbf{AS}(U))$. Therefore we obtain the following:

Corollary 7.4.3. $J^{\langle U,P \rangle}(\mathcal{P}(\mathbf{AS}(U))) = \mathcal{RS}_B(\mathbf{AS}(U)).$

Of course we have to show that this operator and the next ones manipulate Heyting algebras. That is, we have to show that for any Approximation Space $\mathbf{AS}(U)$ and $Z \in \mathbf{AS}(U)$, $\langle \mathcal{RS}_Z(\mathbf{AS}(U)), \vee, \wedge, \neg, \supset, 0, 1 \rangle$ is a complete Heyting algebra. This will be formally stated in *Proposition* 8.3.1, in a more abstract setting.

   Therefore: *any Rough Set System is obtained by filtering a Post algebra of order three.*

   In Frame 10.1 this result will be proved from the point of view of Chain-based lattices.

Example 7.4.5. Rough Set System by filtering a Post algebra
In the diagram below the elements of the Rough Set System $\mathcal{RS}_B(\mathbf{AS}(G))$ presented in Example 7.4.4. are embedded in bold fonts in $\mathcal{P}(\mathbf{AS}(G))$. The double arrows show the congruence $\equiv_{J\langle U,P \rangle}$:

$\langle \mathbf{G}, \mathbf{G} \rangle$

$\langle \mathbf{G}, \{\mathbf{a'}, \mathbf{a'''}\} \rangle$

$\langle G, \{a, a', a''\} \rangle$ $\qquad$ $\langle G, \{a, a'', a'''\} \rangle$

$\langle \{\mathbf{a'}, \mathbf{a'''}\}, \{\mathbf{a'}, \mathbf{a'''}\} \rangle$

$\langle G, \{a'\} \rangle$ $\qquad$ $\langle G, \{a'''\} \rangle$

$\langle \{\mathbf{a}, \mathbf{a'}, \mathbf{a''}\}, \{\mathbf{a}, \mathbf{a'}, \mathbf{a''}\} \rangle$ $\qquad$ $\langle G, \{a, a'\} \rangle$ $\qquad$ $\langle \{\mathbf{a}, \mathbf{a''}, \mathbf{a'''}\}, \{\mathbf{a}, \mathbf{a''}, \mathbf{a'''}\} \rangle$

$\langle \{a', a'''\}, \{a'\} \rangle$ $\qquad$ $\langle \{a', a'''\}, \{a'''\} \rangle$

$\langle \{\mathbf{a}, \mathbf{a'}, \mathbf{a''}\}, \{\mathbf{a'}\} \rangle$ $\qquad$ $\langle G, \emptyset \rangle$ $\qquad$ $\langle \{\mathbf{a}, \mathbf{a''}, \mathbf{a'''}\}, \{\mathbf{a'''}\} \rangle$

$\langle \{a, a', a''\}, \{a, a''\} \rangle$ $\qquad$ $\langle \{a, a'', a'''\}, \{a, a''\} \rangle$

$\langle \{\mathbf{a'}\}, \{\mathbf{a'}\} \rangle$ $\qquad$ $\langle \{a', a'''\}, \emptyset \rangle$ $\qquad$ $\langle \{\mathbf{a'''}\}, \{\mathbf{a'''}\} \rangle$

$\langle \{a, a', a''\}, \emptyset \rangle$ $\qquad$ $\langle \{a, a'', a'''\}, \emptyset \rangle$

$\langle \{\mathbf{a}, \mathbf{a''}\}, \{\mathbf{a}, \mathbf{a''}\} \rangle$

$\langle \{a'\}, \emptyset \rangle$ $\qquad$ $\langle \{a'''\}, \emptyset \rangle$

$\langle \{\mathbf{a}, \mathbf{a''}\}, \emptyset \rangle$

$\langle \emptyset, \emptyset \rangle$
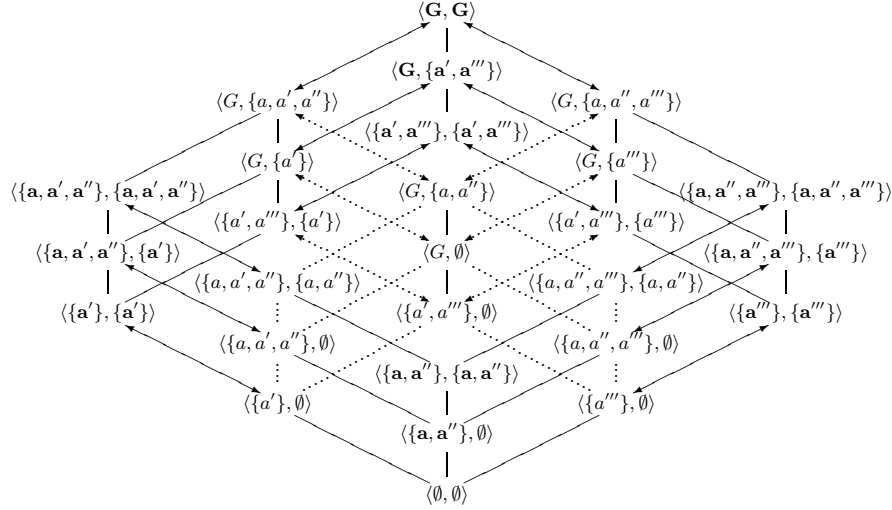
Remember that $P = \neg B = -B$. Hence $P = \{a, a''\}$. The reader can easily see that $\mathcal{RS}_B(\mathbf{AS}(G))$ is isomorphic to $\downarrow \langle G, \neg B \rangle = \downarrow \langle G, \{a, a''\} \rangle$. Let us verify that the operator $J^{\langle U, P \rangle}$ is an isomorphism from $\downarrow \langle G, \neg B \rangle$ to $\mathcal{RS}_B(\mathbf{AS}(G))$, but not from $\mathcal{P}(\mathbf{AS}(G))$. In fact, for instance, $J^{\langle U, P \rangle}(\langle \{a, a'', a'''\}, \{a, a''\} \rangle) \neq J^{\langle U, P \rangle}(\langle \{a, a'', a'''\}, \emptyset \rangle)$ because both elements belong to $\downarrow \langle G, \neg B \rangle$.

On the contrary, $J^{\langle U, P \rangle}(\langle G, \emptyset \rangle) = J^{\langle U, P \rangle}(\langle G, \{a'\} \rangle) = \langle G, \{a', a'''\} \rangle$ because $\{a'\} \cap B \neq \emptyset$, and so on.

Finally, note what is stated in *Lemma* 7.4.1: take for instance the equivalence class of $\langle \{a', a'''\}, \{a'\} \rangle$ modulo $\equiv_{J^{\langle G, \neg B \rangle}}$, i.e. $\{\langle \{a', a'''\}, \{a'\} \rangle, \langle \{a', a'''\}, \emptyset \rangle, \langle \{a', a'''\}, \{a'''\} \rangle, \langle \{a', a'''\}, \{a', a'''\} \rangle\}$. We can notice that the first elements of the members of this equivalence class coincide, as well as the $\neg B-$ part of their second elements. In other words, they are rough top equal and rough bottom equal with respect to B.

Let us see how the Rough Set System $\mathcal{RS}_B(\mathbf{AS}(G))$ is singled out of the Post algebra $\mathcal{P}(\mathbf{AS}(G))$.

We verify some instance of *Corollary* 7.4.3. Let us compute $J^{\langle G, P \rangle}(\langle G, \emptyset \rangle)$: clearly $\langle G, \emptyset \rangle \equiv_{J^{\langle G, P \rangle}} \langle G, \{a'\} \rangle$ because $\langle G, \emptyset \rangle \cap \langle G, P \rangle = \langle G, \{a'\} \rangle \cap \langle G, P \rangle$, This holds of every $\langle X_1, X_2 \rangle$ such that $X_1 = G$ and $X_2 \cap P = \emptyset \cap P = \emptyset$. Since B is the largest element disjoint from P, the largest element of the equivalence class of $\langle G, \emptyset \rangle$ modulo $\equiv_{J^{\langle G, P \rangle}}$ is trivially $\langle G, B \rangle$. In details: $[\langle G, \emptyset \rangle]_{\equiv_{J^{\langle G, P \rangle}}} = \{\langle G, \emptyset \rangle, \langle G, \{a'\} \rangle, \langle G, \{a'''\} \rangle, \langle G, \{a', a'''\} \rangle\}$. Hence $J^{\langle G, P \rangle}(\langle G, \emptyset \rangle) = \bigvee [\langle G, \emptyset \rangle]_{\equiv_{J^{\langle G, P \rangle}}} = \langle G, \{a', a'''\} \rangle = \langle G, B \rangle$.

One can trivially notice that $G \cap B \neq \emptyset \cap B$, so that $\langle G, \emptyset \rangle$ is not a member of $\mathcal{RS}_B(\mathbf{AS}(G))$. Also, $\langle G, \{a'\} \rangle$ is not a legal rough set because $G \cap B = B \neq \{a'\} = \{a'\} \cap B$. On the contrary, both $J^{\langle G, P \rangle}(\langle G, \emptyset \rangle)$ and $J^{\langle G, P \rangle}(\langle G, \{a\} \rangle)$ equal $\langle G, B \rangle$, which is a legal rough set. Again, $\langle \{a, a', a''\}, P \rangle$ is not a legal rough set. Let us compute $J^{\langle G, P \rangle}(\langle \{a, a', a''\}, P \rangle)$: obviously, $\langle \{a, a', a''\}, P \rangle \cap \langle G, P \rangle = \langle \{a, a', a''\}, P \rangle$. Hence, every pair of decreasing elements of $\mathbf{AS}(G)$ $\langle X_1, X_2 \rangle$ such

that $X_1 = \{a, a', a''\}$ and $X_2 \supseteq P$ is in $[\langle\{a, a', a''\}, P\rangle]_{\equiv_{J\langle G, P\rangle}}$. It follows that $\bigvee[\langle\{a, a', a''\}, P\rangle]_{\equiv_{J\langle G, P\rangle}} = \langle\{a, a', a''\}, \{a, a', a''\}\rangle$. Otherwise stated, to X_2 we have to add the "exact part" of X_1 (in this example, $\{a'\}$).

Further, one can verify that $J^{\langle G, P\rangle} \cong \downarrow \langle G, P\rangle$.

The reader should notice that in $\mathcal{P}(\mathbf{AS}(U))$ there is also the element $\langle U, \emptyset\rangle$. This element has the following property:

$$\sim\langle U, \emptyset\rangle = \langle U, \emptyset\rangle.$$

Thus, $\langle U, \emptyset\rangle$ is a *central element* of $\mathcal{P}(\mathbf{AS}(U))$ (see Section 6.2). In the systems we are dealing with, there is at most one central element. Now, we have seen that $\langle U, \emptyset\rangle$ represents a completely uninformed situation in the sense that if $rs(X) = \langle U, \emptyset\rangle$, then X is undefinable in $\mathbf{AS}(U)$: everything could be in X but actually nothing is definitely in it. But this situation is not plausible as soon as we have at least one elementary class which is a singleton. In the general case the *worst situation*, from a knowledge point of view, is given by:

$$\langle U, B\rangle, \text{ that is, } J^{\langle U, P\rangle}(\langle U, \emptyset\rangle).$$

So, the worst solution is $\langle U, \emptyset\rangle$ only when $B = \emptyset$, that is, when the exact region is empty.

We are going to show that if B is the parameter that decides the global logical properties of these systems, $\langle U, B\rangle$ is a pillar in the construction of the modal operators of a number of logics. Moreover, we shall see that in spite of the fact that the element $\langle U, B\rangle$ appears to depend on B, nevertheless it can be singled out, in any Rough Set System $RS(U)$, by means of the lattice-theoretic notion of a *least dense element*.

Chapter 8

Approximation
and Algebraic Logic

8.1 Approximation Operators

Of course, in order to honour the philosophy of Rough Set Theory, we must endow any Rough Set Systems $RS(U)$ with a couple of operators reflecting the approximation features provided by the theory, applied, this time, not to sets but to rough sets. Let us connect the knowledge-oriented interpretation of these operators to their logical properties.

We are looking for two operators M and L such that the following diagrams commute:

$$
\begin{array}{ccc}
\wp(U) & \xrightarrow{\ (uE)\ } & \mathbf{AS}(U) \\
{\scriptstyle rs}\big\downarrow & & \big\downarrow{\scriptstyle rs} \\
RS(U) & \xrightarrow[\ M\ ]{} & rs(\mathbf{AS}(U))
\end{array}
\qquad
\begin{array}{ccc}
\wp(U) & \xrightarrow{\ (lE)\ } & \mathbf{AS}(U) \\
{\scriptstyle rs}\big\downarrow & & \big\downarrow{\scriptstyle rs} \\
RS(U) & \xrightarrow[\ L\ ]{} & rs(\mathbf{AS}(U))
\end{array}
$$

Proposition 8.1.1. *If the above diagrams commute then for any rough set* $\langle (uE)(X), (lE)(X) \rangle$,
1. $L(\langle (uE)(X), (lE)(X) \rangle) = \langle (lE)(X), (lE)(X) \rangle$.
2. $M(\langle (uE)(X), (lE)(X) \rangle) = \langle (uE)(X), (uE)(X) \rangle$.

Proof. For any $X \in \wp(U), (lE)(X) \in \mathbf{AS}(U)$. But for any element A of $\mathbf{AS}(U)$, $rs(A) = \langle A, A \rangle$. Thus $rs((lE)(X)) = \langle (lE)(X), (lE)(X) \rangle$. Similarly for M. QED

So, let us define the two operators in the following manner:

Definition 8.1.1. *For any Rough Set System $RS(U)$, for any $a \in$*
$RS(U)$,
(1) $M(a) = \langle a_1, a_1 \rangle$;   (2) $L(a) = \langle a_2, a_2 \rangle$.

Now, the surprising fact is that the previously defined elements $\langle U, B \rangle$
and its dual $\langle P, \emptyset \rangle$ carry sufficient information for recovering these
modal operators and for describing their algebraic properties. Notice
that $\langle P, \emptyset \rangle = \sim \langle U, B \rangle$.

We frame the definitions of L and M within a wider logico-algebraic
discourse in order to allow the reader to appreciate the particular-
ity of these operators. Moreover in this way we prepare the technical
background for further results.

8.2   Adjointness, Approximations
and the Center of a Rough Set System

FORWARD: This Subsection, although relatively short, is pivotal in the
present Chapter. In fact, here we shall see how all the mathematical
concepts originated from a number of different formal fields and intu-
itive motivations, converge in a single coherent picture.

We have seen in Part I that in any relational structure the following
adjointness relationships hold:

$$(a)\ \langle R \rangle \dashv [R^{\smile}];\ \ (b)\ \langle R^{\smile} \rangle \dashv [R]$$

What happens in the case of indiscernibility relations, that is, when
we apply these results not to a mere partially ordered set but to an
Indiscernibility Space $\langle U, E \rangle$?

Of course, since E is symmetric we do not have the two directions
"left" and "right" any longer (topologically: any open set is a closed
set): $[R]$ and $[R^{\smile}]$ collapse into a single operator $\square$, while $\langle R \rangle$ and
$\langle R^{\smile} \rangle$ collapse into a single operator $\diamond$ as well, with the meaning "it is
necessary" and "it is possible", respectively.

Therefore, from the above results and definitions we have $\square = (lE)$
and $\diamond = (uE)$, so that the following adjointness relations hold:

Corollary 8.2.1. *For any Boolean algebra $\mathbf{A} \subseteq \wp(U)$,*
1. $\diamond \dashv \square$ – when $\mathbf{A}$ is considered as a modal space.

2. $\mathbb{C} \dashv \mathbb{I}$ *– when* **A** *is considered as a 0-dimensional topology.*
3. $(uE) \dashv (lE)$ *– when* **A** *is considered as an Approximation Space.*

This result will have a very interesting consequence. But let us first notice that for any subset X, $X \subseteq \Diamond(X)$ and $X \supseteq \Box(X)$, so that from the adjointness relation the two equalities $\Diamond(X) = X$ and $\Box(X) = X$ imply each other. This is exactly what Rough Set Theory provides: if either $X = (uE)(X)$, or $X = (lE)(X)$ then the two approximations of X coincide. However, notice that $\Diamond$ and $\Box$ are not adjoint operators in general.

Now we claim that our two knowledge-oriented sets B and P are the key parameters for defining the operators L and M and that using the above adjointness properties we are able to make the internal local Boolean behaviour of exact rough sets explicit.

Lemma 8.2.1. *For any Rough Set System $RS(U)$, for all $X \subseteq U$,*
1. $rs(\bigcup[X]_{\simeq}) = \bigvee[rs(X)]_{\simeq} = \langle(uE)(X), (uE)(X)\rangle.$
2. $rs(\bigcap[X]_{\approx}) = \bigwedge[rs(X)]_{\approx} = \langle(lE)(X), (lE)(X)\rangle.$

Proof. For any $Z \subseteq U, Z \in [X]_{\simeq}$ if and only if $(uE)(Z) = (uE)(X)$. Thus $\bigcup[X]_{\simeq} = (uE)(X)$ and $rs(\bigcup[X]_{\simeq}) = \langle(uE)(X), (uE)(X)\rangle$. But for any rough set $z = \langle z_1, z_2\rangle, z_2 \subseteq z_1$ and $z \in [rs(X)]_{\simeq}$ if and only if $z_1 = (uE)(X)$; hence $\bigvee[rs(X)]_{\simeq} = \langle(uE)(X), (uE)(X)\rangle$. By duality we have the second statement.   QED

We need now the Lawvere-Tierney operator J_a:

Definition 8.2.1. *Let* **H** *be a Heyting algebra. Given $a, x \in$ **H**, $J_a(x) = a \vee x$.*

Proposition 8.2.1. *For any Rough Set System $RS(U)$, for any $a \in RS(U)$,*
(1) $\bigvee[a]_{\equiv_{J_{\langle U,B\rangle}}} = M(a);$ *(2)* $\bigwedge[a]_{\equiv_{J_{\langle P,\emptyset\rangle}}} = L(a).$

The proof is an immediate corollary of Lemma 8.2.1 together with the following result:

Lemma 8.2.2. *For any $a, b \in RS(U)$, (1) $a \equiv_{J_{\langle U,B\rangle}} b$ iff $a \simeq b$;*
(2) $a \equiv_{J_{\langle P,\emptyset\rangle}} b$ iff $a \approx b$.

Proof. If $a \equiv_{J_{\langle U,B\rangle}} b$ then $a_1 \wedge U = a_1 = b_1 = b_1 \wedge U$. For the opposite direction (that is: if $a_1 = b_1$ then $a \equiv_{J_{\langle U,B\rangle}} b$) we have to prove that if

$a_1 = b_1$, then $a_2 \cap B = b_2 \cap B$. Obviously, if $a_1 = b_1$, then $a_1 \cap B = b_1 \cap B$, so that in view of the special behaviour of B, $a_2 \cap B = a_1 \cap B = b_1 \cap B = b_2 \cap B$. By a dual reasoning, we obtain the proof for $J_{\langle P, \emptyset \rangle}$.   QED

Proposition 8.2.2. $J^{\langle U,B \rangle}(RS(U))$ and $J_{\langle P,\emptyset \rangle}(RS(U)^{op})$ are order-isomorphic to $\mathbf{AS}(U)$.

Actually we shall prove something stronger. Let us focus on $J^{\langle U,B \rangle}$. This operator filters the elements of $RS(U)$ with respect to the filter $\uparrow \langle U, B \rangle$. So the question arises: what kind of filter $\uparrow \langle U, B \rangle$ is?

Lemma 8.2.3. $\uparrow \langle U, B \rangle$ is the filter of the dense elements of $RS(U)$ qua Heyting algebra.

Proof. We have seen that the pseudo-complement in $RS(U)$, *qua* Heyting algebra, is "$\neg$". Thus an element a is dense in $RS(U)$ if and only if $\neg\neg a = \langle U, U \rangle$, if and only if $a_1 = U$. Hence $\langle U, B \rangle$ is dense. Moreover $x_2 \supseteq B \cap x_1$, for any element $\langle x_1, x_2 \rangle$ of $RS(U)$ and $B \cap U = B$; hence we obtain that $\langle U, B \rangle$ is the least dense element of this Heyting algebra.   QED

But we have seen that given the nature of this filter, $RS(U)/_{\equiv_{J^{\langle U,B \rangle}}}$ is isomorphic to the Boolean algebra of the regular elements of $RS(U)$. The isomorphism is obtained by considering the set $J^{\langle U,B \rangle}(RS(U))$ of the fixed points of the operator $J^{\langle U,B \rangle}$.

Now, since an element a is regular if $\neg\neg a = a$ or $\lrcorner\lrcorner a = a$, we have that $\langle a_1, a_2 \rangle$ is regular if $a_1 = a_2$. Therefore, a regular element of $RS(U)$ has the form $\langle (uE)(X), (uE)(X) \rangle$ (or, equivalently, $\langle (lE)(X), (lE)(X) \rangle$). It follows that the Boolean algebra of the regular elements of $RS(U)$ is the image of $\mathbf{AS}(U)$ via rs. So we are done.

Therefore, in Rough Set Systems the Lawvere-Tierney operators $J^{\langle U,B \rangle}$ and $J_{\langle P,\emptyset \rangle}$ are analogous of the upper and, respectively, lower approximations in Approximation Spaces. What is surprising is that these operators are both defined by means of the pure lattice-theoretic notion of "least dense element" (because $\langle P, \emptyset \rangle = \neg \langle U, B \rangle$), and that, in turn, this element is based on the "empirical" parameter B.

We want to prove the statement by exploiting another interesting property of the Lawvere-Tierney operator $J^{\langle U,B \rangle}$:

Proposition 8.2.3. $J^{\langle U,B \rangle}$ is both multiplicative and additive on $RS(U)$.

Proof. $J^{\langle U,B \rangle}$ is multiplicative since it is a Lawvere-Tierney operator. Now, for any rough set $\langle (uE)(X), (lE)(X) \rangle$, $J^{\langle U,B \rangle}(\langle (uE)(X), (lE)(X) \rangle) = \langle (uE)(X), (uE)(X) \rangle$. By Corollary 8.2.1 (uE) has a right adjoint, hence it must be additive (indeed it is a topological closure operator). Additivity is inherited by the operation $\vee$ between ordered pairs, from its point-wise definition, so we obtain the proof.   QED

From this result a number of interesting consequences follows. First of all, $J^{\langle U,B \rangle}(RS(U))$ is not only the Boolean algebra of the regular elements of $RS(U)$, but we have a particular situation:

Corollary 8.2.2. *For any Approximation Space* $\mathbf{AS}(U)$, *$J^{\langle U,B \rangle}(RS(U))$ is a sublattice of the Rough Set System $RS(U)$.*

Corollary 8.2.3. *$J^{\langle U,B \rangle}(RS(U))$ is the center $\mathcal{CTR}(RS(U))$ of $RS(U)$, that is, the set of complemented elements.*

A complemented element in our framework is any x of the form $\langle x_1, x_1 \rangle$. So we can proceed as we have done above. To sum up, we know that the family of the pairs with form $\langle x_1, x_1 \rangle$ can be made into a Boolean algebra and from the discussion about exact rough sets, it is not a surprise to find this Boolean algebra embedded in $RS(U)$:

Corollary 8.2.4. *For any Rough Set System $RS(U)$,*

1. *$J^{\langle U,B \rangle}(RS(U)) = \mathcal{B}(\mathbf{AS}(U)) = rs(\mathbf{AS}(U))$.*

2. *$\mathbf{B}(\mathbf{AS}(U)) = \langle J^{\langle U,B \rangle}(RS(U)), \wedge, \vee, \sim, 0, 1 \rangle$ is the Boolean algebra of the exact elements of $RS(U)$.*

3. *$\mathbf{B}(\mathbf{AS}(U))$ is isomorphic to $\mathbf{AS}(U)$.*

The same properties hold true of $J_{\langle P, \emptyset \rangle}$, with respect to the opposite Heyting algebra $RS(U)^{op}$. It is worth noticing that the additivity of L (the multiplicativity of M) must be explicitly assumed in systems of modal logic for rough sets (cf. axiom 10a of Section 14.4 of Part III).

Before closing the Section, one more time we must mention that in the abstract case the parameters B and P are actually inessential in order to define L, M and $\mathcal{CTR}(RS(U))$. In fact the notion of a "least dense element" is sufficient and this notion is definable in pure lattice-theoretic terms. This fact will allow us to state some more abstract results.

Example 8.2.1. The Lawvere-Tierney operators $J^{\langle G,B \rangle}$ and $J_{\langle P,\emptyset \rangle}$

In this example we show an application of *Lemma* 8.2.1 to the Rough Set System $\mathcal{RS}_B(\mathbf{AS}(G))$ presented in Example 7.4.4.

Consider the element $\langle \{a,a',a''\}, \{a'\} \rangle$ of $\mathcal{RS}_B(\mathbf{AS}(G))$. We have: $\langle \{a,a',a''\}, \{a'\} \rangle \wedge \langle G,B \rangle = \langle \{a,a',a''\}, \{a'\} \rangle = \langle \{a,a',a''\}, \{a,a',a''\} \rangle \wedge \langle G,B \rangle$, and this equation holds of every pair $\langle X_1, X_2 \rangle$ of decreasing elements of $\mathbf{AS}(G)$, such that $X_1 = \{a,a',a''\}$. Hence, $x \equiv_{J^{\langle G,B \rangle}} \langle \{a,a',a''\}, \{a'\} \rangle$ if x is top equal to $\langle \{a,a',a''\}, \{a'\} \rangle$. Moreover, it follows that the largest element of $[\langle \{a,a',a''\}, \{a'\} \rangle]_{\equiv_{J^{\langle G,B \rangle}}}$ is $\langle \{a,a',a''\}, \{a,a',a''\} \rangle$. Otherwise stated, $J^{\langle G,B \rangle}(\langle \{a,a',a''\}, \{a'\} \rangle) = \langle \{a,a',a''\}, \{a,a',a''\} \rangle = \neg \neg \langle \{a,a',a''\}, \{a'\} \rangle = M(\langle \{a,a',a''\}, \{a'\} \rangle)$.

Moreover, the reader can easily verify that $J^{\langle G,S \rangle}(\mathcal{RS}_B(\mathbf{AS}(G)) \cong \downarrow \langle G,B \rangle$.

Dually, by easy inspection of the diagram of $\mathcal{RS}_B(\mathbf{AS}(G))$ we can see that $\langle \{a,a',a''\}, \{a'\} \rangle \vee \langle P,\emptyset \rangle = \langle \{a'\}, \{a'\} \rangle \vee \langle P,\emptyset \rangle$ and this equation holds of every element $\langle X_1, X_2 \rangle$ such that $X_2 = \{a'\}$ and $X_1 \supseteq \{a'\}$. Actually, $[\langle \{a,a',a''\}, \{a'\} \rangle]_{\equiv_{J_{\langle P,\emptyset \rangle}}} = \{\langle \{a,a',a''\}, \{a'\} \rangle, \langle \{a'\}, \{a'\} \rangle\}$ so that $\bigwedge[\langle \{a,a',a''\}, \{a'\} \rangle]_{\equiv_{J_{\langle P,\emptyset \rangle}}} = \langle \{a'\}, \{a'\} \rangle = \neg \neg \langle \{a'\}, \{a'\} \rangle = L(\langle \{a'\}, \{a'\} \rangle)$.

Finally, the reader is invited to verify that $J_{\langle P,\emptyset \rangle}(\mathcal{RS}_B(\mathbf{AS}(G))) = \uparrow \langle P,\emptyset \rangle$.

8.3 Multi-Valued Logics: A Knowledge-Oriented Interpretation

8.3.1 A Taxonomy of Logical Systems

So far we have developed the discourse along the information-oriented interpretation suggested by Rough Set Theory. Now that the meaning of the parameters B and of the special rough sets $\langle U,B \rangle$ and $\langle P,\emptyset \rangle$ are established, we can change perspective, assuming a more general point of view. In fact, we are going to discover the basic logico-algebraic systems that are defined by means of a functor $\mathcal{RS}_B$, depending on the parameter B, but framing the results in a general setting. After that they shall be interpreted from our usual information-oriented point of view.

So, let $\mathbf{A} = \langle A, \neg, \wedge, \vee, 0, 1 \rangle$ be an arbitrary Boolean algebra and let x and y be elements of A such that $x = \neg y$.

It is obvious that all the results introduced until now continue to hold when we substitute $\mathbf{A}$ for $\mathbf{AS}(U)$, generic elements $x, y \in \mathbf{A}$ such that $x = \neg y$ for B and P and we translate the set-theoretical operations and relations on $\wp(U)$ with the corresponding logical operations and relations on $\mathbf{A}$ (i.e. $\subseteq$ with $\leq$, $\cap$ with $\wedge$ and so on).

In particular, the abstract analogous of the rough-set operator $\mathcal{RS}_B$ is the following:

Definition 8.3.1. *Let $\mathbf{A}$ be a Boolean algebra, and let $x \in A$. Then $\mathcal{RS}_x(\mathbf{A}) = \{\langle a_1, a_2 \rangle \in A^2 : \neg a_2 \vee a_1 = 1$ and $\neg a_1 \vee a_2 \geq x\}$.*

Now we can proceed at the required level.

Proposition 8.3.1. *For any Boolean algebra $\mathbf{A}$, for any $x \in A$,*

1. *$\mathbf{L}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \sim, \phi_1, \phi_2, 0, 1 \rangle$, where $\phi_1 = M$ and $\phi_2 = L$, is a three-valued Łukasiewicz algebra.*

2. *$\mathbf{N}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \sim, \dashv, \longrightarrow, 0, 1 \rangle$ is a semi-simple Nelson algebra.*

3. *$\mathbf{H}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \neg, \supset, 0, 1 \rangle$ is a Heyting algebra.*

The proofs are given in Frame 10.6. Remember that in the above algebraic structures the operations are those defined on ordered pairs, not on $\mathbf{A}$.

Proposition 8.3.2. *For any Boolean algebra $\mathbf{A}$, $\langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \dashv, 0, 1 \rangle$ coincides with the Boolean algebra $\mathbf{B}(\mathbf{A})$ if and only if $x = 1$.*

Proof. If $x = 1$, then $\langle a_1, a_2 \rangle \in \mathcal{RS}_x(\mathbf{A})$ if and only if $a_1 = a_2$. Thus the left to right implication is given by Corollary 8.2.3. For the opposite direction, consider that if $x \neq 1$, then $y \neq 0$ and we can chose two elements $a, b \in A$ such that $b < a, a \wedge x = b \wedge x$ and $a \wedge y \neq b \wedge y$. Then $\langle a, b \rangle \in \mathcal{RS}_x(\mathbf{A})$ and $a \neq b$. It follows that $\mathcal{RS}_x(\mathbf{A}) \neq \mathcal{B}(\mathbf{A})$. QED

Lemma 8.3.1. *For any Boolean algebra $\mathbf{A}$, for any $x \in \mathbf{A}$ and $a, b \in \mathcal{RS}_x(\mathbf{A})$,*

1. *$a \leq b$ if and only if $\sim a \geq \sim b$.*

2. *$L(a) = \sim \dashv a$; $M(a) = \dashv \sim a$.*

3. *$\sim \dashv (a \supset b) = \bigvee \{c \in \mathcal{CTR}(\mathcal{RS}_x(\mathbf{A})) : c \wedge a \leq b\}$. We set $a \overset{c}{\Longrightarrow} b =_{def} \sim \dashv (a \supset b)$.*

4. *$\sim (\sim a \supset \sim b) = \bigwedge \{z \in \mathcal{RS}_x(\mathbf{A}) : z \vee a \geq b\}$. We set $a \subset b =_{def} \sim (\sim a \supset \sim b)$.*

5. *$\dashv \sim (a \subset b) = \bigwedge \{c \in \mathcal{CTR}(\mathcal{RS}_x(\mathbf{A})) : c \vee a \geq b\}$. We set $a \overset{c}{\Longleftarrow} b =_{def} \dashv \sim (a \subset b)$.*

Proof. (**1**) The order reversing property is inherited in $\mathcal{RS}_x(\mathbf{A})$ from the same property of $\neg$ in $\mathbf{A}$. (**2**) By easy computation we have $\sim \lrcorner a =\sim \langle\neg a_2, \neg a_2\rangle = \langle a_2, a_2\rangle$ and $\lrcorner \sim a = \lrcorner\langle\neg a_2, \neg a_1\rangle = \langle a_1, a_1\rangle$. (**3**) As listed as a fact in Table 1, $a \supset b = \bigvee\{z : z \wedge a \leq b\}$; so from 2, $\sim \lrcorner(a \supset b)$ is the largest element of the center below $a \supset b$. (**4**) Because "$\sim$" is order reversing. (**5**) From 2 and 4. QED

It follows that $a \xRightarrow{c} b$ is the largest element of the center of $\mathbf{A}$ whose meet with a is less than or equal to b, while $a \subset b$ is the smallest element whose union with a is greater than or equal to b. Finally, $a \xLeftarrow{c} b$ is the smallest element of the center of $\mathbf{A}$ whose union with a is greater than or equal to b.

Corollary 8.3.1. *For any* $a, b \in \mathcal{RS}_x(\mathbf{A})$,

1. $\lrcorner a = a \subset 1 = \bigwedge\{z : z \vee a \geq 1\}$.

2. $a \xRightarrow{c} b$, $a \subset b$ *and* $a \xLeftarrow{c} b$ *exist.*

3. (i) $L(a) = 1 \xRightarrow{c} a$; (ii) $M(a) = 0 \xLeftarrow{c} a$.

Proof. (**1**) By easy computation we have the equations $a \subset 1 = \lrcorner a \wedge 1 \wedge (\sim \lrcorner \sim a\vee \sim \lrcorner 1) = \lrcorner a$ (we can have a direct proof: if $z \vee a = 1$, then $z_1 \vee a_1 = 1 = z_2 \vee a_2$. But $\neg a_2$ is clearly the least $w \in \mathbf{A}$ such that $w \vee a_2 = 1$. Since $z_1 \geq z_2$, the required element is $\langle\neg a_2, \neg a_2\rangle$, that is, $\lrcorner a$). (**2**) The result is obtained from the fact that $\mathcal{RS}_x(\mathbf{A})$ is closed under the operations $\lrcorner, \neg, \sim$ and $\supset$ and from the definition of $\xRightarrow{c}, \xLeftarrow{c}$ and $\subset$. (**3**) Again by easy computation we have the following two chains of equations (i) $1 \xRightarrow{c} a =\sim \lrcorner(1 \supset a) =\sim \lrcorner \sim 1\vee \sim \lrcorner a \vee (\lrcorner 1 \wedge \lrcorner \sim a) = 0\vee \sim \lrcorner a \vee (0 \wedge \lrcorner \sim a) =\sim \lrcorner a = L(a)$, (ii) $0 \xLeftarrow{c} a = \lrcorner 0 \wedge \lrcorner \sim a \wedge (\lrcorner \sim \lrcorner 0\vee \sim \lrcorner a) = 1 \wedge \lrcorner \sim a \wedge (1\vee \sim \lrcorner a) = \lrcorner \sim a = M(a)$. QED

It should be noticed that the proof of *Corollary* 8.3.1.(1) depends on the fact that $\mathbf{A}$ is a Boolean algebra. The operations $\subset$ and $\lrcorner$ are the relative pseudo-complementation and, respectively, the pseudo-complementation in the Heyting algebra $\mathcal{RS}_x(\mathbf{A})^{op}$ (the reader is invited to compare these results with the identities stated in Epstein & Horn [1974a], for instance: $a \xLeftarrow{c} b =\sim (b \xRightarrow{c} a)$).

Proposition 8.3.3. *For any Boolean algebra* **A**,

$$\mathbf{P2}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, !, e_0, e_1, e_2 \rangle$$

where the pseudo-supplementation ! is L, $e_0 = 0$, $e_1 = \langle 1, x \rangle$ and $e_2 = 1$, is a P_2-lattice of order three, if and only if $x \neq 1$.

Proof. (i) From Corollary 8.3.1(3) (a) it follows that L is indeed the pseudo-supplementation !. Moreover by Corollary 8.3.1(2) this operation is always defined in $\mathcal{RS}_x(\mathbf{A})$. (ii) $e_1 = \langle 1, x \rangle$ derives immediately from the fact that $\langle 1, x \rangle$ is the least dense element in the interval $[\langle 0, 0 \rangle, \langle 1, 1 \rangle]$ and from the result quoted in Frame 10.1. Moreover in view of the discussion about the operator $J^{\langle 1, B \rangle}$, one can prove that $\uparrow \langle 1, x \rangle$ is a Boolean algebra; hence the least (and only) dense element within the interval $[\langle 1, x \rangle, \langle 1, 1 \rangle]$ is $\langle 1, 1 \rangle$. It follows that $\mathbf{P2}(\mathbf{A})$ is of order three. But if $x = 1$ then the least and only dense element of $\mathcal{RS}_x(\mathbf{A})$ would be $\langle 1, 1 \rangle$ and we would obtain a Boolean algebra. (iii) It is easy to check that $e_1 \supset e_0 = e_0$ (the other case, $e_2 \supset e_1$, is trivial). QED

Proposition 8.3.4. *For any Boolean algebra* **A**,

$$\mathbf{P}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \neg, \supset, D_1, D_2, e_0, e_1, e_2 \rangle$$

where $D_1 = M, D_2 = L$, is a Post algebra of order three if and only if $x = 0$. In this case the chain of values is given by $e_0 = 0$, $e_1 = \langle 1, 0 \rangle$ and $e_2 = 1$.

Proof. In view of Proposition 8.3.3, we have only to show that $!e_{n-2} = 0$. Since in our case $e_{n-2} = e_1 = \langle 1, 0 \rangle$, we have: $\sim \lrcorner \langle 1, 0 \rangle =\sim \langle \neg 0, \neg 0 \rangle =\sim \langle 1, 1 \rangle = \langle 0, 0 \rangle$. Moreover it is obvious that if $1 \neq x \neq 0$ then $!e_{n-2}$ is strictly greater than 0, so that $\mathcal{RS}_x(\mathbf{A})$ can be only made into a P_2-lattice of order three; on the contrary, if $x = 1$ then $!e_0 = 0$, but $e_{n-2} = e_0$: in this case $\mathbf{P}(\mathbf{A})$ is indeed a Post algebra, but of order two, that is a Boolean algebra. QED

Proposition 8.3.5. *For any Boolean algebra* **A**, *for any $x \in A$,*

$$\mathbf{PA}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \overset{c}{\Longrightarrow}, \overset{c}{\Longleftarrow}, \supset, \subset, !, ¡, 0, 1 \rangle$$

where the pseudo-supplementation ! is L, the dual pseudo-supplementation ¡ is M, and $\overset{c}{\Longleftarrow}$, $\overset{c}{\Longrightarrow}$ and $\subset$ are the operations defined in Lemma 8.3.1, is a P-algebra.

Proof. In view of Proposition 8.3.1 and Corollary 8.3.1, we have only to check the "linearity property" $(a * b) \vee (b * a) = 1$, for '$*$' any of the operations $\supset, \subset, \stackrel{c}{\Longrightarrow}$ and $\stackrel{c}{\Longleftarrow}$. So let us prove just the cases for $\supset$ and $\stackrel{c}{\Longrightarrow}$ (the others come from duality). In the development of the polynomial $(a \supset b) \vee (b \supset a)$ we obtain an equivalent disjunctive formula in which the two terms $\lrcorner \sim b \vee \sim \lrcorner \sim b$ and $\lrcorner \sim a \vee \sim \lrcorner \sim a$ appear. Since for any z, both $\sim \lrcorner z$ and $\lrcorner \sim z$ are complemented, we have $(a \supset b) \vee (b \supset a) = 1$. Consider now that the operator $\sim \lrcorner$ is additive; thus $(a \stackrel{c}{\Longrightarrow} b) \vee (b \stackrel{c}{\Longrightarrow} a) = \sim \lrcorner (a \supset b) \vee \sim \lrcorner (b \supset a) = \sim \lrcorner((a \supset b) \vee (b \supset a)) = \sim \lrcorner 1 = 1$.   QED

Example 8.3.1. Logico-algebraic structures from Rough Set Systems
As we have seen in Example 7.4.1, the lattices **L1** and **L2** of Example 6.5.1 are isomorphic to the derived lattices $\mathcal{RS}_0(\mathbf{B})$ and, respectively, $\mathcal{RS}_b(\mathbf{B})$ of the Boolean algebra **B**. Therefore, all the considerations about **L1** and **L2** in Example 6.5.1 turns into examples of the previous Propositions.

  As for "concrete" examples, we can refer to the Rough Set Systems of Example 7.4.4:

1. The lattice $\mathcal{RS}_B(\mathbf{AS}(G))$ equipped with the chain $\langle\langle 0, \langle\neg B, \emptyset\rangle, 1\rangle$, i.e. $\langle\langle\emptyset, \emptyset\rangle, \langle\{a, a''\}, \emptyset\rangle, \langle G, G\rangle\rangle$ can be made into a P_0−lattice. Moreover, by setting $e_1 = J^{\langle G, \neg B\rangle}(\langle G, \emptyset\rangle)$, i.e. by setting $e_1 = J^{\langle G, \{a, a''\}\rangle}(\langle G, \emptyset\rangle) = \langle G, \{a', a'''\}\rangle = \langle G, B\rangle$, we obtain a new chain $\langle 0, e_1, 1\rangle$ which makes $\mathcal{RS}_B(\mathbf{AS}(G))$ into a P_2−lattice, because e_1 (alias $\langle G, B\rangle$) is the least dense element in $\mathcal{RS}_B(\mathbf{AS}(G))$. Thus the element $\langle G, B\rangle$ plays here the role of intermediate value, that is the same role played by the central element $\langle G, \emptyset\rangle$ in the case of $\mathcal{P}(\mathbf{AS}(G))$.

2. $\mathcal{RS}_B(\mathbf{AS}(G))$ can be made into a semi-simple Nelson algebra. Indeed, one can verify, for instance, that $\langle\{a, a'', a'''\}, \{a'''\}\rangle \vee \neg\langle\{a, a'', a'''\}, \{a'''\}\rangle = \langle\{a, a'', a'''\}, \{a'''\}\rangle \vee \langle\neg\{a'''\}, \neg\{a'''\}\rangle = \langle\{a, a'', a'''\}, \{a'''\}\rangle \vee \langle\{a, a', a''\}, \{a, a', a''\}\rangle = \langle G, G\rangle$. This, obviously, depends on the fact that $\neg$ is a Boolean complementation in $\mathbf{AS}(G)$.
Symmetrically, we verify that $\langle\{a, a'', a'''\}, \{a'''\}\rangle \wedge \neg\langle\{a, a'', a'''\}, \{a'''\}\rangle$
$= \langle\{a, a'', a'''\}, \{a'''\}\rangle \wedge \langle\neg\{a, a'', a'''\}, \neg\{a, a'', a'''\}\rangle =$
$= \langle\{a, a'', a'''\}, \{a'''\}\rangle \wedge \langle\{a'\}, \{a'\}\rangle = \langle\emptyset, \emptyset\rangle$

3. If we set $\phi_1 = \neg\neg$ and $\phi_2 = \lrcorner\lrcorner$ then $\mathcal{RS}_B(\mathbf{AS}(G))$ can be made into a three-valued Łukasiewicz algebra. For instance, $\phi_1(\langle\{a, a', a''\}, \{a'\}\rangle) = \langle\{a, a', a''\}, \{a, a', a''\}\rangle$ while $\phi_2(\langle\{a, a', a''\}, \{a'\}\rangle) = \langle\{a'\}, \{a'\}\rangle$. Notice that both results are exact rough sets, namely $rs(\{a, a', a''\})$ and, respectively, $rs(\{a'\})$.

The above results describe a mere taxonomy. Indeed if we want to speak of "information-oriented" interpretation, we should be able to

investigate these structures more deeply and to decompose and reconstruct them, from a logical point of view, using our information-based parameters B and P.

8.3.2 Exact and Inexact Information in Logico-Algebraic Systems

In order to understand the two fold nature of the above logico-algebraic structures, we must unravel the double nature of Rough Set Systems. So, though we could continue and develop the reasoning at a more abstract level, as in the previous Subsection, we prefer to frame the next results within the more intuitive setting of Rough Set Theory and to translate them into general terms at the end of the Section.

We have to answer the following question: "What happens if we focus our attention separately on the P-part and on the B-part of a Rough Set System"? In order to answer we have to extract from $RS(U)$ the **local** logical properties of the inexact part and of the exact part. Clearly we shall look again for some suitable Lawvere-Tierney operator. Indeed, in order to discharge the B-part we have to consider $\langle P, P \rangle$ as a (local) top element, whereas in order to discharge the P-part, $\langle B, B \rangle$ has to be a local top element (notice that both elements are legal rough sets).

Let us first deal with the P-part. We use, therefore, the Lawvere-Tierney operator $J^{\langle P,P \rangle}$. In fact it is seen that $J^{\langle P,P \rangle}(RS(U))$ is isomorphic to $\Omega_{J^{\langle P,P \rangle}}(RS(U)) = \{x : x \leq \langle P, P \rangle\}$. We shall obtain $\Omega_{J^{\langle P,P \rangle}}(RS(U))$ step by step, providing it with an information-oriented flavour. First, by definition we have:

Lemma 8.3.2. *For any $a, b \in RS(U), a \equiv_{J^{\langle P,P \rangle}} b$ iff $a_1 \cap P = b_1 \cap P$ and $a_2 \cap P = b_2 \cap P$.*

Thus $\equiv_{J^{\langle P,P \rangle}}$ groups together all rough sets which coincide in the P-part of both elements. It follows that the only difference between $\equiv_{J^{\langle P,P \rangle}}$ equivalent rough sets, rests in the B-parts of their elements. Moreover it follows that for any rough set x, the fixed point $\bigvee[x]_{\equiv_{J^{\langle P,P \rangle}}}$ adds B to x_1 and x_2. In this way B acts no more and the only difference between the fixed points of $J^{\langle P,P \rangle}$ depends uniquely on the P-part, that is the inexact part of $\mathbf{AS}(U)$. It is not difficult to show that the set of fixed points of this operator can be made into a Post algebra of order three with chain $(\langle B, B \rangle, \langle U, B \rangle, \langle U, U \rangle)$.

Now that the B-part is no longer active, we can consider the following transformation of elements $x \in J^{\langle P,P \rangle}(RS(U))$: $h(\langle x_1, x_2 \rangle) = \langle x_1 \cap -B, x_2 \cap -B \rangle = \langle x_1 \cap P, x_2 \cap P \rangle$.

By means of h we discharge the "frozen" B-part of the elements of $J^{\langle P,P \rangle}(RS(U))$ (of course we can directly obtain $h(J^{\langle P,P \rangle}(RS(U)))$ if we consider the dual fixed points $\bigwedge[x]_{J^{\langle P,P \rangle}}$: in this way we subtract B from x_1 and x_2). At this point one can easily prove:

Proposition 8.3.6. *For any Rough Set System $RS(U)$, $h(J^{\langle P,P \rangle}$ $(RS(U))) = \mathcal{RS}_\emptyset(\mathcal{B}^*(P))$, where $\mathcal{B}^*(P)$ is the Boolean algebra whose set of atoms is P^*, i.e. the ideal $\downarrow P$ of $\mathbf{AS}(U)$.*

Readers are invited to prove the exercise.

Corollary 8.3.2. $\mathbf{P}(P) = \langle h(J^{\langle P,P \rangle}(RS(U))), \vee, \wedge, \neg, \supset, L, M, e_0,$ $e_1, e_2 \rangle$ *is a Post algebra of order three with chain of values given by $e_0 = \langle \emptyset, \emptyset \rangle, e_1 = \langle P, \emptyset \rangle, e_2 = \langle P, P \rangle$.*

So the inexact part gives rise to a Post algebra of order three as we expected on the basis of our intuitive considerations.

For the second half of the task, it is to analyse the logical behaviour of the B-part of a Rough Set System. Now we have to collect the elements that coincide in their B-parts and to "freeze" the P-part. It seems intuitive to apply the operator $J^{\langle B,B \rangle}$, since $\langle B, B \rangle = \sim \langle P, P \rangle$:

Lemma 8.3.3. *For any $a, b \in RS(U), a \equiv_{J^{\langle B,B \rangle}} b$ iff $a_1 \cap B = b_1 \cap B$ and $a_2 \cap B = b_2 \cap B$.*

Taking the set of fixed points of this operator we obtain a Boolean algebra with top element $\langle U, U \rangle$ and bottom element $\langle P, P \rangle$. In fact for any x, $\bigvee[x]_{J^{\langle B,B \rangle}}$ adds P to both x_1 and x_2.

Once we have "frozen" the P-part in this way, we must subtract it from each element of $J^{\langle B,B \rangle}(RS(U))$. So we shall use the transformation: $g(\langle x_1, x_2 \rangle) = \langle x_1 \cap -P, x_2 \cap -P \rangle = \langle x_1 \cap B, x_2 \cap B \rangle$. It is easy to prove for any Rough Set System $RS(U)$:

Proposition 8.3.7. $g(J^{\langle B,B \rangle}(RS(U))) = \mathcal{RS}_B(\mathcal{B}^*(B))$, *where $\mathcal{B}^*(B)$ is the Boolean algebra whose set of atoms is B^*, i.e. the ideal $\downarrow B$ of $\mathbf{AS}(U)$ (that is, actually, $\wp(B)$).*

Corollary 8.3.3. $\mathbf{B}(B) = \langle g(J^{\langle B,B \rangle}(RS(U))), \vee, \wedge, \sim, \langle \emptyset, \emptyset \rangle, \langle B, B \rangle \rangle$ *is a Boolean algebra.*

Again one could consider the duals of the fixed points obtaining the resulting structure directly.

Example 8.3.2. The Lawvere-Tierney operators $J^{\langle P,P \rangle}$ and $J^{\langle B,B \rangle}$

Let us have a look of an illustration of the statements of Subsection 8.3.2.

The reader will verify by easy inspection of Figure 8.1 that the following decomposition into a Post algebra of order three and a Boolean algebra holds of $\mathcal{RS}_B$ ($\mathbf{AS}(G)$), where h and g are the maps defined in *Proposition* 8.3.6 and *Proposition* 8.3.7:
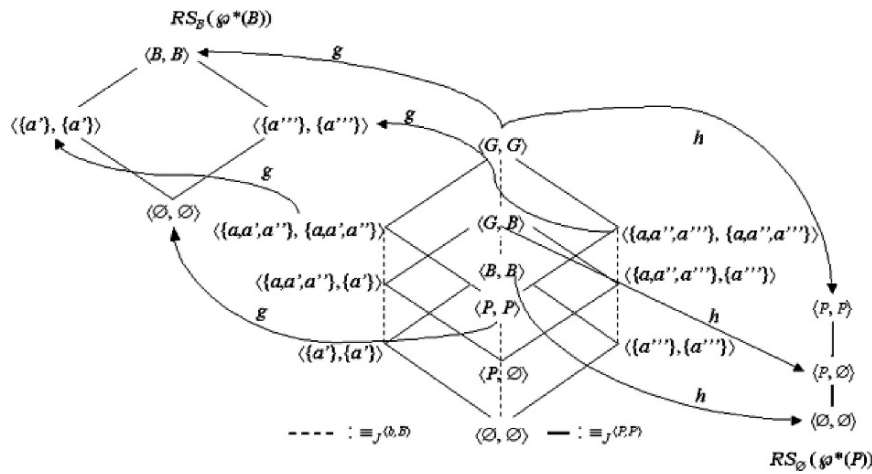

Figure 8.1: Decomposition of a three-valued lattice into a Post algebra of order three and a Boolean algebra

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Now a remark about this last construction is in order. We have seen that to obtain the Boolean elements constituting the center of $RS(U)$, we had to use a double negation topology, namely the Lawvere-Tierney topology $J^{\langle U,B \rangle}(RS(U))$. It is rather interesting to see that the construction of the Boolean algebra $\mathbf{B}(B)$ uses a double negation topology, too. This fact has been hidden by the use of the operator $J^{\langle B,B \rangle}$. But one can prove that in $\mathcal{RS}_B(\mathbf{AS}(U))$ this operator is equivalent to the Lawvere-Tierney operator $B_{\langle P,P \rangle}$ which is defined as follows:

Definition 8.3.2. *For any Heyting algebra* $\mathbf{H}$ *and* $a, p \in \mathbf{H}, B_a(p) = (p \Longrightarrow a) \Longrightarrow a.$

The proof of the equivalence is left as an exercise based on the following Lemma:

Lemma 8.3.4. *If a is an exact element then for all $x, x \implies a = \langle \neg x_1, \neg x_2 \rangle \vee a$.*

Proof. $\sim \lrcorner \sim x \vee a \vee (\lrcorner x \wedge \lrcorner \sim a) = \sim \lrcorner \sim x \vee a \vee (\lrcorner x \wedge a) = \sim \lrcorner \sim x \vee a = \langle \neg x_1, \neg x_1 \rangle \vee a.$ QED

Since $\langle P, P \rangle$ is an exact element, for all a, $B_{\langle P, P \rangle}(a) = (a \implies \langle P, P \rangle) \implies \langle P, P \rangle = (\langle \neg a_1, \neg a_1 \rangle \vee \langle P, P \rangle) \implies \langle P, P \rangle = \langle a_1 \wedge \neg P, a_1 \wedge \neg P \rangle \vee \langle P, P \rangle = \langle P \vee (a_1 \wedge \neg P), P \vee (a_1 \wedge \neg P) \rangle = \langle P \vee a_1, P \vee a_1 \rangle$.
Hence $a \equiv_{B_{\langle P, P \rangle}} b$ if and only if $\langle P \vee a_1, P \vee a_1 \rangle = \langle P \vee b_1, P \vee b_1 \rangle$ if and only if $P \vee a_1 = P \vee b_1$, if and only if $-P \wedge (P \vee a_1) = -P \wedge (P \vee b_1)$, if and only if $B \wedge a_1 = B \wedge b_1$ and (by definition of a rough set) $B \wedge a_2 = B \wedge b_2$. Therefore, the P part is immaterial. It follows that $\bigvee [a]_{\equiv_{B_{\langle P, P \rangle}}} = \langle a_1 \vee P, a_1 \vee P \rangle$.

Exercise 8.1. *Prove that, (i) $\bigvee [a]_{\equiv_{B_{\langle B, B \rangle}}} = \bigvee [a]_{\equiv_{J^{\langle P, P \rangle}}}$ and (ii) $\bigvee [a]_{\equiv_{B_{\langle P, P \rangle}}} = \bigvee [a]_{\equiv_{J^{\langle B, B \rangle}}}$.*

Exercise 8.2. *Prove that $B_{\langle P, P \rangle}$ coincides with the mixed operator $J^{\langle G, B \rangle} \vee B_{\langle P, \emptyset \rangle}$.*

The congruence relation induced by this operator, gives the so-called *Boolean quotient.* The term "Boolean" is justified by the fact that the set of fixed points of the operator B_a is $\{p \in \mathbf{H} : (p \implies a) \implies a \leq p\}$. Hence in the case $a = 0$ we obtain the well-known double negation quotient $REG(\mathbf{H})$.[1] Thus applying $B_{\langle P, P \rangle}$ we obtain the set of regular elements with respect to the local bottom element $\langle P, P \rangle$ (and it is natural to consider $\langle P, P \rangle$ a bottom element: in order to accomplish our construction, we have to forget the P-part).

After this, let us come to some conclusions

From the above results it follows that any Rough Set System $RS(U)$ is isomorphic to the direct product of the Post algebra of order three

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[1] In the case $a = 0$ the set of fixed points of the operator B_a coincides with $\{p \in \mathbf{H} : (p \implies a) \implies p \implies p = 1\}$. But this is precisely Peirce's law, that is, the law which separates the implicative fragment of Classical Logic from the implicative fragment of Intuitionistic Logic.

$\mathbf{P}(P)$ (i.e. $\mathcal{RS}_{\emptyset}(\mathcal{B}^*(P))$), and the Boolean algebra $\mathbf{B}(B)$ (i.e. $\mathcal{RS}_B(\mathcal{B}^*(B))$). The lattice isomorphism is:

$$f : \mathbf{P}(P) \times \mathbf{B}(B) \longmapsto RS(U) : f(\langle\langle p_1, p_2\rangle, \langle b_1, b_2\rangle\rangle) = \langle p_1 \cup b_1, p_2 \cup b_2\rangle.$$

Thus f re-connects B and P and as such it is the reciprocal of g and h.

Notice that if $P = \emptyset$, then $\mathbf{P}(P)$ is a degenerate Post algebra, while $\mathbf{B}(B)$ equals $\mathbf{B}(\mathbf{AS}(U))$ and is isomorphic to $\mathbf{AS}(U)$. Conversely, if $B = \emptyset$, then $\mathbf{B}(B)$ is a degenerate Boolean algebra and $\mathbf{P}(P) = \mathbf{P}(\mathbf{AS}(U))$.


Example 8.3.3. Post $\times$ Boole=Lukasiewicz
Here we apply to $\mathcal{RS}_B(\mathbf{AS}(G))$ the decomposition and reconstruction technique which was examined above.
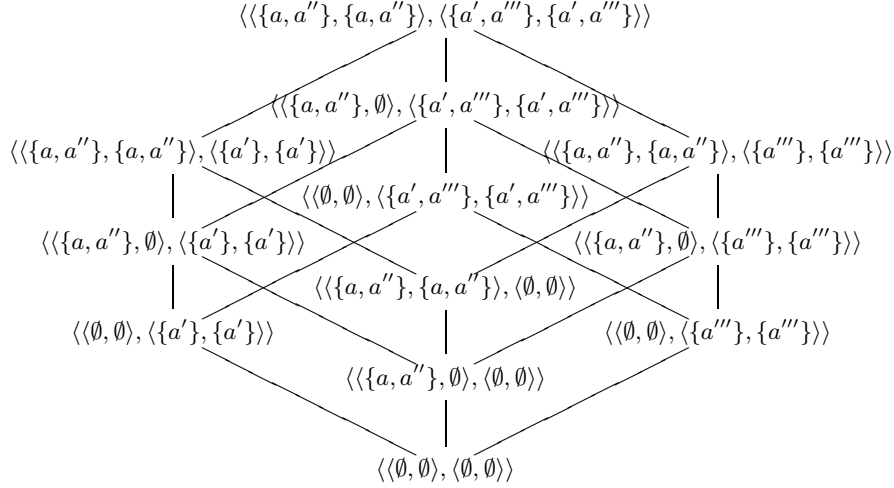Step 1: we separate the inexact from the exact parts of the Approximation Space $\mathbf{AS}(G)$:

- Inexact part: $P^* = \{\{a, a''\}\}$, $P = \bigcup P^* = \{a, a''\}$;
- Exact part : $B^* = \{\{a'\}, \{a'''\}\}$, $B = \bigcup B^* = \{a', a'''\}$.

Step 2: we form the Boolean algebras $\mathcal{B}^*(B)$ and $\mathcal{B}^*(P)$, which has B^* and, respectively, P^* as set of atoms, and then we form the Post algebra $\mathcal{RS}_{\emptyset}(\mathcal{B}^*(P))$ and the Boolean algebra $\mathcal{RS}_B(\mathcal{B}^*(B))$, according to *Corollary* 8.3.2 and *Corollary* 8.3.3:
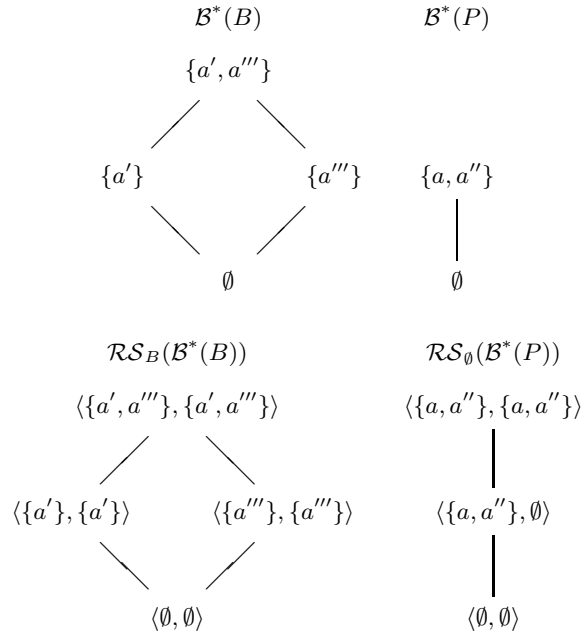
Step 3: now we form the product $\mathcal{RS}_\emptyset(\mathcal{B}^*(P)) \times \mathcal{RS}_B(\mathcal{B}^*(B))$:

$$\langle\langle\{a,a''\},\{a,a''\}\rangle,\langle\{a',a'''\},\{a',a'''\}\rangle\rangle$$

$$\langle\langle\{a,a''\},\emptyset\rangle,\langle\{a',a'''\},\{a',a'''\}\rangle\rangle$$

$$\langle\langle\{a,a''\},\{a,a''\}\rangle,\langle\{a'\},\{a'\}\rangle\rangle \qquad \langle\langle\{a,a''\},\{a,a''\}\rangle,\langle\{a'''\},\{a'''\}\rangle\rangle$$

$$\langle\langle\emptyset,\emptyset\rangle,\langle\{a',a'''\},\{a',a'''\}\rangle\rangle$$

$$\langle\langle\{a,a''\},\emptyset\rangle,\langle\{a'\},\{a'\}\rangle\rangle \qquad \langle\langle\{a,a''\},\emptyset\rangle,\langle\{a'''\},\{a'''\}\rangle\rangle$$

$$\langle\langle\{a,a''\},\{a,a''\}\rangle,\langle\emptyset,\emptyset\rangle\rangle$$

$$\langle\langle\emptyset,\emptyset\rangle,\langle\{a'\},\{a'\}\rangle\rangle \qquad \langle\langle\emptyset,\emptyset\rangle,\langle\{a'''\},\{a'''\}\rangle\rangle$$

$$\langle\langle\{a,a''\},\emptyset\rangle,\langle\emptyset,\emptyset\rangle\rangle$$

$$\langle\langle\emptyset,\emptyset\rangle,\langle\emptyset,\emptyset\rangle\rangle$$

Step 4: Finally, we apply the pair-wise summation to each element.

For instance, the element $\langle\langle\{a,a''\},\{a,a''\}\rangle,\langle\emptyset,\emptyset\rangle\rangle$ is transformed into $\langle\{a,a''\}\cup \emptyset,\{a,a''\}\cup\emptyset\rangle = \langle\{a,a''\},\{a,a''\}\rangle$ and $\langle\langle\{a,a''\},\emptyset\rangle,\langle\{a',a'''\},\{a',a'''\}\rangle\rangle$ is transformed into $\langle\{a,a''\}\cup\{a',a'''\},\emptyset\cup\{a',a'''\}\rangle = \langle G,B\rangle$. So one can notice that the intermediate value of $\mathcal{RS}_B(\mathbf{AS}(G))$ *qua* P_2−lattice, $\langle G,B\rangle$, is obtained by multiplying the intermediate value $\langle P,\emptyset\rangle$ of $\mathcal{RS}_\emptyset(\mathcal{B}^*(P))$ and the "intermediate value" $\langle B,B\rangle$ of $\mathcal{RS}_B(\mathcal{B}^*(B))$ (in a Boolean algebra the intermediate value coincides with the top element). Similarly, the bottom element of $\mathcal{RS}_B(\mathbf{AS}(G))$ is obtained by multiplying the bottom elements of $\mathcal{RS}_\emptyset(\mathcal{B}^*(P))$ and $\mathcal{RS}_B(\mathcal{B}^*(B))$ while the top element is given by the multiplication of the top elements of the two algebras.

The above steps provides us with a logical decomposition/recomposition procedure which is information-oriented. In fact, it is worth emphasizing that as to their algebraic meanings these results are related to the decomposition and representation results of Moisil and Cignoli (see Moisil [1949] and Cignoli [1969]). We shall give a glance at these results in Frame 10.10. For the moment we would like to mention the fact that we have proved that given a finite semi-simple Nelson algebra $\mathbf{A}$ (a finite Łukasiewicz algebra), the decomposition process of $\mathbf{A}$ may be suspended at a well determined point, that is, when we reach the point corresponding to the $P^*/B^* - decomposition$ of the atoms of $\mathbf{A}$, thus exhibiting the "logical core" of the analyzed structure.

We can sum up the discussion of the first part in the following way:

1. When $B = \emptyset$ (that is, when we do not have any precise information), then $RS(U)$ is a Post algebra of order three with

intermediate element $\langle U, \emptyset \rangle$. Alternatively, it is a centered three-valued Łukasiewicz algebra – or a centered semi-simple Nelson algebra – with $\langle U, \emptyset \rangle$ as central element. We have seen that $\langle U, \emptyset \rangle$ reflects a **completely inexact knowledge**.

2. When $B \neq \emptyset$ and $B \neq U$, then $RS(U)$ is a non-centered three-valued Łukasiewicz algebra (a non-centered semi-simple Nelson algebra). Indeed in this case we cannot have any knowledge which is inexact everywhere, since the elements of B^* are completely known. We can have at most a **completely inexact knowledge up to the elements of** B^*: this situation is represented by $\langle U, B \rangle$ which becomes the intermediate value when we interpret $RS(U)$ as a P_2-lattice.

3. On the contrary, when $B = U$ (that is when we have a **complete information** about U), then $RS(U)$ is a Boolean algebra. In this case the two modalities M and L collapse into the identity.

4. Any Rough Set System is isomorphic to the product of the Rough Set System induced by the **exact** part of an information system and the Rough Set System induced by the **inexact** part of the same information system.

So, all these results make it possible to give an intuitive background to the following statement:

Proposition 8.3.8. *Let* **A** *be a complete three-valued Łukasiewicz algebra (or a semi-simple Nelson algebra, or a P_2-lattice of order three), then* **A** *is isomorphic to the direct product* $\mathbf{B_A} \times \mathbf{P_A}$ *of a Boolean sublattice* $\mathbf{B_A}$ *and of a Post sublattice* $\mathbf{P_A}$ *of order three of* **A**.

Proof. In order to apply the Lawvere-Tierney operator J^x, we have to define a relative pseudo-complementation for any two elements $a, b \in$ **A**. In semi-simple Nelson algebras this operation is provided by $\supset$, in Łukasiewicz algebras by the Moisil residuation $\sqsupset$ defined in (6.3.25) and in P_2-lattices by the operation $\ni$, defined in the footnote of Section 6.4. In the cases we are dealing with, we have $a \sqsupset b = a \ni b = a \supset b$, for any element a, b of the given algebra (provided the translations $\phi_1/\neg\neg/\text{¡}$ and $\phi_2/\lnot\lnot/!$). In order to find the smallest dense element δ of **A**, in semi-simple Nelson algebras and in three-valued Łukasiewicz algebras we have to find the least x such that $x \supset 0 = 0$; in P_2-lattices, we

already know that δ coincides with the intermediate value e_1. Once we have δ, following the above techniques and the representation theorems stated in Frame 10.3 we obtain the result.   QED

Therefore any Rough Set System is, actually, a combination of a Boolean subalgebra and a Post subalgebra of order three, each one with a precise informational meaning.

We end this part with a table summing up the relationships between the operators of the different algebraic structures:

semi-simple Nelson alg.	3-valued Łukasiewicz alg.	$P_2 - alg.$	Rough Set Systems
$\neg\neg$	ϕ_1	¡	(uE)
$\lrcorner\lrcorner$	ϕ_2	!	(lE)
$\supset$	$\sqsupset$	$\ni$	

Chapter 9

A Logico-Philosophic Excursus

The logico-algebraic analysis of Rough Set Systems so far carried on shows that one cannot adopt a "pure" classical or a "pure" constructivistic point of view, a "pure" bivalent or a "pure" muti-valued point of view, because things are much more complicated, also in higher level degrees of abstraction. This observation leads us to more general considerations about the status of current logical researches.

Anyway, after illustrating these general considerations, from this new viewpoint we shall smoothly land back to our Rough Set Systems.

9.1 Truth-Oriented and Knowledge-Oriented Approaches in Logic

As we have anticipated in the Introduction, according to the classical point of view the goal of Logic is the discovery of the *laws of truth*, in opposition to other sciences whose goal is to discover truth.

However in modern logic a different position arose as to the claim that Logic has to discover the discovery laws. In other words, the domain of Logic should be the *laws of knowledge*.

These two positions induce different understanding as to what the meaning of a sentence is. The truth-oriented point of view (advocated by Frege and usually called "classical" or "extensional") claims that the meaning of a sentence A is given by its *truth-conditions*, that is, the set of states of affairs that make A true. On the contrary, the

knowledge-oriented position, also called "intensional", maintains that the meaning of A is given, in a sense, by the way A "is used". In both cases the meaning of A is solely given by the meaning of the logical constants of A, but in the truth-oriented case this meaning is given by a notion of "*trueness*", while for the knowledge-oriented position it is connected with the notion of "*how to recognize trueness*" or *verification conditions*.

Indeed, the knowledge-oriented approach points out a serious weakness existing in the truth-oriented position: in order to determine that a sentence A is true with respect to a given state of affairs, we should already know the meaning of A.

9.2 Understanding the Knowledge-Oriented Point of View

Let us analyse the knowledge-oriented point of view. According to it:

(M1) We can recognize the meaning of a sentence on the sole basis of its logical constants.

(M2) The meaning of the logical constants cannot depend on any classical notion of "truth", since this relies on the assumption of an external state of affairs.

(M3) The meaning of a complex sentence depends on the meaning of its components.

In short, the meaning of a sentence α is given by the way α has been syntactically assembled along with some ability to recognize the correctness of this construction. To put it in a slogan, the meaning of α is a (or "the") *verifiable construction* of α. Therefore, we shall also call this position *constructivistic position* or *Constructivism* at large. Actually, this is the core of the so-called BHK interpretation of Intuitionism which, as we shall see, leads to the identification of "*meaning conditions*" with "*verification conditions*" and, in turn, with "*proof conditions*".[1]

Thus, "meaning conditions" on the one hand require a notion of "invariance", because according to (M1) and (M2), the meaning of a

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[1]The acronym BHK is after the names of the logicians Brouwer, Heyting and Kolmogorov.

logical constant must be given solely by the role that it plays in *any* sentence. Moreover, we should be able to distinguish between correct and incorrect uses of logical constants.

These issues deserve a deeper analysis.

In order to be compliant with the latter, we have to base the notion of "knowledge" on the more primitive notion of "*evidence*". But according to the former issue, the notion of an "evidence" should be invariant, that is, it should not depend on the cognitive context, or cognitive domain, in which it is applied. Since it is unlikely to find any notion of an "evidence" completely unaffected by the cognitive context it is used for (otherwise we would speak about some kind of Platonistic truth and not about "evidence"), everything amounts to saying that we are looking for a paradigmatic cognitive context which is general enough to exhibit an invariant notion of "evidence".

Let us consider, for instance, any cognitive context in which the notion of a *"perception"* or "empirical knowledge" is central. Clearly, in this context we can have pieces of evidence – true evidence – that may happen to be false.

For instance, look again at Kanizsa's triangle depicted in the Introduction.

You surely are now perceiving two overlapping triangles. Let us call this epistemic state of yours E_1. Now we show you that actually the triangle you see in the foreground is just given by a cognitive interpolation of the other geometrical figures (cover, for instance one of the black disks). Thus, now you reach a different epistemic state E_2. Nonetheless, when you look again at the figure as it is, you come back to the epistemic state E_1.

On the contrary, suppose you are given a mathematical proof Π and you evaluate Π as correct, so reaching the epistemic state, say, E_3. If I show you later an error in some step of Π – so that I turn your epistemic state into E_4 – you are not able to come back to the epistemic state E_3 any longer.

To sum up, while in an empirical context we accept to apply the term "evidence" also to temporarily acceptable sentences, in Mathematics we reject this possibility. To put it in another way, in Mathematics the "evidence" of a proposition implies a stable condition that we can call "mathematical truth", while in empirical contexts this does not hold.

Logico-philosophical remarks. 1

From this discussion it follows that "evidence" in Mathematics is close to the traditional notion of "knowledge", as it has been elaborated since Plato and which basically states that *knowledge* is:

(K1) A belief (in order to know α, I must be aware of α),

(K2) True (otherwise α would just be an opinion),

(K3) Justified (I am not tossing a coin, but I have justifications to believe α true).

It has been shown that actually conditions (K1), (K2) and (K3) are neither sufficient nor necessary to define knowledge. An analysis of the counterexamples shows that one important weakness of the above definition is the lack of any defined relationship between justifications to believe A true (required by point (K3)) and trueness (required by (K2)). Indeed, trueness is related to states of affairs, and states-of-affairs may vary. So,

(KR1) Are we required to have a unique justification, applicable to all states-of-affairs,
    or

(KR2) For any state-of-affair are we required to find at least a justification?

The tradition seems to support the second reading and, in this way, is not able to avoid counterexamples (cf. Gettier [1963]).

It is not difficult to see that the first reading is connected with the constructivistic position. In fact, a proof Π of A is constructively valid if it is a construction (i.e. a justification) of A for any interpretation of its assumptions. Otherwise we would have a proof that depends on the different states-of-affairs. In other words Π would not be valid just because of its intrinsic construction and, as a consequence, the meaning of A would not be given by conditions (M1), (M2) and (M3). Indeed, (KR1) and (KR2) discriminate between constructive and non constructive proofs in a very precise manner (see Frame 10.17.2 for a striking example).

In any case we have seen the main reasons why almost all knowledge-oriented points of view assume mathematics as a privileged cognitive context in order to construct a good *abstract* model of epistemic activities.

Since mathematical assertions are given evidence by means of proofs, logics, from a knowledge-oriented point of view, is usually characterised as the analysis of the more abstract properties of the construction of proofs.

9.3 Some Problems Arising From the Knowledge-Oriented Point of View

However there are some problems to solve.

(I1) According to points (M1) and (M3) above, we face an impasse when we reach atomic sentences, in the analysis of the construction of a proof. Indeed, what is the meaning of an atomic sentence p according to the constructivistic point of view? Strictly speaking p should not have any meaning, since the meaning of a sentence is given by its logical constants and p does not have any logical constant. In conclusion, as Miglioli et al. [1989] puts it, the identification of verification conditions with proof conditions *"seems to us to be better conceived as a <u>normative</u> proposal than an explicative one"*, so that we are required to consider this identification with some additional care.[2]

(I2) Any logic $\mathcal{L}$ aiming at analysing the validity of a cognitive construction by means of the analysis of its logical constants, should fulfill some basic properties: the Disjunction Property (DP) and the Explicit Definability Property (ED):

DP : If $\alpha \vee \beta$ belongs to $\mathcal{L}$, that is, $\alpha \vee \beta$ is derivable in $\mathcal{L}$ (with possible assumptions Γ), in symbols $\Gamma \vdash_{\mathcal{L}} \alpha \vee \beta$, then either α is derivable in $\mathcal{L}$ or β is derivable in $\mathcal{L}$. Therefore, for instance, if $\Gamma \vdash_{\mathcal{L}} \alpha \vee \neg\alpha$, then either $\Gamma \vdash_{\mathcal{L}} \alpha$ or $\Gamma \vdash_{\mathcal{L}} \neg\alpha$, because the justification for $\alpha \vee \neg\alpha$ is given by its construction and not by the assumption that necessarily either α or $\neg\alpha$ is valid in some Platonistic "objective" world, as advocated by Classical Logic, $\mathcal{CL}$.

ED : If $\Gamma \vdash_{\mathcal{L}} \exists x\alpha(x)$, then $\Gamma \vdash_{\mathcal{L}} \alpha(t)$ for some closed term t. In other words, in order for $\exists x\alpha(x)$ to be derivable we must exhibit a closed term t which is an instance of the claim, which is not true of $\mathcal{CL}$.[3]

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[2] There is an additional problem: the meaning of an implication "$\alpha \longrightarrow \beta$", according to the BHK interpretation, is that we know a construction that makes it possible to transform every proof of α into a proof of β. But as mathematics develops, one could find out a method to prove α that cannot be transformed into a method to prove β.

[3] Indeed, $\vdash_{\mathcal{CL}} \neg\forall x\neg\alpha(x) \longleftrightarrow \exists x\alpha(x)$, while if $\mathcal{L}$ is constructive we just have $\vdash_{\mathcal{L}} \exists x\alpha(x) \longrightarrow \neg\forall x\neg\alpha(x)$. Indeed we could prove that the hypothesis $\forall x\neg\alpha(x)$

Logics fulfilling ED and DP are called "constructive logics" in the usual sense.

Intuitionistic Logic, $\mathcal{INT}$, is considered to be the paradigm of a constructive logic. But, what do we obtain if we add classical but not intuitionistic principles to $\mathcal{INT}$? What is the status of *super-intuitionistic* logics obtained in this way? Are there constructive superintuitionistic logics in the range between $\mathcal{INT}$ and $\mathcal{CL}$ (to be called "intermediate constructive logics")? Jan Łukasiewicz conjectured a negative answer. But nowadays it is well-known that there is plenty – indeed a continuum – of intermediate constructive logics (see Miglioli et al. [1989b], Miglioli [1992]). So, what is the "real" constructive logic (if any)? Finally, if $\mathcal{L}$ is a constructive logic and $\mathcal{T}$ is a set of extra-logical axioms (for instance the axioms of Peano Arithmetic), is $\mathcal{T} + \mathcal{L}$ still constructive?

9.3.1   Possible Solutions 1: Making Classical and Constructive Attitudes Coexist

We can try to solve issue (I1) by admitting that, actually, there is not just one cognitive attitude, but at least two:

(i) The constructivistic cognitive attitude, according to which in order to give α a meaning we must analyse the construction of the proofs for α.

(ii) The classical cognitive attitude, according to which we do not have to analyse α, since we can assume the truth or falsity of α as evident on the basis of an objective state-of-affairs.

Along with this distinction, the constructivistic attitude reflects the point of view of a partial *cognitive subject*, while the classical attitude reflects the point of view of an *complete observer*.

According to these possible choices, we should apply the classical approach to atomic sentences. More generally, it is wise to be able to apply the classical approach in each case where a constructive analysis is not useful (or unmotivated, as for atomic formulae).

For instance, in the context of Computer Science, we must apply the constructivistic attitude when we need to analyse a cognitive object α

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leads to a contradiction, without being able to exhibit any closed term t such that $\alpha(t)$.

in order to extract enough information from the proof of α to specify an algorithmic construction of α, while we shall adopt the classical attitude when we assume α as an un-analysed "data". Indeed, in a program we have both ingredients: algorithms to solve problems and data to initialise algorithms.

To sum up, instead of conceiving the constructivistic attitude and the classical attitude as two approaches that cannot communicate each other, we treat them as "complementary cognitive contexts" that may coexist and cooperate.

9.3.2 Possible Solutions 2: Strengthening $\mathcal{INT}$ With Classical Principles

A second possible approach deals more directly with issue (I2).

In fact, we could augment $\mathcal{INT}$ with a number of non intuitionistic principles which are, however, plausible from a constructivistic point of view and obtain (eventually non-standard) intermediate constructive systems.

The possibility to go in this direction was opened in [Kreisel & Putnam 1957] where it was shown that if one augments $\mathcal{INT}$ with the principle

$$(KP)\ (\neg\alpha \longrightarrow \beta \vee \gamma) \longrightarrow ((\neg\alpha \longrightarrow \beta) \vee (\neg\alpha \longrightarrow \gamma))$$

then the resulting system is still constructive, thus refuting Łukasiewicz' conjecture.[4]

Is there any limit to such a move? Actually there are at least two limits.

The first is connected with the existence of *maximal intermediate constructive logics*, that is, superintuitionistic constructive logics such that if they are augmented with any non derivable principle, then they collapse to non constructive systems.

An interesting example of such a logic is Medvedev Logic, $\mathcal{MV}$, which is a particular but faithful interpretation of Intuitionism in the

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[4]Formula (KP) is not derivable in $\mathcal{INT}$ but it is admissible. A rule $\frac{\alpha}{\beta}$ is called *admissible* in a theory T if β is provable in T whenever α is, while it is called *derivable* if $\alpha \longrightarrow \beta$ is provable in T. In Classical propositional logic $\mathcal{CP}$, the class of admissible rules and the class of derivable rules coincide. On the contrary KP is admissible in $\mathcal{INT}$ but not derivable. Indeed KP can be refuted by the simple Kripke model $0 \leq 1, 0 \leq 2, 0 \leq 3, 1 \models \alpha, 2 \models \beta, 3 \models \gamma$. In fact, $\neg[\![\alpha]\!] = \{2,3\} = [\![\beta \vee \gamma]\!]$. Thus, $[\![\neg\alpha \longrightarrow \beta \vee \gamma]\!] = W$, but $[\![\neg\alpha \Longrightarrow \beta]\!] \cup [\![\neg\alpha \Longrightarrow \gamma]\!] = \{1,2\} \cup \{2,3\} = \{1,2,3\} \lneqq W$.

sense of a logic of uniformly soluble finitary problems (see Frame 10.18).
For a number of reasons, probably Medvedev Logic is something close
to an ideal constructive logic. Unfortunately no axiomatisation for $\mathcal{MV}$
is known.

The second limitation is given by the constructive mutual incompatibility of some superintuitionistic principles. To say, we can have
two principles $\mathcal{A}$ and $\mathcal{B}$ such that both $\mathcal{INT} + \mathcal{A}$ and $\mathcal{INT} + \mathcal{B}$ are
constructive but $\mathcal{INT} + \mathcal{A} + \mathcal{B}$ is no longer constructive. Indeed, the
set of intermediate constructive logics does not enjoy a linear order but
is ramified (cf. Kirk [1982][5]).

9.4 A "Mixed-Radix" Attitude in Logic

Both the possible approaches lead us now discuss what is probably one
of the major novelties in recent logical researches: the combination of
logical systems.

As mentioned in the Introduction, in particular two well-known
attempts to combine different logics emerged in logic-literature as
promising fields of investigation: Gabbay's approach based on *Labelled
Deductive Systems* (see Gabbay [1996, 1997]) and Girard's "Unity of
Logic" based on *Linear Logic* (see Girard [1993]).

While in Gabbay's approach the cognitive context is specified by
labels which mark (hence, control) any deduction step, so that we
can speak of separate logical contexts communicating by means of an
external mechanism which regulates the use of the labels, in Girard's
approach we have a single context which, in a sense, acts also as the
metacontext needed to control the entire mechanism. We can, therefore,
call the former approach "pluralistic program" and the latter "monadic
program" (or "Unity of logic" program).[6]

Within the "Pluralistic program" a large number of different logical systems can be elegantly described and applied using a uniform,
appealing and, let us say, "user-friendly" deduction technique.[7]

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[5]Pay attention that in this paper the thesis is correct, but the proof presents a
slight error while gluing Kripke models.

[6]A different classification of mechanisms for combining logics is discussed in
Blackburn & de Rijke [1995].

[7]Often together with their functional interpretations (see Gabbay & De Queiroz
[1992]).

The "Unity of Logic" program is, clearly, more radical but the notion of a system as a meta-system has a difficult conceptual and technical implementation, as witnessed by U. Eco's understanding of a semiotic code as a meta-code, that we quoted in the Introduction.

But we can also conceive of an intermediate approach which shares, with the pluralistic position, the coexistence of different attitudes, whereas with the unifying program it shares the interest for understanding, so to say, to what extent we can equip constructive systems with classical features without making them collapse into non-constructive systems.

This is the case of Pier Angelo Miglioli's philosophy of logic.

However, P. Miglioli was not interested in de-structuring/re-structuring Classical Logic or Intuitionistic Logic, but the syntactic side of his works was characterised by imposing classical behaviours to specific parts of a constructive system, or by adding superintuitionistic principles to constructive systems.[8] In particular he approached the former problem using "modal" operators to identify the scope of classical behaviours and/or by assuming that atomic formulas must classically behave. The latter method led to the study of non-standard intermediate constructive logics, that is, systems in which the Uniform Substitution principle is not valid (see Frame 10.19.1).[9] Renouncing this "nice property" was by no means a technicality. Instead it was induced by considering the logical status of atomic theories from a constructivistic point of view, as we have already seen and shall detail later on.

Now, consider a logical framework in which the two cognitive contexts are clearly identified but interlaced. We can define it, for instance, by introducing a modality $\mathbf{T}(\ldots)$ within a constructivistic apparatus, such that $\mathbf{T}(\alpha)$ means "α is classically true". Otherwise stated, within the scope of the operator $\mathbf{T}$ the "verification conditions" do not count whereas outside its scope the "truth-conditions" do not count.

Call such a system $\mathcal{E}$ (to be detailed, of course). Therefore, in $\mathcal{E}$ we have a constructive framework hosting a classical context.

Coming back to atomic formulae, its is now natural to make them fall within the scope of the classical cognitive context $\mathbf{T}(\ldots)$. Thus, the

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[8] We mean principles such as KP or Rose's formula, Rose [1953], and similar, or, at a predicate level, the Kuroda principle, Troelstra [1973], the Grzegorczyk principle, Grzegorczyk [1972] or the Markov principle, Markov [1954].

[9] Although they may enjoy restricted forms of substitution.

meaning of an atomic formula p should be given by the relation:

$$(AT - T) \quad \mathbf{T}(p) \longleftrightarrow p,$$

so that p is given the classical meaning provided by the identity truth-function.

This is, however, a very strong assumption. In fact, if $\mathbf{T}$ is to express classical validity, then we should have:

$$(CV) \vdash_{\mathcal{CL}} \alpha \text{ if and only if } \vdash_{\mathcal{E}} \mathbf{T}(\alpha), \text{ for any formula } \alpha.$$

But in the propositional case we have the well-known Gödel-Glivenko theorem, stating:

$$(GGT) \vdash_{\mathcal{CL}} \alpha \text{ if and only if } \vdash_{\mathcal{INT}} \neg\neg\alpha,$$

that is what we have already expressed by saying that classical tautologies are *dense* elements in Heyting algebras (i.e. $\neg\neg[\![\alpha]\!] = 1$ for any classical tautology α).

Therefore, if $\mathcal{E}$ embeds $\mathcal{INT}$, as it probably does in order to provide the constructive context, then in the propositional case the above principle $(AT\text{-}T)$ is equivalent to $\neg\neg p \longleftrightarrow p$ for atomic formulae p, which is valid neither in $\mathcal{INT}$, nor in any intermediate *standard* logic, i.e. intermediate logics enjoying the principle of Uniform Substitution. Indeed, $\mathcal{E}$ and any intermediate system accepting $(AT\text{-}T)$ or analogous principles, turn into *non standard logics*.

At this point, it must be re-stated that the operator $\mathbf{T}$ was introduced to tackle a serious problem arising at the predicative level and dealing, indeed, with Gödel-Glivenko theorem. In fact, according to (GGT), at the propositional level Classical Logic is a "limit logic", because $\vdash_{\mathcal{CL}} \neg\alpha$ if and only if $\vdash_{\mathcal{INT}} \neg\alpha$, so that if we add any non classical principle to $\mathcal{INT}$, we obtain an inconsistent system. But, this is not true at a predicative level any longer.

For instance, $\vdash_{\mathcal{CL}} \forall x(\alpha(x) \vee \neg\alpha(x))$. But consider a Kripke model $K = \langle W, D, \models \rangle$ where:

- $W = \{0, 1, 2, 3, \ldots, n, \ldots\} = \mathbb{N}$

- $D = \{D_0 = \{0\}, D_1 = \{0, 1\}, \ldots, D_n = \{0, 1, \ldots n\}, \ldots\}$

- $1 \models \alpha(0); 2 \models \alpha(0), \alpha(1); \ldots; n \models \alpha(0), \alpha(1), \ldots, \alpha(n-1); \ldots$.

We have that $0 \not\models \alpha(0)$, Nonetheless, $0 \not\models \neg\alpha(0)$ because for all $n \gneqq 0, n \models \alpha(0)$. Henceforth, From $D_0 = \{0\}$ we obtain $0 \not\models \forall x(\alpha(x) \vee \neg\alpha(x))$. More in general, *for all n, $n \not\models \alpha(n)$ and $n \not\models \neg\alpha(n)$*, hence for any n, $n \models \neg\forall x(\alpha(x) \vee \neg\alpha(x))$. It follows that $K \models \neg\forall x(\alpha(x) \vee \neg\alpha(x))$. We can conclude that $\not\vdash_{\mathcal{INT}} \neg\neg\forall x(\alpha(x) \vee \neg\alpha(x))$ and $\mathcal{INT} + \neg\forall x(\alpha(x) \vee \neg\alpha(x))$ is consistent.

Logico-philosophical remarks. 2
From the above discussion, it follows that we can develop $\mathcal{INT}$ towards an "anti-classic" direction. But there is a number of doubts that this is a sound direction, from a philosophical point of view. As a matter of fact, Classical Logic is a "limit logic" since metalanguages uses Classical Logic. For instance, the reasoning to decide if a proof for a formula α is a "true" proof, run by means of classical principles. Also in extremely constructive contexts, such as Computer Science, in which algorithms or abstract data types (i.e. objects that are sort of prototypes of "constructive objects") are at the center of the attention, meta-logic is Classical Logic. Indeed, there is a number of evidences that we should consider Classical Logic as a "limit logic".

In order to obtain a (non-standard) predicative constructive Logic $\mathcal{L}$ such that $\vdash_{\mathcal{CL}} \alpha$ if and only if $\vdash_{\mathcal{L}} \neg\neg\alpha$ we need to add a couple of principles to $\mathcal{INT}$:

(K) $\forall x \neg\neg\alpha(x) \longleftrightarrow \neg\neg\forall x \alpha(x)$, all α – the *Kuroda principle*;
(Reg) $\neg\neg p \longleftrightarrow p$ for all *atomic p*.

Let us call $\mathcal{KURAT}$ the system $\mathcal{INT} + (K) + (Reg)$. Then also in the predicative case it is possible to show:

$$\vdash_{\mathcal{CL}} \alpha \text{ if and only if } \vdash_{\mathcal{KURAT}} \neg\neg\alpha, \text{ all } A.$$

The modality $\mathbf{T}$ is able to amend this "anomaly" at the predicative level.

Indeed, $\mathbf{T}$ is defined by means of the following two simple "$\mathbf{T}$-rules" which aim at grasping the essence of the problematic classical negation and, particularly, of the problematic principle of Proof by Contradiction:

$$(1)\ (\neg\alpha \longrightarrow \bot) \longrightarrow \mathbf{T}(\alpha); \quad (2)\ (\alpha \longrightarrow \bot) \longrightarrow \neg\mathbf{T}(\alpha) \qquad (T)$$

Clearly if we set:

$$(a)\ \mathbf{T}(p) \longrightarrow p; \quad (b)\ \mathbf{T}(\neg p) \longrightarrow \neg p, \quad for\ p\ atomic \qquad (T - Reg)$$

then $(T - Reg)$ together with (T) and the intuitionistic principle $\alpha \longrightarrow \neg\neg\alpha$, amount to $(AT - T)$, which is the *T*-formulation of the principle (Reg) stated in the logico-philosophical box above, and one can prove

$\mathbf{T}(\forall x \alpha(x)) \longleftrightarrow \forall x \mathbf{T}(\alpha(x))$, which is the T-formulation of Kuroda principle.[10]

More in general, it is possible to prove $\models_{\overline{c\mathcal{L}}} \alpha$ if and only if $\models_{\overline{\varepsilon}} \mathbf{T}(\alpha)$, also for any predicate α. Hence, $\mathbf{T}$ is stronger than double intuitionistic negation. Notice that Kuroda principle is equivalent to $\neg\neg\forall x(\alpha(x) \vee \neg\neg\alpha(x))$ and it is validated by Kripke models with constant domains, that is by models in which $D_i = D_j$ for any $D_i, D_j \in D$. In view of the above counterexample this requirement is by no means a surprise. From a philosophical point of view it features a situation closer to an "objective world" than it is done by Kripke models with variable domains which, on the contrary, have a more subjective taste.

9.5   A Maximal Intermediate Constructive Logic

Now let us deal with the second approach.

Although Łukasiewicz' conjecture revealed to be false, there are, indeed, intermediate constructive logics such that we cannot augment them with non derivable principles without making them collapse into a non-constructive system. Such logics are called *maximal*.

We mentioned that the cardinality of the set of all intermediate constructive logics is $2^{\aleph_0}$. But what is surprising is that there is also a continuum of maximal intermediate constructive logics (cf. Miglioli [1992] and Ferrari & Miglioli [1993]).

Maximal intermediate constructive logics are logics which have a high deductive power, very close to that of Classical Logics, but which still satisfy the disjuction property and the principle of explicit definability.

In view of the above notes, one can start understanding a first, very general, connection between the logical issues connected with (maximal) intermediate constructive logics and the logical behaviour of Rough Set Systems. In fact, to some extent Rough Set Systems are three-valued systems equipped with some Classical feature. We shall develop this intuition in what follows.

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[10]It is also possible to prove that $(T - Reg)$ is equivalent to $(AT - T)$, because $\sim \mathbf{T}(\alpha) \equiv \mathbf{T}(\sim \alpha)$ – for this see Frame 10.16.3.

Among maximal constructive logics a prominent position is held by Medvedev's Logic. As anticipated, Medvedev's Logic of Finite Problems, $\mathcal{MV}$, was introduced in Medvedev [1962] as a faithful interpretation of Intuitionistic Logic. As far as now, as we have seen, an axiomatisation of $\mathcal{MV}$ is not known. On the contrary, we know some semantics for $\mathcal{MV}$.

Surprisingly enough, the story of the proof of maximal constructivity of $\mathcal{MV}$ gives a new deeper insight into our algebraic interpretation of Rough Set Systems.

Indeed, let us follow backward the story of this result.

(**step -1**) $\mathcal{MV} = (\mathcal{F}_{\mathcal{CL}})_*$, where: (a) $\mathcal{F}_{\mathcal{CL}}$ is a particular maximal intermediate non-standard constructive logic (which will be specified in the next step); (b) the operator $(\ldots)_*$, is the "stabilisation" of a non-standard logic, that is, $(\ldots)_*$ outputs the part of the logic which is closed under uniform substitution (note that $(\ldots)_*$ preserves constructivity and maximality).

(**step -2**) $\mathcal{F}_{\mathcal{CL}}$ is the system obtained by augmenting the intuitionistic system $\mathcal{INT}$ with (KP) and (Reg).

Instead of adding (KP) and (Reg) to $\mathcal{INT}$, we can add $(T\text{-}Reg)$ and the following axiom, to *Constructive Logic with Strong Negation* (or $\mathcal{CLSN}$, see Frame 10.15.1):

$$(\mathbf{T}(\alpha) \longrightarrow \beta \vee \gamma) \longrightarrow ((\mathbf{T}(\alpha) \longrightarrow \beta) \vee (\mathbf{T}(\alpha) \longrightarrow \gamma)) \qquad (T - KP)$$

where in the definition of the operator $\mathbf{T}$ we use the constructible, or *strong*, negation "$\sim$" inherited by the basic logical system $\mathcal{CLSN}$, instead of the intuitionistic negation "$\neg$".[11]

Now we start understanding a stronger connection between Rough Set Systems, mixed logical behaviours and this story. Indeed the modal operator $\mathbf{T}$ was introduced in order to account for the concept "being classically valid" within a constructive framework. Moreover, the constructive base is given by $\mathcal{CLSN}$ which has Nelson algebras as models, so that *Proposition* 8.3.1.(2) provides us with another link with the story we are presenting.

In order to better follow this story we suggest the reader going through Frames 10.13, 10.15, 10.16, 10.17 and 10.18.

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[11]It is worth recalling that in the quoted papers the strong negation is generally denoted by "$\neg$" while the weak negation is denoted by "$\sim$".

(**step -3**) The full role of **T** is achieved when we assume $(T - Reg)$ and $(T - KP)$. However, this operator was first added alone to $\mathcal{CLSN}$ in order to explore the technical feasibility and soundness of the philosophical intuition about the coexistence of classical and constructive behaviours.

The resulting system, $\mathcal{CLSN} + (T)$ was named $\mathcal{E}_0$ and presents surprising features because the behaviour of the operator **T** influences that of the strong negation a big deal.

Indeed, strong negation acts in $\mathcal{E}_0$ differently than in $\mathcal{CLSN}$. This difference is semantically evident: in Kripke models for $\mathcal{CLSN}$ we can have possible worlds g and atomic formulas p such that neither g nor any possible world which is accessible from g force either p or $\sim p$. On the contrary, in $\mathcal{E}_0$ semantics, from any possible world g and atomic formula p, we can access a possible world g' such that either g' forces p or g' forces $\sim p$. Otherwise stated, eventually we reach a theory (a possible world) which decides any atomic formula (see Frames 10.3.3 and 10.3.4). This decidability propagates to arbitrary well-formed formulae.

$\mathcal{INT} - model$ $\mathcal{CLSN} - model$

$\mathcal{E}_0 - model$
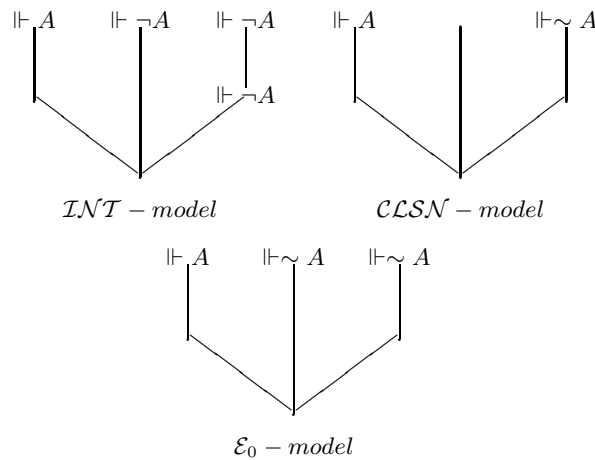
From a formal point of view, the strong negation "$\sim$" is closer to the intuitionistic negation in $\mathcal{E}_0$ than in $\mathcal{CLSN}$. Nevertheless, it keeps enjoying properties like double negation elimination and both De Morgan laws, while the Excluded Middle is not uniformly fulfilled.

We have now all the ingredients to describe the deepest connections between the intuition which lies behind the operator **T** and Rough Set Systems.

The underlying philosophy is synthesized in the following box.

Logico-philosophical remarks. 3
In the preprint of Miglioli et al. [1989] we find an interesting explanation of the basic philosophical idea behind the operator **T**:

*"[...] the classical notion of truth, far from being useless for a theory of meaning (as Intuitionism claims) and far from being the core notion of such a theory (as the Platonists claim), can be explicated as derived from a specific cognitive modality that may be adopted towards the objects of knowledge – a modality which can be intuitively qualified as assumption of cognitive data, and whose particularity is that, when objects are considered under it, it felt as not relevant the fact that they are possibly the result of our constructive activity. The other main cognitive modality is the one we adopt whenever we are interested in analyzing or in synthesizing our knowledge, and under which the objects are seen as complex constructions, whose modes of constitution are to be taken into account. [...] From this point of view it becomes essential the availability of linguistic contexts in which each cognitive modality is in turn relevant. Let us therefore introduce the context "**T** ..." (where "**T**" is a sentential operator to be read as "it is classically true that"), within which the classical modality is relevant: any sentence occurring within that context does not require a constructive analysis, in the sense that only its true value must be taken into account. In the empty context (i.e., out of the scope of **T**), on the contrary, the 'constructive' modality is relevant."*

In view of this discussion, we can explain why $(T - KP)$ is intuitively admissible. Indeed this principle says that if α is classically true, then there is a verification procedure to decide either β or γ. Therefore the right hand part of the principle is nothing but the interpretation of the premise.

9.6 Mixed-Radix Information Systems

Now we are ready to come back to Information Systems and Rough Sets. But first, let us have a deeper insight into the semantics of $\mathcal{E}_0$ using an algebraic interpretation.[12]

9.6.1 Local Validity in Nelson Lattices from Heyting Algebras

In order to let the reader have an understanding of Nelson operations and to get closer to our use of Nelson lattices for representing Rough Set Systems, we present the construction of a Nelson lattice from the material provided by the Heyting algebras.

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[12]By abuse of language, we do not distinguish here between Nelson lattices, Nelson algebras and Nelson models.

The construction of Nelson algebras through Heyting algebras is a sufficiently general technique and deserves to be adequately illustrated, because in this construction we find an algebraic synthesis of a number of topics: exact *vs* inexact information in Rough Set Theory, decidability of particular formulas in constructive systems (notably, atomic formulas), and, finally, the polymorphism of Rough Set Systems. Topics which are linked with each other and which are the *leitmotive* of this Part.

Thus let $\mathbf{H}$ be any Heyting algebra. Let Θ be a congruence induced *via* (7.4.19) by an element b which is less than or equal to the least dense element of $\mathbf{H}$ (i.e. Θ is $\equiv_{J^b}$). Then, in view of the above discussion, Θ is a Boolean congruence on $\mathbf{H}$, that is, $\mathbf{H}/_{\Theta}$ is a Boolean algebra. On the basis of the Boolean congruence Θ we can then collect a family of ordered pairs, which will be denoted by $N_{\Theta}(\mathbf{H})$:

$$N_{\Theta}(\mathbf{H}) = \{\langle a_1, a_2\rangle \in \mathbf{H} \times \mathbf{H} : a_1 \wedge a_2 = 0 \ \ and \ \ a_1 \vee a_2 \Theta 1\} \quad (9.6.1)$$

Then let us define on $N_{\Theta}(\mathbf{H})$ the following operations:

Definition 9.6.1 (Operation on pairs of disjoint elements of a Heyting algebra $\mathbf{H}$)**.**

1. $0 = \langle 0, 1\rangle$, *the bottom element*

2. $1 = \langle 1, 0\rangle$, *the top element*

3. $\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, a_2 \vee b_2\rangle$

4. $\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle = \langle a_1 \vee b_1, a_2 \wedge b_2\rangle$

5. $\langle a_1, a_2\rangle \longrightarrow \langle b_1, b_2\rangle = \langle a_1 \Longrightarrow b_1, a_1 \wedge b_2\rangle$, *weak (or non-extensional) implication*

6. $\langle a_1, a_2\rangle \rightarrowtail \langle b_1, b_2\rangle = (\langle a_1, a_2\rangle \longrightarrow \langle b_1, b_2\rangle) \wedge (\sim \langle b_1, b_2\rangle \longrightarrow \sim \langle a_1, a_2\rangle)$, *extensional implication*

7. $\sim \langle a_1, a_2\rangle = \langle a_2, a_1\rangle$, *strong negation*

8. $\lnot \langle a_1, a_2\rangle = \langle a_1, a_2\rangle \longrightarrow \langle 0, 1\rangle = \langle \neg a_1, a_1\rangle$, *weak negation*

9. $\neg \langle a_1, a_2 \rangle = \sim \backsim \sim \langle a_1, a_2 \rangle = \langle a_2, \neg a_2 \rangle$, *dual weak negation*

where the operations inside the ordered pairs are operations of **H**.

One can prove:

Theorem 9.6.1.

1. $N_\Theta(\mathbf{H})$ *is closed under the operations listed in Definition 9.6.1.*

2. $\mathbf{N}_\Theta(\mathbf{H}) = \langle N_\Theta(\mathbf{H}), \wedge, \vee, \sim, \longrightarrow, 1, 0 \rangle$ *is a Nelson algebra.*

3. $a \longrightarrow b = 1$ *iff* $a_1 \Longrightarrow b_1 = 1$ *in* **H**.

4. *If* $\leq$ *is the lattice order in* $\mathbf{N}_\Theta(\mathbf{H})$, *then* $a \leq b$ *iff* $a \rightarrowtail b = 1$.

5. *Let us define the bi-implication* $a \longleftrightarrow b =_{def} a \longrightarrow b \wedge b \longrightarrow a$. *Then the relation "$\triangleq$" defined as* $a \triangleq b$ *iff* $a \longleftrightarrow b = 1$, *is a congruence on* $(N, \wedge, \vee, \longrightarrow, \backsim, 0, 1)$, *but it is not a congruence for the strong negation* $\sim$ *(hence* $\neg$ *). Clearly* $a \triangleq b$ *iff* $a_1 = b_1$.

6. $\mathbf{N}_\Theta(\mathbf{H})/_{\triangleq}$ *is a Heyting algebra isomorphic to* **H**.

7. $\mathbf{N}_\Theta(\mathbf{H})$ *is* semi-simple *iff* $\mathbf{N}_\Theta(\mathbf{H})/_{\triangleq}$ *is a Boolean algebra, hence iff* **H** *is a Boolean algebra.*

8. *For any* a, $\backsim a = \sim \neg \sim a$.

Exercise 9.1.

Prove Theorem 9.6.1.

Example 9.6.1. Nelson algebras from Heyting algebras
Consider the Heyting algebra **A** of Example 6.1.1. In **A** the element b (which is less than the least dense element d) induces the congruence relation $\equiv_{J^b}$ defined by the equivalence classes $[b, d, e, 1], [0, a, c]$. On the contrary, d is the least dense element and induces the congruence relation depicted in Example 7.4.3. Below we show in bold the resulting Nelson lattice $N_{\equiv_{J^b}}(\mathbf{A})$ embedded into the lattice $N(\mathbf{A})$ of all pairs of disjoint elements of **A**, and $N_{\equiv_{J^d}}(\mathbf{A})$:

$N_{\equiv_{J^b}}(\mathbf{A})$
$\langle \mathbf{1}, \mathbf{0} \rangle$

$N_{\equiv_{J^d}}(\mathbf{A})$
$\langle 1, 0 \rangle$

$\langle \mathbf{e}, \mathbf{0} \rangle$

$\langle e, 0 \rangle$

$\langle c, 0 \rangle \qquad \langle \mathbf{d}, \mathbf{0} \rangle$

$\langle c, b \rangle \qquad \langle d, 0 \rangle$

$\langle \mathbf{c}, \mathbf{b} \rangle \qquad \langle a, 0 \rangle \qquad \langle \mathbf{b}, \mathbf{0} \rangle$

$\langle a, b \rangle \qquad \langle b, a \rangle$

$\langle \mathbf{a}, \mathbf{b} \rangle \qquad \langle 0, 0 \rangle \qquad \langle \mathbf{b}, \mathbf{a} \rangle$

$\langle 0, d \rangle \qquad \langle b, c \rangle$

$\langle \mathbf{0}, \mathbf{b} \rangle \qquad \langle 0, a \rangle \qquad \langle \mathbf{b}, \mathbf{c} \rangle$

$\langle 0, e \rangle$

$\langle \mathbf{0}, \mathbf{d} \rangle \qquad \langle 0, c \rangle$

$\langle 0, 1 \rangle$

$\langle \mathbf{0}, \mathbf{e} \rangle$

$\langle \mathbf{0}, \mathbf{1} \rangle$
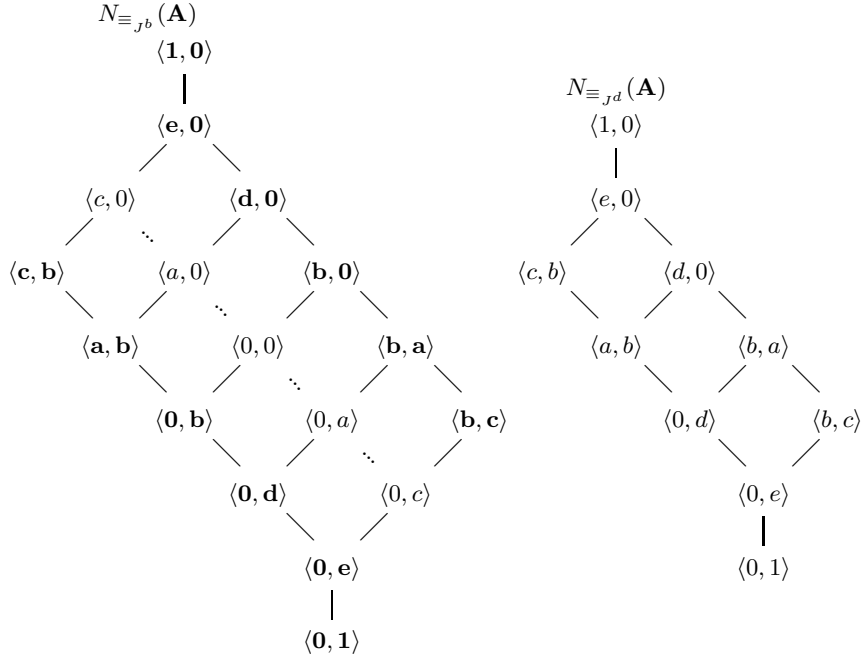
By way of example, it is easy to verify that in $\mathbf{A}$, $0 \wedge b = 0$ and $0 \vee b \equiv_{J^b} 1$, because $0 \vee b = b \in \uparrow b$. Therefore, $\langle 0, b \rangle$ fulfills condition (9.6.1), so that $\langle 0, b \rangle \in N_{\equiv_{J^b}}(\mathbf{A})$. On the contrary, $0 \vee b \not\equiv_{J^d} 1$ because $b \notin \uparrow d$. Thus, $\langle 0, b \rangle \notin N_{\equiv_{J^d}}(\mathbf{A})$. Instead, $a \wedge b = 0$ and $a \vee b \equiv_{J^b} 1$, because $a \vee b = d$ and trivially $d \equiv_{J^d} 1$. It follows that $\langle a, b \rangle$ fulfills condition (9.6.1), so that $\langle a, b \rangle \in N_{\equiv_{J^d}}(\mathbf{A})$.

Notice that the lattice of all ordered pairs of disjoint elements of $\mathbf{A}$, $N(\mathbf{A})$, is obtained by means of the improper filter $\uparrow 0$. That is, $N(\mathbf{A}) = N_{\equiv_{J^0}}(\mathbf{A})$.

The reader may notice that in $N(\mathbf{A})$ there is a central element, namely $\langle 0, 0 \rangle$ and that the strong negation is symmetric with respect to this central element. But in $N_{\equiv_{J^b}}(\mathbf{A})$ the central element (as well as other elements) is not admitted.

Now let us compute some logical function, with some comments.

(i) $\sim \langle a, b \rangle = \langle b, a \rangle$, $\sim \langle b, a \rangle = \langle a, b \rangle$ (ii) $\neg\!\!\lrcorner \langle a, b \rangle = \langle \neg a, a \rangle = \langle b, a \rangle$, $\neg\!\!\lrcorner \langle b, a \rangle = \langle \neg b, b \rangle = \langle c, b \rangle$. Hence, notice that we can immediately affirm that neither $N_{\equiv_{J^b}}(\mathbf{A})$ nor $N_{\equiv_{J^d}}(\mathbf{A})$ are semi-simple. In fact, in semi-simple Nelson algebras the operator $\neg\!\!\lrcorner$ sends any element onto an element of the center. But in the center all the three negations coincide, while $\neg\!\!\lrcorner \langle b, a \rangle \neq \sim \langle b, a \rangle$. It follows that $\neg\!\!\lrcorner$ does not send $\langle a, b \rangle$ onto the center of the algebra.

(iii) $\langle a, b \rangle \vee \langle b, c \rangle = \langle a \vee b, b \wedge c \rangle = \langle d, 0 \rangle$, $\langle a, b \rangle \wedge \langle b, c \rangle = \langle a \wedge b, b \vee c \rangle = \langle 0, e \rangle$.

(iv) $\langle 0, b \rangle \longrightarrow \langle b, a \rangle = \langle 0 \Longrightarrow b, 0 \wedge a \rangle = \langle 1, 0 \rangle$,

(v) $\langle b, 0 \rangle \longrightarrow \langle b, a \rangle = \langle 1, 0 \rangle$. However, $\langle b, 0 \rangle \not\leq \langle b, a \rangle$. Therefore $x \longrightarrow y = 1$ does not imply $x \leq y$. (vi) $\langle b, a \rangle \longleftrightarrow \langle b, c \rangle$, but $\sim \langle b, a \rangle = \langle a, b \rangle \not\longleftrightarrow \langle c, b \rangle = \sim \langle b, c \rangle$. Therefore $\longleftrightarrow$ is not a congruence for the operator $\sim$. Moreover, $\neg \langle b, a \rangle = \langle a, b \rangle \not\longleftrightarrow \langle c, b \rangle = \neg \langle b, c \rangle$. On the contrary, $\neg\!\!\lrcorner \langle b, a \rangle = \langle c, b \rangle \longleftrightarrow \langle c, b \rangle = \neg\!\!\lrcorner \langle b, c \rangle$. (vii) $\langle 0, b \rangle \longleftrightarrow \langle 0, d \rangle$ and $\langle 0, b \rangle \vee \langle b, a \rangle = \langle b, 0 \rangle \longleftrightarrow \langle b, a \rangle = \langle 0, d \rangle \vee \langle b, c \rangle$, and so on.

(viii) $\langle 0, b \rangle \rightarrowtail \langle b, a \rangle = \langle 1, 0 \rangle \wedge (\langle a, b \rangle \longrightarrow \langle b, 0 \rangle) = \langle 1, 0 \rangle \wedge \langle b, 0 \rangle = \langle b, 0 \rangle$. Therefore, $\rightarrowtail$ is not a relative pseudo-complementation.

We have seen that $\langle x_1, x_2 \rangle \triangleq \langle y_1, y_2 \rangle$ if and only if $x_1 = y_1$ (while the second elements may be different). Thus it is not difficult to understand why the filtration $\mathbf{N}_{\equiv_{J^b}}(\mathbf{A})/_{\triangleq}$ is isomorphic to $\mathbf{A}$ (we recover all and only the elements of $\mathbf{A}$ by making the second elements of all pairs collapse that share the same first element, for instance a, 0 and c in $\langle b, a \rangle$, $\langle b, 0 \rangle$ and $\langle b, c \rangle$).

Moreover, suppose we know that a given Nelson algebra is obtained as $\mathbf{N}_{\equiv_{J^x}}(\mathbf{A})$ for some $x \in \mathbf{A}$. It is not difficult to understand what element x is. In fact $\langle x, 0 \rangle$ is the least dense element of the Nelson algebra, so that we have just to find out the least element of the form $\langle z, 0 \rangle$. In our example it is $\langle b, 0 \rangle$.

Further the reader is invited to notice that $\neg$ and $\lnot$ are neither pseudo-complementations nor dual pseudo-complementations. For instance $\langle c, b \rangle \wedge \neg \langle c, b \rangle = \langle c, b \rangle \wedge \langle b, c \rangle = \langle 0, e \rangle$, $\langle c, b \rangle \vee \lnot \langle c, b \rangle = \langle c, b \rangle \vee \langle b, c \rangle = \langle e, 0 \rangle$; moreover the three negations coincide on $\langle c, b \rangle$ and $\langle b, c \rangle$. We call the set of elements where $\sim$, $\neg$ and $\lnot$ coincide, the *pseudo-center* of the lattice.

Exercise 9.2. *Show necessary conditions on $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ in order for $\langle a_1, a_2 \rangle \rightarrowtail \langle b_1, b_2 \rangle$ to be the relative pseudo-complementation of $\langle a_1, a_2 \rangle$ with respect to $\langle b_1, b_2 \rangle$.*

Basically we can define two classes, depending on the Boolean congruence we use:

$$\mathbf{NA} = \{\mathbf{N}_\Theta(\mathbf{H}) : \Theta \text{ is a Boolean congruence on } \mathbf{H}\};$$

$$\mathbf{E} = \{\mathbf{N}_\Delta(\mathbf{H}) : \Delta \text{ is the least Boolean congruence on } \mathbf{H}\}.$$

Obviously $\mathbf{E} \subseteq \mathbf{NA}$. Define on any element of $\mathbf{NA}$ (hence of $\mathbf{E}$) the operation:

$$\mathbf{T}(\langle a_1, a_2 \rangle) = \langle \neg\neg a_1, \neg a_1 \rangle \tag{9.6.2}$$

Any $\mathbf{N}_\Theta(\mathbf{H})$ is called a *Nelson lattice* because it is an algebraic model for $\mathcal{CLSN}$, while any $\mathbf{N}_\Delta(\mathbf{H})$ is an algebraic model for $\mathcal{E}_0$ and will be called an *Effective Lattice*.

For instance, the lattice $N_{\equiv_{J^b}}(\mathbf{H})$ of Example 9.6.1 belongs to $\mathbf{NA}$, while $N_{\equiv_{J^d}}(\mathbf{H}) \in \mathbf{E}$.

Now we notice what follows:

Proposition 9.6.1. *Let $\mathbf{H}$ be a Boolean algebra, $x \in \mathbf{H}$ and let Θ be the congruence induced by the filter $\uparrow x$ (i.e. Θ is $\equiv_{J^x}$). Then $\mathcal{RS}_x(\mathbf{H})$ and $N_\Theta(\mathbf{H})$ are order isomorphic.*

Proof. We prove that the following maps provide a 1-1 correspondence:
$nrs : N_\Theta(\mathbf{H}) \longmapsto \mathcal{RS}_x(\mathbf{H}); nrs(\langle a_1, a_2 \rangle) = \langle \neg a_2, a_1 \rangle$.
$srn : \mathcal{RS}_x(\mathbf{H}) \longmapsto N_\Theta(\mathbf{H}); srn(\langle a_1, a_2 \rangle) = \langle a_2, \neg a_1 \rangle$.

Indeed let $\langle a_1, a_2 \rangle \in \mathcal{RS}_x(\mathbf{H})$ then $a_2 \leq a_1$ so that $a_1 \wedge \neg a_2 = 0$. Moreover, since $\langle a_1, a_2 \rangle \in \mathcal{RS}_x(\mathbf{H})$ we have that $\neg a_1 \vee a_2 \in \uparrow x$, so that trivially $\neg a_1 \equiv_{J^x} a_2$. Hence $srn(\langle a_1, a_2 \rangle) \in N_\Theta(\mathbf{H})$. Vice-versa, if $\langle a_1, a_2 \rangle \in N_\Theta(\mathbf{H})$ then $a_1 \wedge a_2 = 0$, so that $\neg a_2 \geq a_1$. Moreover, since $a_2 \wedge a_1 \in \uparrow x$, obviously $a_2 \in \uparrow x$ and trivially $\neg a_2 \vee a_1 \in \uparrow x$. It follows that $nrs(\langle a_1, a_2 \rangle) \in \mathcal{RS}_x(\mathbf{H})$. Finally, for any $a \in \mathcal{RS}_x(\mathbf{H})$ $nrs(srn(a)) = a$. In fact, $nrs(srn(\langle a_1, a_2 \rangle)) = nrs(\langle a_2, \neg a_1 \rangle) = \langle \neg\neg a_1, a_2 \rangle = \langle a_1, a_2 \rangle$. To show that both nrs and srn are order isomorphisms is now immediate.   QED

Corollary 9.6.1. *Let* $\mathbf{H}$ *be a Boolean algebra,* $x \in \mathbf{H}$ *and let* Θ *be the congruence induced by the filter* $\uparrow x$. *Then* $\langle \mathcal{RS}_x(\mathbf{H}), \wedge, \vee, \sim, \neg, \dashv, \longrightarrow, 0, 1 \rangle$ *and* $\langle N_\Theta(\mathbf{H}), \wedge, \vee, \sim, \neg, \dashv, \longrightarrow, 0, 1 \rangle$ *are isomorphic.*

Exercise 9.3. *Prove that the above structures are isomorphic. [Hints: from Proposition 9.6.1 and using the fact that in a Boolean algebra both De Morgan laws hold]*

It is worth noticing that the above results do not hold if $\mathbf{H}$ is a generic Heyting algebra. Indeed, they strictly depend on the fact that the negation $\neg$ is an involution, in Boolean algebras.

To be sure, since any Boolean algebra is a Heyting algebra, the reader can easily understand that Nelson Lattices are generalisations of Rough Set Systems. Indeed this is not a novelty because we know that Rough Set Systems are semi-simple Nelson Lattices. But what we have to explore is the role of the operator $\mathbf{T}$ and of the congruence Θ in an information oriented context.

The properties of the operations are subjected to some restrictions, here.

Clearly, the three negations "$\sim$", "$\dashv$" and "$\neg$" do not coincide in general. As for the strong negation "$\sim$", it has an ambiguous behaviour. In fact for any Heyting algebra $\mathbf{H}$ we have (by easy inspection):

$$1 \geq a \vee \dashv a \geq a \vee \neg a = a \vee \sim a \tag{9.6.3}$$

$$0 \leq a \wedge \neg a \leq a \wedge \dashv a = a \wedge \sim a \tag{9.6.4}$$

Hence, $\sim$ coincides with the dual weak negation with respect to the *excluded middle* formula, whereas it coincides with the weak negation with respect to the *law of contradiction* formula (notice that the above relations are generalisation of (6.2.22) and (6.2.23)). We can notice

that $\sim a = {}_{\lrcorner}a$ only if $a_2 = \neg a_1$ in $\mathbf{H}$, while $\sim a = {}_{\urcorner}a$ only if $a_1 = \neg a_2$. Therefore, the three negations collapse only if they are applied to elements a such that $a_1 = \neg a_2$ and $a_2 = \neg a_1$. It follows that the three negations coincide when they are applied to elements of the form ${}_{\urcorner}{}_{\urcorner}a$ (i.e. of the form $\langle \neg a_2, \neg\neg a_2 \rangle$) or of the form ${}_{\lrcorner}{}_{\lrcorner}a$ (i.e. of the form $\langle \neg\neg a_1, \neg a_1 \rangle$). Since for any $a \in N_\Theta(\mathbf{A})$, ${}_{\lrcorner}{}_{\lrcorner}{}_{\lrcorner}{}_{\lrcorner}a = {}_{\urcorner}{}_{\urcorner}{}_{\lrcorner}{}_{\lrcorner}a = {}_{\lrcorner}{}_{\lrcorner}a$ and ${}_{\urcorner}{}_{\urcorner}{}_{\urcorner}{}_{\urcorner}a = {}_{\lrcorner}{}_{\lrcorner}{}_{\urcorner}{}_{\urcorner}a = {}_{\urcorner}{}_{\urcorner}a$, in view of the discussion from Subsection 7.3.1 we call ${}_{\urcorner}{}_{\urcorner}a$ a "*regular element*" and ${}_{\lrcorner}{}_{\lrcorner}a$ "*co-regular element*" (or "regular elements" in general).

In particular the three negations collapse if applied to elements a such that a_2 is the Boolean complement of a_1. Clearly, we shall call such kind of elements "*exact elements*" and we know that they belong to the center $\mathcal{CTR}(\mathbf{N}_\Theta(\mathbf{H}))$.

To end this discussion about the three forms of negation, notice that in view of the definition of "${}_{\lrcorner}$" it is clear that for any element $a \in N_\Theta(\mathbf{H})$, ${}_{\lrcorner}\, a$ is the largest element x such that $a \wedge x \triangleq 0$. However, ${}_{\lrcorner}\, a$ is not the pseudo-complement of a because, indeed, in general $a \wedge {}_{\lrcorner}\, a \neq 0$ as, in general, $\longrightarrow$ is not a relative pseudo-complementation, as we have seen in Example 6.2.1. Neither ${}_{\urcorner}$ is, because in general $a \wedge {}_{\urcorner}\, a \neq 0$, as in general neither $\supset$ is a relative pseudo-complementation.

As we have seen, things change in a subtle but important manner when $\mathbf{H}$ is a Boolean algebra (that is, when $N_\Theta(\mathbf{H})$ is semi-simple). In this case, $\supset$ turns into a relative pseudo-complementation so that ${}_{\urcorner}$ turns into a pseudo-complementation. Moreover, in semi-simple Nelson algebras the first inequalities in (9.6.3) and (9.6.3) turn into equalities.[13]

Now we can analyse the differences between $\mathbf{N}_\Delta(\mathbf{H})$ and $\mathbf{N}_\Theta(\mathbf{H})$ by means of the operator $\mathbf{T}$. They may be synthesised as follows:

1. For any $a \in N_\Theta(\mathbf{H})$, ${}_{\lrcorner}{}_{\lrcorner}a = \mathbf{T}(a) \leq {}_{\urcorner}{}_{\urcorner}a$.

2. For any $a \in N_\Theta(\mathbf{H})$, $\sim {}_{\urcorner}{}_{\urcorner}a = {}_{\lrcorner}{}_{\lrcorner} \sim a = \mathbf{T}(\sim a) \leq \sim \mathbf{T}(a) = \sim {}_{\lrcorner}{}_{\lrcorner}a = {}_{\urcorner}{}_{\urcorner} \sim a$.

3. For any $a \in N_\Delta(\mathbf{H})$, $\mathbf{T}(\sim a) = \sim \mathbf{T}(a)$ (so that the inequalities in 1 and 2 turn into equalities).[14]

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[13] And, in fact, we use $\supset$, ${}_{\urcorner}$ and ${}_{\urcorner}$ instead of $\longrightarrow$, ${}_{\lrcorner}$ and ${}_{\urcorner}$.

[14] We could set a new operator $\mathbf{T}'(a) = \langle \neg\neg a_1, \neg\neg a_2 \rangle$. In this case we have: $\mathbf{T}'(\sim a) = \sim \mathbf{T}'(a)$ also in $\mathbf{N}_\Theta(\mathbf{H})$, but ${}_{\lrcorner}{}_{\lrcorner}a \leq \mathbf{T}'(a) \leq {}_{\urcorner}{}_{\urcorner}a$. Evidently, in Effective Lattices the operators $\mathbf{T}'$ and $\mathbf{T}$ coincide.

Let us stress that this difference is induced by the fact that in $\mathbf{N}_\Theta(\mathbf{H})$, $\neg\neg a_1 \leq \neg a_2$ and $\neg\neg a_2 \leq \neg a_1$, any a, while in $\mathbf{N}_\Delta(\mathbf{H})$, $\neg\neg a_2 = \neg a_1$ (hence $\neg\neg a_1 = \neg a_2$).[15] In turn the latter equation depends on the fact that in $N_\Delta(\mathbf{H})$ we filter the ordered pairs by means of the least Boolean congruence Δ of $\mathbf{H}$. In fact, this congruence is induced by the filter $\uparrow d$ generated by the least dense element of $\mathbf{H}$, d, hence $\langle x_1, x_2 \rangle \in N_\Delta(\mathbf{H})$ only if $x_1 \vee x_2 \geq d$.

On the contrary in $N_\Theta(\mathbf{H})$ the congruence Θ may be induced by a filter $\uparrow b$ for $b \lneq d$. But if $a_1 \vee a_2 \ngeq d$ then $d \wedge \neg(a_1 \vee a_2) = \delta$, for some element $\delta \neq 0$. Since in any Heyting algebra $d \leq a_1 \vee \neg a_1$ and $\delta \leq d$ but $a_1 \wedge \delta = 0$, we obtain $\delta \leq \neg a_1$. For a symmetric reason, we have $\delta \leq \neg a_2$.[16]

Thus $\neg\neg a_2 \wedge \delta = 0$ while $\neg a_1 \wedge \delta = \delta$ (symmetrically, $\neg\neg a_1 \wedge \delta = 0$ and $\neg a_2 \wedge \delta = \delta$). It follows that if $a_1 \vee a_2 \ngeq d$ then $\neg\neg a_2 \lneq \neg a_1$ and $\neg\neg a_1 \lneq \neg a_2$.

Therefore in Effective Lattices, $\mathbf{T}(\langle a_1, a_2 \rangle) = \langle \neg a_2, \neg a_1 \rangle$, while in general $\langle \neg a_2, \neg a_1 \rangle$ is not even an element of Nelson lattices because, as already noticed, both $\neg a_1$ and $\neg a_2$ are greater than or equal to $d \wedge \neg(a_1 \vee a_2)$. At a semantic level, this means that if a formula α is interpreted on a, then neither α nor $\sim \alpha$ is decided by δ.

On the contrary, if $a_1 \vee a_2 \geq d$, then $d \wedge \neg(a_1 \vee a_2) = 0$, so that there are no elements below d (except 0) which do not decide either α or $\sim \alpha$. This means that for all $x \in \mathbf{H}$ ($x \neq 0$), there is a $y \leq x$ such that either $[\![\alpha]\!] = y$ or $[\![\sim \alpha]\!] = y$ (where $[\![\alpha]\!]$ is an interpretation of the formula α onto $\mathbf{H}$) (see Frame 10.16.3 for a proof-theoretical and parallel algebraic argument).

9.6.2 Local Validity and Mixed Logical Behaviour

So we have seen that the above differences reflect those between Δ and all the other Boolean congruences Θ of $\mathbf{H}$.

If we remember the relationship between dense elements in Heyting algebras and classical tautologies, then we understand why filtering the ordered pairs via Δ is essential in order to make $\mathbf{T}$ represent a classical validity operator: the prerequisite is that classically valid formulas are mapped onto dense elements. For instance, $\mathbf{T}(a\vee \sim a) = 1$ in $\mathbf{N}_\Delta(\mathbf{H})$, for any a, because $a_1 \vee a_2$ is always a dense element in $\mathbf{N}_\Delta(\mathbf{H})$. In turn,

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[15] Indeed $\neg\neg a_2 = \neg a_1$ iff $\neg\neg a_1 = \neg\neg\neg a_2 = \neg a_2$.

[16] Alternatively we obtain this result by observing that $\delta = d \wedge \neg a_1 \wedge \neg a_2$.

this is true because $a_1 \vee a_2 \cong_\Delta 1$.

Clearly, for any other Boolean congruence this is not uniformly true.

Suppose now that a Boolean congruence Θ on $\mathbf{H}$ is induced by a principal filter $\uparrow b$, with b an arbitrary element of $\mathbf{H}$ less than or equal to d. From a slight generalisation of the previous argument, it follows that in $\mathbf{N}_\Theta(\mathbf{H})$ any classical tautology $\mathcal{T}$ is mapped onto an element $[\![\mathcal{T}]\!] \geq \langle b, 0 \rangle$.

Therefore, $\langle b, 0 \rangle$ is an element that *locally* validates classical tautologies.

Dually, any classical contradiction $\mathcal{C}$ is mapped onto an element $[\![\mathcal{C}]\!] \leq \langle 0, b \rangle$.

Considering the principle of Excluded Middle as a prototype of classical tautologies and its dual as a prototype of contradictions, we have:

$$1 \geq a \vee \lrcorner a \geq a \vee \neg a = a \vee \sim a \geq \langle b, 0 \rangle \tag{9.6.5}$$

$$0 \leq a \wedge \neg a \leq a \wedge \lrcorner a = a \wedge \sim a \leq \langle 0, b \rangle \tag{9.6.6}$$

Otherwise stated, $\langle b, 0 \rangle$ is a *local (classical) top element*, and $\langle 0, b \rangle$ is a *local (classical) bottom element* so that we could think of a "local" application of the operator $\mathbf{T}$, whereas in $\mathcal{E}_0$ we have a uniform and global application since b, in that case, is the least dense element of $\mathbf{H}$.

Example 9.6.2. Inequalities (9.6.5)

In the Nelson lattice $N_{\equiv_{J^b}}(\mathbf{A})$ of Example 9.6.1, we have:
(a) $1 \geq \langle e, 0 \rangle = \langle b, a \rangle \vee \langle c, b \rangle = \langle b, a \rangle \vee \lrcorner \langle b, a \rangle \geq \langle b, a \rangle \vee \neg \langle b, a \rangle = \langle b, a \rangle \vee \langle a, b \rangle = \langle b, a \rangle \vee \sim \langle b, a \rangle = \langle d, 0 \rangle \geq \langle b, 0 \rangle$;
(b) $0 \leq \langle 0, e \rangle = \langle a, b \rangle \wedge \langle b, c \rangle = \langle a, b \rangle \wedge \neg \langle a, b \rangle \leq \langle 0, d \rangle = \langle a, b \rangle \wedge \lrcorner \langle a, b \rangle = \langle a, b \rangle \wedge \sim \langle a, b \rangle \leq \langle 0, b \rangle$.

Finally, notice that since $\mathbf{A}$ is not a Boolean algebra, the lattices $N_{\equiv_{J^b}}(\mathbf{A})$ and $\mathcal{RS}_b(\mathbf{A})$ are not isomorphic (in opposition to *Corollary* 9.6.1). For instance, both $\langle 1, d \rangle$ and $\langle d, d \rangle$ belong to $RS_b(\mathbf{A})$. However, $srn(\langle 1, d \rangle) = srn(\langle d, d \rangle) = \langle d, 0 \rangle$. More precisely, all decreasing pairs in $\mathcal{RS}_b(\mathbf{A})$, $\langle \delta, x \rangle$, with δ a dense element, have the same srn image $\langle x, \emptyset \rangle$ onto $N_{\equiv_{J^b}}(\mathbf{A})$.

In particular, it is not difficult to verify that in $\mathbf{N}_\Theta(\mathbf{H})$ the following holds true:

$$\mathbf{T}(a \vee \sim a) = \mathbf{T}(a \vee \neg a) = \langle \neg \delta, \delta \rangle \leq \langle 1, 0 \rangle = \mathbf{T}(a \vee \lrcorner a) \tag{9.6.7}$$

$$\mathbf{T}(a \wedge \sim a) = \mathbf{T}(a \wedge \neg a) = \mathbf{T}(a \wedge \lrcorner a) = \langle 0, 1 \rangle \tag{9.6.8}$$

Therefore, since $\delta = 0$ in $N_\Delta(\mathbf{H})$, we have again that classical tautologies are mapped by $\mathbf{T}$ into $\langle 1, 0 \rangle$.[17]

It is, however, very interesting to note that the element $\langle b, 0 \rangle$ is the least dense element of $\mathbf{N}_\Theta(\mathbf{H})$, whatever b is. Thus, $\langle b, 0 \rangle$ is indeed an intermediate value in the sense of Chain Based Lattices (Figure 9.1).
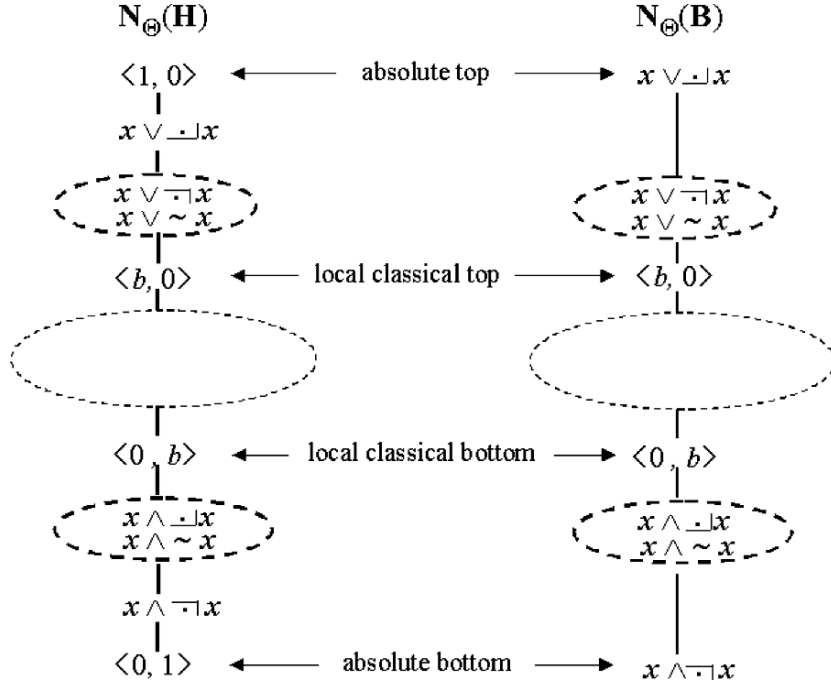

Figure 9.1: Absolute and local values in general, $\mathbf{N}_\Theta(\mathbf{H})$, and in Boolean case, $\mathbf{N}_\Theta(\mathbf{B})$

9.7   Conclusions

To sum up, we can claim that *by means of the analysis of Effective lattices and $\mathcal{E}_0$ system*, the deep logical meaning of Boolean congruences in the construction of Nelson lattices becomes transparent.

We can apply this intuition to account for informational situations in the spirit of the present Part.

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[17]Compare with $\mathbf{T}'(a \vee \sim a) = \mathbf{T}'(a \vee \neg a) = \langle \neg\delta, 0 \rangle \leq \langle 1, 0 \rangle = \mathbf{T}'(a \vee \dashv a)$ and $\mathbf{T}'(a \wedge \sim a) = \mathbf{T}'(a \wedge \dashv a) = \langle 0, \neg\delta \rangle \geq \langle 0, 1 \rangle = \mathbf{T}'(a \wedge \neg a)$.

Imagine $\mathbf{D}$ as any distributive lattice of subsets of a given domain D, representing in some way a cognitive status about D. For any sentence α about this domain, we can distinguish between the set A_1 of elements in D that definitely satisfy α, and the set A_2 of elements in D that definitely do not satisfy α. Since our knowledge may be incomplete, generally $A_1 \cup A_2 \subseteq D$. Indeed, there can be elements x such that we are not (yet) able to decide if $x \in A_1$ or $x \in A_2$. But our cognitive status may be complete *with respect to some elements* of D. Let us denote by B the set of all the elements of D for which we have a complete knowledge. Therefore, (i) $A_1 \cap A_2 = \emptyset$ (A_1 and A_2 must be disjoint), (ii) $A_1 \cup A_2 \subseteq D$ (our knowledge may be incomplete), but (iii) $A_1 \cup A_2 \supseteq B$ (because for any element b of B we are able to *decide* if either $b \in A_1$ or $b \in A_2$). It follows that in order to represent our sentences about D, we cannot take all the ordered pairs $\langle A_1, A_2 \rangle$ of disjoint elements of $\mathbf{D}$, but we have to filter them out, by means of the filter generated by B, $\uparrow B$.

Then $N_{\equiv_{J^b}}(\mathbf{D})$ might be considered well-suited for algebraically modeling this situation.

Unfortunately, as far as we know there are not yet suitable applications of this machinery, when $\mathbf{D}$ is an arbitrary distributive lattice and if by "suitable" we mean a natural description of a cognitive status.

> On the contrary, if $\mathbf{D}$ is a Boolean algebra, we have an application with high degree of "naturalness": Rough Set Systems

Chapter 10

Frames (Part II)

10.1 Frame – Rough Set Systems and Chain-Based Lattices

The equivalence between semi-simple Nelson algebras and three-valued Łukasiewicz algebras was stated in [Monteiro, 1967].

The transformation stated in Example 6.5.1 at point 5 is supported by the following results from [Epstein & Horn, 1974a]):

Lemma 10.1.1. *If* $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0 = e_0 \leq \ldots \leq e_{n-1} = 1 \rangle$ *is a pseudo-complemented* P_0*-lattice, then* $\mathbf{A}$ *has a chain base* $f_0 \leq f_1 \leq \ldots \leq f_{n-1}$ *s.t.* f_1 *is the smallest dense element of* $\mathbf{A}$.

Lemma 10.1.2. *If* $\mathbf{A} = \langle A, \vee, \wedge, \neg, \Longrightarrow, 0 = e_0 \leq \ldots \leq e_{n-1} = 1 \rangle$ *is a* P_0*-lattice and* $\langle A, \vee, \wedge, \neg, \Longrightarrow, 0, 1 \rangle$ *is a Heyting algebra, then there exists a chain base* $0 = f_0 \leq f_1 \leq \ldots \leq f_{n-1} = 1$ *s.t.* $\langle A, \vee, \wedge, \neg, \rightarrow, 0 = f_0 \leq f_1 \leq \ldots \leq f_{n-1} = 1 \rangle$ *is a* P_1*-lattice.*

In order to get the required chain base one can refer to *Lemma* 10.1.1, taking as f_1 the first dense element of $\mathbf{A}$, and inductively as f_{n+1} the smallest dense element of the convex interval $[f_n, 1]$.

Finally, a general relationship between chain-based lattices is stated in the following result:

Lemma 10.1.3. *Every* P_2*-lattice is a principal ideal in a Post algebra.*

As a matter of fact, this result is an instance of *Proposition* 7.4.1.(3) and *Corollary* 7.4.3, as the reader is invited to prove in the following exercise:

Exercise 10.1. *Prove Lemma 10.1.3 in the case of $\mathcal{RS}_x(\mathbf{B})$, $\mathbf{B}$ any Boolean algebra, and the isomorphism stated in Lemma 7.4.1.(3) [hints: first, consider the principal ideal $\downarrow \langle 1, \neg x \rangle$ and define a map p such that $p(a) = min\{b \in \mathcal{RS}_x(\mathbf{B}) \ s.t. \ b \geq a \ in \ \mathcal{P}(\mathbf{B})\}$; second, prove that $p(a) = J^{\langle 1, \neg x \rangle}(a)$; third, define the lower adjoint of p].*

It follows, from *Exercise* 10.1 that for any Boolean algebra $\mathbf{B}$ and $x \in \mathbf{B}$, $\mathcal{RS}_x(\mathbf{B})$, which is isomorphic to the principal ideal $\downarrow \langle 1, \neg x \rangle$ of the Post algebra $\mathcal{P}(\mathbf{B})$, is a P_2-lattice of order three. In $\mathcal{P}(\mathbf{B})$, $e_0 = \langle 0, 0 \rangle$, $e_1 = \langle 1, 0 \rangle$ and $e_2 = \langle 1, 1 \rangle$, while the chain of values in $\mathcal{RS}_x(\mathbf{B})$ is $f_0 = p(e_0) = e_0$, $f_1 = p(e_1) = \langle 1, x \rangle$ and $f_2 = p(e_2) = p(\langle 1, \neg x \rangle) = e_2$, where p is the map defined in *Exercise* 10.1.

Example 10.1.1. P_2-algebras by filtering Post algebras

The reader can notice that the lattice $\mathcal{RS}_b(\mathbf{B})$ of Example 7.4.1 consists of the images of the map p provided by *Exercise* 10.1. In the diagram below, $\mathcal{RS}_b(\mathbf{B})$ is shown embedded in bold fonts in the Post algebra $\mathcal{P}(\mathbf{B})$ (alias $\mathcal{RS}_0(\mathbf{B})$). The arrows show the effect of the map p. The image Im_p of p coincides with the set of fixpoints, in the Post algebra $\mathcal{P}(\mathbf{B})$), of the Lawvere-Tierney operator $J^{\langle 1, a \rangle}$, i.e. $J^{\langle 1, \neg b \rangle}$, and it is isomorphic to the principal ideal $\downarrow \langle 1, a \rangle$ of $\mathcal{P}(\mathbf{B})$, as one can readily see from the diagram:
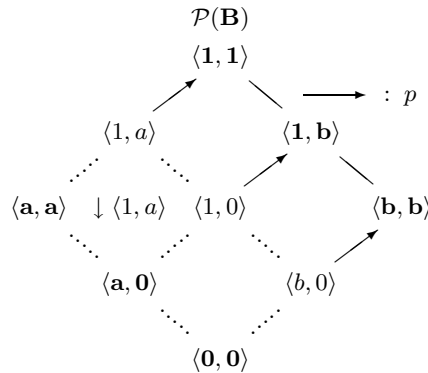
Now let us see an application of the adjoint map p^*: $p^*(\langle 1, b \rangle) = \bigwedge\{p^{\leftarrow}(\uparrow \langle 1, b \rangle)\} = \bigwedge\{p^{\leftarrow}(\{\langle 1, b \rangle, \langle 1, 1 \rangle\})\} = \bigwedge\{\langle 1, 1 \rangle, \langle 1, b \rangle, \langle 1, a \rangle, \langle 1, 0 \rangle\} = \langle 1, 0 \rangle$. Incidentally, notice that $J^{\langle G, \neg B \rangle}(\langle G, \emptyset \rangle) = p(\langle G, \emptyset \rangle)$ and that p^* recovers the intermediate element of the Post algebra, $\langle G, \emptyset \rangle$, from the intermediate value of the P_2-lattice.

Let us notice that the element $\langle 0, 0 \rangle$ does not belong to $\mathcal{RS}_x(\mathbf{B})$ if $\equiv_{J^x}$ is not trivial. The map p provides then the way to recover a new intermediate value f_1 according to the results of *Lemma* 10.1.1 and *Lemma* 10.1.2. Indeed, $p(e_1) = \langle 1, x \rangle$ which is the least dense element in the interval $[\langle 0, 0 \rangle, \langle 1, 1 \rangle]$ of $\mathcal{RS}_x(\mathbf{B})$. It must be noticed that here "dense" must be understood with respect to the operation "$\neg$", that is the pseudo-complementation of $\mathcal{RS}_x(\mathbf{B})$ *qua* Heyting algebra. In fact, $\neg\langle 1, x \rangle = \langle \neg 1, \neg 1 \rangle = \langle 0, 0 \rangle$ and if $\langle \neg a_1, \neg a_1 \rangle = \langle 0, 0 \rangle$ then $\langle a_1, a_2 \rangle \geq \langle 1, x \rangle$, because (i) surely $a_1 = 1$ in order to have $\neg a_1 = 0$ (1 is the only dense element in the Boolean algebra $\mathbf{B}$); (ii) from the filtration condition we have $1 \wedge x = a_2 \wedge x$, so that $x = a_2 \wedge x$. Hence, $a_2 \geq x$.

Exercise 10.2. *Is the operation $\rightarrowtail$ a relative pseudo-complementation in any semi-simple Nelson algebra?*

10.2   Frame – Rough Set Systems as Regular Double Stone Algebras

Definition 10.2.1.

- *A bounded pseudo-complemented lattice* $\mathbf{L} = (L, \wedge, \vee, *, 0, 1)$ *is a* Stone algebra *if* $\forall x \in L$, $x^* \vee x^{**} = 1$.

- *A dual-order pseudo-complemented lattice* $\mathbf{L} = (L, \wedge, \vee, ^+, 0, 1)$ *is a* dual Stone algebra *if* $\forall x \in L$, $x^+ \wedge x^{++} = 0$.

- *A distributive lattice* $\mathbf{L}$ *is a* double Stone algebra *if it is both a Stone and a dual Stone algebra.*

- *A double Stone algebra* L *is said* regular *if* $\forall x, y \in L$, $x \wedge x^+ \leq y \vee y^*$.

From (6.2.22) and (6.2.23) it is immediate to prove that for any Boolean algebra $\mathbf{B}$ and $x \in \mathbf{B}$, $\langle \mathcal{RS}_x(\mathbf{B}), \wedge, \vee, \rightharpoonup, 0, 1 \rangle$ is a Stone algebra and $\langle \mathcal{RS}_x(\mathbf{B}), \wedge, \vee, \neg, 0, 1 \rangle$ is a dual Stone algebra, so that $\langle \mathcal{RS}_x(\mathbf{B}), \wedge, \vee, \rightharpoonup, \neg, 0, 1 \rangle$ is a double Stone algebra.[1]

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[1]Actually, it is well-known that any P_2-lattice (of order n) is a Stone lattice (of order n).

10.3   Frame – Information-Oriented Duality Theorems

10.3.1   Information-Oriented Interpretation of Duality

We have seen in Section 7.2 the duality between partial orders and finite distributive lattices. Now, instead of a partial order let us consider a preordered set $\mathbf{P} = \langle P, \preceq \rangle$. If we take $\mathbf{F}(\mathbf{P})$ we obtain a distributive lattice. However, $\mathbf{J}(\mathbf{F}(\mathbf{P}))$ is not isomorphic to $\mathbf{P}$ but to the quotient space $\mathbf{P}/_{\equiv} = \langle P/_{\equiv}, \preceq' \rangle$, where for all $p, p' \in P$, $p \equiv p'$ if and only if $p \preceq p'$ and $p' \preceq p$ and $[x]_{\equiv} \preceq' [y]_{\equiv}$ if and only if $x \preceq y$.

Vice-versa, if $\mathbf{L} = \langle L, \leq \rangle$ is a lattice of subsets of a given universe U, we can define a preordered set $\mathbf{S}(\mathbf{L}) = \langle U, \preceq \rangle$ where $a \preceq a'$ if and only if for all $A \in L$, $a \in A$ implies $\Rightarrow a' \in A$. Then we have that $\mathbf{F}(\mathbf{S}(\mathbf{L})) = \mathbf{L}$. However, $\mathbf{J}(\mathbf{F}(\mathbf{S}(\mathbf{L})))$ (i.e. $\mathbf{J}(\mathbf{L})$) is not order isomorphic to $\mathbf{S}(\mathbf{L})$, but to $\mathbf{S}(\mathbf{L})/_{\equiv}$.

It has seen observed in the Introduction that if $\mathbf{L}$ is the frame of open subsets of a topology on U, then $\preceq$ is the so-called *specialization preorder*. Moreover $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ is the *soberification* of $\mathbf{L}$ (which coincide with the T_0-ification in the finite case), in the sense that superfluous points are disposed off. The same result can be achieved by (i) taking the specialization preorder $\mathbf{S}(\mathbf{L})$, (ii) factorising $\mathbf{S}(\mathbf{L})$ through $\equiv$, thus obtaining a partial order, and (iii) taking the dual of the resulting partial order.

The informational content of the two maneuvers is the same. In fact, if any member X of L is seen as a property fulfilled (or "inhabited") by some elements of U, then the corresponding element $\mathbf{F}(\mathbf{J}(\mathbf{L}))$, i.e. $\phi(x)$, will be inhabited not by the same number of elements as X, but by "champions" (that is, sets) representing them, so that the elements of U that uniformly fulfill the same properties in $\mathbf{L}$ are represented by the same "champion" in $\mathbf{F}(\mathbf{J}(\mathbf{L}))$. It follows that $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ and $\mathbf{L}$ are isomorphic but not homeomorphic *qua* topological spaces. To put it another way, they have the same geometrical shape but the elements of $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ are less populated than the members of $\mathbf{L}$, because they are inhabited by that strict number of abstract points which is required to distinguish one property from the others.

Analogously, a is specialized by a' (i.e. $a \preceq a'$) if any property fulfilled by a is fulfilled by a' too, so that a' fulfills at least the same properties as a (hence a' is more *determined* than a). Thus $a \equiv a'$ if

and only if they fulfill exactly the same properties. It follows that the factorisation $\mathbf{S}(\mathbf{L})/_\equiv$ makes a and a' collapse into the same "champion" (that is, equivalence class), so that $\mathbf{F}(\mathbf{S}(\mathbf{L})/_\equiv)$ and $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ are not only isomorphic, but also homeomorphic.
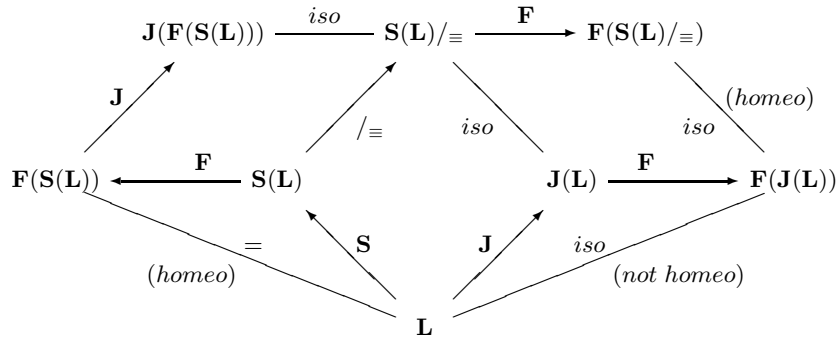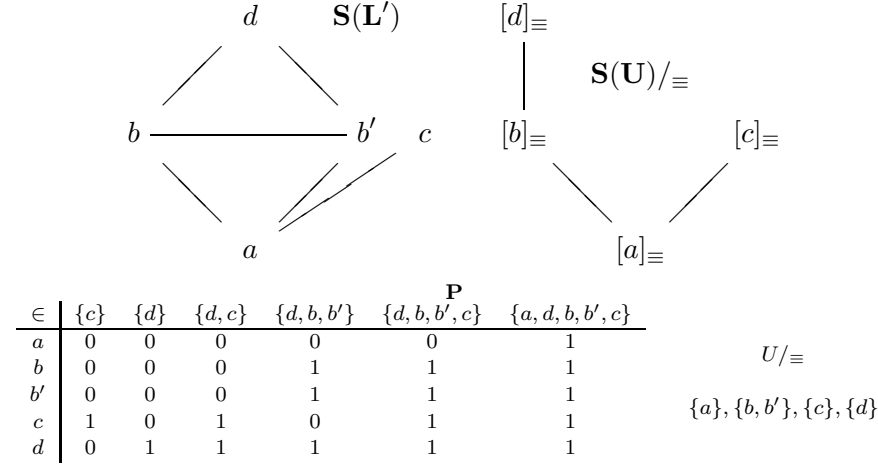

If $\mathbf{L}$ is a Boolean algebra of sets, then the specialization preorder is an equivalence relation. This means that given an Approximation Space $\mathbf{A}$ we can recover the Indiscernibility Space it is induced by, just by taking $\mathbf{S}(\mathbf{A})$. Conversely, if $\mathbf{X}$ is an Indiscernibility Space, then the induced Approximation Space is $\mathbf{F}(\mathbf{X})$.

Moreover, given any finite lattice $\mathbf{L}$ of subsets of a universe U, $\langle U, \equiv \rangle$ is an Indiscernibility Space and $\mathbf{P} = \langle U, \mathbf{L}, \in \rangle$ is a *P-system* such that the quantum relation system $\mathbf{Q}(\mathbf{P}) = \mathbf{L}$, because the information quantum relation induced by $\mathbf{P}$ coincides with the specialization preorder $\preceq$. Clearly, the information quantum relation of its nominalisation, $R_{\mathcal{N}(\mathbf{P})}$ (see Part I), coincides with $\equiv$.

EXAMPLE   Let $U = \{a, b, b', c, d\}$.

d $\mathbf{S(L')}$ $[d]_\equiv$

$\mathbf{S(U)}/_\equiv$

b —————— b' c $[b]_\equiv$ $[c]_\equiv$

a $[a]_\equiv$

$\mathbf{P}$

$\in$	$\{c\}$	$\{d\}$	$\{d,c\}$	$\{d,b,b'\}$	$\{d,b,b',c\}$	$\{a,d,b,b',c\}$
a	0	0	0	0	0	1
b	0	0	0	1	1	1
b'	0	0	0	1	1	1
c	1	0	1	0	1	1
d	0	1	1	1	1	1

$U/_\equiv$

$\{a\},\{b,b'\},\{c\},\{d\}$
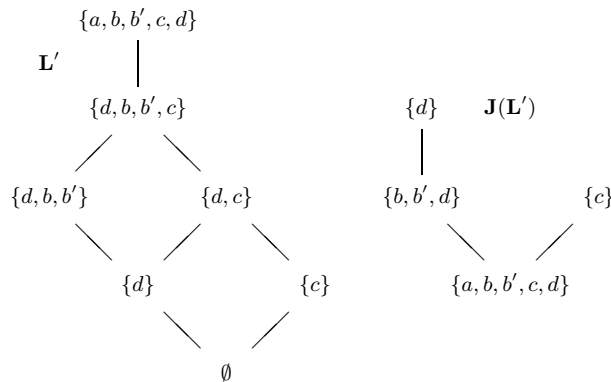
We have: $Q_d^{\mathbf{P}} = \{d\}, Q_c^{\mathbf{P}} = \{c\}, Q_b^{\mathbf{P}} = Q_{b'}^{\mathbf{P}} = \{b,b',d\}$.

10.3.2   Duality of Logico-Algebraic Structures

In Subsection 7.2.1 we have seen, given a Heyting algebra $\mathbf{H}$, how to define an isomorphic Heyting algebra on a topological space and, vice-versa.

Duality for the other algebraic structures is defined in a similar way with the addition of extra ingredients. For instance, if we have to model some form of strong negation, then we need to define an involution on dual spaces. Intuitively the involution provides us with a rule to find the strong negation of each element. Thus we first define *I-spaces*, that is, spaces with an involution (for a history of the following representation techniques see Frame 10.9.2).

Definition 10.3.1. *A finite* I-space *is a pair* $\mathcal{X} = \langle \mathbf{X}, f \rangle$ *where* $\mathbf{X}$ *is a poset and* f *is an involution on* $\mathbf{X}$, *that is, for any* $x \in X$, $f(f(x)) = x$. *If* $A \subseteq X$ *then with* $f(A)$ *we denote the direct image* $f^{\rightarrow}(A) = \{f(a)\}_{a \in A}$.
Consider then the following set of operations on $\wp(X)$:

(i) $1 = X$, *(ii)* $0 = \emptyset$, *(iii)* $A \wedge B = A \cap B$, *(iv)* $A \vee B = A \cup B$, *(v)* $\sim A = X \cap -f(A)$, *(vi)* $A \twoheadrightarrow B = \sim A \cup B$, *(vii)* $A \longrightarrow B = - \downarrow (A \cap f(A) \cap -B)$, *(viii)* $\lrcorner A = - \downarrow (A \cap f(A))$.

Definition 10.3.1 makes it possible to define Kleene and Nelson spaces. As we are going to see, the difference between them is a certain condition on the interrelations between the involution and the partial order.

Definition 10.3.2. *A* Kleene space *is an I-space* $\mathcal{X} = \langle \mathbf{X}, f \rangle$ *s.t.* f *is a linear involutive order-antiautomorphism: if* $\mathbf{X}^{op}$ *is the dual of* $\mathbf{X}$ *then* $f : \mathbf{X} \longmapsto \mathbf{X}^{op}$ *is an order-isomorphism, for all* $x \in X$, $f(f(x)) = x$ *and* $x \leq f(x)$ *or* $f(x) \leq x$.

Given a finite Kleene space $\mathcal{X} = \langle \mathbf{X}, f \rangle$ *we can define its dual Kleene algebra* $\mathbf{K}(\mathcal{X})$ *in the following way:* $\mathbf{K}(\mathcal{X}) = \langle \mathbf{F}(\mathbf{X}), \wedge, \vee, \twoheadrightarrow, \sim, 0, 1 \rangle$, *where for any* $A, B \in \mathbf{F}(\mathbf{X})$ *the operations are defined as in 10.3.1 (i)–(vi).*

On the other way, given a finite Kleene algebra $\mathbf{A} = \langle A, \wedge, \vee, \twoheadrightarrow, \sim, 0, 1 \rangle$ *we can recover its dual Kleene space* $\mathbf{KS}(\mathbf{A})$ *as follows:*

$\mathbf{KS}(\mathbf{A}) = \langle \mathbf{J}(\mathbf{A}), f \rangle$, *where the involution* f *is given by* $f(x) = min(\mathbf{J}(\mathbf{A}) \cap -\{\sim b : b \in \uparrow x\})$.

We have that if $\mathbf{A}$ *is a finite Kleene algebra and* $\mathcal{X}$ *is a finite Kleene space, then* $\mathbf{A} \cong_k \mathbf{K}(\mathbf{KS}(\mathbf{A}))$ *and* $\mathcal{X} \cong_p \mathbf{KS}(\mathbf{K}(\mathcal{X}))$, *where* $\cong_k$ *is the Kleene algebras isomorphism.*

Eventually, we can give the dual construction for Nelson algebras:

Definition 10.3.3. *A finite* Nelson space *is a finite Kleene space* $\mathcal{X} = \langle \mathbf{X}, f \rangle$ *s.t. for any* $a, b \in X$ *the following interpolation property is fulfilled:*

$$\text{(IN) if } a \geq f(a), b \geq f(a), a \geq f(b), b \geq f(b),$$
$$\text{then } \exists c \in X \text{ s.t. } c \leq a, c \leq b, f(a) \leq c, f(b) \leq c.$$

If $\mathbf{A} = \langle A, \wedge, \vee, \longrightarrow, \dot{-}, \sim, 0, 1 \rangle$ *is a finite Nelson Algebra, then we can recover its dual Nelson space* $\mathbf{NS}(\mathbf{A})$ *as before. On the other side, if* $\mathcal{X} = \langle \mathbf{X}, f \rangle$ *is a finite Nelson space then we can define its dual Nelson algebra* $\mathbf{N}(\mathcal{X})$ *as follows:* $\mathbf{N}(\mathcal{X}) = \langle \mathbf{J}(\mathbf{X}), \wedge, \vee, \longrightarrow, \dot{-}, \sim, 0, 1 \rangle$, *where for any* $A, B \in \Omega(\mathbf{X})$ *the operations are defined as in 10.3.1 (i)–(viii).*

We have that if $\mathbf{A}$ *is a finite Nelson algebra and* $\mathcal{X}$ *is a finite Nelson space, then* $\mathbf{A} \cong_n \mathbf{N}(\mathbf{NS}(\mathbf{A}))$ *and* $\mathcal{X} \cong_p \mathbf{NS}(\mathbf{N}(\mathcal{X}))$, *where* $\cong_n$ *is the Nelson algebras isomorphism.*

It is now important to notice that Nelson and Kleene spaces can be split into two parts and that Kleene and Nelson algebras can be constructed as ordered pairs of subsets of elements of just one part:

Definition 10.3.4. *Given an I-space* $\mathcal{X}$, *let* X^+ *be the set* $\{x \in X : x \leq f(x)\}$ *and* $\mathbf{X}^+$ *be* $\langle X^+, \leq \rangle$ *where* $\leq$ *is the partial order inherited from*

X. $\mathbf{X}^+$ *will be called the "positive part" of* $\mathcal{X}$*. Consider the function* k_f:

$$k_f : \wp(X) \longmapsto \wp(X^+) \times \wp(X^+) : k_f(A) = \langle X^+ \cap A, X^+ \cap -f(A) \rangle.$$

Notice that the index recalls that this map depends on the involution f*. On* $\mathbf{Im}_{k_f}$ *we can define operations analogous to those of Definition 9.6.1. Namely:*

(i) $1 =_{def} \langle X^+, \emptyset \rangle$*; (ii)* $0 =_{def} \langle \emptyset, X^+ \rangle$*; (iii)* $\sim \langle X_1, X_2 \rangle =_{def} \langle X_2, X_1 \rangle$*; (iv)* $\langle X_1, X_2 \rangle \wedge \langle Y_1, Y_2 \rangle =_{def} \langle X_1 \wedge Y_1, X_2 \vee Y_2 \rangle$*, (v)* $\langle X_1, X_2 \rangle \vee \langle Y_1, Y_2 \rangle =_{def} \langle X_1 \vee Y_1, X_2 \wedge Y_2 \rangle$*, (vi)* $\langle X_1, X_2 >\!\!\twoheadrightarrow \langle Y_1, Y_2 \rangle =_{def} \langle X_2 \vee Y_1, X_1 \wedge Y_2 \rangle =\sim \langle X_1, X_2 \rangle \vee \langle Y_1, Y_2 \rangle$*; (vii)* $\langle X_1, X_2 \rangle \longrightarrow \langle Y_1, Y_2 \rangle =_{def} \langle X_1 \Longrightarrow Y_1, X_1 \wedge Y_2 \rangle$*; (viii)* $\mathbin{\rotatebox[origin=c]{180}{$\sqsupset$}}\langle X_1, X_2 \rangle =_{def} \langle \neg X_1, X_1 \rangle = \langle X_1, X_2 \rangle \longrightarrow 0$*, where inside the ordered pairs the operations listed in Definition 7.2.1.(i)-(vi) are used.*

We obtain the following result:

Proposition 10.3.1. *Let* $\mathcal{X} = \langle \mathbf{X}, f \rangle$ *be an I-space where* f *is a linear, involutive antiautomorphism. Then:*

1. $\mathbf{kK}(\mathcal{X}) = \langle k_f(\mathbf{F(X)}), \wedge, \vee, \twoheadrightarrow, \sim, 0, 1 \rangle$ *is a Kleene algebra.*

2. *If in addition* f *satisfies* (IN), *then* $\mathbf{kN}(\mathcal{X}) = \langle k_f(\mathbf{F(X)}), \wedge, \vee, \longrightarrow, \sim, \mathbin{\rotatebox[origin=c]{180}{$\sqsupset$}}, 0, 1 \rangle$ *is a Nelson algebra,*

provided that the operations are those of Definition 10.3.4.

Proof. Clearly $\mathbf{X}^+$ is a partial order. Thus the first thing to prove is that if A is an order filter in $\mathbf{X}$ then $A \cap X^+$ is a order filter in $\mathbf{X}^+$. Indeed, let $a \in A \cap X^+$, and $a' \in X^+$; if $a \leq a'$ then $a' \in A$, too, because A is an order filter. Hence $a' \in A \cap X^+$. Moreover from the conditions on the involution f we immediately obtain that $\mathbf{kK}(\mathcal{X})$ is isomorphic to $\mathbf{K}(\mathcal{X})$ and $\mathbf{kN}(\mathcal{X})$ is isomorphic to $\mathbf{N}(\mathcal{X})$.   QED

In view of the above proposition, let $\mathbf{Z} = \langle Z, \leq \rangle$ be a Heyting space. Then in order to obtain a Nelson algebra we can proceed as follows:

Definition 10.3.5. *(a) Let* $S \subseteq maximal(\mathbf{Z})$*. Take an opposite copy (from the point of view of order) of* $Z \cap -S$*,* $\mathbf{Z}' = \langle Z', \preccurlyeq \rangle$*, that is,* $Z' = \{h(x) : x \in Z \cap -S\}$*, and* $h(x) = x'$ *is an anti-order-isomorphism between* x *and* Z'*, for* $x \in Z \cap -S$*.*
(b) Glue $\mathbf{Z}$ *and* $\mathbf{Z}'$ *together by setting* $x \leq x'$ *for any* $x \in Z \cap -S$*.*

(c) Set the following involution:

$$f(x) = \begin{cases} a' & \text{if } x = a \text{ and } a \notin S \\ a & \text{if } x = a \text{ and } a \in S \\ a & \text{if } x = a' \end{cases}$$

(d) Call the new space $\mathcal{X}^S(\mathbf{Z})$. It is easy to verify that it is a Nelson space.

EXAMPLE

On the first line we show the Kleene space dual to the Kleene algebra $\mathbf{C}$ depicted at the beginning of Frame 6.2.1. The involution f is depicted in dashed lines. Here $f(c) = c$. On the second line the Nelson space $\mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))$ is depicted, that is induced by the Heyting algebra $\mathbf{A}$ of Example 6.2.1. In this space for all $x \neq a$, $f(x) = x'$ and $f(x') = x$ while $f(a) = a$. Aside the two spaces we show their positive parts, too. The reader will easily verify that $\mathbf{kN}(\mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))) = \mathbf{N}_{\equiv_{J\{a\}}}(\mathbf{J}(\mathbf{A})) \cong_n \mathbf{N}_{\equiv_{Ja}}(\mathbf{A})$.
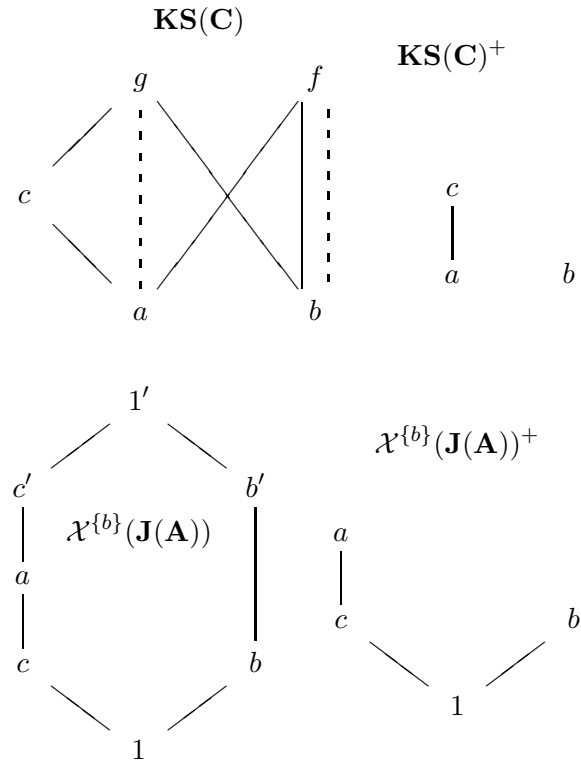
In $\mathbf{KS}(\mathbf{C})$ the interpolation condition (IN) does not hold (in between a, g, b and f there is no interpolating element. On the contrary, (IN) trivially holds in $\mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))$. Now let us define some elements of $\mathbf{K}(\mathbf{KS}(\mathbf{C}))$ and $\mathbf{N}(\mathcal{X})$. From the filter $\{b, g, f\}$ of $\mathbf{KS}(\mathbf{C})$ we have: $k_f(\{b, g, f\}) = \langle \{b, g, f\} \cap \mathbf{KS}(\mathbf{C})^+, \mathbf{KS}(\mathbf{C})^+ \cap -f(\{b, g, f\}) \rangle = \langle \{b, g, f\} \cap \{a, b, c\}, \{a, b, c\} \cap -\{a, b, f\} \rangle = \langle \{b\}, \{c\} \rangle$. From the filter $\{a, c', 1'\}$ of $\mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))$ we obtain: $k_f(\{a, c', 1'\}) = \langle \{a, c', 1'\} \cap \mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))^+, \mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))^+ \cap -f(\{a, c', 1'\}) = \langle \{a\}, \mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))^+ \cap -\{a, c, 1\} \rangle = \langle \{a\}, \{b\} \rangle$.

Exercise 10.3.
(A) *Compute: $k_f(\{c, g\})$ and $k_f(\{c, b, g, f\})$ in $\mathbf{KS}(\mathbf{C})$; $k_f(\{b, b', 1'\})$ and $k_f(\{b, a, 1', c', b'\})$ in $\mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))$.*
(B) *Verify that the Nelson algebra $\mathbf{N}_{\equiv_{J^b}}(\mathbf{A})$ of Example 9.6.1 is isomorphic to $\mathbf{kN}(\mathcal{X}^{\phi(b)}(\mathbf{J}(\mathbf{A})))$ and to $\mathbf{F}(\mathcal{X}^{\phi(b)}(\mathbf{J}(\mathbf{A})))$, where $\phi(b)$ is the image of b in $\mathbf{F}(\mathbf{J}(\mathbf{A}))$ via the isomorphism ϕ of (7.2.7) [hint: draw $\mathcal{X}^{\phi(b)}(\mathbf{J}(\mathbf{A}))$; extend the isomorphism ϕ pairwise to a function γ; define the dual function of k_f].*
(C) *Find the isomorphic image of $\langle b, a \rangle$ in $\mathbf{F}(\mathcal{X}^{\{b\}}(\mathbf{J}(\mathbf{A})))$; find the isomorphic image of $\{a', c', 1'\}$ in $\mathbf{N}_{\equiv_{J^b}}(\mathbf{A})$.*

10.3.3 Collapse of Maximal Elements and Atomic Decidability

One can verify that, for instance, the ordered pair $\langle \{b\}, \emptyset \rangle$ cannot be recovered by k_f from $\mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))$. In fact given any Nelson space $\mathcal{X}$ and any order filter A of $\mathbf{X}$, if $f(x) = x$ we have (i) $x \in X^+$ and (ii) $x \in A$ iff $x \in f(A)$. It follows that if $x \notin A \cap X^+$ then $x \notin f(A)$ so that $x \in X^+ \cap -f(A)$. If $x \notin X^+ \cap -f(A)$ then $x \in f(A)$ and, hence, $x \in A \cap X^+$. In our example $f(a) = a$, so that if $\emptyset$ is the second element of $k_f(A)$ then a must belong to the first element. This proves:

Proposition 10.3.2. *Let $\mathcal{X}$ be a Nelson or a Kleene space such that $x \in X^+$ and $f(x) = x$. Then for any order filter A of $\mathbf{X}$, if $\langle X_1, X_2 \rangle = k_f(A)$ then either $x \in X_1$ or $x \in X_2$.*

Corollary 10.3.1. *For any Heyting space $\mathbf{W} = (W, \leq)$, $S \subseteq$ maximal $(\mathbf{W})$, $\mathbf{kN}(\mathcal{X}^S(\mathbf{W})) = \mathbf{N}_{\equiv_{J^S}}(\mathbf{H}(\mathbf{W}))$, and they are Nelson algebras.*

Proof. (Sketch) Set $D = $ maximal$(\mathbf{W})$. Then D is the least dense element of the Heyting algebra $\mathbf{H}(\mathbf{W})$. Indeed, any order filter of $\mathbf{W}$ has non void intersection with D. Therefore, if $S \subseteq D$ then $\equiv_{J^s}$ is

a Boolean congruence on $\mathbf{H}(\mathbf{W})$. Finally, since in $\mathbf{kN}(\mathcal{X}^S(\mathbf{W}))$, $S = \{x : f(x) = x\}$, from *Proposition* 10.3.2 we obtain the result. QED

In view of Chapter 9 this fact has an important logical meaning.

Indeed, if $x \in W$ and there exists $x' \in W'$ such that $x' = f(x)$, then, intuitively, there may be at least an atomic formula p such that $x \not\models p$ and $x \not\models \sim p$, even if x is maximal. In fact from the definition of k_f we could have an element P such that $x \notin (k_f(P))_1, x \notin (k_f(P))_2$ (for example, in $\mathcal{X}^{\{a\}}(\mathbf{J}(\mathbf{A}))$ if $P = \{a, c', 1', b'\}$, then $k_f(P) = \langle \{a\}, \emptyset \rangle$ and $b \notin (k_f(P))_1, b \notin (k_f(P))_2$). Thus P is the algebraic value of some formula p such that neither $b \models p$ nor $b \models \sim p$. But we have seen that if $f(x) = x$ then either $x \in (k_f(A))_1$ or $x \in (k_f(A))_2$, any A. Call a possible world which is able to decide any atomic formula "complete". From the monotonicity clause of forcing, in order to know what possible worlds are complete, it is sufficient to analyse the set of maximal states of W. Suppose that the set of complete maximal possible worlds is $S \subseteq maximal(\mathbf{W})$, then we have just seen, for any $\langle A_1, A_2 \rangle$, that $A_1 \vee A_2 \equiv_{J^s} W$ only if all the elements of S are distributed between A_1 and A_2. This is tantamount to saying that for each atomic formula p for any element $s \in S$ either $s \models [\![p]\!]$ or $s \models [\![\sim p]\!]$.

If $S = maximal(\mathbf{W})$ then we have $maximal(\mathbf{W}) \subseteq [\![p]\!] \vee \sim [\![p]\!]$. Hence for any formula α, $[\![\alpha]\!] \vee \sim [\![\alpha]\!]$ is a dense element in $\mathbf{N}_{\equiv_{J^s}}(\mathbf{H}(\mathbf{W}))$ (indeed, as we have seen, $maximal(\mathbf{W})$ intersects all filter of $\mathbf{W}$).

10.3.4 Rough Sets, Duality and Decidability

In this subsection we show an example of application of the above duality construction from Approximation Spaces to Rough Set Systems. Consider the Approximation Space $\mathbf{AS}(G)$ of Example 7.4.4. First we have to build the dual space $\mathcal{X}^S(\mathbf{J}(\mathbf{AS}(G)))$ (where now $S = B = \bigcup\{X : card(X) = 1\} = \bigcup\{\{a'\}, \{a'''\}\} = \{a', a'''\}$). Thus:

The space $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))$ is built up of two separate subspaces: $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^- = \{X' : \exists X \in \mathcal{J}(\mathbf{AS}(G)) \ \& \ X \notin\downarrow \{a', a'''\}\}$ and $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+ = \mathcal{J}(\mathbf{AS}(G))$ (this set is, of course, the family of atoms of $\mathbf{AS}(G)$, that is, the family of basic classes of the Approximation Space).

The involution f is given as in *Definition* 10.3.5.

The elements of $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^-$ may be thought of as basic constituents of upper approximations (or closures) while the elements of $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+$ as basic constituents of lower approximations (or interiors).

Thus the dual space of any rough set is an order filter $\uparrow X$ where $X \subseteq \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))$. It follows that in this dual space any rough set takes the form $\overrightarrow{X} = \{X_1', X_2', \ldots, X_n', X_1, X_2, \ldots, X_m^1\}$ such that $0 \le m, 0 \le n$ and if $X_i \in \overrightarrow{X}$ and X_i' exists, then $X_i' \in \overrightarrow{X}$, too. To put it in other words, in a dual rough set space we can have primed elements, X_i' without the corresponding non-primed element, X_i, but not the other way around. This is intuitive, because, topologically, if a basic set Z belongs to $\mathbb{I}(X)$ then $Z \in \mathbb{C}(X)$, but the opposite is not always true (that is, an elementary class which is included in the upper approximation of a set X is not necessarily included in its lower approximation, but the opposite always holds true). However, if Z is a singleton, then $Z \in \mathbb{I}(X)$ if and only if $Z \in \mathbb{C}(X)$. This is the reason why we do not have primed copies of singleton elementary sets.

In order to obtain rough sets in disjoint representation we have to apply the following steps:

(A.1) apply the function k_f to any $\overrightarrow{X}$ (obtaining $\langle \overrightarrow{X_1}, \overrightarrow{X_2} \rangle$);
(A.2) apply the set-theoretic summation component-wise to the result of step (A.1) (thus obtaining $\langle \bigcup \overrightarrow{X_1}, \bigcup \overrightarrow{X_2} \rangle$).

We denote this procedure with $dis(\overrightarrow{X})$.

If we want the decreasing representation we must use, instead of k_f, the following transformation k_f^*: $k_f^*(\overrightarrow{X}) = \langle f(\overrightarrow{X}) \cap \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+, \overrightarrow{X} \cap \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+ \rangle$. Then apply step A.2.

We denote the resulting procedure with $dec(\overrightarrow{X})$.

EXAMPLE
$k_f(\{\{a, a''\}', \{a'\}\}) = \langle \{\{a'\}\}, \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+ \cap -f(\{\{a, a''\}', \{a'\}\}) \rangle = \langle \{\{a'\}\}, \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+ \cap -\{\{a, a''\}, \{a'\}\} \rangle = \langle \{\{a'\}\}, \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+ \cap \{\{a'''\}\} \rangle = \langle \{\{a'\}\}, \{\{a'''\}\} \rangle$.
Hence $dis(\{\{a, a''\}', \{a'\}\}) = \langle \bigcup \{\{a'\}\}, \bigcup \{\{a'''\}\} \rangle = \langle \{a'\}, \{a'''\} \rangle$.
2. $k_f^*(\{\{a, a''\}', \{a'\}\}) = \langle f(\{\{a, a''\}', \{a'\}\}) \cap \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+, \{\{a, a''\}', \{a'\}\} \cap \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+ \rangle = \langle \{\{a, a''\}, \{a'\}\} \cap \mathcal{X}^S(\mathbf{J}(\mathbf{AS}(G)))^+, \{\{a'\}\} \rangle = \langle \{\{a, a''\}, \{a'\}\}, \{\{a'\}\} \rangle$.

Hence $dec(\{\{a, a''\}', \{a'\}\}) = \langle \bigcup \{\{a, a''\}, \{a'\}\}, \bigcup \{\{a'\}\} \rangle = \langle \{a, a', a''\}, \{a'\} \rangle$.

3. $k_f(\{\{a, a''\}, \{a, a''\}', \{a'\}\}) = \langle \{\{a, a''\}, \{a'\}\}, -\{\{a, a''\}, \{a, a''\}', \{a'\}\} \cap \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+ \rangle = \langle \{\{a, a''\}, \{a'\}\}, \{\{a'''\}\} \cap \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))^+ \rangle = \langle \{\{a, a''\}, \{a'\}\}, \{\{a'''\}\} \rangle$.

Hence $dis(\{\{a, a''\}, \{a, a''\}', \{a'\}\}) = \langle \{a, a'', a'\}, \{a'''\} \rangle$.

4. $k_f^*(\{\{a, a''\}, \{a, a''\}', \{a'\}\}) = \langle \{\{a, a''\}, \{a'\}\}, \{\{a, a''\}, \{a'\}\} \rangle$.

Hence $dec(\{\{a, a''\}, \{a, a''\}', \{a'\}\}) = \langle \{a, a'', a'\}, \{a, a'', a'\} \rangle$.

Thus $\{\{a, a''\}, \{a, a''\}', \{a'\}\}$ is the dual representation of an exact rough set.

This transformation makes it possible to put the above machinery another way. In fact, we can notice that primed elements stand for the possibility of being included in the negative part (i.e. second element) of a disjoint representation. Indeed, a primed element X_i' is mapped onto X_i via f so that it can belong to $-f(\overrightarrow{X})$. Thus we have the following cases: Let A be an elementary class.

(Case 1) $A \in \overrightarrow{X}$.

(Case 1.1) $A \notin\downarrow B$. Then A' exists and $A' \in \overrightarrow{X}$, because $\overrightarrow{X}$ is an order filter and $A \leq A'$ in $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))$. Since $A = f(A')$, $A \notin -f(\overrightarrow{X})$ so that A cannot belong to the negative part of $k_f(\overrightarrow{X})$.

(Case 1.2) $A \in\downarrow B$. Then $A = f(A)$ and we have the same conclusion as in the previous case.

(Case 2) $A \notin \overrightarrow{X}$.

(Case 2.1) $A \notin\downarrow B$.

(Case 2.1.1) $A' \in \overrightarrow{X}$. Then according to Case 1.1, A cannot belong to the negative part of $k_f(\overrightarrow{X})$ either.

(Case 2.1.2) $A' \notin \overrightarrow{X}$. Then $A \in -f(\overrightarrow{X})$, so that A belongs to the negative part of $k_f(\overrightarrow{X})$.

(Case 2.2) $A \in\downarrow B$. For $f(A) = A$, $A \in -f(\overrightarrow{X})$, and we are in the same case as above.

(Case 2.2.1) This is the "rough case": $A \notin \overrightarrow{X}$ but $A' \in \overrightarrow{X}$. In this case A belongs neither to the positive nor to the negative part of $\overrightarrow{X}$. But this cannot happen if $A \in\downarrow B$.

In other words, any rough set is decidable with respect to the elements of $\downarrow B$.

Let us now consider the Grothendieck topology induced on $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))$ by the filter $\uparrow \{\{a, a''\}', \{a'\}, \{a'''\}\}$. First notice that $\{\{a, a''\}', \{a'\}, \{a'''\}\}$ is the least dense open set in $\mathbf{F}(\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))$ (when it is considered as a frame of open subsets). To see this, notice that it is the least order filter which includes all the maximal element of $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G)))$. Further, one can notice that $dec(\{\{a, a''\}', \{a'\}, \{a'''\}\}) = \langle G, \{a', a'''\}\rangle = \langle G, B\rangle$. And when we compute $\neg\neg\langle G, B\rangle$ we obtain $\langle G, G\rangle$ (indeed, $\neg\neg\langle G, B\rangle = \neg\langle\neg G, \neg G\rangle = \neg\langle\emptyset, \emptyset\rangle = \langle\neg\emptyset, \neg\emptyset\rangle = \langle G, G\rangle$), which is tantamount to saying that $\langle G, B\rangle$ is dense in algebraic terms. Moreover, as proved in *Lemma* 8.2.3, any dense element must have the form $\langle G, X\rangle$ with $X \supseteq B$ (necessarily, because of the clause of filtration via B). Hence, $\langle G, B\rangle$ is the least dense element of $\mathcal{RS}_B(\mathbf{AS}(G))$.

Thus let us denote $\{\{a, a''\}', \{a'\}, \{a'''\}\}$ with D.

The Grothendieck topology J^D is then:

X	$\{a, a''\}'$	$\{a, a''\}$	$\{a'\}$	$\{a'''\}$
$J^D_{\lceil X\rceil}$	$\{\{\{a, a''\}'\}\},$	$\{\{\{a, a''\}, \{a, a''\}'\}, \{\{a, a''\}'\}\}$	$\{\{\{a'\}\}\}$	$\{\{\{a'''\}\}\}$

For the reader's convenience we show below the lattice $\mathbf{F}(\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G))))$, with the least dense element in bold. Double-arrows show the equivalence relation $\equiv_{J^D}$.
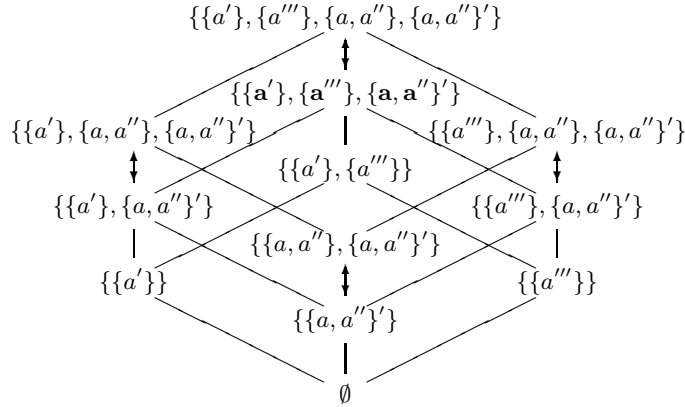
10.3.5   Rough Set Systems, Post Algebras and Total Atomic Undecidability

Suppose that on G we are given an Indiscernibility Space $\mathbf{A}' = \langle G, E\rangle$ with an equivalence relation E giving exactly two equivalence classes,

$\{a, a'\}$ and $\{a'', a'''\}$, then $B = \emptyset$. We show the induced Approximation Space $\mathbf{AS}(G/E)$, the dual Nelson space $\mathcal{X}^{\emptyset}(\mathbf{J}(\mathbf{AS}(G/E)))$ and the induced Rough Set System $\mathcal{RS}_{\emptyset}(\mathbf{J}(\mathbf{AS}(G/E)))$ (that is, the application of function dec to the elements of $\mathbf{F}(\mathcal{X}^{\emptyset}(\mathbf{J}(\mathbf{AS}(G/E))))$):

$$\mathbf{AS}(G/E)$$

$$\{a, a', a'', a''\}$$

$$\mathcal{X}^{\emptyset}(\mathbf{J}(\mathbf{AS}(G/E)))$$

$$\{a, a'\} \qquad \{a'', a'''\}$$

$$\{a, a'\}' \qquad \{a'', a'''\}'$$

$$\{a, a'\} \qquad \{a'', a'''\}$$

$$\emptyset$$

$$\mathcal{RS}_{\emptyset}(\mathbf{AS}(G/E))$$

$$\langle\{a, a', a'', a'''\}, \{a, a', a'', a'''\}\rangle$$

$$\langle\{a, a', a'', a'''\}, \{a, a'\}\rangle \quad \langle\{a, a', a'', a'''\}, \{a'', a'''\}\rangle$$

$$\langle\{a, a'\}, \{a, a'\}\rangle \quad \langle\{a, a', a'', a'''\}, \emptyset\rangle \quad \langle\{a'', a'''\}, \{a'', a'''\}\rangle$$

$$\langle\{a, a'\}, \emptyset\rangle \qquad \langle\{a'', a'''\}, \emptyset\rangle$$

$$\langle\emptyset, \emptyset\rangle$$
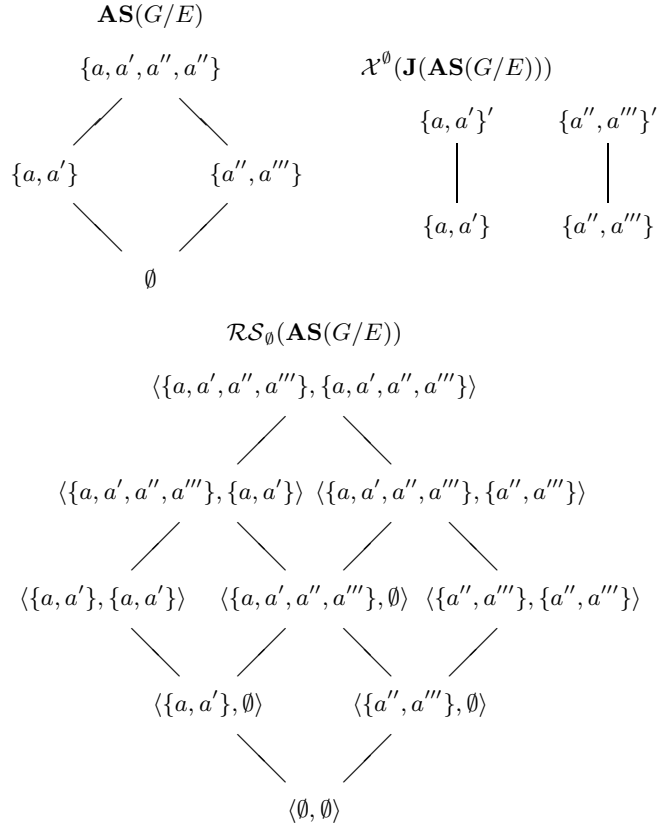
It is immediate to verify that $\mathcal{RS}_{\emptyset}(\mathbf{AS}(G/E))$ is a Post algebra of order three, with central element $\langle\{a, a', a'', a'''\}, \emptyset\rangle$ (cf. Example 7.4.1).

The Grothendieck topology induced by $\uparrow \emptyset$ on $\mathcal{X}^{\emptyset}(\mathbf{J}(\mathbf{AS}(G/E)))$ is:

X	$\{a, a'\}'$	$\{a'', a'''\}'$	$\{a, a'\}$	$\{a'', a'''\}$
$J^{\emptyset}_{[X]}$	$\{\emptyset, \{\{a, a'\}'\}\}$	$\{\emptyset, \{\{a'', a'''\}'\}\}$	$\{\emptyset, \{\{a, a'\}'\}, \{\{a, a'\}, \{a, a'\}'\}\}$	$\{\emptyset, \{\{a'', a'''\}'\}, \{\{a'', a'''\}, \{a'', a'''\}'\}\}$

One can see that $\emptyset$ covers all the elements of $\mathcal{X}^{\emptyset}(\mathbf{J}(\mathbf{AS}(G/E)))$. This testifies the possible "ambiguity" or "roughness" of all sets.

EXCURSUS: **Grothendieck topology on Approximation Spaces**
Consider the Approximation Space **A** described in Example 7.4.4. First
of all, let us see what Grothendieck topology is induced on the poset
$\mathbf{Q}(\mathbf{A})$ by the filter $\uparrow B$ (i.e. $\uparrow \{a', a'''\}$) on $\mathbf{F}(\mathbf{Q}(\mathbf{A}))$ (remember that
$\mathbf{Q}(\mathbf{A}) = \langle G, R_{\mathbf{A}} \rangle$ and $\mathbf{F}(\mathbf{Q}(\mathbf{A})) = \mathbf{AS}(G)$):

x	a	a'	a''	a'''
$J^B_{[x]}$	$\{\emptyset, \{a, a''\}\},$	$\{\{a'\}\}$	$\{\emptyset, \{a, a''\}\}$	$\{\{a'''\}\}$

To verify this, notice that $\mathbf{Q}(\mathbf{A})$ is the following preordered space:

$$a \longleftrightarrow a'' \qquad a' \qquad a'''$$

Thus $\uparrow a = \uparrow a'' = \{a, a''\}$, $\uparrow a' = \{a'\}$ and $\uparrow a''' = \{a'''\}$. Moreover,
$\emptyset \subseteq \uparrow a, \emptyset \subseteq \uparrow a'$ and $\emptyset \equiv_{J^B} \uparrow a, \emptyset \equiv_{J^B} \uparrow a'$.

10.4 Frame – Representation of Three-Valued Łukasiewicz Algebras as Rough Set System

By means of the same decomposition procedure as that shown in Sub-
section 8.3.2 one can prove that given an Approximation Space $\mathbf{AS}(U)$
the center of $\mathcal{RS}_B(\mathbf{AS}(U))$ (hence $\mathbf{AS}(U)$) is lattice isomorphic to
$\mathcal{B}^*(B) \times \mathcal{B}^*(P)$.

This procedure is supported by a general result about duality that
we describe now.

In what follows, given a (Nelson, Heyting, Boolean) space $\mathbf{S}$, the
generic operation resulting in the dual (Nelson, Heyting, Boolean) alge-
bra is denoted by $\mathbf{F}(\mathbf{S})$. Dually, given a (Nelson, Heyting, Boolean)
algebra $\mathbf{A}$, by $\mathbf{X}(\mathbf{A})$ we intend its dual space:

Lemma 10.4.1. (*see [Davey & Duffus, 1982]*) *Let $\mathbf{S}$ and $\mathbf{S}'$ be two
spaces, then*

$$\mathbf{F}(\mathbf{S} \oplus \mathbf{S}') \cong \mathbf{F}(\mathbf{S}) \times \mathbf{F}(\mathbf{S}')$$

where $\oplus$ is the ordinal sum (juxtaposition) of the two spaces.

Clearly, for any Approximation Space $\mathbf{AS}(U)$, $\mathbf{X}(\mathbf{AS}(U)) = \langle atoms(\mathbf{AS}(U)), = \rangle$ and we can partition $\mathbf{X}(\mathbf{AS}(U))$ in the two sub-
spaces $\mathbf{B} = \langle B^*, = \rangle$ and $\mathbf{P} = \langle P^*, = \rangle$, so that $\mathbf{B} \oplus \mathbf{P} = \mathbf{X}(\mathbf{AS}(U))$.
Clearly, $\mathbf{F}(\mathbf{P}) = \mathcal{B}^*(P)$ and $\mathbf{F}(\mathbf{B}) = \mathcal{B}^*(B)$. Thus we obtain $\mathbf{AS}(U) = \mathbf{F}(\mathbf{B} \oplus \mathbf{P}) \cong \mathbf{F}(\mathbf{B}) \times \mathbf{F}(\mathbf{P}) = \mathcal{B}^*(B) \times \mathcal{B}^*(P)$.

To obtain the results of Subsection 8.3.2 we similarly apply the above Lemma and the following one:

Lemma 10.4.2. (*cf. [Balbes & Dwinger, 1974]*) *Let $\mathcal{L}_n$ be the class of n-valued Łukasiewicz algebras and let $\mathbf{A} \in \mathcal{L}_n$. Then any prime ideal of $\mathbf{A}$ is contained in exactly one maximal chain of at most $n-1$ prime ideals.*

Since any Rough Set System $\mathcal{RS}_B(\mathbf{AS}(U))$ can be made into a three-valued Łukasiewicz algebra, by reasoning on prime filters instead of prime ideals and substituting co-prime elements for prime filters (since we are in the finite case), we have that the dual space of $\mathcal{RS}_B(\mathbf{AS}(U))$ is a set of chains of at most two elements. Indeed, it is isomorphic to $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(U)))$, because $N_\Theta(\mathbf{AS}(U)) \cong \mathbf{kN}(\mathcal{X}^B(\mathbf{AS}(U))) \cong \mathbf{F}(\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(U))))$. In view of the construction of $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(U)))$ we can split this space into two parts: $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(U)))^{P^*} = \langle\{X, X' : X \in atoms(\mathbf{AS}(U)) \ \& \ card(X) \geq 2\}, X \leq X'\rangle$ which is a set of chains of exactly two co-prime elements of $\mathbf{AS}(U)$, and $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(U)))^{B^*} = \langle\{X : X \in atoms(\mathbf{AS}(U)) \ \& \ card(X) = 1\}, X \leq X\rangle$ which is a set of chains with exactly one co-prime element.

Therefore, if we denote $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(U)))^{P^*}$ with $\{C'_i\}_{1 \leq i \leq n}$ and $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(U)))^{B^*}$ with $\{C_i\}_{1 \leq i \leq m}$ it is obvious that $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(U))) = C'_1 \oplus \ldots \oplus C'_n \oplus C_1 \oplus \ldots \oplus C_m$, so that we can recover (an isomorphic copy of) $\mathcal{RS}_B(\mathbf{AS}(U))$ by tacking $\mathbf{F}(C'_1) \times \ldots \times \mathbf{F}(C'_n) \times \mathbf{F}(C_1) \times \ldots \times \mathbf{F}(C_m) \cong \mathbf{F}(C'_1 \oplus \ldots \oplus C'_n \oplus C_1 \oplus \ldots \oplus C_m) = \mathbf{F}(\mathcal{X}^B(\mathbf{AS}(U))) \cong \mathbf{F}(C'_1 \oplus \ldots \oplus C'_n) \times \mathbf{F}(C_1 \oplus \ldots \oplus C_m) \cong \mathcal{RS}_\emptyset(\mathcal{B}^*(P)) \times \mathcal{RS}_B(\mathcal{B}^*(B))$.

Now we show that given any three-valued Łukasiewicz algebra $\mathbf{L_3}$, we can define an Approximation Space $\mathbf{A}(\mathbf{L_3})$ such that $\mathbf{L_3} \cong \mathbf{L}(\mathbf{AS}(\mathbf{A}(\mathbf{L_3})))$, where $\mathbf{L}$ is the operator of *Proposition* 8.3.1.

We have immediately that $\mathbf{X}(\mathbf{L_3})$ (i.e. $\langle\mathcal{J}(\mathbf{L_3}), \leq\rangle$) is formed of chains of co-prime elements of at most length 2. Let us set the following binary Information System $\mathbf{I}$:

$\mathbf{I} = \langle G, R\rangle$, where $G = \mathcal{J}(\mathbf{L_3})$ and $\langle a, b\rangle \in R$ iff $a \leq b$ or $b \leq a$ (in other words, just the members of the same chain are mutually related). Obviously, R is an equivalence relation and $\mathbf{I}$ is an Indiscernibility Space, so that $\mathbf{AS}(G/R)$ is an Approximation Space. We claim that $\mathbf{L}(\mathbf{AS}(G/R)) \cong \mathbf{L_3}$.

Let us set $B = \{X : X \in atoms(\mathbf{AS}(G/R)) \ \& \ card(X) = 1\}$. It is obvious that $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G/R)))$ is isomorphic to $\mathbf{X}(\mathbf{L_3})$. Indeed, if

$a \leq b$ is a chain of length 2 in $\mathbf{X}(\mathbf{L_3})$ then $R(\{a\}) = R(\{b\}) = \{a,b\}$, so that $\{a,b\} \leq \{a,b\}'$ is a chain of length 2 in $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G/R)))$ and if $a \leq a$ is a chain of length 1 in $\mathbf{X}(\mathbf{L_3})$ then $R(\{a\}) = \{a\}$ so that $\{a\} \leq \{a\}$ is a chain of length 1 in $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G/R)))$. Vice versa, for any element $\{a,b\} \in \mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G/R)))$, $\{a,b\} \leq \{a,b\}'$ if and only if $R(\{a\}) = R(\{b\}) = \{a,b\}$ if and only if either $a \leq b$ or $b \leq a$ in $\mathbf{X}(\mathbf{L_3})$, and $\{a\} \leq \{a\}$ in if and only if $a \leq a$. It follows that for any chain of length 2 of $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G/R)))$ there is exactly a chain of length 2 in $\mathbf{X}(\mathbf{L_3})$, for any chain of length 1 of $\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G/R)))$ there is exactly a chain of length 1 in $\mathbf{X}(\mathbf{L_3})$, and vice-versa.

We conclude that $\mathbf{F}(\mathbf{X}(\mathbf{L_3})) \cong \mathbf{F}(\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G/R))))$. From this and the facts that $\mathbf{F}(\mathbf{X}(\mathbf{L_3})) \cong \mathbf{L_3}$ and $\mathbf{F}(\mathcal{X}^B(\mathbf{J}(\mathbf{AS}(G/R)))) \cong \mathcal{RS}_B(\mathbf{AS}(G/R))$ we have immediately that $\mathbf{L_3} \cong \mathbf{L}(\mathbf{AS}(G/R))$.

It is worth mentioning that Mohua Banerjee extended these results to the general (infinite) case in [Banerjee, 1997].

EXAMPLE
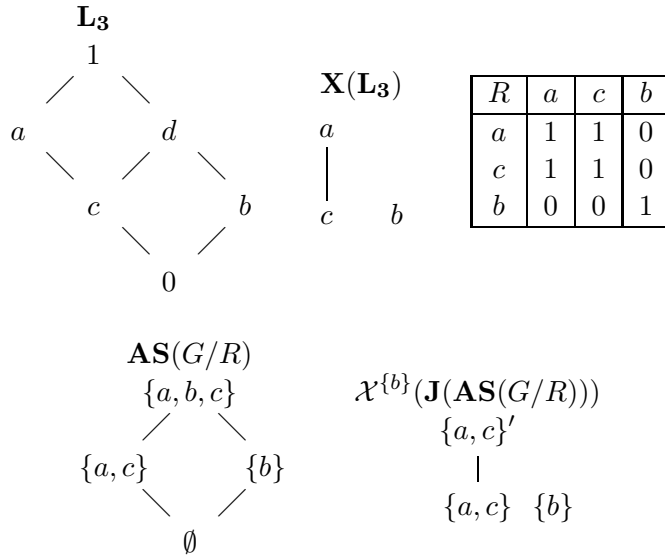
The isomorphic image of 1 in $\mathbf{F}(\mathcal{X}^{\{b\}}(\mathbf{J}(\mathbf{AS}(G/R))))$ is $\{\{a,c\}, \{a,c\}', \{b\}\}$, that of b is $\{\{a,c\}', \{b\}\}$, and so on. The isomorphic image of 1 in $\mathcal{RS}_{\{b\}}(\mathbf{AS}(G/R))$ is $\langle G, G \rangle$, that of b is $\langle \{a,c,b\}, \{b\} \rangle$, and so on.

10.5   Frame – Proof of the Facts Stated in Window 7.1

The only property to be really proved is (a), that is, the adjunction property between $\supset$ and $\wedge$. As a variant we carry on this proof in the disjoint representation (see *Definition* 9.6.1). So, let $\mathbf{A}$ be a Boolean algebra and $a, b \in \mathbf{A}$. In order to fulfill the adjunction property, $a \supset b$ must be an element $\langle c_1, c_2 \rangle$ s.t.

(a) c_2 is the least element, x, of $\mathbf{A}$ s.t. $x \vee a_2 \leq b_2$, while

(b) c_1 must be the greatest element y of $\mathbf{A}$ s.t. $y \wedge a_1 \leq b_1$, and $x \cap y = 0$.

We claim that $c_2 = b_2 \wedge \neg a_2$. In fact, in view of the first requirement, $b_2 \wedge \neg a_2$ is the least element x s.t. $x \wedge a_2 \geq b_2$. Now, in view of the disjunction condition and the requirement of maximization of c_1 , $b_2 \wedge \neg a_2$ is the best solution for c_2. Now, the greatest element y of $\mathbf{A}$ s.t. $y \wedge a_1 \leq b_1$ is $\neg a_1 \vee b_1$ but in order to get the condition of disjointedness we have to subtract c_2 from it obtaining $(\neg a_1 \vee b_1) \wedge \neg(b_2 \wedge \neg a_2)$. Let us then develop this Boolean polynomial:

$$
\begin{aligned}
(\neg a_1 \vee b_1) \wedge \neg(b_2 \wedge \neg a_2) &= (\neg a_1 \vee b_1) \wedge (\neg b_2 \vee a_2) \\
&= (\neg a_1 \wedge (\neg b_2 \vee a_2)) \vee (b_1 \wedge (\neg b_2 \vee a_2)) \\
&= (\neg a_1 \wedge \neg b_2) \vee (\neg a_1 \wedge a_2) \vee (b_1 \wedge \neg b_2) \vee (b_1 \wedge a_2) \\
&= (\neg a_1 \wedge \neg b_2) \vee a_2 \vee b_1 \vee (b_1 \wedge a_2)
\end{aligned}
$$

but since $(a_2 \vee b_1) \geq (b_1 \wedge a_2)$ the last expression reduces to $(\neg a_1 \wedge \neg b_2) \vee a_2 \vee b_1$.

Hence we have:

(*) $c_1 = (\neg a_1 \wedge \neg b_2) \vee (a_2 \vee b_1)$;

(**) $c_2 = b_2 \wedge \neg a_2$.

It follows that $\langle c_1, c_2 \rangle$ is the disjunction of two elements d and e s.t. $d_1 \vee e_1 = (\neg a_1 \wedge \neg b_2) \vee (a_2 \vee b_1)$ and $d_2 \wedge e_2 = b_2 \wedge \neg a_2$.

Again, d is the conjunction of two elements d' and d'' s.t. $d'_1 \wedge d''_1 = \neg a_1 \wedge \neg b_2$ and $(d'_2 \vee d''_2) \wedge e_2 = d_2 \wedge e_2 = b_2 \wedge \neg a_2$, while e is the disjunction of two elements e' and e'' s.t. $e'_1 \vee e''_1 = a_2 \vee b_1$ and $(e'_2 \wedge e''_2) \wedge (d'_2 \vee d''_2) = (e'_2 \wedge e''_2) \wedge d_2 = d_2 \wedge e_2 = b_2 \wedge \neg a_2$.

We are to find a solution with minimal structural complexity.

We claim that $d' = \neg\langle a_1, a_2\rangle$, $d'' = \neg \sim \langle b_1, b_2\rangle$, $e' =\sim \neg \sim \langle a_1, a_2\rangle$ and $e'' = \langle b_1, b_2\rangle$.

In fact on the one hand we have:

$\neg\langle a_1, a_2\rangle \wedge \neg \sim \langle b_1, b_2\rangle = \langle\neg a_1, a_1\rangle \wedge \neg \sim \langle\neg b_2, b_2\rangle = \langle\neg a_1 \wedge \neg b_2, a_1 \vee b_2\rangle$.

On the other hand:

$\sim \neg \sim \langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle = \langle a_2, \neg a_2\rangle \vee \langle b_1, b_2\rangle = \langle a_2 \vee b_1, b_2 \wedge \neg a_2\rangle$.

But $\langle\neg a_1 \wedge \neg b_2, a_1 \vee b_2\rangle \vee \langle a_2 \vee b_1, b_2 \wedge \neg a_2\rangle$ equals

(***) $\langle(\neg a_1 \wedge \neg b_2) \vee (a_2 \vee b_1), (a_1 \vee b_2) \wedge (b_2 \wedge \neg a_2)\rangle$.

Since $(a_1 \vee b_2) \geq b_2 \leq (b_2 \wedge \neg a_2)$, we have $(a_1 \vee b_2) \wedge (b_2 \wedge \neg a_2) = (b_2 \wedge \neg a_2)$.

Hence (***) becomes $\langle(\neg a_1 \wedge \neg b_2) \vee (a_2 \vee b_1), b_2 \wedge \neg a_2\rangle$ as required by (*) and (**). By easy calculation one can verify that the last polynomial is exactly $a \supset b$.   QED

The other facts listed in Window 7.1:
(b) essentially reads $a \wedge b = a$ iff $a \leq b$ iff $a \supset b = 1$; but this is a consequence of (a) while (c), (d) and (e) are verified by easy inspection.
Notice that from (c) and (e) we straightforwardly have: (e') $\sim \neg\neg a = \lrcorner\lrcorner \sim a = \neg a$, (c') $\sim \lrcorner\lrcorner a = \neg\neg \sim a = \lrcorner a$.

Further, (c), (d), (e), (e') and (c') may be interestingly read as follows:

(i) $\sim \Diamond = \Box \sim$; $\neg\Diamond = \Box\neg$. (ii) $\sim \Box = \Diamond \sim$; $\lrcorner\Box = \Diamond\lrcorner$.
(iii) $\lrcorner\Diamond \leq \Box\lrcorner$. (iv) $\neg\Box \geq \Diamond\neg$.

Exercise 10.4. *Recall that $\rightarrowtail$ is defined as $a \longrightarrow b\wedge \sim b \longrightarrow\sim a$. Prove that in semi-simple Nelson algebras $a \rightarrowtail 0 =\sim a$.*

10.6   Frame – Proof of Proposition 8.3.1

Proposition 10.6.1. *Let $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ be a Boolean algebra and $x \in A$. Then, $\mathbf{H}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \neg, \lrcorner, \supset, \subset, 0, 1\rangle$ is a bi-Heyting algebra.*

Proof. Immediately from the fact that $\supset$ is a relative pseudo-complementation, $\neg$ is a pseudo-complementation (see Frame 10.5), $\subset$ is a co-relative pseudo-complementation (see *Lemma* 8.3.1.(4)) and that $\lrcorner$ is a co-pseudo-complementation (see *Corollary* 8.3.1.(1)).   QED

Lemma 10.6.1.
1. $\neg\neg$ and $\lrcorner\lrcorner$ are endomorphisms in the lattice $\mathbf{H}(\mathbf{A})$.

2. $\neg\neg$ *and* $\lrcorner\lrcorner$ *distribute over* $\wedge$ *and* $\vee$.

Proof. (1) Lifting the abstraction level from Boolean algebras of sets to generic Boolean algebras, from *Lemma* 8.2.3 and *Proposition* 7.3.4 we have that the operator induced by $J^{\langle 1,x\rangle}$ is $\neg\neg$. Thus from *Corollary* 8.2.2 we obtain that $\neg\neg$ is an endomorphism in $\mathbf{H}(\mathbf{A})$. By duality we obtain the same result as to $\lrcorner\lrcorner$. (2) From *Proposition* 8.2.3 and duality. QED

Lemma 10.6.2. *Let* $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ *be a Boolean algebra and* $x \in A$. *Then,* $\mathbf{D}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \sim, 0, 1\rangle$ *is a De Morgan lattice.*

Proof. We have just to prove the two De Morgan rules. But $\sim (a \wedge b) = \langle \neg(a_2 \wedge b_2), \neg(a_1 \wedge b_1)\rangle = \langle \neg a_2 \vee \neg b_2, \neg a_1 \vee \neg b_1\rangle = \sim a \vee \sim b$, because $\neg$ is a Boolean complementation. Dually for the other De Morgan rule. QED

Proposition 10.6.2. *Let* $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ *be a Boolean algebra and* $x \in A$. *Then,* $\mathbf{N}(\mathbf{A}) = \langle \mathcal{RS}_x(\mathbf{A}), \wedge, \vee, \lrcorner, \sim, \longrightarrow, 0, 1\rangle$ *is a semi-simple Nelson algebra.*

Proof. In view of the above *Lemma* 10.6.2 we have just to prove properties (6.2.15) and (6.2.17) and semi-simplicity. The first property is equivalent to $a \longrightarrow b = a \supset (\sim a \vee b)$, which is proved straightforwardly as follows: $a \supset (\sim a \vee b) = \sim \lrcorner \sim a \vee \sim a \vee b \vee (\lrcorner a \wedge \lrcorner \sim (\sim a \vee b)) = \sim a \vee b \vee (\lrcorner a \wedge (\lrcorner a \vee \lrcorner \sim b)) = \sim a \vee b \vee \lrcorner a = \langle \neg a_2 \vee b_1, \neg a_1 \vee b_2\rangle \vee \langle \neg a_2, \neg a_2\rangle = \langle \neg a_2 \vee b_1, \neg a_2 \vee b_2\rangle = a \longrightarrow b$. The second property is immediate from the trivial fact that $a \longrightarrow b = \lrcorner a \vee b$ and the fact that, by duality, the two De Morgan laws hold of $\lrcorner$. Hence $(a \wedge b) \longrightarrow c = \lrcorner(a \wedge b) \vee c = \lrcorner a \vee \lrcorner b \vee c = a \longrightarrow (b \longrightarrow c)$. Now we have to prove that $x \vee \lrcorner x = 1$. Indeed, $x \vee \lrcorner x = \langle x_1, x_2\rangle \vee \langle \neg x_2, \neg x_2\rangle = \langle 1, 1\rangle$, because $x_2 \leq x_1$ and $\neg x_2$ is the Boolean complement of x_2. QED

Proposition 10.6.3. *Let* $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ *be a Boolean algebra and* $x \in A$. *Set* $\phi_1 = \neg\neg$ *and* $\phi_2 = \lrcorner\lrcorner$. *Then* $\mathbf{L}(\mathbf{A}) = \langle \mathcal{RS}_x(A), \vee, \wedge, \sim, \phi_1, \phi_2, 0, 1\rangle$ *is a Łukasiewicz algebra of order three.*

Proof. In view of *Lemma* 10.6.2 it is sufficient to verify the identities of *Definition* 6.3.1.

1. From Lemma 10.6.1.

2. $\phi_i(x) \wedge \phi_j(x) = \phi_j(x)$, $(1 \leq i, j \leq 3 - 1)$: from (6.2.24) $\lrcorner\lrcorner \leq \neg\neg$.

3. $\phi_i(x) \vee \sim \phi_i(x) = 1; \phi_i(x) \wedge \sim \phi_i(x) = 0$, $(1 \leq i \leq 3 - 1)$: from *Lemma 8.3.1.(2)* and the fact that $\lrcorner\lrcorner a = \sim \lrcorner a$ and $\neg\neg a = \sim \neg a$.

4. $\phi_i(\sim x) = \sim \phi_{n-i}(x)$, $(1 \leq i \leq 3 - 1)$: see Frame 10.5.

5. $\phi_i(\phi_j(x)) = \phi_j(x)$, $(1 \leq i, j \leq 3 - 1)$: because the codomain of both ϕ_1 and ϕ_2 is the center of the algebra and $\lrcorner$ and $\neg$ are involutions on the center.

6. $x \vee \phi_1(x) = \phi_1(x); x \wedge \phi_2(x) = \phi_2(x)$, because $\neg\neg x \geq x$ and $\lrcorner\lrcorner x \leq x$.

7. $\phi_i(0) = 0; \phi_i(1) = 1$, $(1 \leq i \leq n - 1)$: because $\lrcorner\lrcorner(1) = \neg\neg(1) = 1$ and $\lrcorner\lrcorner(0) = \neg\neg(0) = 0$.

8. $\sim x \wedge \phi_2(x) = 0; \sim x \vee \phi_1(x) = 1$: from easy computation (or the fact that $\lrcorner\lrcorner x$ is the largest element of the center below x and $\neg\neg x$ is the smallest element of the center above x).

9. $y \wedge (x \vee \sim \phi_1(x) \vee \phi_2(y)) = y$: by easy inspection.   QED

We can also verify, by straightforward inspection, that the Moisil residuation $\sqsupset$ defined in (6.3.25) coincides with the operation $\supset$ of semi-simple Nelson algebras.

On the other side, given a three-valued Łukasiewicz algebra **L**, we can introduce two new operations $\neg$ and $\longrightarrow$ in such a way that the resulting structure is a semi-simple Nelson algebra:

Lemma 10.6.3. *Let* **L** *be a three-valued Łukasiewicz algebra and* ϕ_1, ϕ_2 *its two endomorphisms. Define* $\forall a, b \in L$ *the following two operations:*

(1) $\lrcorner a = \sim \phi_2(a)$;   *(2)* $a \longrightarrow b = \sim \phi_2(a) \vee b$.
Then $\mathbf{L}^+ = (L, \vee, \wedge, \sim, \lrcorner, \longrightarrow, 0, 1)$ *is a semi-simple Nelson algebra.*

Proof. By an easy verification of the axioms of semi-simple Nelson algebras or exploiting the relation between ϕ_2 and the centre of **L**. QED

10.7   Frame – Grothendieck Topologies and Lawvere-Tierney Operators

Let $\langle X, \Omega(X) \rangle$ be a topological space, $Y \subset X$ and $\mathcal{O} \subseteq \wp(X)$ a family of open subsets. If $\bigcup \mathcal{O} = Y$, then $\mathcal{O}$ is called an *open covering* of Y. In 1960 Alexander Grothendieck generalised the notion of an open covering to that of an *étalé* covering (or *stalk space covering*) in order

to define cohomology theory. The axiomatisation of the properties of *étalé* covering lead to the notion of a "Grothendieck topology".

Somewhat more precisely, consider the *sheaf* of continuous real-valued functions defined on X which associates with every open subset $U \subseteq X$ the set $F(U)$ of real-valued continuous functions defined on U. If $U \subseteq V \subseteq X$ we have a "restriction map" from $F(V)$ to $F(U)$.

If $\{V_i\}_{i \in I}$ is an open covering of the set U, and we are given mutually compatible elements of $\{F(V_i)\}_{i \in I}$, then there exists precisely one element of $F(U)$ that restricts to all the given ones. This is the notion which is basically axiomatised by a Grothendieck topology.

Some years later, F. William Lawvere and Miles Tierney were able to axiomatise *Elementary topos theory*, in order to provide a foundation for differential geometry. They showed that *sheaf theory* could be developed axiomatically by starting with a topos **T** with global element Ω and a morphism $j : \Omega \longmapsto \Omega$, whose properties are categorical versions of the multiplicative, additive and inflationary requirements for an operation on a lattice. The pair $(\mathbf{E}, j)$ was called a *site*. In [Lawvere, 1970] it was openly stated that "A Grothendieck topology appears most naturally as a modal operator of the nature it is locally the case that". This is the intuition we used in the present Part.

The definitions and propositions in this Part are slight modifications of those that can be found for instance in [Goldblatt, 1984], [Mac Lane & Moerdijk, 1992] and [Fourman & Scott, 1979].

10.8 Frame – Representation of Rough Sets

The increasing representation of rough sets (that is, by means of pairs $\langle X, Y \rangle$ such that $X \subseteq Y$, representing $\langle (lE)(X), (uE)(X) \rangle$), was adopted, for instance, in [Iwinski, 1987].

In the present book we have preferred the decreasing representation since, in a precise sense, this reading is linked to the notion of a *refinement of an approximation* and it is consistent with the interpretation of some *multi-valued logics* which were proposed, for instance, in [Traczyk, 1963] and [Epstein & Rasiowa, 1987] (see also [Rasiowa, 1987]).

The first representation of Rough Sets as pairs $\langle (lE)(X), -(uE)(X) \rangle$ of disjoint elements was proposed in [Pagliani, 1993b], in order to deal with Nelson algebras. According to this interpretation $\sim \langle X_1, X_2 \rangle = \langle X_2, X_1 \rangle$, so that the filtration *via* $\uparrow B$ receives a probably more

intuitive interpretation: an ordered pair $X = \langle X_1, X_2 \rangle$ is a rough set only if $\langle B, \emptyset \rangle \leq (X \vee \sim X)$. That is, the *excluded middle* must be locally valid on $\langle B, \emptyset \rangle$.

10.9   Frame – Rough Sets and Non Classical Logico-Algebraic Systems

The first researches on the relationships between Rough Set Systems and the logico-algebraic structures that we developed in this Part can be found in the following seminal works: [Obtułowicz, 1987], [Pomykala & Pomykala 1988] where an interpretation of Rough Set Systems as Stone algebras was proposed, [Comer 1991 and 1993] where Double Stone algebras were proposed, [Banerjee & Chakraborty, 1993], in which modal-algebraic structures were used (for this topic and related literature we address the reader to Part III), [Pagliani, 1993b] where semi-simple Nelson algebras, hence three-valued Łukasiewicz algebras, were used and [Pagliani, 1998d], where Rough Set Systems were represented as semi-simple Nelson algebras, three-valued Łukasiewicz algebras, Post algebras of order three, semi-Post algebras, Stone algebras and Chain Based Lattices.

More recently these studies have been revamped by [Düntsch, 1997] who introduced a logic for Rough Sets by working on Rough Set Systems represented as Katrinak algebras (essentially, regular double Stone algebras), [Pagliani, 1997b] (mixed logico-algebraic behaviour of Rough Set Systems), [Iturrioz, 1999] (Rough Set Systems and three-valued structures). As to a more recent study, we record [Milne, 2004] where Rough Sets Systems are connected to De Finetti algebras (i.e. centered three-valued Łukasiewicz algebras).

It is important to underline that several deductive systems have been studied for Rough Set Systems. We quote [Düntsch et al., 2000] where relational proof theory was used for many-valued information structures, because relational deductive systems reveal to be flexible and powerful in the realm of non-classical logics (see Part III). Notably, [Sen & Chakraborty, 2002] presents sequent calculi for a variety of topological algebras connected to Rough Sets Systems and for Wajsberg algebras, as well as a connection between some of these logics and Linear Logic. Recently [Dai et al., 2005] introduced a sequent calculus for Stone logic connected to Rough Set Systems.

10.9.1   Rough Sets and Brouwer-Zadeh Lattices

It is worth noticing that while we arrived at structures connected to Quantum Logic by investigating Rough Set Systems (see Part I, Frames 4.6.1, and 15.10 of Part III), from the other way around Giampiero Cattaneo and Davide Ciucci, by investigating models of Quantum Logics (cf. [Cattaneo et al., 1993]), arrived at structures that are able to link Rough Sets and Fuzzy Sets. More precisely, Rough Set Systems have been interpreted within the framework of Brouwer-Zadeh Lattice and Heyting-Wajsberg algebras (see [Cattaneo, 1998] and [Cattaneo & Ciucci, 2002b]). Shortly, a structure $\langle X, \vee, \wedge, \neg, \sim, 0 \rangle$ is a distributive Brouwer-Zadeh lattice if:

1. $\langle X, \vee, \wedge, 0 \rangle$ is a distributive lattice, with minimum 0.

2. $\sim$ is a Kleene orthocomplementation, that is: (i) $\sim \sim a = a$, (ii) $\sim (a \vee b) = \sim a \wedge \sim b$, (iii) $a \wedge \sim a \leq b \vee \sim b$, any $a, b \in X$.

3. $\neg$ is a Brouwer orthocomplementation, that is: (i') $a \wedge \neg\neg a = a$, (ii') $\neg(a \vee b) = \neg a \wedge \neg b$, (iii') $a \wedge \neg a = 0$.

4. The two orthocomplementations are linked by the following equation: $\sim \neg a = \neg\neg a$.

The mapping $\sim$ is also called a Łukasiewicz or fuzzy or Zadeh orthocomplementation .

It is clear that for any Approximation Space $\mathbf{AS}(U)$, the Rough Set System $RS(U)$ is a Brouwer-Zadeh lattice. Moreover, by adding or subtracting properties on the two sides of these algebraic structures, one can generalise or specialize them, so as to be able to deal with information structures based on preclusivity, similarity and other relations (see [Cattaneo & Ciucci, 2002a]).

10.9.2   Lattices and Non-Classical Logics

The logical systems here discussed have different origins and motivations.

Indeed, Emil Post aimed mainly at a generalization of the logical system of Whitehead and Russell's "Principia Mathematica" (see [Post, 1920]). Post algebras where analysed in [Traczyk, 1963], [Cignoli 1969, 1972] and [Balbes & Dwinger, 1971].

Jan Łukasiewicz focused on the fact that some usually accepted modal principles lead to disagreeable consequences in a two-valued setting (see [Łukasiewicz & Borkowski, 1970]). Arend Heyting tried to formalise the ideas of Intuitionistic Logics in [Heyting, 1930], while David Nelson introduced his system in order to circumvent the non-constructive features of intuitionistic negation (cf. [Nelson, 1949]).

Heyting algebras, Łukasiewicz algebras and Post algebras provided the starting point for developing Chain Based Lattices (viz. P_0-*lattices*, P_1-*lattices* and P_2-*lattices*) and P-algebras. These lattices were studied by George Epstein and Afred Horn within the theory of multi-valued signal processing and programming languages in [Epstein & Horn 1974a, b].

Systems connected to Nelson algebras have been used in Logic Programming by [Pearce & Wagner, 1990] (see Frame 10.12.3), in program synthesis by [Miglioli et al., 1989]) and in the framework of information systems (cf., for instance, [Wansing, 1993] and [Pagliani, 1997a], see also Frame 10.20 of Part II).

10.9.3   Lattices with Strong Negation

The algebraic companions of Nelson logic were studied by the Polish mathematicians A. Białynicki-Birula and H. Rasiowa, who named their algebraic models "Quasi-pseudo Boolean algebras" (see [Białynicki-Birula & Rasiowa, 1958]) and by the Latino-American school of logic (see [Monteiro, 1963a; the interpolation property of *Definition* 10.3.3 was introduced, in [Cignoli, 1986]), that represents the first thorough study of the relations between Kleene, Łukasiewicz and Nelson algebras. Notably, in this paper is given an example of complete and centered Nelson algebra which is not a Heyting algebra.[2]

Relations among Nelson algebras, Post algebras, Łukasiewicz algebras, Chain Based Lattices and P-algebras have been studied in [Monteiro, 1967], [Balbes & Dwinger, 1974], [Priestley, 1984], [Cignoli, 1986], [Boicescu et al., 1991], [Pagliani, 1998d].

---

[2]Take $\mathbf{kN}(\mathbf{H}(\mathbb{R}))$, where $\mathbf{H}(\mathbb{R})$ is the complete Heyting algebra of open sets of the real line $\mathbb{R}$. Consider the elements $\alpha = \langle]0, 1[,] - 1, 0[\cup]1, 2[\rangle$ and $\beta = \langle]1/3, 3/2[,] - 1, 1/2[\cup]3/2, 2[\rangle$. Then $\alpha \implies \beta$ does not exists in $\mathbf{kN}(\mathbf{H}(\mathbb{R}))$. Indeed, consider the family of elements $\Gamma = \{\gamma_n = \langle] - 1, -1/n]\cup]1/2, 2[\rangle\}_{n \in N}$, then $\alpha \wedge \gamma_n \leq \beta$, all $n \geq 1$. Suppose Γ has a greatest element $\gamma = \langle \gamma_1, \gamma_2 \rangle$. Then $\gamma_n \leq \gamma$, for each n, so that $[0, 1/2] \supseteq \gamma_2 \supseteq] - 1/n, 1/2[$ for each $n \geq 1$, which is impossible.

Some of them have been discussed in this Part where, moreover, we have shown that the "strange" formal phenomena which appear when considering such relations, can be explained in terms of "information", at least when we are dealing with structures of order three.

A comprehensive study of all these kinds of algebraic structures is [Cornish, 1986].

For different accounts of the logical and mathematical foundations of Rough Set Theory we address the reader to [Demri & Orłowska, 2002] and [Polkowski, 2002].

10.10   Frame – Representation Theorems and Decomposition of Distributive Lattices

Representation theorems for the algebraic structures presented in this Part, have been proved mostly by the authors quoted above.

In particular, we have to mention that Post algebras of order n are represented by means of *Post fields of sets* that we briefly describe.

Definition 10.10.1. *A topological space $\langle X, \Omega(X) \rangle$ is said to be a* Post space *of order $n \geq 2$ if*

1. *$X = \bigcup \{X_1, \ldots, X_{n-1} : X_i \cap X_j = \emptyset$, for $1 \leq i, j \leq n-1\}$.*

2. *There exists a Boolean algebra $\mathbf{B}$ with dual space $\mathbf{X}(\mathbf{B})$ and homeomorphisms $g_i : X_i \longmapsto \mathbf{X}(\mathbf{B})$ of X_i onto $\mathbf{X}(\mathbf{B})$, $1 \leq i \leq n-1$.*

3. *The family $B(X) = \{\bigcup_{i=1}^{n-1} g_i^{-1}(C) : C$ is a clopen subset of $\mathbf{X}(\mathbf{B})\}$ is a basis for $\langle X, \Omega(X) \rangle$.*

This Post space is denoted by $\mathbf{X} = \langle \{X_i, g_i\}_{1 \leq i \leq n-1}, B(X) \rangle$.

Intuitively, to use a geological metaphor, a Post space $\mathbf{X}$ is a *regular stratification* of disjoint homeomorphic Boolean spaces. Any element of $B(X)$ is the *core* drilled, throughout all the strata, by a drilling ring with section equal to a clopen set of $\mathbf{B}$.

For instance, the space $\mathcal{X}^{\emptyset}(\mathbf{J}(\mathbf{AS}(G/E)))$ of Frame 10.3.5 is a Post space made up of the Boolean space $X_1 = \{\{a, a'\}, \{a'', a'''\}\}$, the first stratum, and the Boolean space $X_2 = \{\{a, a'\}', \{a'', a'''\}'\}$, the second stratum. The dual space $\mathbf{X}(\mathbf{B}) = (\{a, a'\}, =)$ of the Boolean algebra $\mathbf{B}$ of Example 7.4.1 is clearly homeomorphic to both X_1

and X_2, *via* the isomorphisms $g_1(\{a,a'\}) = a, g_1(\{a'',a'''\}) = b$ and $g_2(\{a,a'\}') = a, g_2(\{a'',a'''\}') = b$. In this space all the subsets of $\{a,b\}$ are trivially clopen. It follows that, for instance, $\{\{a,a'\},\{a,a'\}'\} = \bigcup(g_1^{-1}(a), g_1^{-1}(b))$ and $\{\{a'',a'''\},\{a'',a'''\}'\} = \bigcup(g_2^{-1}(a), g_2^{-1}(b))$ are elements of $B(X)$.

We recall that in any Post space $\mathbf{X}$:

10.10.0.1.

1. $\mathbf{B}^*(X) = \langle B(X), \cap, \cup, -, \Longrightarrow, \emptyset, X\rangle$ is the field of all simultaneously open and closed subsets of $\mathbf{X}$.

2. $e_0 = \emptyset, e_1 = X_1, e_2 = X_1 \cup X_2, \ldots, e_{n-1} = X$, is a chain.

3. By $P(X)$ is intended the class of all subsets of X of the form $Y = Y_1 \cap e_1 \cup \ldots \cup Y_{n-1} \cap e_{n-1}$ where $Y_i \in B(X)$ for $1 \le i \le n-1$.

4. There exist operations $D_i : P(X) \longmapsto B(X)$, for $1 \le i \le n - 1$, which with every $Y \in P(X)$ associate uniquely determined coefficients $D_i(Y) \in B(X)$ such that Y has a unique monotonic representation $Y = D_1(Y) \cap e_1 \cup \ldots \cup D_{n-1}(Y) \cup e_{n-1}$, where $D_1(Y) \supseteq D_2(Y) \supseteq \ldots \supseteq D_{n-1}(Y)$.

5. The following condition holds:
$$D_i(e_j) = \begin{cases} 1 & \text{for } 1 \le i \le j \le n-1 \\ 0 & \text{for } n-1 \ge i > j \ge 0 \end{cases}$$

Continuing our "geological explanation", e_i is the *accumulation* of all the strata up to the $i^{th} - level$ (enumerating from the surface), while any element Y is an *accumulation* of *samples of various strata* such that any sample of the ith stratum is greater than or equal to any sample of the $i + 1^{th}$ stratum. Finally $D_i(Y)$ is a call for a *core drilling* with section of the same dimension as the ith sample forming the element Y.

In our example from Frame 10.3.5 we have:

$e_0 = \emptyset, e_1 = \{\{a,a'\}', \{a'',a'''\}'\}$ and $e_2 = \{\{a,a'\}, \{a'',a'''\}, \{a,a'\}', \{a'',a'''\}'\}$ and it is worth noticing that $dec(e_1) = \langle\{a,a',a'',a'''\}, \emptyset\rangle$, while $dis(e_1) = \langle\emptyset, \emptyset\rangle$. Moreover, $\{\{a,a'\}, \{a,a'\}', \{a'',a'''\}\}$ is a member of $P(X)$ given by $\{\{a,a'\}', \{a'',a'''\}', \{a,a'\}, \{a'',a'''\}\} \cap e_1 \cup \{\{a,a'\}', \{a,a'\}\} \cap e_2$, while $\{\{a,a'\}'\}$ is given by $\{\{a,a'\}', \{a,a'\}\} \cap e_1 \cup \emptyset \cap e_2$. Further, the element $\{\{a,a'\}'\}$ is given by $\{\{a,a'\}'\} \cap e_1 \cup \emptyset \cap e_2 = \{\{a,a'\}' \cup \emptyset\}$. Hence $D_2(\{\{a,a'\}'\}) = \emptyset$ and $D_1(\{\{a,a'\}'\}) = \{\{a,a'\}, \{a,a'\}'\}$.

We have the following Representation Theorem (see [Dwinger, 1966], [Rasiowa, 1974], [Traczyk, 1963]):

Lemma 10.10.1. *Every Post algebra* $\mathbf{P} = \langle P, +, \bullet, -, \Longrightarrow, D_1, D_2, \ldots, D_{n-1}, 0 = e_0, \ldots, e_{n-1} = 1\rangle$ *is isomorphic to a Post field* $\mathbf{P}(X) = \langle P(X), \cup, \cap, \neg, \supset, D_1, D_2, \ldots, D_{n-1}, 0 = e_0, \ldots, e_{n-1} = 1\rangle$ *of subsets of a Post space* $\mathbf{X} = (\{X_i, g_i\}_{1 \leq i \leq n-1}, B(X))$, *where, for all* $Z, Y \in P(X)$:

1. $C(Y) = X \cap -D_1(Y)$.

2. $Z \supset Y = (CD_1(Z) \cup D_1(Y)) \cap e_1 \cup ((CD_1(Z) \cup D_1(Y)) \cap (CD_2(Z) \cup D_2(Y))) \cap e_2 \cup \ldots \cup ((CD_1(Z) \cup D_1(Y)) \cap \ldots \cap (CD_{n-1}(Z) \cup D_{n-1}(Y))) \cap e_{n-1}$.

3. $\neg Z = Z \supset 0$.

As we have seen, representation theorems are connected with decomposition results of these lattices, given in terms of products of chains. On the other side, the classical construction of these kinds of lattices is given in terms of sublattices of the direct products of their centers (see [Balbes & Dwinger, 1974] and the references quoted there).

EXAMPLE
Consider the three-valued Łukasiewicz algebra $\mathbf{L_3}$ of Frame 10.4. The center $\mathcal{C}TR(\mathbf{L_3})$ is the Boolean algebra $\mathbf{B}$ of Example 7.4.1. In Figure 10.1 the hypercube $\mathcal{C}TR(\mathbf{L_3}) \times \mathcal{C}TR(\mathbf{L_3})$ is depicted and the isomorphic copy of $\mathbf{L_3}$ is embedded in the hypercube. From the hypercube we can recover a number of other sublattices isomorphic to $\mathbf{L_3}$, as shown in Figure 10.2. In Figure 10.2 the sublattice $\mathbf{L2}$ includes the element
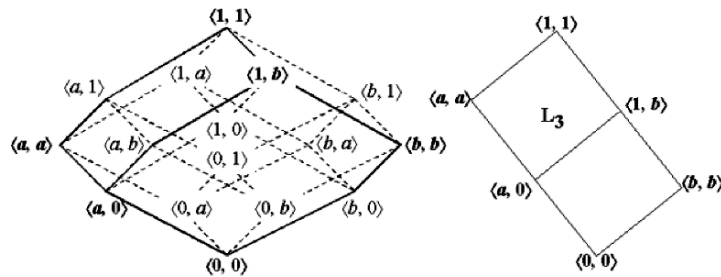
Figure 10.1: The hypercube $\mathcal{C}TR(\mathbf{L_3}) \times \mathcal{C}TR(\mathbf{L_3})$ with embedded an isomorphic copy of $\mathbf{L_3}$
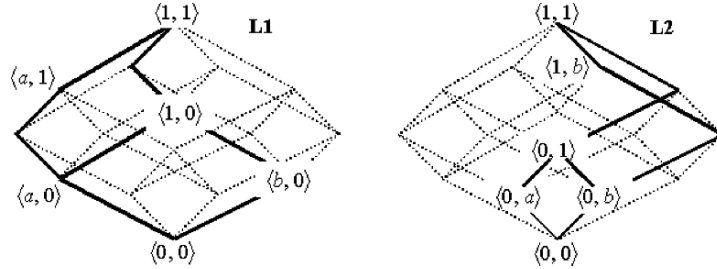
Figure 10.2: Lattice **L1** and **L2** isomorphic to **L₃**

$\langle 1, b \rangle$, as like as **L₃**, but the intermediate value is $\langle 0, 1 \rangle$. Each of them has symmetric twins (substitute $\langle b, 1 \rangle$ for $\langle a, 1 \rangle$ in **L1** and $\langle 1, b \rangle$ for $\langle 1, a \rangle$ in **L2**).

Other sublattices isomorphic to **L₃** can be singled out. Notice that the sublattice **L1** has $\langle 1, 0 \rangle$ as an intermediate value as like as the Post algebra $\mathcal{R}S_0(\mathbf{B})$ of Example 7.4.1 which is a sublattice of the hypercube, too. Thus a certain formal shape of the chain of values is preserved. However, by following this method we can miss the logico-informational content of the construction.

Indeed, as already pointed out, the result proved in this Part does not state an algebraic decomposition (it would not be sufficient since it does not give a complete decomposition) but a logical decomposition which is achieved through the construction by means of a filter $\uparrow x$, that is more suitable for an information-oriented analysis.

We end this Frame by recalling that a Post algebra of order n is also obtained as the co-product, $\mathbf{B} \coprod \mathbf{n}$ of a Boolean algebra $\mathbf{B}$ and a chain $\mathbf{n}$ of n elements. By duality $\mathbf{B} \coprod \mathbf{n} \cong \mathbf{F}(\mathbf{X}(\mathbf{B}) \times \mathbf{X}(\mathbf{n}))$. If we consider the Post algebra $\mathcal{R}S_0(\mathbf{B})$, we obtain:

$$\mathbf{X}(\mathbf{B}) \times \mathbf{X}(\mathbf{n}) \qquad [\cong \mathcal{X}^{\emptyset}(\mathbf{J}(\mathbf{B}))]$$

$$\mathbf{F}(\mathbf{X}(\mathbf{B}) \times \mathbf{X}(\mathbf{n})):$$

$$\{\langle a, 1\rangle, \langle b, 1\rangle, \langle a, 0\rangle, \langle b, 0\rangle\}$$

$$\{\langle a, 1\rangle, \langle 1, 0\rangle, \langle b, 1\rangle\} \quad \{\langle b, 1\rangle, \langle b, 0\rangle, \langle a, 1\rangle\}$$

$$\{\langle a, 1\rangle, \langle a, 0\rangle\} \quad \{\langle a, 1\rangle, \langle b, 1\rangle\} \quad \{\langle b, 1\rangle, \langle b, 0\rangle\}$$

$$\{\langle a, 1\rangle\} \quad \{\langle b, 1\rangle\}$$

$$\emptyset$$
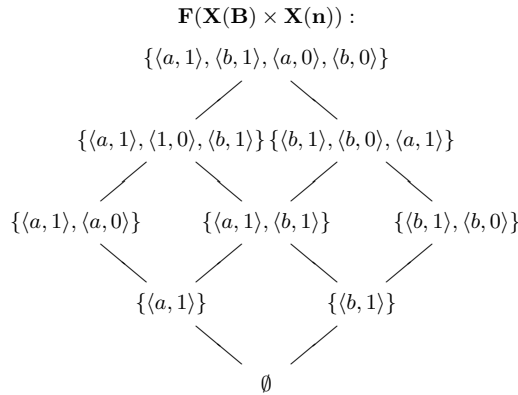
10.11 Frame – Representation of Logical Values by Ordered Pairs

The function k_f introduced in *Definition* 10.3.4 of Frame 10.3.2 is a specialization of that used in the "Polarity Theorem" of [Dunn, 1966]). Independently, but many years later, it was introduced in [Pagliani, 1990] in the framework of the representation theorem of logic $\mathcal{E}_0$. A few months later, John Michael Dunn wrote a letter to P. Pagliani in which Dunn's Dissertation was recalled together with a number of notes about the use of ordered pairs to model De Morgan lattices, paraconsistent logics and logics of entailment. Michael Dunn enclosed also a copy of [Dunn, 1986] and a manuscript.

On the basis of these documents and other sources we can sketch the following history.

Functor $N(\mathbf{D})$, for $\mathbf{D}$ a distributive lattice, was introduced in [Kalman, 1958] in order to construct De Morgan lattices. On the basis of [Białynicki-Birula & Rasiowa, 1957] and influenced by Rudolf Carnap and Bar-Hillel's concepts of a "content" (states that make a proposition false) and an "information" (states that make a proposition true), in his dissertation J. M. Dunn gave another proof of Kalman's results by means of lattices of ordered pairs $\langle A^+, A^-\rangle$. However Dunn did not require $A^+ \cap A^- = \emptyset$, thus opening the possibility to paraconsistency.[3]

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[3]Actually, this story is far from being complete if we do not mention S. Halldén's **The Logic of Nonsense** (Uppsala Univ. Arsskr) where in 1949 a semantics by (disjoint) ordered pairs was substantially introduces in order to interpret the notions of "α is *true* at state t ($t \in [\![\alpha]\!]^+$)", "α is *false* at state t ($t \in [\![\alpha]\!]^-$)" and "α is *acceptable* at state t ($t \notin [\![\alpha]\!]^-$)".

As to strong negation, later on the technique of ordered pairs was used in [Vakarelov, 1977] (where Białynicki-Birula and Rasiowa's work is cited, but not Dunn's), in [Pagliani, 1990] (where [Białynicki-Birula & Rasiowa, 1957] was used and [Vakarelov, 1977] was cited) and in [Sendlewski, 1990] which is essentially based on Vakarelov's work and where the general construction of the class **NA** was introduced. In [Sendlewski, 1990] the fact that **E** is a subvariety of the class **NA** is pointed out, but there is no reference to the logical properties of **E**. The logical meaning of the difference between **E** and **NA** was explored in [Pagliani, 1999] (here cf. Frames 10.3, 10.16 and 10.12.3).

In [Pagliani, 1993b] the construction $\mathbf{N}_\Theta(\mathbf{H})$ was applied in the case **H** is a Boolean algebra, to represent Rough Set Systems as semi-simple Nelson algebras (the result was already presented two years before at the University of Warsaw and at the Technische Hochschule of Darmstadt). In [Pagliani, 2000] the construction was restated in terms of Lawvere-Tierney operators and Grothendieck topologies.

10.12   Frame – Negation

In this Part we have dealt with different kinds of negation (namely, Intuitionistic, Classical, co-Intuitionistic and strong negations).

Indeed "negation" is one of the most controversial operator in Logic. Actually it is polymorphic and controversial even in Natural Language.

In [Westbury] we can find a classification of negations in Natural Language, form a psychological point of view. This is a slight modification:

Type	Use
Rejection	To reject or signal displeasure with an undesirable situation
Refusal	To signal a refusal to comply a request
Imperative	As a directive to other to act differently
Cognitive comment	To comment on his/her failure to achieve an intended goal
Scalar predication	Used for the concept of non-existence or to compare scalar values
Denial of propositions	To deny a stated utterance

The latter form is the fully cognitive linguistic use of the negation. Nonetheless, even in the use of this form we found, within the same group of Indoeuropean languages, such as Italian and English, different attitudes. For instance, in Italian the sentence "Non e' ne' bravo ne' bello", literally "He is not neither clever nor handsome", is a negative sentence where the second negative terms, "ne'" reinforces the first, "non". In English the literal translation is grammatically incorrect, but in any case, two negative terms would form an affirmative one, not a reinforcement of a negation.

In Formal Logic, as we have seen, we can distinguish different kinds and degrees of negation, but, first of all, we have to distinguish the syntactical view and the semantical view of negation.

As to the syntactical view, in the framework of Gentzen's Sequent Calculi, the negation is a change of position: from the "active" zone on the right, to the "passive" zone on the left of the sequent symbol:

$$\frac{\alpha, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\alpha} \qquad \frac{\Gamma \vdash \Delta, \alpha}{\neg\alpha, \Gamma \vdash \Delta}$$

It is of importance to note that in these rules we use sequents with multiple conclusions, that which is not allowed in Intuitionistic sequents.

This behaviour of negation has been interpreted from a game-theoretical point of view as swapping of roles between players. This is, indeed, a privileged interpretation in Linear Logic (cf. [Girard, 1989], [Lafont & Streicher, 1991], or [Abramsky & Jagadeesan, 1992]; for some critical issues see [Blass, 1994]).[4]

Therefore, if the negation of α, $\alpha^\perp$ in Linear Logic notation (the "dual of α"), is to be interpreted as the swapping from the role of proponent to the role of opponent, it follows that negation is involutive, i.e. $\alpha^{\perp\perp} \equiv \alpha$, because the proponent gives the opponent the move and the opponent gives it back to the proponent. In addition it is possible to show that the linear implication $\alpha \multimap \beta$ is equivalent to $\alpha^\perp \parr \beta$ so that $\alpha \multimap \alpha \equiv \alpha^\perp \parr \alpha$. But $\alpha \multimap \alpha$ is a thesis of Linear Logic, so it is a thesis $\alpha \parr \alpha^\perp$. Since $\parr$ is the intensional or multiplicative version of "or", we have the "intensional" version of the Excluded Middle. This fact can be explained in terms of Game Theory. Indeed, $\alpha \parr \beta$ may be interpreted as a protocol consisting of interleaved runs of the protocol for α and for β. Hence, suppose α is a move of Kasparov against Paul (a beginner),

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[4]Maybe this interpretation may be credited to Hintikka's **Logic, Language Games, and Information** (Claredon, 1973).

known as White, and $\alpha^{\perp}$ is a move of Short against Paul, as Black. Then the connective $\wp$ allows Paul to apply the following strategy: Paul repeats, against Short, any move of Kasparov against Paul, and duplicate against Kasparov any move of Short against himself. This is called the "copy-cat" strategy. Clearly, Paul is bound to win whoever, Kasparov or Short, wins.

10.12.1 Classifying Formal Negations

First of all, we refer the interested reader to the following notable collections of papers about negation: [Gabbay & Wansing, 1999] and [Wansing, 1996].

Summing-up the story, according to the elaboration of [Restall, 2002] on [Dunn, 1999], we can list the following properties involving negation:

Name	Abbrev.	Formalisation	Reference in the present Part
Contraposition	Cont	$\phi \vdash \psi \Rrightarrow \neg\psi \vdash \neg\phi$	(6.2.19)
Sub De Morgan	Sub dM	$\neg(\phi \vee \psi) \vdash \neg\phi \wedge \neg\psi$ $\neg\phi \vee \neg\psi \vdash \neg(\phi \wedge \psi)$	(6.1.8)
Constructive De Morgan	Const dM	$\neg\phi \wedge \neg\psi \vdash \neg(\phi \vee \psi)$	
Constructive contraposition	Const cont	$\phi \vdash \neg\psi \Rrightarrow \psi \vdash \neg\phi$	
Constructive double negation	Const $\neg\neg$	$\phi \vdash \neg\neg\phi$	(6.1.7)
Absurdity	Ab	$\phi \wedge \neg\phi \vdash \chi$	(6.2.22)
Classical De Morgan	Class dM	$\neg(\phi \wedge \psi) \vdash \neg\phi \vee \neg\psi$	(6.1.9)
Classical contraposition	Class cont	$\neg\phi \vdash \psi \Rrightarrow \neg\psi \vdash \phi$	
Classical double negation	Class $\neg\neg$	$\neg\neg\phi \vdash \phi$	(6.2.12)
Excluded middle	Ex	$\chi \vdash \phi \vee \neg\phi$	(6.1.11)

Then we can set the following table of systems (the negations in bold have been introduced by G. Restall):

	cont	sub dM	const dM	const cont	const ¬¬	ab	class dM	class cont	class ¬¬	ex
Subminimal (HL)	√	√								
Preminimal	√	√	√							
Galois	√	√	√	√	√					
Minimal (HM)	√	√	√	√	√					
Intuitionistic (HJ)	√	√	√	√	√	√				
De Morgan	√	√	√	√	√		√	√	√	
Ortho	√	√	√	√	√	√	√	√	√	√
Preminimal'	√	√					√			
Galois'	√	√					√	√	√	
Minimal'	√	√					√	√	√	
Paraconsistent	√	√					√	√	√	√

Some explanations and comments are in order.

Galois negation, or *split* negation, is a pair of two negation operators ◁ and ▷, which are defined in the presence of a Galois connection **O** ⊣$^{◁,▷}$ **O^{op}**. Hence we have for all α, β, $\alpha \vdash \triangleright\beta$ if and only if $\beta \vdash \triangleleft\alpha$ and $\alpha \vdash \triangleleft \triangleright \alpha$, $\alpha \vdash \triangleright \triangleleft \alpha$. The components of a split negation individually satisfy all the checked properties except *const cont* and *class cont* which here are read as bi-implications, and *const* ¬¬ and *class* ¬¬, where we have to substitute ◁▷ and ▷◁ for ¬¬.[5]

The Orthonegation is a classical negation, to all ends. What is missed in ortholattices (and orthologic) is distributivity (cf. also Part I, Frame 4.6.1).

A negation $\sim$ is called "strong", with respect to a negation $\neg$, if for all α, $\sim\alpha \implies \neg\alpha$. Once we set $\neg\alpha =_{def} \alpha \implies\sim \alpha$ we find in [Zeman, 1968] the following classification starting from the system HA of positive Intuitionistic logic, were the negations above the dotted line $S \cdots S$ are strong:

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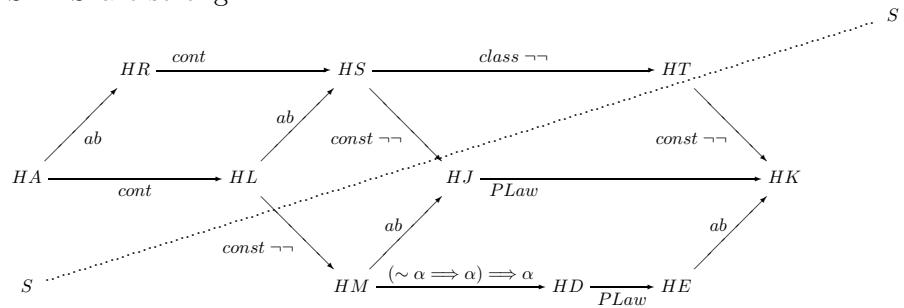
[5]Galois negation was introduced in [Dunn, 1991] (cf. also [Dunn & Hartonas, 1993] and the quoted collections).

where: (i) $PLaw$ is the Peirce's law $((\alpha \implies \beta) \implies \beta) \implies \beta$, (ii) $HD =$ strict negation (Johansson), (iii) $HE =$ classical refutability (Kripke), (iv) $HK =$ Classical negation.

10.12.2 A Geometric Interpretation of Negation

Post n-valued negation, Kleene strong three-valued and Łukasiewicz negations are analysed in terms of geometrical manipulations in [Varzi & Warglien, 2003]. Indeed, we can basically perform two kinds of inversions:
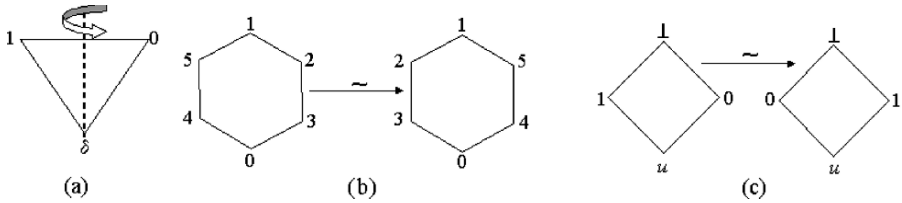
Figure 10.3: Symmetric inversion applied to (a) three values, (b) six values and (c) four values
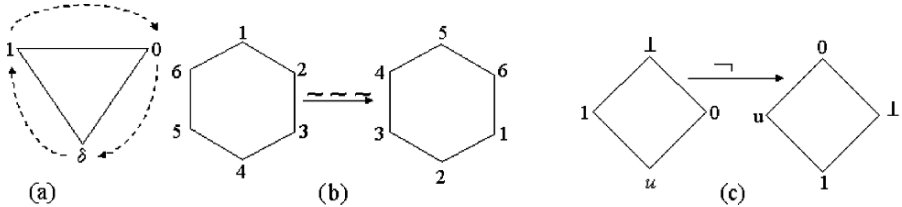
Figure 10.4: Cyclic inversion applied to (a) three values, (b) six values and (c) after applying a symmetric inversion to four values

In Figure 10.3 the three-valued lattice is a model of Kleene's strong three-valued logic, while the four-valued lattice is a model of Belnap's four-valued logic (cf. also bilattices in Frame 15.12 of Part III). In Figure 10.4, case (a) is a model of Post's three-valued logic case (c) corresponds to the convolution defined by Fitting on bilattices.

One can straightforwardly notice that in n-valued logics an n-fold cyclic negation is tantamount to an affirmation, while a 2-fold symmetric negation corresponds to an affirmation.

10.12.3 Strong Negations and Knowledge States in Artificial Intelligence

It is worth noticing that since the operator $\mathbf{T}$ was introduced in order to analyse the notion "it is classically true" within a constructive environment, so that $\mathbf{T}(\alpha)$ is provable in $\mathcal{E}_0$ if and only if α is provable in $\mathcal{CL}$, the classical negation has to be represented by the strong negation "$\sim$".

Probably for a similar reason, strong negation is called "classical" in [Gelfond & Lifshitz, 1990]. Usually, Logic Programming provides a (meta) negation as failure *not*: if the program cannot prove α then we can assume *not* α. This assumption is admissible under the Closed World Hypothesis which states that what is declared is all what exists. Thus, suppose in a program we have the clause $ingest(X) : - \ not \ side\_effect(Y, X)$, expressing that one can ingest a drug X if it does not have the side effect Y. Suppose there is no information about the presence of Y as a side effect of X, so that $not \ side\_effect(Y, X)$ is proved. However it may be very dangerous to Paul to intake X if he has a severe allergy to Y. We need, in this case an explicit fact $\sim (side\_effect(Y, X))$. Also, we should need a clause of the form $investigate(X, Y) : - not \ side\_effect(Y, X), not \sim (side\_effect(Y, X))$, which suggests some additional investigations in case of doubts.

This kind of reason suggests introducing an explicit negation which, although called "classic" in the quoted paper, it is recognized to be a Nelson negation also within the Logic Programming community – see, for instance, [Pearce & Wagner, 1990].

Systems with strong negation were "empirically" explored within Artificial Intelligence. An interesting example is the generation of hypotheses in machine learning. For instance, in [Delgrande, 1988] with each hypothesis two subsets are associated which denote the ground instances that we know to satisfy the hypothesis and the ground instances that we know to not satisfy it. This way an algebraic logic for forming conjectures is described as much as like a lattice of the kind $N_\Theta(\mathbf{B})$. Curiously enough, in the paper there is reference to Kalman's and Kleene's construction but no reference to the fact that the resulting structure is a three-valued Łukasiewicz algebra.

Delgrande's work is close to the approaches by Fagin, Halpern and Levesque to epistemic and doxastic logics. For instance in [Fagin & Halpern, 1985] and [Levesque, 1984] two "support relations" $\vDash^t$ and $\vDash^f$ are used, in a fashion that can be compared with Thomason's

relational models, once $w \vDash \sim \alpha$ is translated $w \vDash^f \alpha$ (see Sub-frame 10.16.2 and footnote 10.11).

In [Akama, 1987] the consequences on knowledge representation of the specific clauses of Kripke models for $\mathcal{E}_0$ are explicitly exploited to cope with some problems in Artificial Intelligence (but without any reference to the work of Miglioli's group). Indeed, Akama studies an equivalent system endowed with modal operators to face the "frame problem" in knowledge bases. In that paper, intuitively, it is required that any search for *complete information* must be successfully accomplished. On the basis of our previous discussion it is easy to understand why Akama satisfies this request by postulating that each maximal chain of possible worlds ends with a greatest element fulfilling a Boolean forcing. Hence the set of these elements is dense.

In any case, the Boolean behaviour captured by all these systems is, in our terminology, an internal local one. In fact, we have already noticed that the concept of "density" is definable by means of mere lattice theoretic notions. But we have seen that when $\mathbf{H}$ is a Boolean algebra, internal local behaviours drastically lose attraction, because $\mathbf{N}_\Delta(\mathbf{H})$ is isomorphic to $\mathbf{H}$.

However we have not to be confined to internal local behaviours: indeed we have shown in Frame 10.3, that in a Kripke model $\mathbf{K} = \langle \mathbf{W}, \leq, \vDash \rangle$ for Nelson logic if B is the set of maximal worlds with a Boolean forcing relation and if Θ is $\equiv_{J\phi(B)}$, then the element $\langle \phi(B), 0 \rangle$ is the greatest **locus** (the local top) in $\mathbf{N}_\Theta(\mathbf{F}(\mathbf{W}))$ in which Boolean identities hold, where $\phi(B)$ is the element of $\mathbf{F}(\mathbf{W})$ corresponding to B by duality. This is the way we uses $\mathbf{N}_\Theta$ in Rough Set Theory (see [Pagliani, 1998d]). Since in this case B does not have a lattice-theoretic individuality, a two-sorted deductive system characterized by a local (sorted) version of the operator $\mathbf{T}$ was suggested in [Pagliani, 1994].

10.12.4   Negations, Bodies and Boundaries

The operation $\lrcorner$ was probably introduced for the first time by the polish mathematician Cecylia Rauszer in [Rauszer, 1974]. Two years later, the properties of $\lrcorner$ within Heyting-Brouwer and three-valued Łukasiewicz algebras were studied by Luisa Iturrioz who in [Iturrioz, 1982] expanded her researches towards *Symmetric Heyting algebras of order n*, or SHn algebras. SHn algebras were introduced to give an algebraic account for Moisil's symmetrical modal propositional calculus, that is, an intu-

itionistic calculus augmented with an involutive negation satisfying the contraposition law. These algebras are strictly linked with Post algebras and can be seen as Łukasiewicz algebras equipped with a generalised negation (in [Iturrioz & Orłowska, 1996] a completeness theorem of SHn logics with respect to Kripke semantics was proved).

To give the reader a taste of the recent developments, we mention that Rauszer's system, renamed "Subtractive Logic" and with its semantics revisited as a *bi-Cartesian closed category*, was investigated in the framework of λ-calculus and Curry-Howard isomorphism, in [Crolard, 2004].

The negation provided by bi-Heyting algebras was independently exploited in [Lawvere, 1982] to give a logical account for the notions of a "boundary", "essential core of a body" and "sub-body", in the context of Continuum Physics.

Given an element a of a co-Heyting algebra **CH**, Lawvere calls $\lrcorner\lrcorner a$ the *regular core* of a. Generally $\lrcorner\lrcorner a \leq a$. It is claimed that a part a may be considered a *sub-body* (or shortly a *body*) if and only if $\lrcorner\lrcorner a = a$.

Everything is based on the fact that in co-Heyting algebras we can recapture the geometrical notion of a "boundary". Indeed Lawvere points out that this notion is definable by means of the co-intuitionistic negation, $\lrcorner$, in the following manner (for a belonging to any co-Heyting algebra):

$$\partial(a) = a \wedge \lrcorner a.$$

First of all, $\partial(a)$ is the boundary of a in a topological sense: if the given co-Heyting algebra is the system of all closed sets of a topological space, then a is a closed set. Thus $a \cap \lrcorner(a) = \mathbb{C}(a) \cap -\mathbb{I}(a)$, which is exactly the topological boundary, $\mathbb{B}(a)$, of a.

More generally, $\partial(a)$ is a boundary since for any a, b of any co-Heyting algebra, it formally fulfills the rules:

(1) $\partial(a \wedge b) = (\partial(a) \wedge b) \vee (a \wedge \partial(b))$;
(2) $\partial(a \wedge b) \vee \partial(a \vee b) = \partial(a) \vee \partial(b)$.

The first equation is called "Leibniz formula" by W. Lawvere who emphasizes that though its validity for boundaries of closed sets is supported by our space intuition (think of two partially overlapping ovals), nevertheless it is virtually unknown in general topology literature.

Indeed, we can notice that it is essentially the usual Leibniz rule for differentiation of a product (but see also the Grassmann rule). More-

over, Lawvere notices that any element a in a co-Heyting algebra is the join of its core and its boundary: $a = \lrcorner\lrcorner a \vee \partial(a)$.

In view of Lawvere's work, the notion of a co-Heyting boundary was exploited in [Pagliani, 1998a] in the context of Rough Set analysis, where Lawvere's intuitions fit perfectly.

In fact, given an Approximation Space $\mathbf{AS}(U)$, if $a = rs(X)$ for $X \subseteq U$, then $rs^{-1}(\lrcorner\lrcorner a) = (lE)(X)$, that is the *necessary* part of X, (in a literal sense when $\mathbf{AS}(U)$ is interpreted as an $\mathbf{S}5$ modal space – see Part III). Thus the notion of a "sub-body" coincides in Rough Set Systems with that of *greatest exact rough set* included in $rs(X)$, that is, rough sets "(deductively) closed" and "perfect", hence without any boundary.[6] In turn, the boundary of X is given by $rs^{-1}(a \wedge \lrcorner a)$ (or, equivalently, $rs^{-1}(a \wedge \sim a)$.

The above relationships suggest that it is possible to connect Rough Set Theory to other interesting topics in mathematics and in physics (see below Frame 10.12.5).

10.12.5   Negations, Modalities and Stalk Spaces

In this Part we have mostly seen negations as kinds of modality. Relationships between negations and modalities have been studied in [Došen, 1986], [Dunn, 1991] (and other papers), [Pagliani, 1990], [Restall 1997, 1998] and by some other authors.

Notably, in [Reyes & Zolfaghari, 1996], it is shown that bi-Heyting algebras feature some interesting general properties. Let us define two operators $\Box$ and $\Diamond$ as follows:

Definition 10.12.1. *Given an additionally complete bi-Heyting algebra* $\mathbf{BH}$, *$\forall a \in \mathbf{BH}$,*
(i) $\Box_0 = \Diamond_0 = Id$; (ii) $\Box_{n+1} = \neg\lrcorner\Box_n$, $\Diamond_{n+1} = \lrcorner\neg\Diamond_n$.
(iii) $\Box(a) = \bigwedge_{i=1}^{n} \Box_i(a)$; (iv) $\Diamond(a) = \bigvee_{i=1}^{n} \Diamond_i(a)$.

Then it is shown that for any $a, \Box(a)$ is the largest complemented element of $\mathbf{BH}$ below a, while $\Diamond(a)$ is the smallest complemented element above a.[7]

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[6]Not by chance we are using the terminology introduced by Leibniz to describe the notion of "individual substance" (*Discourse on Metaphysics*).

[7]The interested reader must take great care that in the quoted paper the co-intuitionistic negation $\lrcorner$ is denoted by $\sim$.

EXAMPLE

In order to avoid drawing too complicate Hasse diagrams of lattices, it is often more comfortable to work with their simpler dual spaces. Thus consider the Heyting space $\mathbf{W}$ depicted below:
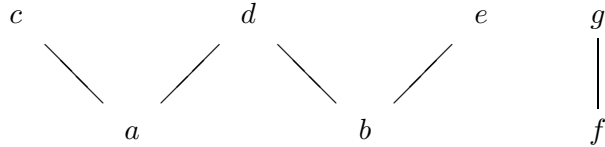
Given an order filter $A \in \mathscr{F}(\mathbf{W})$, we know that in the bi-Heyting algebra $\mathbf{H}(\mathbf{W})$, $\neg A$ is the smallest element B of $\mathscr{F}(\mathbf{W})$ such that $A \cup B = W$. Thus it is straightforward to verify that $\neg A =\uparrow -A$ (in fact $A \cup -A = W$, but $-A$ is an order ideal in $\mathbf{W}$, so that we have to take the smallest order filter containing $-A$). From *Definition* 7.2.1.(ii) we know that $\neg A = - \downarrow A$. With the above hint how to compute $\neg$ and $\neg$ we can immediately verify that in $\mathbf{H}(\mathbf{W})$, $\neg\{c\} = \{b, d, e, g, f\}$, $\neg\{b, d, e, g, f\} = \{a, c, d\}$, $\neg\{a, c, d\} = \{e, f, g\}$ and, finally, $\neg\{e, f, g\} = \{a, b, c, d, e\}$. Since $\neg\{a, b, c, d, e\} = \{f, g\}$ and $\neg\{f, g\} = \{a, b, c, d, e\}$, we have that $\neg\neg\{a, b, c, d, e\} = \{a, b, c, d, e\}$. Hence $\{a, b, c, d, e\}$ is the smallest complemented element greater than or equal to $\{c\}$ (indeed $\neg$ and $\neg$ coincide on $\{a, b, c, d, e\}$). Thus the sequence $\Diamond_i$ stabilizes in two steps.

As to $\Box$, $\neg\{e, f, g\} = \{a, b, c, d, e\}$, $\neg\{a, b, c, d, e\} = \{f, g\}$ and $\neg\{f, g\} = \{a, c, d, e\}$. Hence $\{f, g\}$ is the largest complemented element less than or equal to $\{e, f, g\}$. In this case the sequence $\Box_i$ stabilizes in one step. On the contrary, the sequence $\Box_i$ stabilizes in two steps if applied to $\{b, d, e\}$: $\Box_2\Box_1(\{b, d, e\}) = \Box_2(\{e\}) = \emptyset$.

We know that in Rough Set Systems $\Box = \Box_1$ and $\Diamond = \Diamond_1$. In other words both sequences $\Box_i$ and $\Diamond_i$ stabilize at step 1. This fact, as pointed out by Reyes and Zolfaghari for the general case, is related to the two De Morgan laws that, respectively, fail in Heyting and in co-Heyting algebras, generally:

Let $\mathbf{H}$ and $\mathbf{CH}$ be a Heyting and, respectively, a co-Heyting algebra. Then we say:

1. $\mathbf{H}$ satisfies the De Morgan law for $\neg$, if $\neg(x \wedge y) = \neg x \vee \neg y$, $\forall x, y$.

2. $\mathbf{CH}$ satisfies the De Morgan law for $\neg$, if $\neg(x \vee y) = \neg x \wedge \neg y$, $\forall x, y$.

One can show that in bi-Heyting algebras the De Morgan law for $\neg$ implies $\Box(a) = \neg\lrcorner a$ and that the law for $\lrcorner$ implies $\Diamond(a) = \lrcorner\neg a$. The reverse of these implications does not hold in general, as we are going to see in the next example.

EXAMPLE
Consider the two Heyting spaces $\mathbf{W}$ and $\mathbf{W}'$ below:
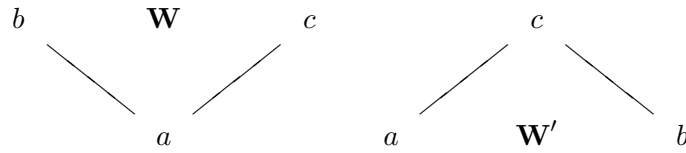
For any element $A \neq W$ of $\mathbf{H}(\mathbf{W})$, $\lrcorner A = W$ while $\lrcorner W = \emptyset$. It follows that $\Box(A) = \emptyset$ if $A \neq W$ and $\Box(W) = W$, so that the sequence $\Box_i$ always stabilizes at step 1. However the De Morgan law for $\neg$ does not hold: $\neg(\{b\} \cap \{c\}) = \neg\emptyset = W \neq \neg\{b\} \cup \neg\{c\} = \{c\} \cup \{b\} = \{c, b\}$.

Symmetrically, for any element $A \neq \emptyset$ of $\mathbf{H}(\mathbf{W}')$, $\neg A = \emptyset$, while $\neg\emptyset = W$. It follows that $\Diamond(A) = W$ for all $A \neq \emptyset$ and $\Diamond(\emptyset) = \emptyset$, so that the sequence $\Diamond_i$ always stabilizes at step 1. However the De Morgan law for $\lrcorner$ does not hold: $\lrcorner(\{a, c\} \cup \{b, c\}) = \lrcorner W = \emptyset \neq \lrcorner\{a, c\} \cap \lrcorner\{b, c\} = \{b, c\} \cap \{a, c\} = \{c\}$.

However, one can verify that both De Morgan laws actually hold in Rough Set Systems.

The general reason is discussed in [Johnstone, 1977] where it is proved that in a Heyting algebra $\mathbf{H}$ the De Morgan law for $\neg$ is equivalent to the fact that $Reg(\mathbf{H})$ is a sublattice of $\mathbf{H}$. Dually for $\lrcorner$ in co-Heyting algebras. But this is precisely the case for $\mathcal{RS}_x(\mathbf{A})$, $\mathbf{A}$ any Boolean algebra, x any element of $\mathbf{A}$.

In [Reyes & Zolfaghari, 1996] the above arguments are presented within the framework of presheaf topoi.

This is not a surprise. Indeed from *Proposition* 8.3.5 we have that the maps ϕ_s and η_s from $\mathcal{CTR}(\mathcal{RS}_x(\mathbf{A}))$ to $\mathcal{RS}_x(\mathbf{A})$, defined as $\phi_s(e) = e \wedge s$ and $\eta_s(e) = e \vee s$, are residuated. It happens that algebraic structures with these residuals enjoy a *stalk-space* representation (see [Crown et al. 1996]).

In these notes we can just say that any rough set is a *section* of this stalk space.

10.13   Frame – Intuitionistic Logic: Natural Deduction System $\mathcal{INT}$

Here we introduce the deductive systems of Heyting logic. We assume the Predicative Natural Deduction Calculus for $\mathcal{INT}$ (see, for instance, [Prawitz, 1965]).

$$\alpha \quad (assumption). \qquad \frac{\alpha \quad \beta}{\alpha \wedge \beta} \ (\wedge - int); \qquad \frac{\alpha \wedge \beta}{\alpha} \qquad \frac{\alpha \wedge \beta}{\beta} \ (\wedge - elim);$$

$$\frac{\begin{array}{c}[\alpha]\\ \vdots\\ \bot\end{array}}{\neg\alpha} \ (\neg - int); \quad \frac{\alpha}{\alpha \vee \beta} \qquad \frac{\beta}{\alpha \vee \beta} \ (\vee - int); \quad \frac{\alpha \vee \beta \quad \begin{array}{c}[\alpha]\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}[\beta]\\ \vdots\\ \gamma\end{array}}{\gamma} \ (\vee - elim);$$

$$\frac{\alpha(p)}{\forall x\alpha(x)}(\forall - int); \qquad \frac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array}}{\alpha \longrightarrow \beta} \ (\longrightarrow -int); \qquad \frac{\alpha \quad \alpha \longrightarrow \beta}{\beta} \ (\longrightarrow -elim);$$

$$\frac{\forall x\alpha(x)}{\alpha(t)}(\forall - elim); \qquad \frac{\alpha(t)}{\exists x\alpha(x)}(\exists - int); \qquad \frac{\exists x\alpha(x) \quad \begin{array}{c}[\alpha(p)]\\ \vdots\\ \gamma\end{array}}{\gamma} \ (\exists - elim);$$

$$\frac{\neg\alpha \quad \alpha}{\beta} \ (contr);$$

Notice that

- The parameter p in $\forall - int$ cannot be free in any non discharged assumption that $\alpha(p)$ depends on. The parameter p in $\exists - elim$ cannot be free in $\alpha(x)$, in γ or in any assumption that γ depends on, except $\alpha(p)$.

- The assumptions within brackets in a rule, for instance $[\alpha]$, are discharged after the application of the rule.

- In the rule for the introduction of *negation*, $\bot$ is any contradiction. Actually $(\neg - int)$ is an instance of $(\longrightarrow -int)$.

- In the restricted version of the rule (*contr*), the consequence β is any atomic formula. However, the unrestricted version is derivable from the restricted one. This prove that (*contr*) does not introduce formulas of arbitrary complexity.

10.14   Frame – Classical Logic: Natural Deduction System $\mathcal{CL}$

The classical calculus $\mathcal{CL}$ is obtained by adding to $\mathcal{INT}$ the following rule:

$$
\begin{array}{c}
[\neg\alpha] \\
\vdots \\
\dfrac{\bot}{\alpha} \ (CCR);
\end{array}
$$

In this rule a formula of arbitrary complexity may be introduced. This gives $\mathcal{CL}$ a particular strength together with the particular non-constructive flavour which characterises Classical Logic.

10.14.1   Deriving the Principle of Excluded Middle from $\mathcal{CL}$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\alpha_{[1]}}{\alpha\vee\neg\alpha} \qquad \neg(\alpha\vee\neg\alpha)_{[3]}}{\bot}}{\neg\alpha}\ dis.1 \qquad \cfrac{\cfrac{\cfrac{\neg\alpha_{[2]}}{\alpha\vee\neg\alpha} \qquad \neg(\alpha\vee\neg\alpha)_{[3]}}{\bot}}{\alpha}\ dis.2}{\cfrac{\bot}{\neg\alpha\vee\alpha}\ CCR\ dis.3}
$$

The expression "*dis.n*" on the right of an inference step, says that after the step we discharge the assumption(s) marked with $[n]$.

The reader here should note that the proof strictly depends on the application of the Classical Contradiction Rule, CCR (last step). In Frame 10.17.2 we shall constructively analyse this proof from the point of view of the Evaluation Form Semantics.

10.15   Frame – Nelson Logic: Natural Deduction System $\mathcal{CLSN}$

10.15.1   Constructive Logics with Strong Negation

$\mathcal{CLSN}$ is also termed "Nelson Logic for Constructible Falsity". As we have already seen, Nelson noted that in $\mathcal{INT}$ negation is not constructive in that we can deduce $\neg(\alpha\wedge\beta)$ without being able to deduce either $\neg\alpha$ or $\neg\beta$. Indeed, the De Morgan rule $\neg(\alpha\wedge\beta)\equiv\neg\alpha\vee\neg\beta$ fails in $\mathcal{INT}$. Henceforth Nelson introduced a strong negation $\sim$ for which both De

Morgan rules are valid although $\alpha \vee \sim \alpha$ still fails. Actually, the validity of both De Morgan rules is connected with the already discussed requirement that in a real constructive approach, the verification of a proof for α should recursively account for the sub-proofs of any structural component of α, even if α is negative. The original motivations of $\mathcal{CLSN}$ are in [Nelson, 1949].

The adoption of a strong form of negation seems to come straightforwardly from the constructivistic approach. In fact, if the interpretation of a formula α is given by its provability-conditions, then it is not wise to assume (as Intuitionism does) that proving $\neg \alpha$ amounts to proving that α is not provable. On the contrary we have to explicitly analyse both positive and negative verification-conditions for α. Hence any step in a proof of α must be analysed as to its negated or non negated ingredients. Thus, while in $\mathcal{INT}$ the introduction of $\neg$ is uniformly given by the rule stating that if from α we obtain a contradiction then we deduce $\neg \alpha$, regardless to the specific structure of α, on the contrary in $\mathcal{CLSN}$ for any connective we have to list both its positive and negative introduction and elimination rules in order to be able to analyse the structure of any proof step as recursively related to the structural complexity of the proved formula (a requirement which motivates Thomason's relational models).

10.15.2   Natural deduction system for $\mathcal{CLSN}$

We now introduce the deductive system for Constructive Logic with Strong Negation ($\mathcal{CLSN}$). The system $\mathcal{CLSN}$ is obtained by adding the following rules to $\mathcal{INT}$:

$$\frac{\sim\alpha}{\sim(\alpha \wedge \beta)} \qquad \frac{\sim\beta}{\sim(\alpha \wedge \beta)}\ (\sim \wedge -int); \qquad\qquad \frac{\sim(\alpha \wedge \beta) \quad \overset{[\sim\alpha]}{\overset{\vdots}{\gamma}} \quad \overset{[\sim\beta]}{\overset{\vdots}{\gamma}}}{\gamma}\ (\sim \wedge -elim);$$

$$\frac{\sim\alpha \quad \sim\beta}{\sim(\alpha \vee \beta)}\ (\sim \vee -int); \qquad\qquad \frac{\sim(\alpha \vee \beta)}{\sim\alpha} \qquad \frac{\sim(\alpha \vee \beta)}{\sim\beta}\ (\sim \vee -elim);$$

$$\frac{\alpha \quad \sim\beta}{\sim(\alpha \longrightarrow \beta)}\ (\sim \longrightarrow -int); \qquad\qquad \frac{\sim(\alpha \longrightarrow \beta)}{\alpha} \qquad \frac{\sim(\alpha \longrightarrow \beta)}{\sim\beta}\ (\sim \longrightarrow -elim);$$

$$\frac{\alpha}{\sim\sim\alpha}(\sim - int); \qquad\qquad \frac{\sim\sim\alpha}{\alpha}(\sim \sim -elim); \qquad \frac{\sim\alpha \quad \alpha}{\beta}(contr);$$

$$\frac{\exists x \sim \alpha(x)}{\sim\forall x\alpha(x)}(\sim\forall - int); \qquad\qquad \frac{\sim\forall x\alpha(x)}{\exists x \sim \alpha(x)}(\sim\forall - elim);$$

$$\frac{\forall x \sim \alpha(x)}{\sim\exists x\alpha(x)}(\sim\exists - int); \qquad\qquad \frac{\sim\exists x\alpha(x)}{\forall x \sim \alpha(x)}(\sim\exists - elim);$$

It is worth noticing that the law of contraposition does not hold for constructive negation. Hence also the law of substitution of equivalents fails. Here is an example:

(i) $\vdash_{\overline{\mathcal{CLSN}}} \sim\sim (\alpha \longrightarrow \alpha) \longleftrightarrow (\alpha \longrightarrow \alpha)$ and (ii) $\vdash_{\overline{\mathcal{CLSN}}} \alpha \longrightarrow \alpha$. Moreover,
(iii) $\vdash_{\overline{\mathcal{CLSN}}} \sim (\alpha \longrightarrow \alpha) \longleftrightarrow (\alpha \wedge \sim \alpha)$ and (iv) $\vdash_{\overline{\mathcal{CLSN}}} \sim (\alpha \wedge \sim \alpha) \longleftrightarrow (\alpha \vee \sim \alpha)$.
However, we cannot conclude $\vdash_{\overline{\mathcal{CLSN}}} (\alpha \longrightarrow \alpha) \longleftrightarrow (\alpha \vee \sim \alpha)$, because $\nvdash_{\overline{\mathcal{CLSN}}} \sim (\alpha \wedge \sim \alpha) \longleftrightarrow \sim\sim (\alpha \longrightarrow \alpha)$. Otherwise stated, we cannot substitute $\alpha \wedge \sim \alpha$ for $\sim (\alpha \longrightarrow \alpha)$ in (i). In general, indeed, $\nvdash_{\overline{\mathcal{CLSN}}} (\alpha \longrightarrow \beta) \longrightarrow (\sim \beta \longrightarrow \sim \alpha)$.

Incidentally, this proves that the Excluded Middle is not derivable even for strong negated formulae, in spite of the fact that both the De Morgan rules hold.

Look at the algebraic evidence:

$a \longrightarrow a = \langle a_1 \Longrightarrow a_1, a_1 \wedge a_2 \rangle = \langle 1, 0 \rangle$,
$a \wedge \sim a = \langle a_1 \wedge a_2, a_2 \vee a_1 \rangle = \langle 0, a_2 \vee a_1 \rangle$,
$a \vee \sim a = \langle a_1 \vee a_2, a_2 \wedge a_1 \rangle = \langle a_1 \vee a_2, 0 \rangle$.
It follows that $\sim (a \wedge \sim \alpha) = (a \vee \sim a)$.
On the contrary, $\sim (a \wedge \sim a) \leq (a \longrightarrow a)$, because $(a_1 \vee a_2) \leq 1 = (a_1 \Longrightarrow a_1)$.

Finally, because of these weaker properties of the implication, the rule $(\neg - int)$ does no longer define an intuitionistic negation in this contest, but the weaker negation "$\lrcorner$". To recover the power of $\neg$ we need to define an operator which checks both the positive and the negative part of a formula (cf. Section 9.6). That which we achieve with system $\mathcal{E}_0$.

10.16   Frame – The System $\mathcal{E}_0$

$\mathcal{E}_0$, obtained by adding the two **T**-rules to $\mathcal{CLSN}$, was introduced by P. Miglioli in 1979 and its properties were studied together with M. Ornaghi, G. Usberti and P. Pagliani in the same year. It was almost immediately clear that P. Miglioli was aiming at a particular and rather new direction. Outside Miglioli's group, the first scholar who fully appreciated the strength of the **T**-rules, was Helena Rasiowa, three years later, during a meeting with P. Pagliani. Indeed, how peculiar

and strong they were, it is testified by the radical changes in the meaning of the logical constants that they induce in $\mathcal{CLSN}$ – as we were able to appreciate from the algebraic discussion in the present Part.

10.16.1 The Natural Deduction System $\mathcal{E}_0$

To obtain propositional $\mathcal{E}_0$ we have to add the two **T**-rules to $\mathcal{CLSN}$:

$$\cfrac{\begin{array}{cc} [\sim\alpha] & [\sim\alpha] \\ \vdots & \vdots \\ \beta & \sim\beta \end{array}}{\mathbf{T}(\alpha)} \; (\mathbf{T}-int); \qquad \cfrac{\begin{array}{cc} [\alpha] & [\alpha] \\ \vdots & \vdots \\ \beta & \sim\beta \end{array}}{\sim\mathbf{T}(\alpha)} \; (\sim\mathbf{T}-int)$$

where "$\sim$" is the constructive negation.

10.16.2 Relational Models for Nelson Logic

A relational model $\mathbf{K} = \langle W, \leq, \models \rangle$ for Nelson logic is a Kripke model for Intuitionistic logic with, in addition, a set up for strongly negated formulas with the following two clauses: (i) no $w \in W$ can force both p and $\sim p$, (ii) if $w \Vdash \sim p$ and $w \leq w'$ then $w' \Vdash \sim p$. The clauses for strongly negated formulas are ([Thomason, 1969]):

$w \models \sim (\alpha \vee \beta)$ iff $w \models \sim \alpha$ and $w \models \sim \beta$;
$w \models \sim (\alpha \wedge \beta)$ iff $w \models \sim \alpha$ or $w \models \sim \beta$;
$w \models \sim (\alpha \longrightarrow \beta)$ iff $w \models \alpha$ and $w \models \sim \beta$;
$w \models \sim\sim \alpha$ iff $w \Vdash \alpha$.

10.16.2.1 Relational Models for $\mathcal{E}_0$

A relational model for $\mathcal{E}_0$ is a Kripke model for Nelson logic in which it is postulated that for any atomic formula p and for any possible world w, there exists a maximal possible world $w' \geq w$ such that $w' \models p \vee \sim p$. That is, the set of worlds for which the forcing relation $\models$ is Boolean is dense in these models. This is exactly what is discussed in Frame 10.3 and it is the reason why one has to use N_Δ for modeling $\mathcal{E}_0$. In fact, Δ is induced by the filter of all and only the dense elements of the dual lattice $\mathbf{F}(\mathbf{W})$.

10.16.3 Logic and Algebra in Partnership

The analysis carried on so far precisely explicates the fundamental links between the syntactic level and the topo-algebraic level. In the following

schema we resume these links, for a Heyting algebra $\mathbf{H}$ dual to a partial order $\mathbf{W} = \langle W, \leq \rangle$, used as Kripke frame:

Algebra	Topology	Relational Models	Logic
$S = maximal(\mathbf{W})$; Δ = minimal Boolean congruence on $\mathbf{H}(\mathbf{W})$	$\mathbb{I} - (A_1 \cup A_2) = \emptyset$	for any state s $s \not\models \neg(\alpha \vee \sim \alpha)$ any α	$\vdash_{\mathcal{E}_0} \mathbf{T}(\alpha \vee \sim \alpha)$
$\uparrow S$ is the filter of all dense elements of $\mathbf{H}(\mathbf{W})$; $A_1 \vee A_2 \Delta 1$ iff $A_1 \vee A_2$ is dense	$A_2 = \emptyset$ iff A_1 is dense	at any maximal state s $s \models \alpha$ for any classical tautology α	If α is a classical tautology then $\vdash_{\mathcal{E}_0} \mathbf{T}(\alpha)$
$\mathbf{H}(\mathbf{W})/_\Delta$ is isom. to the Boolean algebra of the regular elements of $\mathbf{H}(\mathbf{W})$	$\mathbb{IC}(A_2) = \mathbb{I}(-A_1)$	at any maximal state s $s \models \alpha$ or $s \models \sim \alpha$ (any α)	$\vdash_{\mathcal{E}_0} \neg\neg\alpha \Longleftrightarrow \mathbf{T}(\alpha)$ $\vdash_{\mathcal{E}_0} \sim \mathbf{T}(\alpha) \Longleftrightarrow \mathbf{T} \sim (\alpha)$

where:

- α is any sentence and $\langle A_1, A_2 \rangle$ is the interpretation of α into $\mathbf{N}_\Delta(\mathbf{H}(\mathbf{W}))$.

- $Maximal(\mathbf{W})$ is the set of maximal elements in $\mathbf{W}$.

- $\mathbb{I}$ and $\mathbb{C}$ are the interior and, respectively, closure operators of $\mathbf{H}(\mathbf{W})$ *qua* frame of open subsets of a topological space on W.

- $\alpha \rightarrowtail \alpha =_{def} \alpha \longrightarrow \alpha \wedge \sim \beta \longrightarrow \sim \alpha$ and $\alpha \Longleftrightarrow \beta =_{def} \alpha \rightarrowtail \beta \wedge \beta \rightarrowtail \alpha$.

On this basis we can analyse syntactical proofs from an algebraic point of view.

10.16.4   Algebraic Analysis of the Characteristic Proofs of $\mathcal{E}_0$

Taken singularly, the two $\mathbf{T}$-rules resemble the Intuitionistic rule $(\neg - int)$ of Frame 10.13. More precisely, in view of our discussion about the weaker properties induced by the rules $(\longrightarrow -int)$ and $(\neg - int)$ in $\mathcal{CLSN}$, $(\mathbf{T} - int)$ looks like a "$\lrcorner \sim$" introduction, while $(\sim \mathbf{T} - int)$ looks like a "$\lrcorner$" introduction (otherwise stated, what in $\mathcal{INT}$ is $(\neg - int)$, in $CLSN$ turns into a $(\lrcorner - int)$ rule).

We shall see that, on the contrary, the two rules together define a very peculiar system that cannot be grasped by the two negations, $\lrcorner$

and $\sim$, at our disposal. We prove this claim by a parallel proof/algebraic analysis.

The equivalence $\sim\mathbf{T}(\alpha) \Longleftrightarrow \mathbf{T}(\sim\alpha)$ can be proved only by means of both the two $\mathbf{T}$-rules and can be regarded as a characteristic tautology of system $\mathcal{E}_0$. In this section we shall follow the steps of the natural deduction of this formula, translating them into relevant algebraic features that will prove that *Effective Lattices* constitute the appropriate algebraic structures for $\mathcal{E}_0$.

Let us then analyze this natural deduction: it is achieved by means of four subproofs (points A^*, B^*, C^* and D^* mark the critical steps):

Proposition 10.16.1. $\mathcal{E}_0 \vdash \sim \mathbf{T}(\alpha) \Longleftrightarrow \mathbf{T} \sim (\alpha)$.

Proof.

$$
A \qquad
\frac{
  \dfrac{
    \dfrac{\sim(\sim\alpha)_{[1]} \qquad \sim\alpha_{[2]}}{\sim(\sim\alpha) \qquad \sim\alpha}
  }{
    \dfrac{\sim\mathbf{T}(\sim\alpha)_{[3]} \qquad \mathbf{T}(\sim\alpha)}{\mathbf{T}(\alpha)}
  }
}{
  \sim\mathbf{T}(\sim\alpha) \longrightarrow \mathbf{T}(\alpha)
}
\qquad
\begin{array}{l}(dis.1)(A^*)\\(dis.2)\\(dis.3)\end{array}
$$

$$
B \qquad
\frac{
  \dfrac{
    \dfrac{\alpha_{[1]} \qquad (\sim\alpha)_{[2]}}{\sim(\sim\alpha) \qquad (\sim\alpha)}
  }{
    \dfrac{\mathbf{T}(\alpha)[3] \qquad \sim\mathbf{T}(\alpha)}{\sim\mathbf{T}(\sim\alpha)}
  }
}{
  \mathbf{T}(\alpha) \longrightarrow \sim\mathbf{T}(\sim\alpha)
}
\qquad
\begin{array}{l}(dis.1)(B^*)\\(dis.2)\\(dis.3)\end{array}
$$

$$
C \qquad
\frac{
  \dfrac{
    \dfrac{\sim(\sim\alpha)_{[1]} \qquad \sim\alpha_{[2]}}{\sim(\sim\alpha) \qquad \sim\alpha}
  }{
    \dfrac{\sim\mathbf{T}(\alpha)_{[3]} \qquad \mathbf{T}(\alpha)}{\mathbf{T}(\sim\alpha)}
  }
}{
  \sim\mathbf{T}(\alpha) \longrightarrow \mathbf{T}(\sim\alpha)
}
\qquad
\begin{array}{l}(dis.2)(C^*)\\(dis.1)\\(dis.3)\end{array}
$$

$$
D \qquad
\frac{
  \dfrac{
    \dfrac{\alpha_{[1]} \qquad (\sim\alpha)_{[2]}}{\sim(\sim\alpha) \qquad (\sim\alpha)}
  }{
    \dfrac{\mathbf{T}(\sim\alpha)_{[3]} \qquad \sim\mathbf{T}(\sim\alpha)}{\sim\mathbf{T}(\alpha)}
  }
}{
  \mathbf{T}(\sim\alpha) \longrightarrow \sim\mathbf{T}(\alpha)
}
\qquad
\begin{array}{l}(dis.2)(D^*)\\(dis.1)\\(dis.3)\end{array}
$$

Notice that the following algebraic analysis of the above proof can be regarded as a part of the completeness theorem for $\mathcal{E}_0$ with respect to Effective Lattices.

First of all it is worth pointing out that in these proofs the notations "$\sim (\sim\alpha)$" and "$(\sim\alpha)$", redundant from a formal point of view, underline the "encapsulation" of the strongly negated formula α in order to suitably apply the rules $(\mathbf{T} - int)$ and $(\sim\mathbf{T} - int)$ (since the $\mathbf{T}$-rules are defined for any formula these encapsulations are legal).

The crucial points in the proof are $(A^*), (B^*), (C^*)$ and (D^*): if we consider that the two $\mathbf{T}$ rules are a "$\lrcorner$-introduction" and a "$\lrcorner \sim$-introduction" rule, in view of $\sim\sim\alpha = \alpha$ we can synthesize these crucial points by means of the two following diagrams:
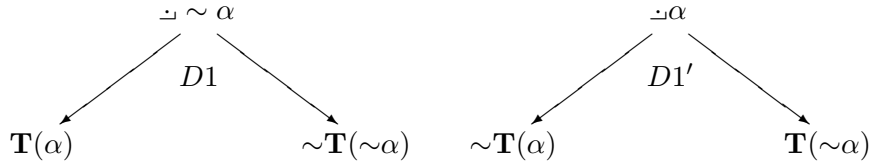
Diagram $D1$ represents the branching performed by step C^* and, respectively, D^* while discharging hypothesis [2] (i.e. "$\sim\alpha$").

Diagram $D1'$ represents the branching performed by step B^* and, respectively, A^* while discharging hypothesis [1] (i.e. "α").

Now, given an interpretation v s.t. $v(\alpha) = a = \langle a_1, a_2\rangle$, for a belonging to a Nelson algebra $\mathbf{N}_\Theta(\mathbf{H})$, for some Heyting algebra $\mathbf{H}$ and Boolean congruence Θ, if we assume semantically $\mathbf{T}(a) = \lrcorner \sim a$ (which at first sight could look like a good candidate in view of rule $(\mathbf{T}\text{-int})$), then from the algebraic fact $\sim \lrcorner a \longrightarrow \lrcorner \sim a = 1$, we obtain the following diagrams (the arrows represent the weak implication in $\mathbf{N}_\Theta(\mathbf{H})$):
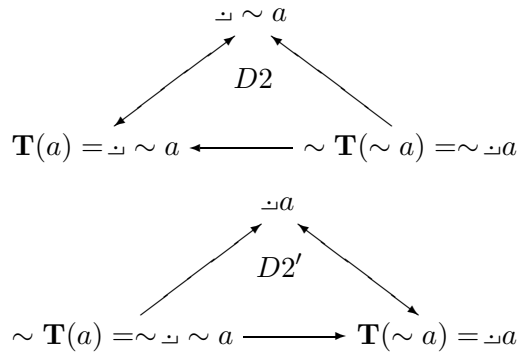
In order to interpret the diagrams $D1$ and $D1'$ in $D2$ and, respectively, $D2'$, we have to add the reverse of the arrows originated in $\sim \urcorner a$ and, respectively, in $\sim \urcorner \sim a$. Henceforth we must have $\sim \urcorner a \longleftrightarrow \urcorner \sim a$ and, respectively, $\urcorner a \longleftrightarrow \sim \urcorner \sim a$. Thus we must have $\neg a_2 = a_1$ and $\neg a_1 = a_2$.

Hence $\neg\neg a_2 = \neg a_1 = a_2$ and $\neg\neg a_1 = \neg a_2 = a_1$. So the intuitionistic negation $\neg$ must be an involutive pseudo-complementation in the underlying Heyting algebra $\mathbf{H}$. It follows that we can suitably complete the diagrams $D2$ and $D2'$ just in the case of $\mathbf{H}$ Boolean and Θ minimal (that is, if the resulting Nelson algebra is itself a Boolean algebra).

If we interpret $\mathbf{T}$ as $\sim \urcorner$ we obtain the diagrams (dual to the latter two):

$$
\begin{array}{c}
\urcorner \sim a \\[4pt]
\nearrow \quad \searrow \quad D3 \\[6pt]
\mathbf{T}(a) =\sim \urcorner a \longrightarrow \sim\mathbf{T}(\sim a) = \urcorner \sim a
\end{array}
$$

$$
\begin{array}{c}
\urcorner a \\[4pt]
\nearrow \quad \searrow \quad D3' \\[6pt]
\sim\mathbf{T}(a) = \urcorner a \longleftarrow \mathbf{T}(\sim a) =\sim \urcorner \sim a
\end{array}
$$
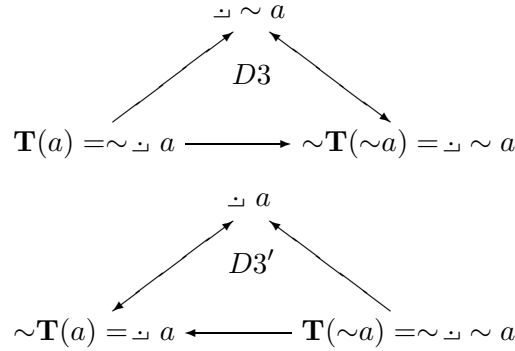
and the preceding considerations about the completion of the diagrams apply again.

If we assume semantically $\mathbf{T}(a) = \urcorner \urcorner a$ we obtain the diagrams:

$$
\begin{array}{c}
\urcorner \sim a \\[4pt]
\nearrow \quad \searrow \quad D4 \\[6pt]
\mathbf{T}(a) = \urcorner\urcorner a \longleftarrow \sim\mathbf{T}(\sim a) =\sim \urcorner\urcorner \sim a
\end{array}
$$

$$
\begin{array}{c}
\urcorner a \\[4pt]
\nearrow \quad \searrow \quad D4' \\[6pt]
\sim\mathbf{T}(a) =\sim \urcorner\urcorner a \longleftarrow \mathbf{T}(\sim a) = \urcorner\urcorner \sim a
\end{array}
$$
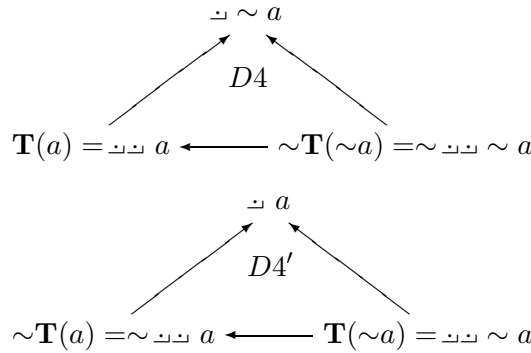
To complete these diagrams with arrows from $D1$ and $D1'$ we must have $\neg a_2 = \neg\neg a_1$ and $\neg\neg a_2 = \neg a_1$. In other words, if we assume $\mathbf{T}(a) = \urcorner\urcorner a$ we have $\langle \neg a_1, \neg\neg a_1 \rangle = \langle \neg\neg a_2, \neg a_2 \rangle$.

Therefore we can suitably complete the diagrams $D4$ and $D4'$ in the case of Θ minimal and $\mathbf{H}$ any Heyting algebra: in fact from the hypothesis of minimality of Θ it follows that $\neg a_1 = \neg\neg a_2$, henceforth $\neg\neg a_1 = \neg\neg\neg a_2 = \neg a_2$. Thus, if we assume Θ to be minimal, we can define $\mathbf{T}(a) = (a \longrightarrow 0) \longrightarrow 0 = {\lrcorner}{\lrcorner}a = \langle \neg\neg a_1, \neg a_1 \rangle = \langle \neg a_2, \neg\neg a_2 \rangle = \urcorner\urcorner a$.

On the other hand, if we assume $\mathbf{T}(a) = \langle \neg a_2, \neg\neg a_2 \rangle$ the algebraic relationships corresponding to the rules $(\mathbf{T} - int)$ and $(\sim\mathbf{T} - int)$ (that is $a \longrightarrow 0 =\sim \mathbf{T}(a)$ and $\sim a \longrightarrow 0 = \mathbf{T}(a)$) are fulfilled only if Θ is minimal.

For instance, we can inspect the lattices $N_{\equiv_{J^d}}(\mathbf{A})$ and $N_{\equiv_{J^b}}(\mathbf{A})$ of Example 9.6.1. We know that $N_{\equiv_{J^d}}(\mathbf{A})$ is an Effective Lattice and equals $N_{\equiv_{J^b}}(\mathbf{H})$ minus the elements $\langle 0, b \rangle$ and $\langle b, 0 \rangle$. We can verify that $\mathbf{T}(\langle 0, b \rangle) = {\lrcorner}{\lrcorner}\langle 0, b \rangle = \langle 0, 1 \rangle \leq \langle c, b \rangle = \urcorner\urcorner\langle 0, b \rangle$ and $\sim\mathbf{T}(\langle 0, b \rangle) =\sim {\lrcorner}{\lrcorner}\langle 0, b \rangle = \langle 1, 0 \rangle \geq \langle b, c \rangle = \urcorner\urcorner\langle 0, b \rangle = \mathbf{T}(\sim\langle 0, b \rangle)$.

On the contrary, for all the elements of $N_{\equiv_{J^d}}(\mathbf{H})$ all these inequalities turn into equalities.

Finally, let us recall that any three-valued Łukasiewicz algebra $\mathbf{L}$ is equipped with an endomorphism ϕ_1 that projects any element $a \in \mathbf{L}$ onto the largest element of the centre of $\mathbf{L}$ less than or equal to a as well as an endomorphism ϕ_2 that fprojects any element $a \in \mathbf{L}$ onto the least element of the centre of $\mathbf{L}$ greater than or equal to a. Since the centre of $\mathbf{L}$ is the family of elements fulfilling the Boolean property of Excluded Middle it makes sense to ask if there is some relationship between these two operators and the operator $\mathbf{T}$.

Indeed we have shown in this Part that in a three-valued Łukasiewicz algebra $\mathbf{L}$, ϕ_1 corresponds to "$\sim {\lrcorner}$" and ϕ_2 corresponds to "${\lrcorner} \sim$" in semi-simple Nelson algebras.

From the above discussion, we obtain that if a three-valued Łukasiewicz algebra $\mathbf{L}$ (a semi-simple Nelson algebra) is to fulfill the $\mathbf{T}$ features, then for any $a \in \mathbf{L}$, $\phi_1(a)$ must equal $\phi_2(a)$; that is, $\mathbf{L}$ must collapse to a Boolean algebra.

10.17   Frame – The Logic $\mathcal{F}_{\mathcal{CL}}$

We have seen in Section 9.5 that $\mathcal{F}_{\mathcal{CL}}$ (or, more precisely, $\mathcal{E}^*$) is obtained by augmenting $\mathcal{E}_0$ with $(T\text{-}Reg)$ and $(T\text{-}KP)$. Actually, the axioms for $\mathcal{F}_{\mathcal{CL}}$ were introduced as Natural Calculus-like rules in [Miglioli et al., 1989]. However, we have to note that since the rule for $(T\text{-}KP)$ does not

have an inverse, we obtain a pseudo-natural calculus with a restricted form of normalisation.

It is worth stressing that $\mathcal{F}_{\mathcal{CL}}$ enjoys a particular semantics: the "evaluation form" interpretation. This semantics is one of the most interesting steps towards a suitable interpretation of the Intuitionistic spirit, since, according to it, any formula is interpreted by means of its own set of possible proofs.

10.17.1 The Natural Deduction System $\mathcal{E}^*$ – Alias $\mathcal{F}_{\mathcal{CL}}$

The whole calculus $\mathcal{E}^*$ is given by adding to $\mathcal{E}_0$ the rules:

$$\frac{\mathbf{T}(\alpha) \longrightarrow \beta \vee \gamma}{(\mathbf{T}(\alpha) \longrightarrow \beta) \vee (\mathbf{T}(\alpha) \longrightarrow \gamma)} \quad (T - KP);$$

$$\frac{\mathbf{T}(\alpha) \longrightarrow \exists x \beta(x) C}{\exists x (\mathbf{T}(\alpha) \longrightarrow \beta(x))} \quad (T - KPP);$$

$$\frac{\mathbf{T}(p)}{p} \qquad \frac{\mathbf{T}(\sim p)}{\sim p}, \text{ for } p \text{ atomic } (T - REG)$$

10.17.2 Evaluation Form Semantics

The semantics of Evaluation Forms, EFS, intuitively associates with every well formed formula α a set of "possible proofs", or "possible justifications" or "not yet interpreted constructions" of α. These objects are then given an interpretation when we evaluate their atomic formulae.

Otherwise stated, after having recursively associated evaluation forms with a formula, they must be filtered, or interpreted (intuitively, we have to discriminate between true and false forms, or between "proofs" and "non-proofs").

It follows that EFS differs from Kripke-style semantics in that it does not directly assign an interpretation to formulae, and it differs from the BHK semantics in that EFS rejects a notion of an "abstract proof" for atomic formulae while for the BHK semantics it makes no sense to introduce an interpretation different from the constructions themselves, even for atomic formulae.

Interpretations of evaluation forms may be classical (with co-domain $\{0, 1\}$) or intuitionistic (over Kripke models).

Definition 10.17.1. *Given a formula α, $\mathcal{F}(\alpha)$ denotes the set of evaluation forms (EFs) associated with α and $\widehat{\alpha}$ will denote any element of $\mathcal{F}(\alpha)$. Here are the inductive clauses:*

1. $\mathcal{F}(\perp) = \{\perp\}$.

2. $\mathcal{F}(p) = \{p\}$, *for every atomic p.*

3. $\mathcal{F}(\sim p) = \{\sim p\}$, *for every atomic p.*

4. $\mathcal{F}(\mathbf{T}(\alpha)) = \{\mathbf{T}(\alpha)\}$.

5. $\mathcal{F}(\sim\mathbf{T}(\alpha)) = \{\sim\mathbf{T}(\alpha)\}$.

6. $\mathcal{F}(\alpha \wedge \beta) = \left\{ \dfrac{\widehat{\alpha} \quad \widehat{\beta}}{\alpha \wedge \beta} : \widehat{\alpha} \in \mathcal{F}(\alpha) \ and \ \widehat{\beta} \in \mathcal{F}(\beta) \right\}$.

7. $\mathcal{F}(\alpha \vee \beta) = \left\{ \dfrac{\widehat{\alpha}}{\alpha \vee \beta} : \widehat{\alpha} \in \mathcal{F}(\alpha) \right\} \cup \left\{ \dfrac{\widehat{\beta}}{\alpha \vee \beta} : \widehat{\beta} \in \mathcal{F}(\beta) \right\}$.

8. $\mathcal{F}(\sim(\alpha \wedge \beta)) = \left\{ \dfrac{\widehat{\sim\alpha}}{\sim(\alpha \vee \beta)} : \widehat{\sim\alpha} \in \mathcal{F}(\sim\alpha) \right\} \cup$

   $\qquad\qquad\qquad \left\{ \dfrac{\widehat{\sim\beta}}{\sim(\alpha \vee \beta)} : \widehat{\sim\beta} \in \mathcal{F}(\sim\beta) \right\}$.

9. $\mathcal{F}(\sim(\alpha \vee \beta)) = \left\{ \dfrac{\widehat{\sim\alpha} \quad \widehat{\sim\beta}}{\sim(\alpha \vee \beta)} : \widehat{\sim\alpha} \in \mathcal{F}(\sim\alpha) \ and \ \widehat{\sim\beta} \in \mathcal{F}(\sim\beta) \right\}$.

10. $\mathcal{F}(\alpha \longrightarrow \beta) = \mathcal{F}(\beta)^{\mathcal{F}(\alpha)}$.

11. $\mathcal{F}(\sim(\alpha \longrightarrow \beta)) = \left\{ \dfrac{\widehat{\alpha} \quad \widehat{\sim\beta}}{\sim(\alpha \longrightarrow \beta)} : \widehat{\alpha} \in \mathcal{F}(\alpha) \ and \ \widehat{\sim\beta} \in \mathcal{F}(\sim\beta) \right\}$.

12. $\mathcal{F}(\alpha) = \left\{ \dfrac{\widehat{\alpha}}{\sim\sim\alpha} : \widehat{\alpha} \in \mathcal{F}(\alpha) \right\}$.

We remind that $\neg\alpha$ is short for $\alpha \longrightarrow \perp$.

A classical interpretation of EFs is an assignment $CI : PVs \longmapsto \{0,1\}$ of one of the two classical values to every propositional variable together with the following inductive definition:

1. $CI(\perp) = 0$.

2. $CI(\widehat{p}) = CI(p)$, for any atomic p.

3. $CI(\widehat{\sim p}) = 1$, iff $CI(p) = 0$.

4. $CI(\widehat{\mathbf{T}(\alpha)}) = CI(\mathbf{T}(\alpha))$, where $CI(\mathbf{T}(\alpha)) = \mathbf{T}(\alpha)$ and $\mathbf{T}(True) = 1, \mathbf{T}(False) = 0$.

5. $CI(\widehat{\sim \mathbf{T}(\alpha)}) = 1$ iff $CI(\mathbf{T}(\alpha)) = 0$.

6. $CI\left(\dfrac{\widehat{\alpha} \quad \widehat{\beta}}{\alpha \wedge \beta}\right) = 1$ iff $CI(\widehat{\alpha}) = 1$ and $CI(\widehat{\beta}) = 1$.

7. $CI\left(\dfrac{\widehat{\sim \alpha} \quad \widehat{\sim \beta}}{\sim(\alpha \vee \beta)}\right) = 1$ iff $CI(\widehat{\sim \alpha}) = 1$ and $CI(\widehat{\sim \beta}) = 1$.

8. $CI\left(\dfrac{\widehat{\sim \alpha}}{\sim(\alpha \wedge \beta)}\right) = CI(\widehat{\sim \alpha}); CI\left(\dfrac{\widehat{\sim \beta}}{\sim(\alpha \wedge \beta)}\right) = CI(\widehat{\sim \beta})$.

9. $CI\left(\dfrac{\widehat{\alpha}}{\alpha \vee \beta}\right) = CI(\widehat{\alpha}); CI\left(\dfrac{\widehat{\beta}}{\alpha \vee \beta}\right) = CI(\widehat{\beta})$.

10. $CI\left(\dfrac{\widehat{\alpha} \quad \widehat{\sim \beta}}{\sim(\alpha \longrightarrow \beta)}\right) = 1$ iff $CI(\widehat{\alpha}) = 1$ and $CI(\widehat{\sim \beta}) = 1$.

11. $CI(\widehat{\alpha \longrightarrow \beta}) = 1$ iff $\forall \widehat{\alpha} \in \mathcal{F}(\alpha)$ such that $CI(\widehat{\alpha}) = 1$ the corresponding $\widehat{\beta} \in \mathcal{F}(\beta)$ is such that $CI(\widehat{\beta}) = 1$.

12. $CI\left(\dfrac{\widehat{\alpha}}{\sim\sim\alpha}\right) = CI(\widehat{\alpha})$.

One can prove:

Proposition 10.17.1. *For any interpretation CI, for any formula α, $CI(\alpha) = 1$ if and only if there exists $\widehat{\alpha} \in \mathcal{F}(\alpha)$ such that $CI(\widehat{\alpha}) = 1$.*

Since CI is a classical interpretation, we immediately obtain:

Corollary 10.17.1. *A formula α is classically valid, $\models_{\overline{\mathcal{CL}}} \alpha$, if and only if $\forall CI, \exists \widehat{\alpha} \in \mathcal{F}(\alpha)$ such that $CI(\widehat{\alpha}) = 1$.*

The difference between classical validity and constructive validity is given by swapping the position of the two quantified formulas (hence, of the two quantifiers):

Definition 10.17.2. *A formula α is constructively valid, $\models_{\overline{\mathcal{CN}}} \alpha$, if and only if $\exists \widehat{\alpha} \in \mathcal{F}(\alpha)$ such that $\forall CI, CI(\widehat{\alpha}) = 1$.*

We can restate the above results in terms of "knowledge" , to keep on the discussion of Section 9.2 (specially the box "Logico-philosophical remarks.1"). Let us substitute "state of affairs" for "interpretation" and "justification" for "evaluation form". Then a formula α is classically valid, according to *Proposition* 10.17.1, if for all states of affairs about α we have a justification to believe α true. On the contrary, α is constructively valid, according to *Definition* 10.17.2, if we have a justification which applies to any state of affair about α. In the classical case justifications depends on states of affaires, in the constructivistic case they do not (see below an example).

It is not difficult to show that Disjunction Property is constructively valid. Indeed, suppose $\models_{\overline{c_N}} \alpha \vee \beta$. Then there is a form $\widehat{\mathfrak{a}} \in \mathcal{F}(\alpha \vee \beta)$ such that $\widehat{\mathfrak{a}}$ is satisfied by every interpretation CI. But either $\widehat{\mathfrak{a}} = \dfrac{\widehat{\alpha}}{\alpha \vee \beta}$ or $\widehat{\mathfrak{a}} = \dfrac{\widehat{\beta}}{\alpha \vee \beta}$. Thus, by definition of $\models_{\overline{c_N}}$, at least one of the two forms are satisfied by any interpretation. This means that $\models_{\overline{c_N}} \alpha$ or $\models_{\overline{c_N}} \beta$.

We can immediately verify that $p \vee \sim p$ is not constructively validated by EFS. In fact, since p is atomic, $\mathcal{F}(p) = \{p\}$ and $\mathcal{F}(\sim p) = \{\sim p\}$.

Therefore:

$$\mathcal{F}(p \vee \sim p) = \left\{ \frac{\widehat{p}}{p \vee \sim p} : \widehat{p} \in \mathcal{F}(p) \right\} \cup \left\{ \frac{\widehat{\sim p}}{p \vee \sim p} : \widehat{\sim p} \in \mathcal{F}(\sim p) \right\}$$

$$= \left\{ \frac{p}{p \vee \sim p}, \frac{\sim p}{p \vee \sim p} \right\} .$$

Thus there are exactly two forms associated with $p \vee \sim p$. But the first is falsified by every interpretation CI such that $CI(p) = 0$ and the second is falsified by every interpretation CI such that $CI(p) = 1$. Therefore, there is not a unique justification applicable to all interpretations. The same holds for the weak negation "$\neg$". This is evident from the proof-tree reported above. Indeed, if $CI(\alpha) = 0$ we have to choose the branch starting from assumption $\neg\alpha$, as a justification for $\alpha \vee \neg\alpha$, while if $CI(\alpha) = 1$ we have to choose the other branch. Hence, for every interpretation we have a justification for $\alpha \vee \neg\alpha$, but not the other way around.

Finally one can prove the following completeness theorem:

Proposition 10.17.2. *For all formula α, $\vdash_{\mathcal{F}_{cl}} \alpha$ if and only if $\models_{\overline{\mathcal{CN}}} \alpha$.*

While the interpretation of evaluation forms over classical models leads to a completeness theorem for $\mathcal{F}_{\mathcal{CL}}$, the interpretation over intuitionistic models (Kripke models) completely determines a logic called $\mathcal{F}_{\mathcal{INT}}$.

At present, the latter system does not have the same importance as $\mathcal{F}_{\mathcal{CL}}$ in logical researches, probably because $\mathcal{F}_{\mathcal{CL}} = (\mathcal{F}_{\mathcal{INT}})^*$, where given a logic $\mathcal{L}$, $(\mathcal{L})^* = \mathcal{L} + D(\{\neg\neg p \longrightarrow p : p\ atomic\})$, where $D(\mathcal{L})$ is the closure of $\mathcal{L}$ with respect to Modus Ponens.

In any case, it is interesting to notice that $\mathcal{F}_{\mathcal{INT}}$ seems to be strictly connected to the Logic of Union Types developed in [Dezani & Ciancaglini, 1991].

10.18 Frame – Medvedev's Logic of Finite Problems

$\mathcal{F}_{CL}$ was introduced to grasp the syntax of Medvedev Logic of finite problems, $\mathcal{MV}$. Medvedev conceived intuitionistic validity of a formula α as "*solvability, by means of a uniform method, of any complex problem obtained by substituting problems for the propositional variables of the formula.*"

Medvedev's proposal, in order to formally treat the notions of a "problem" and a "solution", is to consider any problem as a dilemma and a solution as a possibility admitted by that dilemma. Therefore, any problem may be characterised as a finite set F of "admissible possibilities" and by a subset $X \subseteq F$ (possibly empty) of "solutions". It follows that any problem is represented by a pair $\langle F, X \rangle$. Given two problems, $U_1 = \langle F_1, X_1 \rangle$ and $U_2 = \langle F_2, X_2 \rangle$, one can define the following operations:

($\wedge$) $U_1 \wedge U_2 = \langle F_1 \times F_2, X_1 \times X_2 \rangle$,

($\vee$) $U_1 \vee U_2 = \langle F_1 \uplus F_2, X_1 \uplus X_2 \rangle$, where $\uplus$ is the disjoint union,

($\longrightarrow$) $U_1 \longrightarrow U_2 = \langle F_2^{F_1}, \{f : F_1 \longmapsto F_2 : x \in X_1\ \ implies\ \ f(x) \in X_2\} \rangle$,

($\neg$) $\neg U_1 = U_1 \longrightarrow \langle F, \emptyset \rangle$, where F is an arbitrary finite set.

Given a formula α, we associate it with a pair $\langle a(\alpha), i(a(\alpha)) \rangle$, where a is any function mapping any propositional variable of α onto a finite set F and i is any function mapping any finite set F onto a subset of F itself. Clearly $i(a(\alpha))$ depends on $a(\alpha)$ and $a(\alpha)$ depends on α.

Notice a main difference between Medvedev's approach and Evaluation Forms semantics: in EFS with any propositional variable a singleton is associated and not a finite set, as in $\mathcal{MV}$.

One says that α is *a-solvable* if and only if there exists $x \in a(\alpha)$ such that for all i, $x \in i(a(\alpha))$, or otherwise stated, if there is a "possible solution" which is validated by every interpretation. We say that α is *identically solvable*, if it is $a - solvable$ for all a.

On this basis one can show for instance, that every intuitionistic thesis is identically solvable, on the contrary KP is an identically solvable formula which is not an intuitionistic thesis.

A few facts are known about $\mathcal{MV}$: (i) a semantics was found by ([Jankov, 1969]) and generalised by Miglioli; (ii) we know that $\mathcal{MV}$ is a maximal constructive logic ([Maksimova, 1986] and, independently, [Miglioli et al., 1989b] by means of different mathematical techniques); (iii) it coincides with the part of $\mathcal{F}_{CL}$ which is closed under uniform substitution (see [Miglioli et al., 1989b]); (v) we do not know any system of axioms for $\mathcal{MV}$; (iii) we know that this logic is not finitely axiomatisable ([Maksimova et al., 1979])).

Finally, we notice that Medvedev Logic is stronger than Recursive Realisability, as proved by Jankov.

10.19   Frame – Atomic Decidability
and Non-Standard Systems

The treatment of atomic formulae in a classical manner is both a philosophical key point and a technical problem of constructivism.

As a matter of fact, an atomic formula p cannot be analysed in terms of "proofs", as required by the BHK interpretation. Indeed, a proof of p has the form $\emptyset \vdash p$, that is, all assumptions must be discharged. But according to the Intuitionistic calculus (see Frame 10.13) the only way to discharge assumptions is the $\Longrightarrow -int$ rule. Thus the minimal obtainable proof involving p must have the form $p \Longrightarrow p$ (but this possibility is questionable – see [Gabbay & De Queiroz, 1992], page 1334, footnote 23).

Miglioli and co-workers provide the following philosophical account of atomic decidability:

"*Since the notion of a proof of an atomic sentence is by itself meaningless, the intuitionistic explanation of the meaning of such sentences cannot fail to conceive them as formulas which, even if are left unanalyzed, nevertheless belong to some concrete mathematical theory, so that the problem of specifying what is the proof of an atomic formula comes down to the question whether there is any mathematical evidence warranting the assertibility of the sentences of some specific theory. However, while the identification of the atomic formulas of a propositional language with the (unanalyzed) sentences of some theory is surely correct, the intuitionistic explanation fails to take into account the other crucial aspect of the problem, i.e., the fact that, when we are concerned with logical validity, we want just to make abstraction from the mathematical content of the sentence we leave unanalyzed, making reference only to the abstract element shared by such contents: the true value of each sentence. [. . .] within our framework, logic and mathematical theories do not need to be reduced to each other, but represent different and irreducible levels of analysis of the notion of meaning. [. . .] (From this it follows that) if A is an atomic sentence, A and* $\mathbf{T}(A)$ *are equivalent: 'classical' truth and 'constructive' truth coincide in the case of atomic sentences.*"

Technically, atomic decidability plays an important role in relation to the notion of an *isoinitial model*.[8]

Roughly speaking, a theory $\mathcal{T}$ formalises an isoinitial model if there is a model $M_\mathcal{T}$ such that to all models $M'_\mathcal{T}$ there is a unique isomorphic immersion from $M_\mathcal{T}$ (*viz.* there is a submodel $M''_\mathcal{T}$ of $M'_\mathcal{T}$ such that $M_\mathcal{T}$ and $M''_\mathcal{T}$ are isomorphic in the usual model-theoretic sense). A well-known example of theory which formalises an isoinitial model is Arithmetic, since the standard model of natural numbers is an isoinitial model. Theories with isoinitial models fulfills constructive features for a number of extralogical axioms (for instance, $\exists\forall$-formulae, Harrop formulae, induction or descending chain principles), even in the case the logical "inference engine" is superintuitionistic.

Isoinitial models are unique up to isomorphisms.

In general, it happens that if a theory *Theor* has a model whose

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[8]Introduced in [Bertoni et al., 1979] with the name "monoinitial model" (which, however, usually denotes a slightly different concept).

elements are represented by the closed terms of the language of *Theor* and if for any closed atomic formula p of the language of *Theor*, either *Theor* $\vdash p$ or *Theor* $\vdash\sim p$, then *Theor* completely formalises an isoinitial model. Moreover, if *Theor* completely formalises an isoinitial model, then using a constructive logic augmented with Kuroda principle and KP, we can decide any closed quantifier-free formula of *Theor*.

This is a reason to maintain atomic decidability as a "good principle".

10.19.1 Atomic Decidability and the Failure of Uniform Substitution

In spite of the philosophical appeal of atomic decidability, this principle induces non trivial technical problems, at least if it is framed in intuitionistic-like constructivism.

The most immediate casualty of atomic decidability is the *principle of uniform substitution of formulae* (*USF*).

A logical system $\mathcal{L}$ enjoys the uniform substitution property (*USF*) if

$$\vdash_{\mathcal{L}} \alpha(p_1, \ldots, p_n) \equiv \vdash_{\mathcal{L}} \alpha(\sigma(p_1), \ldots, \sigma(p_n)) \qquad (USF)$$

where $p_1, \ldots, p_n$ are all the propositional variables of the formula α and σ is a function which maps propositional variables onto formulae.

If we add (*Reg*) to $\mathcal{INT}$ it immediately follows that $p \longleftrightarrow \neg\neg p$ for any atomic formula and *USF* does not hold any longer. For instance, in $\mathcal{F}_{\mathcal{CL}}$ we have, $\vdash_{\mathcal{F}_{\mathcal{CL}}} \neg\neg p \longrightarrow p$ but given $\sigma(p) = \alpha \vee \neg\alpha$, we have $\nvdash_{\mathcal{F}_{\mathcal{CL}}} \neg\neg(\alpha \vee \neg\alpha) \longrightarrow (\alpha \vee \neg\alpha)$.

Algebraically, atomic formulas should be interpreted on regular elements of Heyting algebras (or a particular class of Heyting algebras). Since, in general, the set of regular elements of a Heyting algebra is not additive, we have the algebraic counterpart of the failure of uniform substitution.

One should wonder about the "naturalness" of logics without *USF*. Actually, we have already seen very interesting objects exhibiting "jumps" in the presence of disjunctions, namely closure systems induced by Galois connections (cf. Part I).

In a wider philosophical setting, one may argue whether "nice properties" that guarantee a sort of "continuity", may or may not be that evident and "natural" (for instance, the concept of "synergy" implies some non-additive mechanism).

10.20 Frame – An Applications of the Algebraic Approach to Partial Information Systems

In Knowledge Discovery in databases, a central topic is the analysis of the dependencies among sets of attributes. Roughly speaking, given an *A-system*, $\mathbf{I} = (U, At, Val)$, we say that the set of attributes B depends on the set of attributes A, $A \twoheadrightarrow B$, if and only if the values taken for the attributes of B can be univocally determined whenever we are given the values taken by the objects for all the attributes of A.

From *Definition* 3.2.1 and *Definition* 3.2.2, of Chapter 3, we can prove:

Proposition 10.20.1.

1. *B is dependent on A at point g, $A \twoheadrightarrow_g B$, if and only if $[g]_A \subseteq [g]_B$ viz. if for any $g' \in [g]_A$, $(g'(a_1) = v_{a_1} \wedge \ldots \wedge g'(a_n) = v_{a_n}) \Rightarrow (g'(b_1) = v_{b_1} \wedge \ldots \wedge g'(b_m) = v_{b_m})$.*

2. *B is dependent on A, $A \twoheadrightarrow B$, if and only if $\forall g \in U, A \twoheadrightarrow_g B$.*

REMARKS. (cf. [Pawlak, 1991]) In Rough Set Analysis the set of objects g such that $A \twoheadrightarrow_g B$ holds, is called A-positive region of B, $POS_A(B)$, and defined by $POS_A(B) = \bigcup\limits_{X \in Ind(B)} (lE_A)(X)$, where E_A is the indiscernibility relation induced by A. Otherwise stated, we compute the lower approximation with respect to E_A of each equivalence class of U modulo E_B and then take the union of the results. Moreover, in Rough Set Analysis it is used the concept of "partial dependence" or "degree of dependence" $A \twoheadrightarrow_k B$, for k a rational number:

$$A \twoheadrightarrow_k B \text{ if and only if } k = \frac{card(POS_A(B))}{card(U)}$$

If $k = 1$ then B is said to be totally dependent on A. Obviously, $A \twoheadrightarrow_1 B$ if and only if $A \twoheadrightarrow B$.

If $\mathbf{I}$ is a *P-system*, $\langle G, M, \Vdash \rangle$, then a set of properties B is dependent on a set of properties A, if and only if an arbitrary object fulfills B whenever it fulfills A, or, equivalently, it is not the case that an object fulfills A but not B.

Given a *P-systems* it is now convenient to set the following total function:

$$i : G \times M \longmapsto \{0, 1\}; i(\langle g, m \rangle) = 1 \; \text{iff} \; g \Vdash m \qquad (10.20.1)$$

Therefore, let $A, B \subseteq M$, $g \in G$. We have:

1. B is said to be *dependent on A at point g, $A \mapsto_g B$*, if

$$(\forall a \in A(i(\langle g, a \rangle) = 1)) \Rightarrow (\forall b \in B(i(\langle g, b \rangle) = 1)).$$

2. B is said to be *dependent on A, $A \mapsto B$*, iff $\forall g \in G, A \mapsto_g B$.

Notice that dependence in *P-systems* and in *A-systems* are rather different concepts. This is related to the intended meaning of *P-systems* and *A-systems*. In fact, in *P-systems* 0 and 1 are not analysed as particular values, but as values of the characteristic functions of the set of validity of properties. It follows that inclusion of equivalence classes modulo values is a meaningless criterion for computing dependence. Here is an example:

EXAMPLE 1. A complete *P-system*

P	a_1	a_2	b_1	b_2
g_1	1	1	0	1
g_2	1	1	0	1
g_3	0	1	1	1
g_4	0	1	1	0
g_5	1	0	1	1

Let us set $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Thus we have $A \mapsto_{g_3} B$ and $A \mapsto_{g_4} B$ though $[g_3]_A = \{g_3, g_4\} \nsubseteq \{g_3, g_5\} = [g_3]_B$. Therefore, dependence at a point in the sense of *P-systems* does not imply dependence at a point in the sense of *A-systems*. Moreover, $[g_1]_A = \{g_1, g_2\} \subseteq \{g_1, g_2\} = [g_1]_B$. Hence $A \rightarrow_{g_1} B$. On the contrary, neither $A \mapsto_{g_1} B$ nor $A \mapsto_{g_2} B$. Hence, dependence at a point in *A-systems* does not imply dependence at a point in *P-systems*.

Let us then discover what kind of set-theoretic relations stay behind the concept of a dependence in *P-systems*. The search is not really difficult, but needs some explanation.

If $\mathbf{I} = \langle G, M, i \rangle$ is a *P-system*, for any property a we set $a^+ = \{x \in G : i(\langle x, a \rangle) = 1\}$ and $a^- = \{x \in G : i(\langle x, a \rangle) = 0\}$. Let us extend this definition to sets of properties, A:

$$A^+ =_{def} \bigcap \{a^+ : a \in A\} \ and \ A^- =_{def} \bigcup \{a^- : a \in A\} \qquad (10.20.2)$$

The definition of A^+ is quite obvious: if $A = \{a_1, \ldots, a_n\}$, then A^+ is the set of elements that enjoy a_1 *and* a_2 *and* $\ldots$ *and* a_n. This choice depends on the principle of "*ecceitas*", stating that an object is a synthesis of all its properties, so that we consider a set of properties as a logical product of its components.

The definition of A^-, on the contrary, may sound somehow odd: why the union and not the intersection? The reason, in this case, is exquisitely logical in nature, once we have assumed the definition of A^+. Actually, for any property a, a^+ is the extension of a, $[\![a]\!]$. But what is a^-? We can think of a^- as the extension of the *complementary property* of a, $\overline{a}$. As we know from *Definition* 4.8.1 the complementary property $\overline{a}$ is the negative mirror of a, in that for any $g \in G$, $i(\langle g, a \rangle) = 1$ if and only if $i(\langle g, \overline{a} \rangle) = 0$.

Notice that a complementary property is more than an incompatible property. Indeed a and b are incompatible at a point g if and only if $(i(\langle g, a \rangle) \wedge i(\langle g, b \rangle)) = 0$. They are incompatible if they are incompatible at every point (to characterise incompatible properties, only the left to right implication of the definition of a complementary property is required). For instance the attribute a and b below are complementary, while c and d are incompatible.

EXAMPLE 2. Incompatible and dichotomic attributes:

Dichotomic	a	b	Incompatible	c	d
g_1	0	1	g_1	0	1
g_2	1	0	g_2	1	0
g_3	0	1	g_3	0	0

We can extend this notion to pairs $\{A, B\}$ of sets of properties, tacking into account the logical products of their evaluations, $i(A_g) = \bigwedge\{i(\langle g, a \rangle) : a \in A\}$ and $i(B_g) = \bigwedge\{i(\langle g, b \rangle) : b \in B\}$ and saying that A and B are incompatible at point g if $i(A_g) \wedge i(B_g) = 0$.

Is there a difference between not fulfilling a property a and fulfilling the complementary property $\overline{a}$? As far as we deal with a single property

a there are no doubts: $\llbracket \overline{a} \rrbracket = -\llbracket a \rrbracket$. However, when we deal with a set $A = \{a_1, \ldots, a_n\}$ of properties we must take some care. What is actually the complementary property $\overline{A}$? Are we really interested to A as a composite property? If this is the case, an object g fulfills the composition of a_1, a_2, \ldots, a_n if and only if $g \in \llbracket a_1 \rrbracket$ *and* $g \in \llbracket a_2 \rrbracket$ *and*, \ldots, *and* $g \in \llbracket a_n \rrbracket$. This is exactly the meaning of the definition $A^+ =_{def} \bigcap \{a^+ : a \in A\}$. But since in this way we take A as a whole, as a synthesis of properties, A^- should represent the complementary property of this synthesis and not of the single components of A. This is the reason of definitions (10.20.2).

But $\overline{A}$ is defined as $\llbracket \overline{a_1} \rrbracket \cap \ldots \cap \llbracket \overline{a_n} \rrbracket$, that is, $\bigcap \{a^- : a \in A\}$, which is the composition of the complementary properties of each single element of A, not the complementary property of their composition.

This is the reason why we set:

$$A^- = a_1^- \cup \ldots \cup a_n^- = \bigcup \{a^- : a \in A\} = -\llbracket A \rrbracket = -A^+ \qquad (10.20.3)$$

Now we are in position to express dependencies between sets of properties. It is straightforward that in a *P-system*,

$$A \mapsto B \ iff \ \llbracket A \rrbracket \subseteq \llbracket B \rrbracket \ iff \ -\llbracket A \rrbracket \cup \llbracket B \rrbracket = U \qquad (10.20.4)$$

which exactly expresses the sentence "B depends on A if and only if any object x which fulfills all the properties from A, fulfills all the properties from B, too". Hence we can set:

$$A \mapsto_g B \ iff \ g \in A^- \cup B^+ \qquad A \mapsto B \ iff \ A^- \cup B^+ = U \qquad (10.20.5)$$

In the *P-system* of Example 1 above, $A^+ = \{g_1, g_2\}$, $A^- = \{g_3, g_4, g_5\}$, $B^+ = \{g_3, g_5\}$ and $A^- \cup B^+ = \{g_3, g_4, g_5\}$.

Therefore, we have $A \mapsto_{g_3} B$, $A \mapsto_{g_4} B$ and $A \mapsto_{g_5} B$.

It is worth observing that $A \mapsto_g B$ if and only if $i(A_g) = 1$ and $i(B_g) = 0$. In this case $i(\overline{B_g}) = 1$.

Let us notice the difference between $\mapsto$ and $\twoheadrightarrow$: in the above *P-system*, we can verify that $[g_2]_A = \{g_1, g_2\} = [g_2]_B$. Thus we have $A \twoheadrightarrow_{g_1} B$ and $A \twoheadrightarrow_{g_2} B$, while $g_1, g_2 \notin A^- \cup B^+$. On the contrary, $[g_3]_A \nsubseteq [g_3]_B$, thus $A \nrightarrow_{g_3} B$ and $A \nrightarrow_{g_4} B$, while $A \mapsto_{g_3} B$ and $A \mapsto_{g_4} B$.

Things becomes more interesting and more complicated if the information function i is not totally defined.

10.20.1 *P-Systems* with Partial Information

In [Burmeister 1989 and 1993] partial *P-systems* are taken into account as to the problem of how to compute functional dependency in the presence of information gaps.

In those papers a Kleene evaluation is applied to the predicate $A \mapsto B$ (B depends on A). The non deterministic evaluations are provided there using two extreme Boolean completions $\mathbf{I}^0$ and $\mathbf{I}^1$ of a given partial *P-system*: $\mathbf{I}^0$ has all the partial values equated to $\mathbf{0}$ while $\mathbf{I}^1$ has all the partial values equated to $\mathbf{1}$. It is straightforward comparing how this approach is reflected by our algebraic construction.

In [Pagliani, 1997a] a new approach to this topic was introduced that we shall describe here below with some details, and in which the transformation of a partial *P-system* into the required Boolean *P-systems* can be performed locally and run-time.

This will be the topic of the present Frame.

In total *P-systems*, for any $g \in G$, the function i of (10.20.1) is totally defined. On the contrary, in a partial *P-system* i is a partial function. In this case in $\mathbf{I}$ there are lacks of information for some objects. Now, if $g \in G$ and $m \in M$ are such that i is not defined for $\langle g, m \rangle$ then, by definition, for all $B \subseteq M, \{m\} \mapsto_g B$ holds true vacuously. However, if there is a $b \in B$ such that $i(\langle g, b \rangle) = 0$, the preceding statement is a critical commitment: in fact if, by means of some additional information, $i(\langle g, m \rangle)$ is later given the value 1, then we have to retract our evaluation. This is not a drama, since it is a typical example of non-monotonic reasoning. But if we want to keep monotonicity in our evaluation, we must say also that the dependence is undefined at point g. But if we are to accept this interpretation, we must also accept that if $i(\langle g, m \rangle)$ is undefined then for all $B \subseteq M, B \mapsto_g \{m\}$ can never be evaluated "false". So we must be able to distinguish the following cases:

(a) $A \mapsto_g B = \mathbf{T}$: the dependence is definitely true at point g.

(b) $A \mapsto_g B = \mathbf{F}$: the dependence is definitely false at point g.

(c) $A \mapsto_g B = \mathbf{U}$: a definite truth value for the dependence at point g cannot be established given the current information in $\mathbf{I}$.

Derived important cases are:

(d) $A \mapsto_g B \in \{\mathbf{T}, \mathbf{U}\}$.

(e) $A \mapsto_g B \in \{\mathbf{F}, \mathbf{U}\}$.

In order to grasp the intended meaning we must set:

Definition 10.20.1. *Let* $\mathbf{I} = \langle G, M, i \rangle$ *be a partial P-system,* $A, B \subseteq M$, $g \in G$. *Then,*

1. $A \mapsto_g B = \mathbf{T}$ *iff* $(\exists a \in A, i(\langle g, a \rangle) = 0)$ *or* $(\forall b \in B, i(\langle g, b \rangle) = 1)$.

2. $A \mapsto_g B = \mathbf{F}$ *iff* $(\forall a \in A, i(\langle g, a \rangle) = 1)$ & $(\exists b \in B, i(\langle g, b \rangle) = 0)$.

3. $A \mapsto_g B = \mathbf{U}$ *iff* $\neg((A \mapsto_g B = \mathbf{T})$ *or* $(A \mapsto_g B = \mathbf{F}))$.

   Thus

4. $A \mapsto_g B = \mathbf{U}$ *iff*
   $not[(\exists a \in A, i(\langle g, a \rangle) = 0)\ or\ (\forall b \in B, i(\langle g, b \rangle) = 1)]$ & $[not(\forall a \in A, i(\langle g, a \rangle) = 1)$ & $(\exists b \in B, i(\langle g, b \rangle) = 0)]$,
   iff $(\forall a \in A, i(\langle g, a \rangle) \neq 0)$ & $(\forall b \in B, i(\langle g, b \rangle) \neq 1)$ & $[(\exists a \in A, i(\langle g, a \rangle) \neq 1)$ *or* $(\forall b \in B, i(\langle g, b \rangle) \neq 0)]$.

Now we shall try to provide the logico-algebraic operations making a "property explorer" be able to have these kinds of information at a glance, in spite of their structural complexity.

For convenience, given a partial *P-system*, we extend i to a total function $G \times M \longmapsto \{0, 1, ?\}$ in the following manner:

$$i'(\langle g, a \rangle) = \begin{cases} 1 & if\ i(\langle g, a \rangle) = 1 \\ ? & if\ i(\langle g, a \rangle)\ is\ not\ defined \\ 0 & if\ i(\langle g, a \rangle) = 0 \end{cases}$$

EXAMPLE 3. Consider the partial *P-system*:

C	a	b	c	d	e
1	?	0	1	?	?
2	0	0	1	1	0
3	1	?	?	0	1
4	0	0	1	1	0
5	?	0	1	?	?

Let us understand the new framework.

First of all, in the presence of lack of information, given a set of properties A, $A^- \subseteq -A^+$. For example, consider a in the *P-system* **C** above. As one can easily observe, $a^+ = [\![a]\!] = \{3\}$ and $a^- = \{2, 4\}$, while $-a^+ = \{1, 2, 4, 5\}$: for 1 and 5 we do not have enough information.

Therefore, A^+ and A^- must be considered in partnership at a peer. It follows that instead of $\lceil a \rceil$ it is convenient to work with a pairing function

$$\pi : \wp(M) \longmapsto \wp(G) \times \wp(G); \ \pi(A) = \langle A^+, A^- \rangle, \ \ for \ any \ \ A \subseteq M.$$
$$(10.20.6)$$

The difference with the previous situation is that now $A^+ \cup A^- \subseteq G$. If $A^+ \cup A^- = G$, then we say that A is *complete* or *total*. We can have also the extreme case: $A^+ = A^- = \emptyset$.

Let us set $1 = \langle G, \emptyset \rangle$. From the above discussion, the definitions for conjunction and negation are as follows:

(conj) $\pi(A) \wedge \pi(B) =_{def} \langle A^+ \cap B^+, A^- \cup B^- \rangle$.

(neg) $\sim \pi(a) =_{def} \langle A^-, A^+ \rangle$, because by $\sim$ we want to put in evidence the complementary property of A.

Some operations, then, come straightforwardly:

(zero) $0 =\sim 1 = \langle \emptyset, G \rangle$
(or) $\pi(A) \vee \pi(B) =_{def} \sim (\sim \pi(A) \wedge \sim \pi(B)) = \langle A^+ \cup B^+, A^- \cap B^- \rangle$,
(mat-imp) $\pi(A) \to \pi(B) =_{def} \sim \pi(A) \vee \pi(B) = \langle A^- \cup B^+, A^+ \cap B^- \rangle$.

Definition 10.20.2. *Let us denote by $\pi(M)_{\{op_1,...,op_n\}}$ the inductive closure of the set $\pi(M) = \{\pi(a_i)\}_{a_i \in M}$ under the operations, $\{op_1, \ldots, op_n\}$, specified in the index (and, of course, their derived operations).*

From Chapter 6, we have immediately:

Proposition 10.20.2. *Let $\mathbf{I} = \langle G, M, i \rangle$ be a partial P-system. Then the structure*

$$\mathbf{L}^K(\mathbf{I}) = \langle \pi(M)_{\{1,\wedge,\sim\}}, \vee, \wedge, \sim, \to, 1, 0 \rangle$$

is a Kleene algebra.

Kleene algebras provide us with enough machinery to compute the *definite* cases of *Definition* 10.20.1, that is, (1) and (2). From now on given a pair $\langle x, y \rangle$ we set $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$.

Proposition 10.20.3. *For any partial P-system $\mathbf{I} = \langle G, M, i \rangle$, $\forall A, B \subseteq M, \forall g \in G$:*

1. $A \mapsto_g B = \mathbf{T}$ iff $g \in \pi_1(\pi(A) \to \pi(B))$.
2. $A \mapsto_g B = \mathbf{F}$ iff $g \in \pi_2(\pi(A) \to \pi(B))$.

3. $A \mapsto B = \mathbf{T}$ *iff* $(\pi(A) \to \pi(B)) = 1$.
4. $A \mapsto B = \mathbf{F}$ *iff* $(\pi(A) \to \pi(B)) = 0$.

Proof 1. $A \mapsto_g B = \mathbf{T}$ *iff* $(\exists a \in A, i(\langle g, a \rangle) = 0)$ *or* $(\forall b \in B, i(\langle g, b \rangle) = 1)$
iff $g \in \pi_1(\pi(\sim A))$ *or* $g \in \pi_1(\pi(B))$ *iff* $g \in (\pi_2(\pi(A)) \cup \pi_1(\pi(B)))$ *iff*
$g \in \pi_1(\pi(A) \to \pi(B))$.
2. $A \mapsto_g B = \mathbf{F}$ *iff* $(\forall a \in A, i(\langle g, a \rangle) = 1)$ & $(\forall b \in B, i(\langle g, b \rangle) = 0)$
iff $g \in \pi_1(\pi(A))$ & $g \in \pi_1(\pi(\sim B))$ *iff* $g \in \pi_1(\pi(A))$ & $g \in \pi_2(\pi(B))$ *iff*
$g \in \pi_2(\pi(A) \to \pi(B))$.
3. $A \mapsto B = \mathbf{T}$ *iff* $\forall g \in G(A \mapsto_g B = \mathbf{T})$ *iff* $\pi_1(\pi(A) \to \pi(B)) = G$ *iff*
$\pi(A) \to \pi(B) = \langle G, \emptyset \rangle$.
4. Similar.   QED

However, in order to compute the indefinite cases provided by *Definition* 10.20.1.(3), Kleene algebras are not enough. Intuitively, we have to go beyond the definite field, by endowing Kleene algebras with some additional suitable tool.

For instance, noting that $\sim \pi(A) = \langle A^-, A^+ \rangle$, in Kleene algebras we cannot explore the domain in between A^- and A^+ (namely $G \cap -(A^- \cup A^+)$). So let us set:
(co intuitionistic negation) $\neg \pi(A) =_{def} \langle -A^+, A^+ \rangle$.

As we know from Chapter 6 we are now in position to define:

(weak implication)

$$\pi(A) \longrightarrow \pi(B) =_{def} \neg \pi(A) \vee \pi(B) = \langle -A^+ \cup B^+, A^+ \cap B^- \rangle.$$

Again from Chapter 6, we have:

Proposition 10.20.4. *For any partial P-system* $\mathbf{I} = \langle G, M, i \rangle$*, the structure* $\mathbf{L}^N(\mathbf{I}) = \langle \pi(M)_{\{\wedge, \sim, \neg\}}, \vee, \wedge, \sim, \longrightarrow, \neg, 1, 0 \rangle$ *is a semi-simple Nelson algebra.*

Proposition 10.20.5. *For any partial P-system* $\mathbf{I} = \langle G, M, i \rangle$, $\forall A, B \subseteq M, \forall g \in G$:

1. $A \mapsto_g B \neq \mathbf{F}$ *iff* $g \in -A^+ \cup -B^-$.
2. $A \mapsto_g B \neq \mathbf{T}$ *iff* $g \in -A^- \cap -B^+$.

Proof. 1. $A \mapsto_g B \neq \mathbf{F}$ *iff* $\neg ((\forall a \in A, I(\langle g, a \rangle) = 1)$ & $(\forall b \in B, I(\langle g, b \rangle) = 0))$ *iff* $(\exists a \in A, I(\langle g, a \rangle) \neq 1)$ *or* $(\exists b \in B, I(\langle g, b \rangle) \neq 0)$ *iff* $g \in -\pi_1(\pi(A))$ *or* $g \in -\pi_2(\pi(B))$ *iff* $g \in -A^+ \cup -B^-$.
2. is dual of (1).   QED

Now we can define two operations reflecting faithfully the intended implicational character of the dependency relation $\mapsto$ and of the metalinguistic negation *not* (i.e. operations able to cope with the indefinite cases).

Corollary 10.20.1. *For any partial P-system* $\mathbf{I} = \langle G, M, i \rangle$, $\forall A, B \subseteq M$, *let us define the following enteilment relations:*

(a) $\pi(A) \gg \pi(B) =_{def} \lnot \sim \pi(A) \to \sim \lnot \pi(B)$,
(b) $\pi(A) \ggg \pi(B) =_{def} \sim \lnot \pi(A) \to \lnot \sim \pi(B)$.
Then $\forall g \in G$,
(i) $A \mapsto_g B = \mathbf{T}$ *iff* $g \in \pi_1(\pi(A) \gg \pi(B))$;
(ii) $A \mapsto_g B \neq \mathbf{T}$ *iff* $g \in \pi_2(\pi(A) \gg \pi(B))$;
(iii) $A \mapsto_g B = \mathbf{F}$ *iff* $g \in \pi_2(\pi(A) \ggg \pi(B))$;
(iv) $A \mapsto_g B \neq \mathbf{F}$ *iff* $g \in \pi_1(\pi(A) \ggg \pi(B))$.

Corollary 10.20.2. *For any partial P-system* $\mathbf{I} = \langle G, M, i \rangle$, $\forall A, B \subseteq M$ *(i)* $\pi(A) \gg \pi(B) = \sim \lnot(\pi(A) \to \pi(B))$; *(ii)* $\pi(A) \ggg \pi(B) = \lnot \sim (\pi(A) \to \pi(B))$.

As we know, in this way we are dealing with modalities. Indeed, $\sim \lnot(X)$ equals $L(X)$, while $\lnot \sim (X) = M(X)$.

Therefore we have:

$$\pi(A) \gg \pi(B) = M(\pi(A)) \to L(\pi(B)) = L(\pi(A) \to \pi(B)) \quad (10.20.7)$$

$$\pi(A) \ggg \pi(B) = L(\pi(A)) \to M(\pi(B)) = M(\pi(A) \to \pi(B)) \quad (10.20.8)$$

By easy calculation, one can verify that in (10.20.7) and (10.20.8) we can substitute the Nelson implication $\longrightarrow$ for the Kleene implication $\to$.

EXAMPLE 4.
Consider the *P-system* depicted in Example 3. Let us compute the dependence $\{c\} \mapsto \{a\}$: in the Kleene algebra $\mathbf{L}^K(\mathbf{C})$ we have $\pi(\{c\}) = \langle \{1, 2, 4, 5\}, \emptyset \rangle$ and $\pi(\{a\}) = \langle \{3\}, \{2, 4\} \rangle$; thus applying the Kleene implication '$\to$' we obtain: $\pi(\{c\}) \to \pi(\{a\}) = \sim \pi(\{c\}) \vee \pi(\{a\}) = \langle \emptyset, \{1, 2, 4, 5\} \rangle \vee \langle \{3\}, \{2, 4\} \rangle = \langle \emptyset \cup \{3\}, \{1, 2, 4, 5\} \cap \{2, 4\} \rangle = \langle \{3\}, \{2, 4\} \rangle$.

Thus we know where this dependence definitely holds and where it definitely does not hold.

In $\mathbf{L}^N(\mathbf{C})$ we can apply the augmented implications $\gg$ and $\ggg$:

$$\pi(\{c\}) \gg \pi(\{a\}) = {\lrcorner} \sim \pi(\{c\}) \to\sim {\lrcorner}\pi(\{a\}) = \langle\{3\},\{1,2,4,5\}\rangle.$$

Thus, according to *Corollary* 10.20.1 (i) and (ii), $1, 2, 4$ and 5 are the points where $\{c\} \mapsto \{a\}$ is or *could* be $\mathbf{F}$, while 3 is the only element that *necessarily* supports this dependence. In the same way, by computing $\pi(\{c\}) \ggg \pi(\{a\})$ we get $\langle\{1,3,5\},\{2,4\}\rangle$; so, according to *Corollary* 10.20.1 (iii) and (iv), the above dependence is or *could* be $\mathbf{T}$ at points $1, 3$ and 5, while it is *necessarily* $\mathbf{F}$ at points 2 and 4. It is worth noticing the *modal* reading of the results.

Observe, incidentally, that there can be partial *P-systems* $\mathbf{K}_1$ and $\mathbf{K}_2$ such that their induced Kleene algebras are different, while the induced Nelson algebras are equal.

EXAMPLE 5. Consider the partial *P-systems*:

$\mathbf{I}_1$	a
1	?
2	0
3	1

$\mathbf{I}_2$	a	b
1	?	1
2	0	0
3	1	0

Thus $\mathbf{L}^N(\mathbf{I}_1) = \mathbf{L}^N(\mathbf{I}_2) = \mathbf{L}^K(\mathbf{I}_2)$ but $\mathbf{L}^K(\mathbf{I}_1)$ is a strict sublattice of $\mathbf{L}^K(\mathbf{I}_2)$. In the following Hasse diagram, $\mathbf{L}^K(\mathbf{I}_1)$ is drawn in dotted lines and embedded in $\mathbf{L}^K(\mathbf{I}_2)$ (i.e. $\mathbf{L}^N(\mathbf{I}_2)$, $\mathbf{L}^N(\mathbf{I}_1)$):
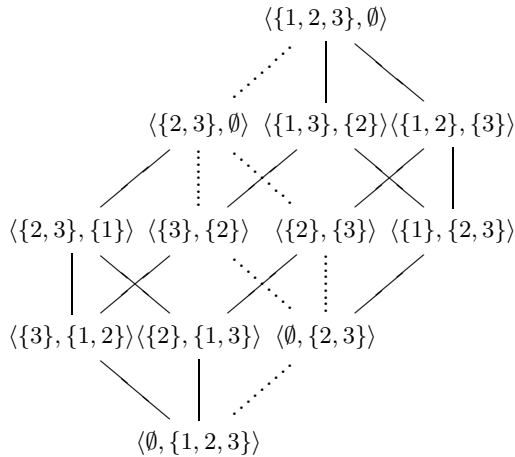
10.20.2 Adequacy of the Kleene Fragment w.r.t. Definite Answers

As we have seen, in complete *P-systems*, the relation $A \mapsto_g B$ is exhaustively defined by the classical material implication. Therefore an enteilment operation intended to compute answers about dependencies in partial *P-systems* must not conflict with the classical material implication. Roughly speaking it has to fulfill two intuitive properties:

(*inv*) If **I** is a partial *P-systems* then intuitively we are allowed to say that $A \mapsto_g B = \mathbf{T}$ or $A \mapsto_g B = \mathbf{F}$ only if for any more specified "release" of **I**, these evaluations do not change. Thus the enteilment operation has to reflect this fact.

(*cons*) If the arguments are sets of complete properties, then the enteilment operation must coincide with the classical material implication.

We call condition (*inv*) *invariance condition* and (*cons*) *consistency condition*. If these conditions are fulfilled by a semantic system $\mathcal{S}$, then $\mathcal{S}$ is said to be "adequate to Classical Logic".

From the above discussion it follows that an algebraic model for partial *P-systems* is good if and only if it is induced by a homomorphism from the algebra of formulas generated by a language $\mathcal{L}$ into the algebra of truth values induced by a semantic adequate to Classical Logic. Indeed, our Kleene algebra $\mathbf{L}^K(\mathbf{I})$ is induced by $\mathcal{L}^K(\mathbf{I}) = \pi(M)_{\{1,\wedge,\sim\}}$ and Kleene Strong Semantics for partial recursive functions (see [Kleene, 1952]). Since it is well-known that Kleene Strong Semantics characterizes a three-valued logic semantically adequate to Classical Logic, our Kleene fragment is adequate for definite answers.

Indeed, let us confine our attention to a partial *P-system* $\mathbf{I}^{1\times 2} = \langle \{g\}, \{a,b\}, i \rangle$. On varying the range of i on $\{1,?,0\}$, this *P-system* reproduces the situation in which two atomic formulas are three-evaluated on a single world.

Then for any atomic $\gamma = \{x\}$, for $x \in \{a,b\}$, we have: $\pi(\gamma) = \langle \{g\}, \emptyset \rangle$ *iff* $i(\langle g,x \rangle) = 1$ and $\pi(\gamma) = \langle \emptyset, \{g\} \rangle$ *iff* $i(\langle g,x \rangle) = 0$; we have $\pi(\gamma) = \langle \emptyset, \emptyset \rangle$ otherwise. Moreover in any algebra induced by $\mathbf{I}^{1\times 2}$ the top element $\mathbf{1}$ is $\langle \{g\}, \emptyset \rangle$ while $\langle \emptyset, \{g\} \rangle$ is the bottom element $\mathbf{0}$. We denote the element $\langle \emptyset, \emptyset \rangle$ by δ. Hence the operations for $\mathbf{I}^{1\times 2}$ can be represented by the following tables:

∧	1	δ	0		→	1	δ	0		~	
1	1	δ	0		1	1	δ	0		1	0
δ	δ	δ	0		δ	1	δ	δ		δ	δ
0	0	0	0		0	1	1	1		0	1

(By easy verification: for instance $1 \wedge \delta = \langle\{g\}, \emptyset\rangle \wedge \langle\emptyset, \emptyset\rangle = \langle\{g\}\cap\emptyset, \emptyset\cup \emptyset\rangle = \langle\emptyset, \emptyset\rangle = \delta$; $\delta \to 0 = \langle\emptyset, \emptyset\rangle \to \langle\emptyset, \{g\}\rangle = \langle\emptyset\cup\emptyset, \{g\}\cap\emptyset\rangle = \langle\emptyset, \emptyset\rangle = \delta$). It happens that these are precisely the tables of the Kleene Strong Semantics.

However, we have seen that in order to cope with the cases $A \mapsto_g B \neq \mathbf{F}$ and $A \mapsto_g B \neq \mathbf{T}$ we have to move from the Kleene fragment to the Nelson fragment. The Nelson fragment on $\mathbf{I}^{1\times 2}$ is given by the tables:

⟶	1	δ	0		≫	1	δ	0		⋙	1	δ	0			⌐
1	1	δ	0		1	1	0	0		1	1	1	0		1	0
δ	1	1	1		δ	1	0	0		δ	1	1	1		δ	1
0	1	1	1		0	1	1	1		0	1	1	1		0	1

But this semantics is not adequate to Classical Logic because (*cons*) is fulfilled while (*inv*) is not. In fact all Nelson operations are consistent with Classical Logic but they make too strong commitments as to the evaluation of indefinite arguments. For instance $\delta \longrightarrow 0 = 1$; but if our information evolves so that the antecedent becomes 1, then we break the rule for classical material implication. $\dashv\delta = 0$, but if δ evolves into 0 we have $\dashv 0 = 0$. The derived operations $\gg$ and $\ggg$ show the same problem.

As a matter of fact the application of the operators $\dashv \sim$ and $\sim \dashv$ to the arguments of $\to$ is a sort of "probabilistic challenge" to Classical Logic and it is not a surprise that operations that are to provide non-definite answers are not adequate to Classical Logic.

However, what is the concrete meaning of the above operations?

If A is a set of properties and g is an object such that $i(A_g) = ?$ (according to table of $\wedge$ above), then $g \notin A^+$ and $g \notin A^-$, but g will belong to the first element of $M(\pi(A))$ (i.e. $-A^-$; that is: when we apply M to $\pi(A)$ we assume that $i(\langle g, a\rangle) = 1$, for all $a \in A$. On the other side g will belong to the second element of $L(\pi(A))$ (i.e. $-A^+$); that is: when we apply L to $\pi(A)$, we assume that $i(\langle g, a\rangle) = 0$, for all $a \in A$.

Since $\pi(A) \gg \pi(B)$ equals, from (10.20.7), $M(\pi(A)) \to L(\pi(B))$, for $A, B \subseteq M$, then it amounts to applying the operation $\to$ to a pair

of "completed" arguments A^1 and B^0: A^1 is completed by equating to 1 all the undetermined evaluations, while B^0 is completed by equating them to 0.

The calculus of $\pi(\{A\}) \ggg \pi(\{B\})$ is dual (from 10.20.8).

Hence when we apply $\gg$ we assume a "sceptical" point of view, while we assume an "optimistic" point of view when we apply $\ggg$.

10.21 Frame – Logical Operations in a Pure Algebraic Setting

In this section we shall recover the operations so far discussed, from a more general picture construed by means of pure logico-algebraic relationships.

Indeed, we must consider the fact that any semi-simple Nelson algebra can be made into a three-valued Łukasiewicz algebra, and that three-valued Łukasiewicz logic enjoys a limited form of weakening, namely, $\frac{A,A,A \vdash B}{A,A \vdash B}$ (this fact was already noticed by Nelson). The limitation or the suppression of structural rules makes it possible to split the meaning of previously unified operations. So, moving back from syntax to semantics, we can argue whether one can define a multiplication operation, say $\otimes$, such that $\otimes$ has an interesting meaning different from $\wedge$.

Let g and f be monadic operators (we recall that a dyadic operation can be transformed, via Currying, into a family of monadic operators). We call the operator $g(f(g(x))$ the $g-$dual of f. For instance, in a Boolean algebra, let g be $\sim$ and f be $a\wedge$. Then $g(f(g(x)) =\sim (a\wedge \sim x) =\sim a\vee \sim\sim x =\sim a \vee x = a \rightarrow x$.

With this example in mind, let us consider the algebraic structure $\mathbf{RS}_x(\mathbf{A}) = \langle \mathcal{R}S_x(A), \wedge, \vee, \neg, \sim, \dashv, \longrightarrow, \supset, 0, 1 \rangle$, for $\mathbf{A}$ a Boolean algebra and $x \in \mathbf{A}$. Let $a, b \in \mathcal{R}S_x(\mathbf{A})$ (that is, $a = \langle a_1, a_2 \rangle, b = \langle b_1, b_2 \rangle$), and proceed as follows:

Definition 10.21.1. *Let $\odot$ be a binary operation. Let us call:*
1. the operation $\dashv(a \odot \dashv b)$: "co-intuitionistic dual" of $\odot$, denoted by $\dashv_d(\odot)$
2. the operation $\sim (a\odot \sim b)$: "De Morgan dual" of $\odot$, denoted by $\sim_d (\odot)$
3. the operation $\neg(a\odot\neg b)$: "intuitionistic dual" of $\odot$, denoted by $\neg_d(\odot)$

Moreover, whenever a binary operation $\odot$ has its left operand modalised by Mod_1 and its right operand modalised by Mod_2, for $Mod_1, Mod_2 \in \{L, M\}$, then we shall denote it by $_{Mod_1}\odot_{Mod_2}$. Finally, in order to apply a duality transformation to a modalised operation $_{Mod_1}\odot_{Mod_2}$ apply the negation sign to the argument of Mod_2, not to the modality (for instance $\lnot_d(L(a) \land M(a)) = \lnot(L(a) \land M(\lnot a))$).

Now we can notice that $\sim_d (\longrightarrow)$ is not commutative. Indeed, in order to have a commutative multiplication, the choice is essentially between $\lnot_d(\longrightarrow)$ and $\neg_d(\supset)$ (remember that $\lnot a = a \longrightarrow 0$ and $\neg a = a \supset 0$). So let us start with the first option:

Proposition 10.21.1. *The equation* $\lnot_d(\longrightarrow) =_L \land_L$ *holds in* **RS**(U).

Proof. (we show the easy deduction just to exhibit a prototype of the proofs of the subsequent propositions, that will be omitted):

$\lnot(a \longrightarrow \lnot b) = \lnot(\lnot a \lor \lnot b) = \lnot\lnot a \land \lnot\lnot b = L(a) \land L(b).$
The first equation derives from the equation $a \longrightarrow b = \lnot a \lor b$. The second from the fact that $\lnot$ inherits both De Morgan laws from the Boolean negation which defines $\lnot$. The third comes from the fact that $L(a) = \lnot\lnot(a)$ (for all these facts, cf. the previous Frame 10.20).   QED

Proposition 10.21.2.

1. *(i)* $\lnot_d(_L\land_L) =_L\longrightarrow_L$; *(ii)* $\sim_d (_L\land_L) = \neg_d(_L\land_L) =_L\longrightarrow_M$.

2. *(i)* $\lnot_d(_L\longrightarrow_M) =\sim_d (_L\longrightarrow_M) =_L \land_L$; *(ii)* $\neg_d(_L\longrightarrow_M) =_L \land_M$.

3. *(i)* $\lnot_d(_L\longrightarrow_L) =_L \land_L$; *(ii)* $\sim_d (_L\longrightarrow_L) = \neg_d(_L\longrightarrow_L) =_L \land_M$.

4. *(i)* $\lnot_d(_L\land_M) =\sim_d (_L\land_M) =_L\longrightarrow_L$; *(ii)* $\neg_d(_L\land_M) =_L\longrightarrow_M$.

Now the second option:

Proposition 10.21.3. *The equation* $\neg_d(\supset) =_M \land_M$ *holds in* **RS**(U).

Proposition 10.21.4.

1. *(i)* $\lnot_d(_M\land_M) =\sim_d (_M\land_M) =_M\longrightarrow_L$; *(ii)* $\neg_d(_M\land_M) =_M\longrightarrow_M$.

2. *(i)* $\lnot_d(_M\longrightarrow_L) =_M \land_L$. *(ii)* $\sim_d (_M\longrightarrow_L) = \neg_d(_M\longrightarrow_L) =_M \land_M$.

3. *(i)* $\lnot_d(_M\longrightarrow_M) =\sim_d (_M\longrightarrow_M) =_M \land_L$; *(ii)* $\neg_d(_M\longrightarrow_M) =_M \land_M$.

4. *(i)* $\lnot_d(_M\land_L) =_M\longrightarrow_L$; *(ii)* $\sim_d (_M\land_L) = \neg_d(_M\land_L) =_M\longrightarrow_M$.

As for the remaining cases, we can notice that $\neg_d(\longrightarrow) =_L \wedge_M, \neg_d(\supset)$ $=_M \wedge_L, \sim_d (a \longrightarrow b) = L(a) \wedge b$ and $\sim_d (a \supset b) = M(a) \wedge b$.

So, we have found operations that apply to the *center* of the above algebraic structures, that is, on operands modalised by means of L or M.

Since the center has a Boolean behaviour, it follows that when $\wedge$ is applied to modalised arguments (as like as $_L\wedge_M$), it behaves as a classical "and", thus enjoying all the structural rules. Moreover, any implication of the form $_{Mod_1} \longrightarrow_{Mod_2}$ works also when we substitute the Nelson implication $\longrightarrow$ with the material implication $\rightarrow$. In particular, we have recovered the two implications, $\gg$ and $\ggg$ that are used in Frame 10.20 to deal with functional dependency in incomplete information systems, namely, the operation $_M \rightarrow_L$ (by means of the $\neg_d-$dual of the $\neg_d-$dual of the relative pseudo-complementation $\supset$) and, respectively, the operation $_L \rightarrow_M$ (by means the $\neg_d-$dual of the $\neg_d-$dual of the Nelson implication $\longrightarrow$). What follows is more interesting.

The implication $_M \longrightarrow_L$ (i.e. $_M \rightarrow_L$) is also the $\neg_d-$dual of $_M\wedge_L$. In turn, $_M\wedge_L$ is the $\neg_d-$dual of $\longrightarrow$ (easy exercise). Therefore, $_M \rightarrow_L$ is the $\neg_d\neg_d$-transformation of $\longrightarrow$. Dually we can observe that the implication $_L \longrightarrow_M$ (i.e. $_L \rightarrow_M$) is also the $\neg_d-$dual of $_L\wedge_M$. In turn, $_L\wedge_M$ is the $\neg_d-$dual of $\longrightarrow$ (easy exercise, too). Therefore, $_L \rightarrow_M$ is the $\neg_d\neg_d-$transformation of $\longrightarrow$. These transformations are implicitly stated in *Corollary* 10.20.2 of Frame 10.20.

In other words, $\neg_d(\neg_d(a \rightarrow b)) = \neg\neg(a \rightarrow b) = a \gg b$ and $\neg_d(\neg_d(a \rightarrow b)) = \neg\neg(a \rightarrow b) = a \ggg b$.

10.22   Solutions

- Exercise 7.1 $\mathbb{I}(-A \cup B) = -\mathbb{C} - (-A \cup B) = -\mathbb{C}(A \cap -B)$.

- Exercise 7.2 Since $a \wedge x \leq x$, we have $x \leq a \Longrightarrow x$. From the Currying property of $\Longrightarrow$, $a \Longrightarrow (a \Longrightarrow x) = a \wedge a \Longrightarrow x$. Finally, $\Longrightarrow$ is upper adjoint, hence it preserves meets.

- Exercise 7.3 The hypotheses give: $b \wedge s' \leq s'$, $c \wedge s' \leq s'$, $b \wedge s' \leq b$; moreover $s' \leq s$ so that $c \wedge s' \leq c \wedge s = b \wedge s \leq b$. It follows that $b \wedge s' \leq (b \wedge s') \vee (c \wedge s') \leq b \wedge s'$, from which we obtain $b \wedge s' = (b \wedge s') \vee (c \wedge s')$ and hence $b \wedge s' = c \wedge s'$.

- Exercise 7.4

  (a) 7.4.16 and 7.4.17 are equivalent. Let $X \subseteq\uparrow p$ and $p \in J(X)$. Then from the definition of $J(X)$, $\uparrow p \cap X \in J_p$. It follows that $X \in \{Z \cap \uparrow p : p \in J(Z)\}$ (or, more directly, use the fact that if $X \subseteq\uparrow p$ then $X \cap \uparrow p = X$). Vice-versa, if $X \in \{\uparrow p \cap Z : p \in J(Z)\}$, then X is an order filter (since a meet of order filters) and $X \subseteq\uparrow p$. Now if $p \in X$ then $p \in J(X)$, trivially. Otherwise, $p \notin Z$. Thus it must exist Z' such that $p \in Z'$ and $Z \equiv_J Z'$. Suppose that $\uparrow p \not\equiv_J X$. This means $\uparrow p \not\equiv_J Z \cap \uparrow p$, which in turn, since $p \in Z'$ implies $\uparrow p \subseteq Z'$, is tantamount to $Z' \cap \uparrow p \not\equiv_J Z \cap \uparrow p$, which is impossible because $\equiv_J$ is a congruence. (A more explicit prove of the last fact runs as follows. Assume J is induced by $\uparrow S$, for S an element of the lattice. Then $Z \cap S = Z' \cap S$. Assume that $\uparrow p \cap S \neq X \cap S$ – so that $\uparrow p \not\equiv_J X$. For $X \subseteq\uparrow p$, we have that there must be a g such that $g \in\uparrow p \cap S$ but $g \notin X \cap S$. Hence, $g \notin X$ and, as a consequence, $g \notin Z$ (because $g \in\uparrow p$). But $g \in Z'$, because $\uparrow p \subseteq Z'$, and $g \in S$. It follows that $g \in Z' \cap S$ but $f \notin Z \cap S$, in contradiction with the fact that $Z \equiv_J Z'$).

  (b) 7.4.17 and 7.4.18 are equivalent. Let $X \subseteq\uparrow p$ and $p \in J(X)$. If $p \in X$, then $\uparrow p \subseteq X$ and we have $X =\uparrow p$, hence $X \equiv_J\uparrow p$, trivially. If $p \notin X$ then there exists Y such that $p \in Y$ and $Y \equiv_J X$. From $p \in Y$ we obtain $\uparrow p \subseteq Y$ which gives $X \subseteq\uparrow p \subseteq Y$. From this, straightforwardly, $\uparrow p \equiv_J X$. Vice-versa, if $X \subseteq\uparrow p$ and $X \equiv_J\uparrow p$ then $\uparrow p \subseteq J(X)$ so that $p \in J(X)$.

- Exercise 7.5

  (A) Assume $p \in \neg X$. Clearly $\neg X \cap X = \emptyset = \emptyset \cap X$. Hence $\neg X \equiv_{J^X} \emptyset$ from which we obtain $\neg X \subseteq J^X(\emptyset)$, so that $p \in J^X(\emptyset)$. It follows that $\emptyset \in J^X_{[p]}$ (because, trivially, $\emptyset \subseteq\uparrow p$). Vice-versa, if $\emptyset \in J^X_{[p]}$ then $\emptyset \equiv_{J^X}\uparrow p$. It follows that $\uparrow p \cap X = \emptyset \cap X = \emptyset$. Hence $\uparrow p \subseteq \neg X$ and $p \in \neg X$.

  (B) From the above result, trivially, because if X is greater than or equal to the least dense element, then $\neg X = \emptyset$, so that for all x, $x \notin \neg X$.

- Exercise 8.1

  (i) With a slight modification of *Lemma* 8.3.4, one can prove that for $a \equiv_{B_{\langle B,B \rangle}} b$ to hold, the B part must be immaterial. Therefore, $\mathcal{RS}_B(\mathbf{AS}(U))/_{\equiv_{B_{\langle B,B \rangle}}} = \mathcal{RS}_B(\mathbf{AS}(U))/_{\equiv_{J_{\langle P,P \rangle}}}$. (ii) $a \equiv_{J_{\langle B,B \rangle}} b$ if

and only if $a \wedge \langle B, B \rangle = b \wedge \langle B, B \rangle$, if and only if $a_1 \wedge B = b_1 \wedge B$ and $a_2 \wedge B = b_2 \wedge B$.

- Exercise 8.2
  It is not difficult to verify that for all a, b, $J^a \vee B_b = B_{a \longrightarrow b}$. Moreover, $\langle G, B \rangle \Longrightarrow \langle P, \emptyset \rangle = \langle G, B \rangle \Longrightarrow \sim \langle G, B \rangle = \sim \sim \lrcorner \langle G, B \rangle = \lrcorner \langle G, B \rangle = \langle \neg B, \neg B \rangle = \langle P, P \rangle$.

- Exercise 9.1
  (1.1) Since the operations inside each pair of $N_\Theta(\mathbf{H})$ are operations in $\mathbf{H}$, surely each output is a pair of elements of $\mathbf{H}$. (1.2) If $a_1 \wedge a_2 = 0$ and $b_1 \wedge b_2 = 0$ then $a_1 \wedge b_1 \wedge (a_2 \vee b_2) = (a_1 \wedge b_1 \wedge a_2) \vee (a_1 \wedge b_1 \wedge b_2) = 0 \wedge 0 = 0$. Same for $\vee$. Trivial for $\sim$. As to $\longrightarrow$, from $a_1 \Longrightarrow b_1 \wedge a_1 \leq b_1$ and $b_1 \wedge b_2 = 0$ we obtain $(a_1 \Longrightarrow b_1) \wedge b_2 = 0$. (1.3) The congruence condition $a_1 \vee a_2 \Theta 1$ holds in any resulting pair because of the definition of a "congruence" in Heyting algebras.

- Exercise 9.2 $\langle a_1, a_2 \rangle \rightarrowtail \langle b_1, b_2 \rangle = \langle (a_1 \Longrightarrow b_1 \wedge b_2 \Longrightarrow a_2), a_1 \wedge b_2 \rangle$. Thus, $\langle a_1, a_2 \rangle \rightarrowtail \langle b_1, b_2 \rangle \leq \langle b_1, b_2 \rangle$ only if $(a_1 \Longrightarrow b_1 \wedge b_2 \Longrightarrow a_2) \leq b_1$ and $a_1 \wedge b_2 \geq b_2$. But $a_1 \Longrightarrow b_1 \geq b_1$. It follows that we must have: (a) $a_1 \geq b_2$ and (b) $b_2 \Longrightarrow a_2 \leq b_1$.

- Exercise 9.3 Let us only verify some cases.
  (a) $nrs(\div a) = \div nrs(a)$: $nrs(\sim \neg \sim a) = nrs(\langle a_2, \neg a_2 \rangle) = \langle a_2, a_2 \rangle = \div nrs(a)$.
  (b) $nrs(a \longrightarrow b) = nrs(a) \longrightarrow nrs(b)$:

  $$
  \begin{aligned}
  nrs(a \longrightarrow b) &= nrs(\langle \neg a_1 \vee b_1, a_1 \wedge b_2 \rangle) \\
  &= \langle \neg a_1 \vee \neg b_2, \neg a_1 \vee b_1 \rangle = \langle \neg a_1, \neg a_1 \rangle \vee \langle \neg b_2, b_1 \rangle \\
  &= \langle \neg \neg a_2, a_1 \rangle \vee \langle \neg b_2, b_1 \rangle = \langle \neg a_2, a_1 \rangle \longrightarrow \langle \neg b_2, b_1 \rangle \\
  &= nrs(a) \longrightarrow nrs(b).
  \end{aligned}
  $$

  (c) $nrs(a \supset b) = nrs(a) \supset nrs(b)$; from the previous results or directly as follows:

  $$
  \begin{aligned}
  nrs(a) \supset nrs(b) &= \langle \neg a_2, a_1 \rangle \supset \langle \neg b_2, b_1 \rangle \\
  &= \langle \neg a_2 \Longrightarrow \neg b_2, (\neg a_2 \Longrightarrow \neg b_2) \cap (a_1 \Longrightarrow b_1) \rangle \\
  &= \langle \neg \neg a_2 \cup \neg b_2, (\neg \neg a_2 \cup \neg b_2) \cap (\neg a_1 \cup b_1) \rangle \\
  &= \langle a_2 \cup \neg b_2, (a_2 \cup \neg b_2) \cap (\neg a_1 \cup b_1) \rangle;
  \end{aligned}
  $$

but

$$(a_2 \cup \neg b_2) \cap (\neg a_1 \cup b_1) = ((a_2 \cup \neg b_2) \cap \neg a_1) \cup ((a_2 \cup \neg b_2) \cap b_1)$$
$$= (a_2 \cap \neg a_1) \cup (\neg b_2 \cap \neg a_1) \cup (a_2 \cap b_1) \cup (\neg b_2 \cap b_1)$$
$$= (\neg b_2 \cap \neg a_1) \cup (a_2 \cup b_1).$$

Thus we have $nrs(a) \supset nrs(b) = \langle a_2 \cup \neg b_2, (\neg b_2 \cap \neg a_1) \cup (a_2 \cup b_1) \rangle$.

$$nrs(a \supset b) = nrs((\neg a \wedge \neg \sim b) \vee \sim \neg \sim a \vee b)$$
$$= nrs(\langle ((-a_1 \cap -b_2) \cup a_2 \cup b_1), -a_2 \cap b_2 \rangle$$
$$= \langle a_2 \cup -b_2, (-b_2 \cap -a_1) \cup (a_2 \cup b_1) \rangle.$$

- Exercise 10.1

  Let us denote the element $\langle 1, \neg x \rangle$ with g. Define the map $p :$ $\mathcal{P}(\mathbf{B}) \longmapsto \mathcal{RS}_x(\mathbf{B}); p(a) = \langle a_1, a_2 \vee (a_1 \wedge x) \rangle$. It is obvious that $p(a)$ is the least element b of $\mathcal{RS}_x(\mathbf{B})$ s.t. $b \geq a$ in $\mathcal{P}(\mathbf{B})$. Indeed, for any $a \in \mathcal{P}(\mathbf{B})$, $p(a) \in \mathcal{RS}_x(\mathbf{B})$ because $p(a)_1 \wedge x = x \wedge p(a)_2$. Clearly, for any $c \in \mathcal{RS}_x(\mathbf{B})$, if $c \geq a$ then $c_1 \geq a_1$, $c_2 \geq a_2$, $c_2 \leq c_1$, $c_2 \wedge x = x \wedge c_1 \geq x \wedge a_1$, so that $c_2 \geq a_2 \cup (x \wedge a_1)$. Now we have to prove that p determines an order isomorphism between $\downarrow g$ and $\mathcal{RS}_x(\mathbf{B})$. Suppose $a, b \in \downarrow g$ and $a = \langle a_1, a_2 \rangle \neq \langle b_1, b_2 \rangle = b$. If $a_1 \neq b_1$ then $p(a) \neq p(b)$, because for all z, $p(z)_1 = z_1$. Suppose $a_1 = b_1$ but $a_2 \neq b_2$. Then $a_2 \vee (a_1 \wedge x) = a_1 \wedge (a_2 \vee x) \neq b_1 \wedge (b_2 \vee x) = b_2 \vee (b_1 \wedge x)$ because $a_2 \wedge x = b_2 \wedge x = 0$. Thus $p \upharpoonright \downarrow g$ is injective. Moreover, $J^g(a) = g \Longrightarrow a = \bigvee \{b : b \wedge g \leq a\}$. It follows that $J^g(a)$ is the least b such that $\langle b_1 \wedge 1, b_2 \wedge \neg x \rangle \leq \langle a_1, a_2 \rangle$. Then, of course, $b_1 = a_1$. Therefore, b_2 is the maximal element such that $b_2 \leq a_1$ and $b_2 \wedge \neg x \leq a_2$. Thus $b_2 \wedge x = a_1 \wedge x$, because $b_2 \wedge x \wedge \neg x = 0$, so that $(a_1 \wedge x) \vee a_2$, i.e. $p(a)_2$ is the maximal element y such that $y \wedge \neg x \leq a_2$. Hence, $J^g(a) = p(a)$, any a. Since p is, therefore, multiplicative, it has a lower adjoint $p^*(a) = \bigwedge \{p^{\leftarrow}(\uparrow a)\}$. We have immediately that $p^*(a)_1 = a_1$, because $p(a)_1 = a_1$. Then $p^*(a)_2 = a_2 \wedge \neg(a_1 \wedge x) = (a_2 \wedge \neg a_1) \vee (a_2 \wedge \neg x)$. But since $a_2 \leq a_1$ we have $a_2 \wedge \neg a_1 = \emptyset$ so that $p^*(a)_2 = a_2 \wedge \neg x$. To sum up: $p^*(a) = \langle a_1, a_2 \wedge \neg x \rangle$. It follows that for any a, $p^*(a) \in \downarrow g$. Thus $p^* \dashv p$, but we can readily verify that $p \dashv p^*$ holds, too. Thus $p(p^*(a)) = a = p^*(p(a))$ and p is onto. From the properties of adjunction relations (or, more simply, from the definition of p), p is order preserving.

- Exercise 10.2 The answer is "No". For example, in the semi-simple Nelson algebra $\mathcal{RS}_B(\mathbf{B})$ of Subframe 7.4.1, $\langle a, 0 \rangle \rightarrowtail \langle b, b \rangle = \langle 1, b \rangle$. In fact, in semi-simple a Nelson algebra the relative pseudo-complementation is $\supset$.

- Exercise 10.3
  (A) (a) $k_f(\{c, g\}) = \langle \{c\}, \{b\} \rangle$; (b) $k_f(\{c, b, f\}) = \langle \{c, b\}, \emptyset \rangle$ in $\mathbf{KS(C)}$; (c) $k_f(\{b, b', 1'\}) = \langle \{b\}, \{a, c\} \rangle$, (d) $k_f(\{b, a, 1', c', b'\}) = \langle \{b, a\}, \emptyset \rangle$ in $\mathcal{X}^{\{a\}}(\mathbf{J(A)})$.

  (B) The required isomorphism γ is given by $\gamma(\langle a_1, a_2 \rangle) = \langle \phi(a_1), \phi(a_2) \rangle$; $\psi(\langle a_1, a_2 \rangle) = \phi(a_1) \cup f(X^+ \cap -\phi(a_2))$, where ϕ is the isomorphism defined in (7.2.7). Indeed we can show that $\gamma(\langle a_1, a_2 \rangle) = \langle \phi(a_1), \phi(a_2) \rangle = k_f(\psi(\langle a_1, a_2 \rangle))$.
  (C) $\gamma(\langle b, a \rangle) = \langle \phi(b), \phi(a) \rangle = \langle \{b\}, \{a\} \rangle$.

  $\psi(\langle \{b\}, \{a\} \rangle) = \{b\} \cup f(-\{a\}) = \{b\} \cup f(\{1, c, b\}) = \{b\} \cup \{1', c', b\} = \{1', c', b\}$; $k_f(\{1', c', b\}) = \langle X^+ \cap \{1', c', b\}, X^+ \cap -f(\{1', c', b\}) \rangle = \langle \{b\}, X^+ \cap -\{1, c, b\} \rangle = \langle \{b\}, X^+ \cap \{a, a', c', 1'\} \rangle = \langle \{b\}, \{a\} \rangle$; $k_f(\{a', c', 1'\}) = \langle X^+ \cap \{a', c', 1'\}, X^+ \cap -f(\{a', c', 1'\}) \rangle = \langle \emptyset, X^+ \cap -\{a, c, 1\} \rangle = \langle \emptyset, \{b\} \rangle$; $\gamma^{-1}(\langle \emptyset, \{b\} \rangle) = \langle 0, b \rangle$.

- Exercise 10.4 $\langle a_1, a_2 \rangle \rightarrowtail \langle 0, 0 \rangle = \langle a_1, a_2 \rangle \longrightarrow \langle 0, 0 \rangle \wedge \langle 1, 1 \rangle \longrightarrow \langle \neg a_2, \neg a_1 \rangle = \langle \neg a_2, \neg a_2 \rangle \wedge \langle \neg a_2, \neg a_1 \rangle$. But from $a_2 \leq a_1$ we have $\neg a_1 \leq \neg a_2$, so that $\langle \neg a_2, \neg a_2 \rangle \wedge \langle \neg a_2, \neg a_1 \rangle = \langle \neg a_2, \neg a_1 \rangle = \sim \langle a_1, a_2 \rangle$.

Part III

The Modal Logic
of Rough Sets

Chapter 11

Modality and Knowledge

"In any act of inference or scientific method we are engaged about a certain identity, sameness, similarity, likeness, resemblance, analogy, equivalence or equality apparent between two objects". W. S. Jevons, The Principles of Science. A Treatise on Logic and Scientific Method, *London, 1877, page 1.*

11.1 Foreword

We have seen that the principal concept upon which the notion of a rough set relies, is "rough equality". We recall that two subsets A and B of U are *roughly equal* in an Approximation Space $\mathbf{AS}(U/E)$ if and only if $(lE)(A) = (lE)(B)$ and $(uE)(A) = (uE)(B)$, indicating that relative to the given partition of the domain, one is unable to discern between the concerned sets. The final intention in the present Part is to look at rough equality syntactically and semantically in the framework of modal logical systems. Indeed, we shall obtain this syntactical view passing through the semantic notion of a *monadic quasi-Boolean algebra* (mqBa).

This notion links together two mathematical fields which, in turn, relate to both the logical aspects we need: algebra, which models the non-modal logical operations, and topology, which models the modal operators that, in Rough Set Systems, are induced by the approximation operators. The resulting structure will be a representation of what we may propose as the *modal logical system of Rough Sets*.

As for the first part of this picture – the non-modal logical operations – in Part II we have seen that in Rough Set Systems roughness induces operations that are slightly but definitely different from the Boolean operations which lead to structures varying from Boolean Algebras to Post Algebras of order three passing through three-valued Łukasiewicz Algebras (or Chain Based Lattices of order three). In the present context roughness will lead to the notion of a "quasi-Boolean algebra".

As for the second part, we have already seen that two modal operators L and M can be defined that, in a precise sense, embed in Rough Set Systems the features of the lower and, respectively, the upper approximation.

The operators L and M exhibit well-defined topological properties. These properties together with quasi-Boolean operations will lead us to the notion of a "monadic quasi-Boolean algebra".

Before entering the technical details of this construction, we shall discuss why a modal perspective has a perspicuous meaning in Perception Systems. Then we shall analyse the specific properties of modalities in Rough Set Systems. Indeed, in what follows we shall explain:

- What does a modality mean in the present context.

- What kind of modalities are we dealing with.

- The topological interpretation of upper and lower approximations.

- The practical effect of this interpretation.

- The modal interpretation of upper and lower approximations.

- The philosophical implications of the modal interpretation.

- How these modalities are inherited by Rough Set Systems.

- What kind of logical system do we obtain by interpreting Rough Set Systems from a modal point of view.

- The syntactical properties of this system.

11.2   Modalities and Assertions

The term "modality" comes from the Latin word "*modus*" that means "manner" or "way". So, by this term we intend any operator which adds information about the way a sentence has to be intended. Typically, such a way may:

- Modulate the "force" of a sentence (its is *possible* that ..., it is *necessary* that ..., it is *impossible* that ..., it is *contingent* that ...) – in this way we have the so-called "Alethic modal logics", or "modal logics" *tout court.*

- Specify the "time" a sentence is asserted about (*always in the past*, *always in the future*, *sometime in the past*, *sometime in the future*) – in this way we obtain "tense logics".

- Concern with the prescription implied by a sentence (it is *allowed* that ..., it is *prohibited* that ...) – in this way we obtain "deontic logics".

- Specify the status of a sentence as related to the mental attitude of a subject (subject S *knows* that ..., subject S *believes* that ...) – this leads to "doxastic logics" (dealing with the operator "*S beliefs* that ...") and "epistemic logics" (dealing with the operator " S *knows* that ...).

- Specify the status of a sentence within a theory $\mathcal{T}$ (it is *coherent in* $\mathcal{T}$ that ..., it is *provable in* $\mathcal{T}$ that ...) – and we have the so-called "provability logic".[1]

- Specify what happens during or after an action or a computation A (it is *always true during* A that .., it is *sometimes true during* A *that* .., it is *always true after* A *that* .., it is *sometimes true after* A that ...). These modalities are taken into account by the so-called "dynamic logic".

Depending on the particular features of the modalizer, any kind of modal logic has different variants. For instance we may think of situations in which time "branches", so that we obtain a *ramified tens logic*; on the contrary, one may have a linear conception of time. Or one

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[1] $\mathcal{T}$ contains Peano Arithmetic, as like as the theories that Gödel's results refer to.

may think that in order to truly assert the possibility of an event, this event must be true in some conceivable state of affairs; in this case we speak of *Diodorean modalities*, which, in turn, may have variants. These variants will depend on the properties connected to the way a "state of affair" may be conceived from the others.

One could add to the above list an "approximation modal logic", but we think that this addition risks to be of no use since there are well-known modalities that give approximation operations a sufficiently meaningful interpretation.

We recall, for instance, that tense operators were already used in Part I in order to provide an intuitive setting for the discussion about the adjunction properties of the perception operators. And we recall, also, that at the end of that discussion we found two Alethic modal operators. And, of course, these two Alethic operators will be found at the end of the present analysis, too, but framed in a more general picture.

REMARKS. In what follows we shall develop our analysis of the classical approximation operators (uE) and (lE), while a similar analysis for pairs of generalised operators such as $\langle R_{\mathbf{S}} \rangle$ and $[R_{\mathbf{S}}]$ or int and cl will be sketched in the Frame section of the present Part.

Let us then, for convenience, recall some concepts. Let $\langle U, E \rangle$ be an Indiscernibility Space induced by an Information System; let P be a property and $[\![P]\!]$ its range of validity. Then, whenever $x \in (lE)([\![P]\!])$ we say that x *necessarily* fulfills P, while whenever $x \in (uE)([\![P]\!])$ we say that x *possibly* fulfills P. In fact, in the first case all the elements that are indiscernible from x from the point of view of our information system, enjoy property P, so that we have no example of elements that enjoy the same basic properties as x but do not enjoy P. In the second case, we only know that we have examples of elements which enjoy the same basic properties as x and enjoy P, too.[2] Moreover, the rough set images of lower and upper approximations have a particular status in Rough Set Systems: they are elements that are reachable by means of the operator L and, respectively, M. Namely, given $[\![P]\!]$, $rs((lE)([\![P]\!]) = L(r([\![P]\!])) = \lrcorner \lrcorner (r([\![P]\!]))$ and $rs((uE)([\![P]\!]) = M(r([\![P]\!])) = \neg \neg (r([\![P]\!]))$.

---

[2] The adverb "only" is appropriate in a strict sense when $x \in (uE)([\![P]\!]) \cap -(lE)([\![P]\!])$, that is, when x belongs to the boundary of $[\![P]\!]$.

This is the reason why we are interested in modal systems as related to Rough Set Systems.

We shall start analysing modalities in a very general setting, distilling the modal features of the two approximation operations step by step.

It will be seen that all the modal operators that are introduced in the list above, share a common set of modelling techniques, that are based mostly on the so-called "Kripke models" approach which was introduced in Frame 4.13 of Part I. These techniques will be further discussed within this Part, after an introduction aimed at analysing modal systems at a more abstract level.

We shall see that upper and lower approximations represent "possibility" and, respectively, "necessity" operators of Diodorean type, with distinguished properties which make Approximation Spaces into models for the so-called **S**5 modal system. These modalities are coherently connected to our conception of an Approximation Space as a *perception system*. However, in order to explain this point, we have to introduce another, less usual but subtle, kind of distinction: that between "*internal modalities*" and "*external modalities*".

11.3   Internal Modalities *vs* External Modalities

Let **S** be a generic logical model, that is, a structure equipped with an evaluation function between well formed formulas of some language and the elements of **S**. A modality Mod is an *internal modality* if, for any formula α, we can evaluate $Mod(\alpha)$ in **S** without the need of any other structure **S**′. On the contrary, Mod is an *external modality* if, in order to evaluate $Mod(\alpha)$, we have to contrast the evaluation of α in **S** against another structure **S**′:

$$
\begin{array}{ccc}
\alpha & \xrightarrow{\;interpretation\;} & \mathbf{S} \\
\Big\downarrow{\scriptstyle deduction} & & \Big\downarrow{\scriptstyle evaluation} \\
L(\alpha) & \xrightarrow{\;interpretation\;} & \mathbf{S}
\end{array}
$$

INTERNAL MODALITY PROCESS

$$\begin{array}{ccc}
\alpha & \xrightarrow{\quad interpretation \quad} & \mathbf{S} \\
\Big\downarrow{\scriptstyle deduction} & & \Big\downarrow{\scriptstyle evaluation} \\
L(\alpha) & \xrightarrow{\quad interpretation \quad} & \mathbf{S}'
\end{array}$$
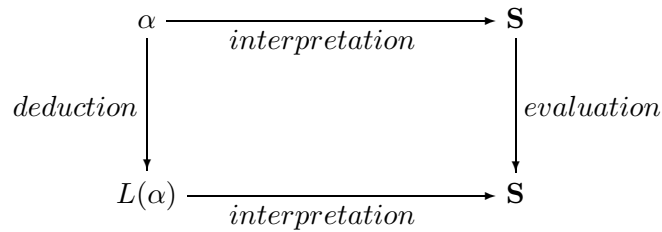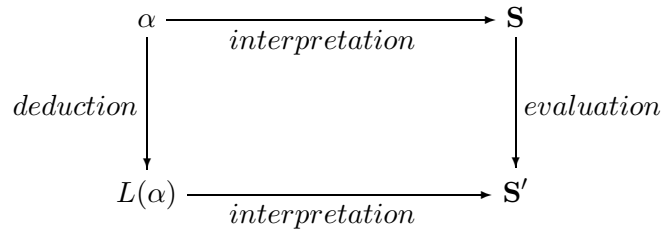
<div align="center">EXTERNAL MODALITY PROCESS</div>

In particular we shall deeply analyse the case in which $\mathbf{S}$ is the Boolean algebra $\mathbf{B}(U)$ of the subsets of a set U, and $\mathbf{S}'$ is a subalgebra of $\mathbf{B}(U)$, namely an Approximation Space $\mathbf{AS}(U/E)$, which is an instance of external modalisation process:

$$\begin{array}{ccc}
\alpha & \xrightarrow{\quad interpretation \quad} & \mathbf{B}(U) \\
\Big\downarrow{\scriptstyle deduction} & & \Big\downarrow{\scriptstyle evaluation} \\
L(\alpha) & \xrightarrow{\quad interpretation \quad} & \mathbf{AS}(U/E)
\end{array}$$
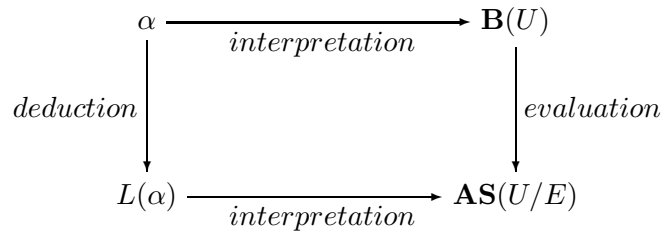
<div align="center">SPECIFIC EXTERNAL MODALITY PROCESS</div>

Now we explain the difference between these two approaches by developing in some details the notion of a *forcing over an algebraic structure.*

By the term "(propositional) forcing over an algebraic structure" we denote a relation between elements of a logico-algebraic structure and formulas of a propositional modal language. If p is an element of the logico-algebraic structure and α is a formula, we say that "*p forces (the validity) of α*" whenever p is part of the domain of validity of α. This situation tells us, intuitively, that in p there is enough information in order to state that α is true. We shall denote this relation with "$p \models \alpha$":

KRIPKE-JOYAL SEMANTICS

Let $\mathcal{L}$ be a propositional modal language, possibly without the implication sign, and let **S** be a complete ortholattice, a complete Boolean algebra or a complete Heyting algebra, with lattice order $\leq$. Let $\widehat{\phi} : \mathcal{L} \longmapsto$ **S** be a map from *atomic* sentences of $\mathcal{L}$ to elements of **S**, called a *set-up*.

We extend this *set-up* to an evaluation map ϕ of the sentences of $\mathcal{L}$, as follows:

(a) $\phi(\alpha \wedge \alpha') = \phi(\alpha) \wedge \phi(\alpha')$.
(b) $\phi(\alpha \vee \alpha') = \phi(\alpha) \vee \phi(\alpha')$.
(c) $\phi(\neg\alpha) = \neg\phi(\alpha)$, where $\neg\alpha$ denotes the orthocomplement, the complement or, respectively, the pseudo complement of α.
(d) $\phi(\alpha \implies \alpha') = \phi(\alpha) \implies \phi(\alpha')$, where $\implies$ denotes the relative pseudo-complementation in Heyting or Boolean algebras.

Let us define a *forcing relation* $\models$ as follows, for $p \in$ **S**$, \gamma, \alpha, \alpha' \in \mathcal{L}, \gamma$ atomic:

1. $p \models \gamma$ iff $p \leq \widehat{\phi}(\gamma)$.
2. $p \models \alpha \wedge \alpha'$ iff $p \models \alpha$ & $p \models \alpha'$.
3. $p \models \alpha \vee \alpha'$ iff $\exists p', p''(p' \vee p'' = p$ & $p' \models \alpha$ & $p'' \models \alpha')$.
4. $p \models \neg\alpha$ iff $\forall p' \leq p(p' \models \alpha \Rightarrow p' \leq \neg p)$.

These clauses can be extended to languages with an implication sign "$\implies$", as follows:

5. $p \models \alpha \implies \alpha'$ iff $\forall p' \leq p(p' \models \alpha \Rightarrow p' \models \alpha')$.

PERSISTENCE PROPERTY

We say that **S** enjoys the *Persistence Property* if for all $p \in$ **S**$, \alpha \in \mathcal{L}$ the following holds:

(Pers). If $p \models \alpha$ then $\forall p' \leq p, p' \models \alpha$.

Window 11.1. **Forcing over algebraic structures**

REMARKS. The extension to the implication $\implies$ (vital for analyzing Heyting algebras) is not important in Boolean algebras, since in this case the relative pseudo-complement $a \implies b$ reduces to the implication by cases $\neg a \vee b$ (in ortholattices, implication with nice properties is problematic because of the lack of distributivity).

Proposition 11.3.1. *(see Bell [1983]) Let $\mathbf{S}$ be a complete ortholattice, Heyting algebra or Boolean algebra. Let $\mathcal{L}$ be a language possibly without the implication sign, α any sentence of $\mathcal{L}$ and β any sentence not containing the connective $\vee$. Let $\models$ be a forcing relation from $\mathcal{L}$ to $\mathbf{S}$ based on a setup $\widehat{\phi}$, and ϕ the extension of $\widehat{\phi}$. Then,*

1. *$\phi(\alpha) = 1$ iff $1 \models \alpha$.*

2. *$0 \models \alpha$.*

3. *For all $p \in \mathbf{S}$, (i) $p \models \beta$ iff $p \leq \phi(\beta)$; (ii) $\phi(\beta) = \bigvee\{p : p \models \beta\}$.*

4. *For all $p \in \mathbf{S}$, if $\mathbf{S}$ is a Heyting algebra or a Boolean algebra, then (i) $p \models \alpha$ iff $p \leq \phi(\alpha)$; (ii) $\phi(\alpha) = \bigvee\{p : p \models \alpha\}$.*

From *Proposition* 11.3.1.(4) we have that the Persistence Property holds in general for Heyting and Boolean algebras; on the contrary, from *Proposition* 11.3.1.(3) in ortholattices it holds restricted to formulas without disjunctions. As we are going to see, this distinction is important for the definition of the forcing clauses for modalised formulas.

In fact, now we have to decide how to define modalities. Intuitively we can think of elements of $\mathbf{S}$ as information grains so that if $p' \leq p$, then the granularity of p' is finer than the granularity of p. Otherwise stated, p' is the result of zooming into p or of adding information to p. Therefore, we may say that p bears sufficient information to claim that α is *necessarily true*, if any refinement, any specialization of the information in p forces us to consider α valid, anyway. Along this intuition we shall say that for any $p \in \mathbf{S}$, for any $\alpha \in \mathcal{L}, p \models L(\alpha)$ if and only if for any $p' \leq p, p' \models \alpha$.

In this way we obtain an "internal characterisation" of the modality L. We call it "internal", because it uses solely the lattice structure of $\mathbf{S}$ and, obviously, the forcing relation.

Of course it is not guaranteed that this definition is meaningful for any structure. Surely, this is not the case for structures enjoying the Persistence Property. In such structures the above definition is simply immaterial since it does not add any value to the information already provided by the forcing clauses for non modalised formulas, because from the Persistence Property, if $p \models \alpha$ and $p' \leq p$, then $p' \models \alpha$.

Example 11.3.1. Ortholattices and internal forcing
Consider an ortholattice **O** isomorphic to that of Frame 4.6.1 of Part I:
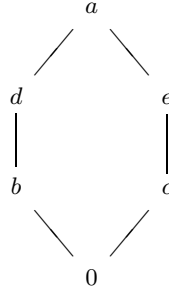
Let A and B be atomic formulas. Let $b \models A, c \models B$ (so that also $0 \models A$ and $0 \models B$ from *Clause 1* of Kripke-Joyal forcing). Therefore, since $a = b \vee c$, from *Clause 3* of Kripke-Joyal forcing we obtain $a \models A \vee B$. Moreover, $d = d \vee 0$ and $d = d \vee d$ but neither $d \not\models A$ nor $d \not\models B$, so that $d \not\models A \vee B$, because the forcing clause for the disjunction of A and B does not apply. For an analogous reason, $e \not\models A \vee B$.

   Therefore, the disjunction $A \vee B$ is not downwards inherited.

   Therefore we have:

- $a \models A \vee B$.
- $d, e \leq a$.
- $d \not\models A \vee B$, $e \not\models A \vee B$.

It follows that the Persistence Property does not hold in **O**.

11.4   Knowledge and Information

However, this characterisation of modal forcing is not the only alternative. In fact, we may also think that the lattice order of **S** delivers, on its own, only an "information order", while we need also a "knowledge order" to evaluate sentences such as "it is *possible* that ..." or "it is *necessary* that ...". In other words, with any element p of **S** we have to explicitly associate the elements which we consider as knowledge counterparts of the information embedded in p. Hence, we shall adopt a "knowledge relation" $K \subseteq \mathbf{S} \times \mathbf{S}$, so that if $\langle x, y \rangle \in K$ we shall say that the information carried by y is part of a conceptual pattern connected with the information carried by x.

   At this point, one could assume that the information-driven lattice order of **S**, $\leq$, and the knowledge-driven relation are unlinked. Under this hypothesis $\leq$ has no role in the identification of $K(p) = \{p' : \langle p, p' \rangle \in K\}$. If this is the case, the information order does not induce

any knowledge order. But this is a rather odd assumption, according to our intuition. It is true that we are framing the elements of **S** into a new structure which is more perspicuous (conceptual) than the given lattice structure; however this new point of view cannot forget that the lattice order represents, anyway, the structuring frame of the information we are dealing with (and this fact is reflected by the forcing clauses); and one cannot forget that knowledge is not independent of information. If we think that this is a reliable picture, then the knowledge order has to be coherent with the information order, so that given $p \in \mathbf{S}$ instead of a set of counterparts we can associate p just with a single element of the neighborhood $K(p)$, namely its largest element with respect to the information order (and provide that such an element exists). Following this assumption, instead of a knowledge relation K we can consider a *knowledge association function* k, so that if $b = k(p)$, then we shall say that b is the representative of the conceptual pattern associated with p. Therefore, b will be called a "knowledge counterpart" of p. This function will have the following properties:

1. *0-preservation*: $k(0) = 0$: from *Proposition* 11.3.1, 0 forces all formulas. Therefore, 0 is a fixed-point for the knowledge association function.

2. *Join-preservation*: $k(p \vee p') = k(p) \vee k(p')$, for any $p, p' \in \mathbf{S}$. This property reflects a sort of "continuity" that we require for conceptual pattern formation. Knowledge is built up from information in a continuous way. Although this request is avoidable in other contexts,[3] nonetheless it makes sense when speaking of "knowledge", in view of the knowledge paradoxes cited in the Introduction. From this property we derive:

3. *Order-preservation*: if $p \leq p'$, then $k(p) \leq k(p')$: if $p \leq p'$, then the information in p is stronger than the information in p'. In fact, at least for atomic formulas, p forces more formulas than p'. It follows that the knowledge associated to p is more refined than the knowledge associated to p'.

We should wonder whether an element belongs to its own conceptual pattern, or not. If yes, we must have $p \leq k(p)$. Moreover, what about

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[3]We already know that Concept Lattices do not fulfill this sort of continuity (cf. Part I).

$k(k(p))$? Is $k(p)$ to be a fixed point of k (i.e. $k(k(p)) = k(p)$)? We answer yes if we think that $k(p)$ is able to "drill down", in one shot, all the conceptual counterparts of the conceptual counterparts of p, and so. Both these properties are acceptable in a number of contexts and they say that k is a closure operator. However, they are not acceptable in general. So, we do not add them to the list of the basic properties 1–3 of k.

If the situation is satisfactory with this picture, we consolidate it in the following definition:

Definition 11.4.1. *Let* **S** *be a lattice. By a knowledge map, we intend any join-preserving 0-automorphism on* **S**.

REMARKS. This is not the only solution. In the Frame section, the reader will see examples in which knowledge order and information order are coherent in a different sense.

Using the above approach, we say that p forces $L(\alpha)$ if α is forced by any information that is more refined than the knowledge counterpart of p.

Dually, for the modality M we must consider, for any element p of **S**, the largest element whose knowledge counterpart contains p. This information is given by a function g definable from k and that enjoys the same properties.[4]

The applications $k(\mathbf{S}) = \{k(p)\}_{p \in \mathbf{S}}$ and $g(\mathbf{S}) = \{g(p)_{\ p \in \mathbf{S}}$ determine two sublattices $\mathbf{S}'$ and $\mathbf{S}''$ of **S**. Therefore, according to our terminology, modalities defined by means of $\mathbf{S}'$ or $\mathbf{S}''$ will be called "external modalities".

Let $\mathcal{L}$ be a propositional modal language and let **S** be a complete ortholattice, a complete Boolean algebra or a complete Heyting algebra, with lattice order $\leq$.

Let $\models$ be a Kripke-Joyal forcing relation over an algebraic structure **S** based on a setup $\widehat{\phi}$, and let ϕ be the extension of $\widehat{\phi}$.

Window 11.2. **Forcing for modalised formulas**

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[4] $g : \mathbf{S} \longmapsto \mathbf{S}; g(p) = \bigvee\{p' : \exists p''(p'' \in k^{\leftarrow}(\downarrow p)\ \&\ p' \wedge p'' \neq 0)\}.$

INTERNAL FORCING

For any $p \in \mathbf{S}$ and $\alpha \in \mathcal{L}$ we set the following *internal forcing clause* for modalised formulas:

1. $p \models L(\alpha)$ iff $\forall p'(p' \leq p \Rightarrow p' \models \alpha)$.

It can be proved that if we set $M(\alpha) = \neg L(\neg \alpha)$, then (see Frame 15.10):

2. $p \models M(\alpha)$ iff $\exists p'(p' \geq p \ \& \ p' \models \alpha)$.

EXTERNAL FORCING

Let k and g be two join-preserving 0-automorphisms $\mathbf{S} \longmapsto \mathbf{S}$.

For any $p \in \mathbf{S}, \alpha \in \mathcal{L}$, we set the following *external forcing clauses* for modalised formulas:

3. $p \models L_k(\alpha)$ iff $\forall p'(p' \leq k(p) \Rightarrow p' \models \alpha)$.
4. $p \models M_g(\alpha)$ iff $\exists p'(g(p') \geq p \ \& \ p' \models \alpha)$.

FACTS ABOUT EXTERNAL FORCING

5.1. $p \models L_k(\alpha)$ iff $k(p) \leq \phi(\alpha)$; 5.2. $\phi(L_k(\alpha)) = \bigvee\{p : k(p) \leq \phi(\alpha)\}$;
6. $\phi(M_g(\alpha)) = \bigvee\{p : \exists p'(g(p') \geq p \ \& \ p' \leq \phi(\alpha)\}$.

Window 11.2. **Continued**

REMARKS.

- For the proof of *Clause 2* of *Window* 11.2, see the Frame section. The three facts are left as exercises.

- In *Clause 3* of *Window* 11.2, the proviso " $\forall p' \leq k(p) \ldots$ " is immaterial in Boolean and Heyting algebras because of the Persistence Property. Indeed, $k(p) \leq k(p)$, therefore if $k(p) \models \alpha$ and $p' \leq k(p)$, then $p' \models \alpha$, so that we can more simply state: $p \models L_k(\alpha)$ iff $k(p) \models \alpha$. Anyway, we have used it in the defining clause for L for the sake of generality, because if we extend k to a function from $\mathbf{S}$ to a larger lattice $\mathbf{S}'$, then we have to state the definition as follows: $p \models L_k(\alpha)$ iff $\forall p' \in \mathbf{S}(p' \leq_{\mathbf{S}'} k(p) \Rightarrow p' \models \alpha)$, where $\leq_{\mathbf{S}'}$ is the lattice order of $\mathbf{S}'$. Moreover, *Clause 3* better stresses the analogies and differences with the corresponding internal forcing clause and with some formula defining approximation operators we have studied in Part I

- If k is a closure operator, then *Clause 3* is equivalent to $\forall p'(k(p') \leq k(p) \Rightarrow p' \models \alpha)$.

TERMINOLOGY AND NOTATION. Since g is derivable from k, we may focus our attention to pairs $\langle \mathbf{S}, k(\mathbf{S}) \rangle$, where $\mathbf{S}$ is a lattice and where an operator L_k is definable by means of *Clause 3* of *Window* 11.2 above. Moreover, from now on the operator M_g (if defined by $M_g(\alpha) = \neg L_k(\neg \alpha)$) will be denoted by M_k, for the same reason.

11.5 Knowledge and Modal Systems

Definition 11.5.1. *Let* $\mathbf{S}$ *be a lattice and* k *a join-preserving 0-automorphism on* $\mathbf{S}$*. Then the pair* $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ *is called a "k-modal system".*

Because of *Facts 5.1, 5.2* and *6.* of *Window* 12.1, we can work directly on the elements of $\mathbf{S}$, without mentioning explicitly any formula but understanding any $a \in \mathbf{S}$ as an evaluation $\phi(\alpha)$ for some formula α. Therefore, we shall define two operators L_k and M_k on $\mathbf{S}$, as follows:

Definition 11.5.2. *Let* $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ *be a k-modal system. For any* $a \in \mathbf{S}$*,*

1. $L_k(a) = \bigvee \{p : k(p) \leq a\}$.

2. $M_k(a) = \bigvee \{p : \exists p'(g(p') \geq p \ \& \ p' \leq a)\}$.

Proposition 11.5.1. *Let* $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ *be a k-modal system. Then for all* $a \in \mathbf{S}$*,* $L_k(a) \in \{p : k(p) \leq a\}$*.*

Proof: Suppose $k(x) \leq a$ and $k(x') \leq a$. Therefore $k(x) \vee k(x') \leq a$. By additivity of k, $k(x) \vee k(x') = k(x \vee x')$. Thus $k(x \vee x') \leq a$. It follows that if $x, x' \in \{p : k(p) \leq a\}$, then $x \vee x' \in \{p : k(p) \leq a\}$. Hence $\bigvee \{p : k(p) \leq a\} \in \{p : k(p) \leq a\}$. QED

REMARKS. The above result says that $L_k(a) = \max\{p : k(p) \leq a\}$. One should not confuse it with the equation $L_k(a) = \max\{k(p) : k(p) \leq a\}$. This equation is one of the main concerns of this chapter and we shall see that it holds under very specific constraints.

The external forcing for modalised formulas is a sort of second level evaluation.[5]

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[5]However, we avoid the tempting term "superevaluation" for historical reasons, because it has been used in order to denote a rather different modelling technique (see van Fraassen [1969]).

The first level acts as an external interface collecting pieces of information and framing them into an information structure. The information structure must subsequently be processed by a second evaluation step, *via* the forcing relation determined by the "knowledge structure" $\mathbf{S}' = k(\mathbf{S})$. This is the reason why we term this technique "external forcing" (see Figure 11.1).
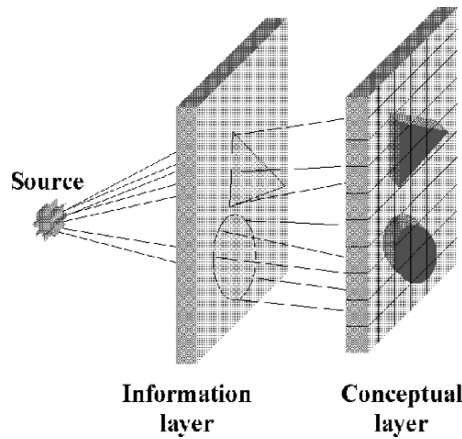
Figure 11.1: The triplane of external modalities

On the contrary, according to internal forcing, modal evaluations are processed at a single level using a single structure. In a sense, we ought to use internal forcing whenever we think that information and knowledge coincide and we should use external forcing otherwise.

However, as already mentioned, we cannot freely choose one approach or the other: if $k(p) \leq p$ internal forcing does not issue any effect in structures enjoying the Persistence Property, since in this case $\phi(\alpha)$ and $\phi(L_k(\alpha))$ coincide, so that we "must" use, in this case, external forcing (or some other technique).

As for the main topic, if we recall the intuitive picture of *Approximation Spaces* given in the Introduction namely the roles played by foreground and background spaces, then it is immediate to realize that we must adopt the external forcing approach in order to give *Approximation Spaces* a suitable modal interpretation. Indeed, a background space plays the role of a second level structure $\mathbf{S}'$ and a foreground space plays the role of a first level structure $\mathbf{S}$.

This is not a surprise, at all. Indeed we already know that the two approximations are determined by contrasting perceptions against a conceptual interpretation.
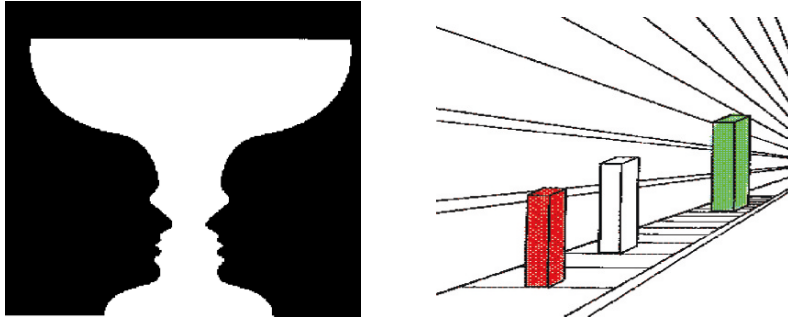
Figure 11.2: Examples of figures which require a conceptual interpretation

The meaning of the figure on the left depends on two main factors: our intention and how we are accustomed to perceive backgrounds (white or black).

The perception of the three blocks (actually aequo-dimensional) depends on how perspective has been historically represented in flat pictures, which in turn, depends on conceptual patterns (at least in Western art).

Therefore, the structure $\langle \mathbf{B}(U), \mathbf{AS}(U/E) \rangle$ over a universe U, is to be qualified as a *modal system with external modalities.* Since $\mathbf{B}(U)$ is a Boolean algebra (namely the Boolean algebra of $\wp(U)$) and the Approximation Space $\mathbf{AS}(U/E)$ is a Boolean subalgebra of $\mathbf{B}(U)$, in the present more abstract setting we shall say that an Abstract Approximation System is any pair $\langle \mathbf{B}, \mathbf{B}' \rangle$ such that $\mathbf{B}$ is a Boolean algebra and $\mathbf{B}'$ is a Boolean subalgebra of $\mathbf{B}$, exactly as we have explained in Part II:

Definition 11.5.3. *An* Abstract Approximation System *is a pair $\langle \mathbf{B}, \mathbf{B}' \rangle$ such that $\mathbf{B}$ is a Boolean algebra and $\mathbf{B}'$ is a Boolean subalgebra of $\mathbf{B}$ and such that the lower approximation $(l\mathbf{B}')$ and the upper approximation $(u\mathbf{B}')$ are defined, for all $a \in \mathbf{B}$, by*

1. *$(l\mathbf{B}')(a) = \max\{c \in \mathbf{B}' : c \leq a\}$.*

2. *$(u\mathbf{B}')(a) = \min\{c \in \mathbf{B}' : c \geq a\}$.*

Now we shall interpret $(l\mathbf{B}')$ and $(u\mathbf{B}')$ as dyadic operations. In this way we shall see that they are instances of a generalisation of some

operations that we have already met. Indeed, one can notice that $(l\mathbf{B}') = \bigvee\{c \in \mathbf{B}' : c \wedge 1 \leq b\}$ and $(u\mathbf{B}') = \bigwedge\{c \in \mathbf{B}' : c \vee 0 \geq b\}$, so that one obtains:

Proposition 11.5.2. *For any Abstract Approximation System* $\langle \mathbf{B}, \mathbf{B}' \rangle$, *for any* $a \in \mathbf{B}$,

1. $(l\mathbf{B}')(a) = 1\overset{\mathbf{B}'}{\Longrightarrow}a,$

2. $(u\mathbf{B}')(a) = 0\overset{\mathbf{B}'}{\Longleftarrow}a,$

   where, given a pair of lattices $\langle \mathbf{S}, \mathbf{S}' \rangle$, *the two operations* $\overset{\mathbf{S}'}{\Longrightarrow}$ *and* $\overset{\mathbf{S}'}{\Longleftarrow}$ *are defined, for any* $a, b \in \mathbf{S}$, *by:*

(i) $\overset{\mathbf{S}'}{\Longrightarrow} : \mathbf{S} \longmapsto \mathbf{S}'; a\overset{\mathbf{S}'}{\Longrightarrow}b = \max\{c \in \mathbf{S}' : c \wedge a \leq b\}$ *(generalised relative pseudo-supplementation); we denote* $1\overset{\mathbf{S}'}{\Longrightarrow}a$ *with* $(l\mathbf{S}')(a)$.

(ii) $\overset{\mathbf{S}'}{\Longleftarrow} : \mathbf{S} \longmapsto \mathbf{S}'; a\overset{\mathbf{S}'}{\Longleftarrow}b = \min\{c \in \mathbf{S}' : c \vee a \geq b\}$ *(generalised dual relative pseudo-supplementation); we denote* $0\overset{\mathbf{S}'}{\Longleftarrow}a$ *with* $(u\mathbf{S}')(a)$.

So, we have eventually found a generalisation of the operations exploited in Part II, when Rough Set Systems are interpreted as Chain Based Lattices. In that case we used the notions of "relative pseudo-supplementation", "pseudo-supplementation", "dual relative pseudo-supplementation" and "dual pseudo-supplementation". Indeed, they are nothing but the above operations in the specific case when $\mathbf{S}$ is a distributive lattice and $\mathbf{S}'$ is the center $\mathcal{CTR}(\mathbf{S})$ of $\mathbf{S}$.

Also, through these notions we have reached a sufficient abstract degree that is packed in the following definition:

Definition 11.5.4. *Let* $\mathbf{S}$ *be a bounded lattice and* $\mathbf{S}'$ *a sublattice of* $\mathbf{S}$ *such that both* $1\overset{\mathbf{S}'}{\Longrightarrow}p$ *and* $0\overset{\mathbf{S}'}{\Longleftarrow}p$ *are defined for any element* p *of* $\mathbf{S}$. *Then the pair* $\langle \mathbf{S}, \mathbf{S}' \rangle$ *is called a* modal system.[6]

Example 11.5.1. Generalised pseudo-supplementations and Modalities
Let us consider the following lattice $\mathbf{L}$ (incidentally, this is a Post lattice of order three):

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[6]This is equivalent to the notion proposed in Iturrioz [1990].

L
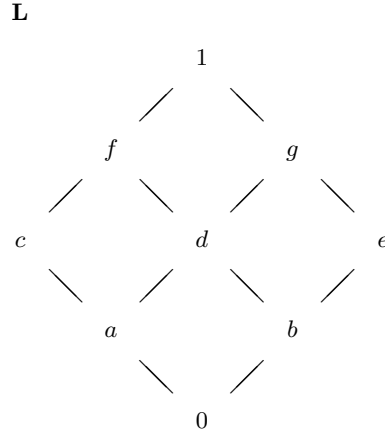
Now, imagine that the following sublattice $\mathbf{L}'$ plays the relevant role in modalising the elements of $\mathbf{L}$ (for instance, $\mathbf{L}'$ could be the image of $\mathbf{L}$ along the following knowledge map k: $k(f) = k(c) = k(1) = 1$, $k(d) = k(g) = g$, $k(b) = k(e) = e$, $k(a) = a$, $k(0) = 0$).

$\mathbf{L}'$
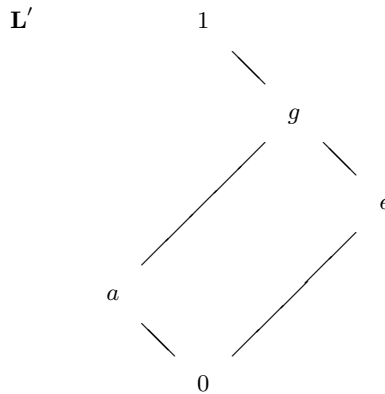
Let us then see, in the modal system $\langle \mathbf{L}, \mathbf{L}' \rangle$, some examples of application of the generalised relative pseudo supplementation, $\overset{\mathbf{L}'}{\Longrightarrow}$, and of the generalised relative dual pseudo supplementation, $\overset{\mathbf{L}'}{\Longleftarrow}$:

(1) $b \overset{\mathbf{L}'}{\Longrightarrow} a$:

(1.a) Compute the set $\{x : x \wedge b \leq a\}$, obtaining $\{0, a, c\}$;

(1.b) Then compute the intersection of this set with $\mathbf{L}'$:
$\{x : x \wedge b \leq a\} \cap \mathbf{L}' = \{o, a\}$;

(1.c) Finally take the maximum in the resulting set:
$\bigvee \{\{x : x \wedge b \leq a\} \cap \mathbf{L}'\} = \bigvee \{o, a\} = a$.

To sum up, $b \overset{\mathbf{L}'}{\Longrightarrow} a = a$ (while $b \Longrightarrow a = c$).

Similarly we obtain, for instance, $e \overset{\mathbf{L}'}{\Longrightarrow} d = a$ (while $e \Longrightarrow d = f$).

However, $c \overset{\mathbf{L}'}{\Longrightarrow} d = c \Longrightarrow d = g$.

(2) $d \overset{\mathbf{L}'}{\Longleftarrow} f$:

(2.a) Compute the set $\{x : x \vee d \geq f\}$, obtaining $\{c, f, 1\}$;

(2.b) Then compute the intersection of this set with $\mathbf{L}'$: $\{x : x \vee d \geq f\} \cap \mathbf{L}' = \{1\}$;

(2.c) Finally take the infimum in the resulting set:

$\bigwedge\{\{x : x \vee d \geq f\} \cap \mathbf{L}'\} = \bigwedge\{1\} = 1$.

To sum up, $d \overset{\mathbf{L}'}{\Longleftarrow} f = 1$ (while $d \Longleftarrow f = c$).

Similarly, we obtain, for instance, $e \overset{\mathbf{L}'}{\Longleftarrow} f = 1$ (while $e \Longleftarrow f = c$).

However, $e \overset{\mathbf{L}'}{\Longleftarrow} d = e \Longleftarrow d = a$.

When we consider modalities, we obtain, for instance:

(3) $(l\mathbf{L}')(d) = \bigvee\{x \in \mathbf{L}' : x \leq d\} = \bigvee\{0, a\} = a = \bigvee\{x : x \wedge 1 \leq d\} \cap \mathbf{L}' = 1 \overset{\mathbf{L}'}{\Longrightarrow} d$.

Moreover, $(l\mathbf{L}')(d) = (l\mathbf{L}')(c) = (l\mathbf{L}')(f) = (l\mathbf{L}')(a)$.

Therefore, any Approximation System is a modal system. It follows that the philosophical discussion about internal and external modalities, together with the evidence that *Approximation Systems* are systems with external modalities because we apply a second level knowledge order to interpret data, makes us focalise on those cases in which the notions of a k-modal system and a modal system coincide. This happens when in a k-modal system $\langle \mathbf{S}, k(\mathbf{S}) \rangle$, the element $\max\{p : k(p) \leq a\}$ coincides with $\max\{k(p) : k(p) \leq a\}$, for any $a \in \mathbf{S}$. In fact in this case $L_k(a) = \max\{x \in k(\mathbf{S}) : x \leq a\} = 1 \overset{k(\mathbf{S})}{\Longrightarrow} a$.

Therefore in what follows we shall mainly be interested in situations where a modal system may be represented by a k-modal system, according to the following definition:

Definition 11.5.5. *Let $\langle \mathbf{S}, \mathbf{S}' \rangle$ be a modal system and $\langle \mathbf{T}, k(\mathbf{T}) \rangle$ a k-modal system. We say that $\langle \mathbf{T}, k(\mathbf{T}) \rangle$ is a* representation *of $\langle \mathbf{S}, \mathbf{S}' \rangle$ if and only if they are component-wise isomorphic and $\mathbf{T}$ is a lattice of set.*

The next paragraphs are devoted to show, in sequence:

1. That certain kinds of modal systems are representable by k-modal systems such that the knowledge map k is connected with some relation R

2. That the properties of this relation R induces specific features of the modal system

3. That if specific properties are enjoyed by the relation R, then the connected modal system enjoys particular pre-topological or topological properties

4. That abstract Approximation System are particular cases of representable modal systems

5. That any finite abstract Approximation System $\langle \mathbf{B}, \mathbf{B}' \rangle$ is representable by a k-modal system $\langle \mathbf{B}(U), k(\mathbf{B}(U)) \rangle$ such that the knowledge map k is connected with an indiscernibility relation E. Therefore:

6. That any Approximation Space $\langle \mathbf{B}(U), \mathbf{AS}(U/E) \rangle$ is actually a k-modal system.

As a matter of fact, to accomplish this job we must analyze the conditions under which the operators L_k and M_k, as defined in *Definition* 11.5.2, coincide with the operators $(l\mathbf{S}')$ and, respectively, $(u\mathbf{S}')$ as defined in *Definition* 11.5.2.

To do this, we have to lower the abstract level and to work with the more "concrete" notion of a "forcing over a frame"

Anyway, let us remain for a while at the higher abstract level used as far as now, in order to see, briefly, how internal forcing works and appreciate its difference with the external variant.

11.5.1 Internal Modalities and Non-Distributivity

In ortholattices we may have elements p and formulas α such that $p \models \alpha$ while for some $p' < p$, $p' \not\models \alpha$. The lack of the Persistence Property is connected with the fact that the lattice order in an ortholattice is not generally represented by its co-prime elements, in the sense that we cannot recover an ortholattice $\mathbf{L}$ from the set $J(\mathbf{L})$ alone. In fact, we need the Dedekind-MacNeille completion of $J(\mathbf{L}) \cup M(\mathbf{L})$ (see Frame 4.14.2 of Part I). In fact, this happens because of the lack of distributivity in ortholattices. More precisely, in view of the forcing clause for disjunctions in Kripke-Joyal semantics, one easily sees that if $\mathbf{L}$ is an ortholattice, then we can have elements $p, p' \in \mathbf{L}$ and formulas α and α' such that $p \models \alpha \vee \alpha'$, $p' < p$, but $p' \not\models \alpha \vee \alpha'$. It is this particular feature that does not make $\phi(\alpha)$ and $\phi(L(\alpha))$ collapse.

Example 11.5.2. Ortholattices and internal forcing – II

Consider Example 11.3.1. Since $d, e \not\models A \vee B$, although $a \models A \vee B$, $a \not\models L(A \vee B)$ so that the validity of a formula and the validity of its necessitation do not coincide. Moreover, $0, b, c \models A \vee B$ (because $b = 0 \vee b$, $c = 0 \vee c$, $b \models A$, $c \models B$, $0 \models B$, $0 \models A$). Hence, $\{x : x \models L(A \vee B)\} = \{0, b, c\}$. But we cannot have a least upper bound of the elements forcing $L(A \vee B)$, since $b \vee c \vee 0 = a$, but $a \not\models L(A \vee B)$. [See also the Frame section].

In particular, if **L** is a complete quantum assemblage induced by a Proximity Space $\langle U, R \rangle$ (see Frame 4.6.1 of Part I), then internal modalities make sense in **L** and they are able to model significant phenomena. In the Frame section the reader will find some remarks about the exploitation of internal modalities in Quantum Logic.

On the contrary, if we start from an Indiscernibility Space $\langle U, E \rangle$ everything changes, as we have seen in Part I: the complete quantum assemblage of $\langle U, E \rangle$ turns into a distributive ortholattice, hence a Boolean algebra, namely the Approximation Space **AS**(U/E), so that the Persistence Property holds. Therefore in any Approximation System, internal modalities do not make sense any longer and we have to apply the external forcing.

Then a question naturally arises: "*Why does this (apparently minimal) move from a similarity relation to an equivalence relation induce such a drastic logical change?*"

A first technical insight was provided in Part I. Here we shall develop this *leitmotiv*. At this aim we have to investigate external forcing a little further.

11.5.2 External Modalities, Distributivity and Possible Worlds Semantics

If the structure **S** we are dealing with is a complete Boolean algebra or a complete Heyting algebra, then distributivity makes the Persistence Property hold. So, according to our discussion above, in order to deal with a meaningful notion of "necessity", we need to identify a suitable substructure of **S**, say **S**′, which, intuitively, represents the way in which things must be considered in order to decide for their necessity (or possibility).

This intuition is made explicit by the forcing clauses between points and formulas: the so-called *Kripke semantics*. We underline the difference: Kripke-Joyal semantic deals with forcing relations between elements of an algebraic structure and formulas, while Kripke semantic defines forcing relations between points and formulas. Clearly, between the two kinds of forcing there are well-established mathematical connections, so that, in a sense, Kripke semantic allows us to look into the elements of a lattice **S** and to understand the "concrete" relation that is represented by the abstract structure $\mathbf{S}'$.[7]

Kripke semantic is well suited for the modal analysis of concrete *Approximation Spaces*, because any such space is a Boolean algebra of sets of elements belonging to a "concrete" Information System.

So, let us start moving from the level of abstract Approximation Systems (abstract modal systems), towards their representations in terms of algebras of subsets of a universe U and relations between the elements of this universe. However, we aim at keeping a certain degree of generalisation.

In order not to start too far from the goal of this Chapter, let us consider, instead of arbitrary structures, modal systems $\langle \mathbf{S}, \mathbf{S}' \rangle$ such that **S** is a finite distributive lattice and $\mathbf{S}'$ a distributive sub-lattice of **S** – we shall call them "*(finite) distributive modal systems*". This is a significantly general framework which, anyway, is sufficiently close to our main goal.

11.5.3   Representing Modal Systems

Finite distributive modal systems can be represented, in the sense of *Definition 11.5.5*, by exploiting Birkhoff duality augmented with some additional features.

We now list below the steps leading to the representation of finite distributive modal systems.

Procedure 11.5.1. Procedure for representing a finite distributive modal system $\langle \mathbf{S}, \mathbf{S}' \rangle$ as a k-modal system (cf. Introduction Section 5.1 and Subsection 7.2.1 of Chapter 7)

1. Take the dual space $\mathbf{J}(\mathbf{S})$ of **S**. $\mathbf{J}(\mathbf{S})$ is the partially ordered set of elements of $\mathcal{J}(\mathbf{S})$ (set of co-prime elements of **S**).

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[7]Actually, this is true in a number of cases, but fails to be true in general. See the Frame section for a brief survey.

2. Take the lattice $\mathbf{F}(\mathbf{J}(\mathbf{S}))$ dual of $\mathbf{J}(\mathbf{S})$. Clearly $\mathbf{F}(\mathbf{J}(\mathbf{S}))$ is isomorphic to $\mathbf{S}$, via the isomorphism ϕ defined in (7.2.7) of Section 7.2 of Chapter 7. Moreover, the elements of $\mathbf{F}(\mathbf{J}(\mathbf{S}))$ are subsets of $\mathcal{J}(\mathbf{S})$.

3. Take the sublattice of $\mathbf{F}(\mathbf{J}(\mathbf{S}))$ that is isomorphic to $\mathbf{S}'$. Call it $\mathbf{L_{S'}}$ and let ϕ' the restriction $\mathbf{S}' \longmapsto \mathbf{L_{S'}}$. Since $\mathbf{S}'$ is embedded in $\mathbf{S}$, via a 1-1 map e, we have $\mathbf{L_{S'}} = \phi(e(\mathbf{S}')) = \phi'(e^{-1}(\mathbf{S}))$ (that is, we obtain $\mathbf{L_{S'}}$ by pulling back e^{-1} along ϕ').

4. Take the preordered set $\mathbf{P} = \langle T^*, \preceq \rangle$ where *(a)* $T^* = \bigcup\{X : X \in \mathbf{L_{S'}}\}$, *(b)* $\preceq$ is the specialization preorder induced by $\mathbf{L_{S'}}$ on T^*, i.e. $x \preceq y$ if and only if for any $X \in \mathbf{L_{S'}}$, if $x \in X$ then $y \in X$. Notice that: *(i)* $T^* = \mathcal{J}(\mathbf{S})$ because $0 \overset{\mathbf{S}'}{\Longleftarrow} 1$ must exist and coincide with 1, so that $e(1) = 1$ and $\phi(1) = \mathcal{J}(\mathbf{S})$; *(ii)* the domain and the range of $\preceq$ coincide, since any preorder is reflexive; *(iii)* $\mathbf{L_{S'}} \subseteq \wp(T^*)$; *(iv)* $\mathbf{L_{S'}} = \mathbf{F}(\mathbf{P})$, where $\mathbf{P} = \langle \mathscr{F}, \subseteq \rangle$ and $\mathscr{F} = \{\uparrow_{\preceq} X : X \in \wp(T^*)\}$, and the order filter $\uparrow_{\preceq} X$ is induced by the preorder $\preceq$.

5. Define a map $k^* : \mathbf{F}(\mathbf{J}(\mathbf{S})) \longmapsto \wp(T^*)$ as $k^*(X) = \uparrow_{\preceq} X$. Hence, $k^*(X) \in \mathscr{F}(\mathbf{P})$, for any X. Moreover, since $T^* = \mathcal{J}(\mathbf{S})$ and $\mathbf{L}'_{\mathbf{S}}$ is a sublattice of $\mathbf{F}(\mathbf{J}(\mathbf{S}))$, we have that k^* is total and onto $\mathscr{F}(\mathbf{P})$ and we can straightforwardly prove that $k^*(\mathbf{F}(\mathbf{J}(\mathbf{S}))) = k^*(\phi(\mathbf{S})) = \mathbf{L_{S'}}$.

TERMINOLOGY AND NOTATION. We shall call k^* a "*standard knowledge map*".

Proposition 11.5.3. *Let $\langle \mathbf{S}, \mathbf{S}' \rangle$ be a finite distributive modal system. Then the pair $\langle \mathbf{F}(\mathbf{J}(\mathbf{S})), k^*(\mathbf{F}(\mathbf{J}(\mathbf{S}))) \rangle$ obtained by the* Procedure *11.5.1 above, is a representation of $\langle \mathbf{S}, \mathbf{S}' \rangle$.*

Proof. The proof comes directly from the above procedure and is left to the reader (remember that you have also to prove that k^* is a knowledge map).   QED

In order to obtain the corresponding map $k : \mathbf{S} \longmapsto \mathbf{S}'$, that is a map k such that $k^*(\phi(\mathbf{S})) = \phi'(k(\mathbf{S}))$, we set $k(\mathbf{S}) = \phi'^{-1}(k^*(\phi(\mathbf{S})))$; otherwise stated, once we have obtained $\mathbf{L_{S'}}$ from $\mathbf{S}$ via $\phi \circ k^*$, we reach $\mathbf{S}'$ by means of the pre-image of $\mathbf{L_{S'}}$ via ϕ'. In this way we obtain a k-modal system at the same abstraction level as $\mathbf{S}$, namely $\langle \mathbf{S}, k(\mathbf{S}) \rangle$.

Example 11.5.3. Representing modal systems as k-modal systems

Consider the two lattices $\mathbf{L}$ and $\mathbf{L}'$ of Example 11.5.1. It is easy to find the dual space $\mathbf{J}(\mathbf{L})$ of co-primes elements, the dual lattice $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ and the isomorphic copy $\mathbf{L_{L'}}$ of $\mathbf{L}'$ embedded in $\mathbf{F}(\mathbf{J}(\mathbf{L}))$, with the isomorphism $\phi : \phi(x) = \uparrow x$, where $\uparrow$ refers to the order of $\mathbf{J}(\mathbf{L})$:
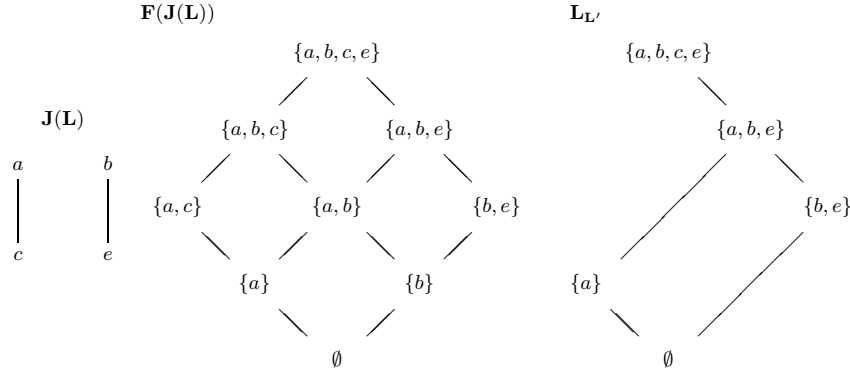
The isomorphism ϕ' between $\mathbf{L}'$ and $\mathbf{L_{L'}}$ is an obvious restriction of ϕ.

The specialization preorder $\preceq$ induced on T^* by the structure of $\mathbf{L_{L'}}$ is given in the following table and diagram:

$\preceq$	a	b	c	e
a	1	0	0	0
b	0	1	0	1
c	1	1	1	1
e	0	1	0	1
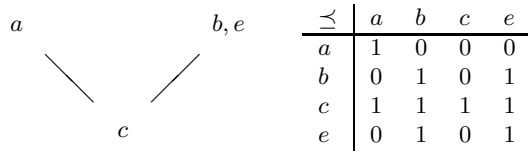
In fact, $\{a, b, c, e\}$ is the only element of $\mathbf{L_{L'}}$ containing c, so that there is no element X of $\mathbf{L_{L'}}$ such that $c \in X$ but $a, b, e \notin X$. Therefore, $c \preceq a, b, e$. The opposite does not hold (a, b and e belong to $\{a, b, e\}$, but c does not belong to $\{a, b, e\}$). Moreover, the element $\{a\}$ isolates a, while b and e cannot be separated by distinct elements of $\mathbf{L_{L'}}$. Let us set $\mathbf{P} = \langle T^*, \preceq \rangle$.

First, the reader is invited to verify that $\mathbf{F}(\mathbf{P}) = \mathbf{L_{L'}}$.

Second, by means of the preorder $\preceq$ we can compute the standard knowledge map $k^* : \mathbf{F}(\mathbf{J}(\mathbf{L})) \longmapsto \mathbf{L_{L'}}; k^*(X) = \uparrow_\preceq X$, namely:

$k^*(\emptyset) = \emptyset, k^*(\{a\}) = \{a\}, k^*(\{b\}) = \{b, e\}, k^*(\{a, b\}) = \{a, b, e\}, k^*(\{a, c\}) = \{a, b, c, e\}$ and so on (it is sufficient to inspect the rows of the above table).

By applying ϕ'^{-1} after $\phi \circ k^*$ we obtain a function $k : \mathbf{L} \longmapsto \mathbf{L}'$ which gives the following transformations:

$0 \mapsto 0, a \mapsto a$ [in fact (i) $\phi(a) = \{a\}$, (ii) $k^*(\phi(\{a\})) = k^*(\{a\}) = \{a\}$, (iii) $\phi'^{-1}(k^*(\phi(a))) = \phi'^{-1}(\{a\}) = a$], $b \mapsto e$ [in fact, (i) $\phi(b) = \{b, e\}$, (ii) $k^*(\phi(\{b\})) = k^*(\{b, e\}) = \{b, e\}$, (iii) $\phi'^{-1}(k^*(\phi(b))) = \phi'^{-1}(\{b, e\}) = e$], $e \mapsto e, d \mapsto g, g \mapsto g, c \mapsto 1, f \mapsto 1, 1 \mapsto 1$, that is another way to represent the knowledge map k of Example 11.5.1.

Exercise 11.1.

(a) Consider the following sublattice $\mathbf{L}''$ of the lattice $\mathbf{L}$ depicted in Example 11.5.1:

$$g$$

$$e$$

$$\mathbf{L}'' \quad a$$

$$0$$
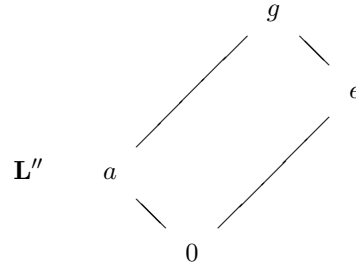
(i) Verify that $T^* \subsetneq \mathcal{J}(\mathbf{L})$. (ii) Compute, however, the function $k :$ $\mathbf{L} \longmapsto \mathbf{L}'', k = \phi \circ k^* \circ \phi^{-1}$, where ϕ and k^* are defined as above. Is k a knowledge map?

(b) Consider the following lattices $\mathbf{A}$ and $\mathbf{A}'$:

$$1$$

$$\mathbf{A} \quad a \qquad\qquad \mathbf{A}' \quad \begin{array}{c} 1 \\ \\ 0 \end{array}$$

$$0$$
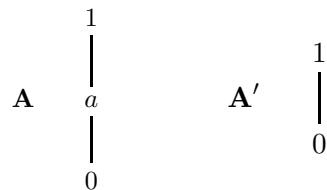
and the function $g : \mathbf{A} \longmapsto \mathbf{A}'$ defined by $g(0) = 0, g(a) = 0, g(1) = 1$. (i) Is g a knowledge map? (ii) If yes, does it coincide with the function k defined as $k = \phi \circ k^* \circ \phi^{-1}$?

(c) Consider the following substructures $\mathbf{L}'''$ and $\mathbf{L}''''$ of the lattice $\mathbf{L}$ depicted in Example 11.5.1:

$$1$$

$$g$$

$$e$$

$$\mathbf{L}''' \quad a \qquad b$$

$$0$$

$$\begin{array}{c} 1 \\ g \\ \mathbf{L}'''' \quad d \qquad e \\ 0 \end{array}$$
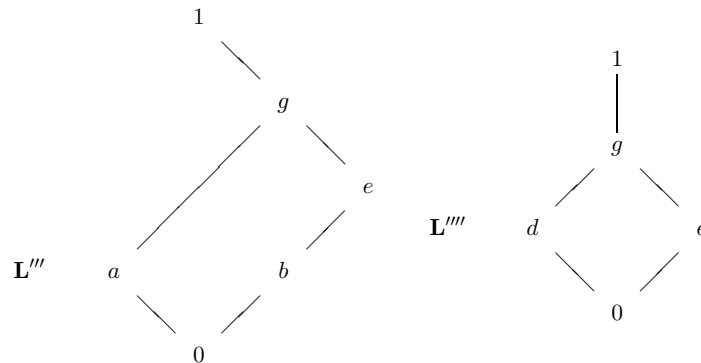
(i) Verify that it is not possible to obtain a standard knowledge map k^*. (ii) Verify that this is due to the lack of distributivity. (iii) Verify if it is possible to define a knowledge map $k : \mathbf{L} \longmapsto \mathbf{L}'''$, though. (iv) Is there a standard knowledge map definable from $\mathbf{L}''''$? (v) Is it possible to define a knowledge map $k : \mathbf{L} \longmapsto \mathbf{L}''''$?

Now, if we apply this procedure to an arbitrary finite abstract Approximation System $\langle \mathbf{B}, \mathbf{B}' \rangle$, we obtain a "concrete" version of $\langle \mathbf{B}(U), k^*(\mathbf{B}(U)) \rangle$, *where the specialization preorder is, actually, an equivalence relation.* Therefore it is worthwhile getting a deeper and wider view of the topic.

Chapter 12

Modalities and Relations

12.1 Modal Systems and Binary Relations

Definition 12.1.1. *Let U be a set and R a binary relation on U, a map $f : \wp(U) \longmapsto \wp(U)$ is said to be* connected with R *if for any $X \in \wp(U)$, $f(X) = R(X) = \{y \in U : \exists x(x \in X \,\&\, \langle x, y \rangle \in R)\}$.*

In view of this definition we can say that finite distributive modal systems can be made into isomorphic k-modal systems and can be represented by k-modal systems where the standard knowledge map k^* is connected with a preorder relation. In fact, $k^*(X) = \uparrow_{\preceq} X = \preceq (X)$.

In this way we have, partially, answered the questions:

(A) "Given an abstract modal system $\langle \mathbf{S}, \mathbf{S}' \rangle$ is there a k-modal system $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ isomorphic to it?"

(B) "Given a k-modal system $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ is there a representation $\langle \mathbf{A}, k^*(\mathbf{A}) \rangle$ such that $\mathbf{A}$ is an algebra of subsets of a universe U and the knowledge map k^* is connected with a binary relation R on U?"

Now we reverse the starting point:

(A') "Given an algebra $\mathbf{A}$ of subsets of a universe U, a binary relation R on U, and a function f connected with R, is the pair $\langle \mathbf{A}, f(\mathbf{A}) \rangle$ a k-modal system? If yes, how do its modal properties vary, depending upon the properties of R?".

(B') "Given a k-modal system $\langle \mathbf{A}, k(\mathbf{A}) \rangle$ such that $\mathbf{A}$ is an algebra of subsets of a universe U, is there a relation R on U such that the knowledge map k is connected with R?".

In other terms, we want to know (i) what relationships exist between binary relations and knowledge maps, (ii) what relationships exist

between the properties of a binary relation and the k-modal system connected with it (if any).

In view of our main topic, Approximation Spaces, we shall solve limited instances of the above problems, namely when **A** is a Boolean algebra of sets.

Therefore, henceforth, if not otherwise stated, the role of **A** will be played by the Boolean algebra of sets $\mathbf{B}(U) = \langle \wp(U), \cap, \cup, -, U, \emptyset \rangle$ for some universe U and R will denote a binary relation on $U : R \subseteq U \times U$.

TERMINOLOGY AND NOTATION. From now on the entities that populate the elements of an algebra of sets will be called "points" (or "elements" when they appear in sentences mentioning the set they belong to). The set of all points will be denoted by means of our familiar notation U (for "universe of discourse"; indeed, U plays the role of G in Part I. Here instead of the set of "Gegenstände", we prefer the more abstract notion of a "universe").

Now it is worthwhile recalling some properties of R-neighborhoods, $R(\ )$ i.e. $\langle R^\smile \rangle$, from Part I:

Proposition 12.1.1.

1. *Given a binary relation $R \subseteq U \times U$, the R-neighborhood $R(\ ) = \langle R^\smile \rangle$ is lower adjoint of $[R]$ with respect to the structure (small category) $\langle \wp(U), \subseteq \rangle$. Hence,*

2. *$R(\ )$ is continuous: $R(X) \cup R(Y) = R(X \cup Y)$,*

3. *R is normal: $R(\emptyset) = \emptyset$,*

4. *R is isotonic: $X \subseteq Y$ implies $R(X) \subseteq R(Y)$. Moreover,*

5. *R is co-discontinuous: $R(X \cap Y) \subseteq R(X) \cap R(Y)$.*

Point 2 is a direct consequence of *Proposition* 1.4.8.(2), because $R(\ )$ is a lower adjoint.

Obviously, the same holds for $R^\smile-$neighborhoods, i.e. $\langle R \rangle$.

In modal contexts, points are usually called *possible worlds*, *information states* or *states of affairs*.[1] The binary relation R, generally has the following meaning: if $\langle x, x' \rangle \in R$, then x' is a possible evolution of

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[1]Or, sometimes, "knowledge states" (of a subject). However we use the term "knowledge" to denote a particular pattern of information states.

the state x (or x' represents a world that is conceivable from x, or an enrichment of the information in x). Usually if $\langle x, x' \rangle \in R$, then we say that x' is *accessible* from x. However, we shall see interpretations which are more perspicuous for our context. We shall develop this point later on. By now, consider that R induces a particular *geometry* on the set of points U, which is well represented by the space $\langle U, R \rangle$, which, for historical reasons, is called a *Kripke frame* (cf. Frame 4.13 of Part I).[2]

We hardly can deny, at this point, that at the end of this story we shall find not a generic relation R, but our familiar equivalence relation E. This is obvious. However it is better we reach that point gradually, passing through some preliminary steps illustrating how an equivalence relation is only one among a number of other interesting possibilities.

From the definitions of R-neighborhoods and $R^{\smile}$-neighborhoods, it is clear that for any $X \subseteq U$ there is only one X' such that $X' = R(X)$ and only one X'' such that $X'' = R^{\smile}(X)$. So we can define two functions from $\wp(U)$, *qua* carrier of $\mathbf{B}(U)$, to $\wp(U)$, *qua* range of the relation R, as follows:

- $f : \wp(U) \longmapsto \wp(U), f(X) = R(X)$ – that is, f is connected with R.

- $h : \wp(U) \longmapsto \wp(U), g(X) = R^{\smile}(X)$ – that is, h is connected with $R^{\smile}$.

Now, the first three points of *Proposition 12.1.1* tell us that any function connected with a binary relation is a knowledge map. In view of this fact, we can restate the definitions of the modal operators L_k and M_k using $R-$neighborhoods to provide these operators with a specific meaning based on the properties exhibited by the binary relation which is connected with k.[3]

Lemma 12.1.1. *Let $\langle \mathbf{B}(U), k(\mathbf{B}(U)) \rangle$ be a k-modal system such that k is connected with some relation $R \subseteq U \times U$, i.e. $k(\mathbf{B}(U)) = \{R(X) :$*

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[2]For some specific purposes, also ternary relations are used (cf. [Allwein-Dunn 1993] as to Kripke models for Linear Logic or [Anderson et al. 1992] as to Kripke models for Relevant Logics). In these cases, the sentence "$\langle x, y, z \rangle \in R$" usually reads: "the information in x combined with the information in y, outputs the information in z". That is, $z = x \circ y$ where "$\circ$" is a monoidal operator. Also, this approach is connected with Phase Semantics for Linear Logic (see [Abrusci 1991]).

[3]If we do not assume $\mathbf{S} = \mathbf{B}(U)$, but we let $\mathbf{S}$ be a sublattice of $\mathbf{B}(U)$, then $f : S \longmapsto \wp(U)$ and it may happen, for some $X \subseteq U$, that $R(X) \notin S$ or $R^{\smile}(X) \notin S$ (we recall that S is the carrier of $\mathbf{S}$). So these definitions must be generalised. For instance, we can adopt (i) $f(X) = \bigcup \{X' \in S : X' \subseteq R(X)\}$ and (ii) $g(X) = \bigcup \{X' \in S : X' \subseteq R^{\smile}(X)\}$.

$X \subseteq U\}$, and g is connected with $R^\smile$. Then, $\forall X, Y, Z \in \mathbf{B}(U), \forall x, y \in U$:

1. $L_k(X) = \bigcup\{Z : R(Z) \subseteq X\}$.

2. $x \in L_k(X)$ iff $\forall y \in U(\langle x, y \rangle \in R \Rightarrow y \in X)$.

3. $M_k(X) = \bigcup\{\{x\} : R(\{x\}) \cap X \neq \emptyset\}$.

4. $x \in M_k(X)$ iff $\exists y \in U(\langle x, y \rangle \in R \ \& \ y \in X)$.

Proof. (**1**): From *Definition* 11.5.2.(1), by substituting X for a, Z for p and $\subseteq$ for $\leq$.

(**2**): From (1) we obtain $L_k(X) = \bigcup\{Z : \forall x \in Z, \forall y \in U(\langle x, y \rangle \in R \Rightarrow y \in X)\}$. Since we have no restrictions on Z, we get $L_k(X) = \{x : \forall y \in U(\langle x, y \rangle \in R \Rightarrow y \in X)\}$. Hence the thesis. (**3**) and (**4**): From *Definition* 11.5.2.(2), $M_k(X) = \bigcup\{Z : \exists Z'(g(Z') \supseteq Z \ \& \ Z' \subseteq X)\} = \bigcup\{Z : \exists Z'(R^\smile(Z') \supseteq Z \ \& \ Z' \subseteq X)\}$. But if $Z' \subseteq X$, from monotonicity $R^\smile(Z') \subseteq R^\smile(X)$, so that if $Z \subseteq R^\smile(Z')$ then $Z \subseteq R^\smile(X)$. Therefore, $M_k(X) = \bigcup\{z : \forall z(z \in Z \Rightarrow \exists x(\langle z, x \rangle \in R \ \& \ x \in X))\}$. But the condition on z is equivalent to $\forall z(z \in Z \Rightarrow R(z) \cap X \neq \emptyset)$. Henceforth we have: $M_k(X) = \bigcup\{\{x\} : R(\{x\}) \cap X \neq \emptyset\}$ and $x \in M_k(X)$ iff $\exists y \in U(\langle x, y \rangle \in U \ \& \ y \in X)$.[4]   QED

Corollary 12.1.1. $\forall X \in \mathbf{B}(U)$:
(a) $L_k(X) = [R](X)$; (b) $M_k(X) = \langle R \rangle(X)$ *(where $[R]$ and $\langle R \rangle$ are the operators defined in Section 2.1.2 of Chapter 2).*

Definition 12.1.2. *Given a modal system connected with a relation R, the modal operators L_k and M_k will be denoted by L_R and M_R or $[R]$ and, respectively, $\langle R \rangle$. Moreover, the modal operators will be denoted by L and M when any reference to the relation R is understood or irrelevant.*

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[4]In the general case, that is, when we deal with a system $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ where $\mathbf{S}$ is a sublattice of $\mathbf{B}(U)$, not every subset of U is an element of $\mathbf{S}$. Hence we can have elements Z such that although $R(Z)$ is included in X, Z is not an element of $\mathbf{S}$. Therefore points *1* and *3* of the *Lemma* are valid only with the additional constraint: $\bigcup\{Z : Z \in \mathbf{S} \ \& \ \dots\}$, and so on. Moreover, in this case points *2* and *4* are valid only from left to right:
  (2') If $x \in L_k(X)$ then $\forall y \in U(\langle x, y \rangle \in R \Rightarrow y \in X)$.
  (4') If $x \in M_k(X)$ then $\exists y \in U(\langle x, y \rangle \in R \ \& \ y \in X)$.

From the results of Part I and *Corollary* 12.1.1 we have:

Corollary 12.1.2. *For any* $X \in \mathbf{B}(U), -M_R(-X) = L_R(X);$ $-L_R(-X) = M_R(X).$

Now we translate the forcing clauses over algebraic structures into forcing clauses over Kripke frames. Since we are dealing with modal systems where the algebraic operations form a Boolean algebra (of sets) $\mathbf{B}(U)$, we shall use *Proposition* 11.3.1.(4) in order to understand what happens to forcing at a point level. Therefore, given an evaluation ϕ from a modal language $\mathcal{L}$ to $\mathbf{B}(U)$, the translation will be lead by the obvious idea that a point x forces a formula α, in symbols $x \Vdash \alpha$ if x belongs to an element X of $\mathbf{B}(U)$ that algebraically forces α, $X \vDash \alpha$.

The translation will be accomplished through two Lemmata: the first will link the algebraic operations induced by an evaluation ϕ with the forcing relation $\Vdash$ between points and formulas, the second Lemma will use this link to list the forcing clauses of $\Vdash$ for any logical constants. The result will be summed up in *Window* 12.1.

TERMINOLOGY AND NOTATION. From now on, by $\mathcal{L}$ we shall intend a propositional language with Boolean constants $\wedge, \vee, \neg, \rightarrow, 0, 1$ and modal constants L and M, while $\alpha, \alpha', \beta, \beta'$ and so on, will vary over well-formed formulas. Notice that results on the material implication $\rightarrow$ will be sometimes omitted since it fulfills the definition $\alpha \rightarrow \beta =_{def} \neg\alpha \vee \beta$.

Lemma 12.1.2. *Let* $\langle U, R \rangle$ *be a Kripke frame,* ϕ *an evaluation map from a modal language* $\mathcal{L}$ *to a k-modal system* $\langle \mathbf{B}(U), k(\mathbf{B}(U)) \rangle$, *and let* k *be connected with* R. *For any element* $x \in U$, *for any formula* $\alpha \in \mathcal{L}$, *let us set:* $x \Vdash \alpha$ *if and only if there is an element* X *of* $\mathbf{B}(U)$ *such that* $x \in X$ *and* $X \vDash \alpha$. *Then, for any formula* $\alpha, \alpha' \in \mathcal{L}$, *for all* $x \in U$:

1. $x \Vdash \alpha \wedge \alpha'$ *iff* $x \in \phi(\alpha) \cap \phi(\alpha')$.

2. $x \Vdash \alpha \vee \alpha'$ *iff* $x \in \phi(\alpha) \cup \phi(\alpha')$.

3. $x \Vdash \neg\alpha$ *iff* $x \in -\phi(\alpha)$.

4. $x \Vdash \alpha \rightarrow \alpha'$ *iff* $x \in -\phi(\alpha) \cup \phi(\alpha')$.

5. $x \Vdash L_R(\alpha)$ *iff* $x \in \{x' : \forall y \in U(\langle x', y \rangle \in R \Rightarrow y \in \phi(\alpha))\}$.

6. $x \Vdash M_R(\alpha)$ *iff* $x \in \{x' : \exists y \in U(\langle x', y \rangle \in R \ \& \ y \in \phi(\alpha))\}$.

Proof.

(A) Boolean part of the proof:

by means of *Proposition* 11.3.1.(4) and the definition of function ϕ (see *Window* 11.1), we obtain the result straightforwardly. The detailed proof is left to the reader. (*Hints*: first, notice that the thesis' assumption reads $x \Vdash \alpha$ iff $x \in \bigcup\{X : X \subseteq \phi(\alpha)\}$, so that from *Proposition* 11.3.1.(4), after substituting $\subseteq$ for $\leq$ we obtain $x \Vdash \alpha$ iff $x \in \phi(\alpha)$. Therefore, for instance, from the definition of function ϕ in *Window* 11.1, $x \Vdash \alpha \wedge \alpha'$ iff $x \in \phi(\alpha \wedge \alpha')$, iff $x \in \phi(\alpha) \cap \phi(\alpha')$, and so on. The reader must only pay attention that in $\mathbf{B}(U)$ "$\neg$" is the set-theoretical complementation).

(B) Modal part of the proof (actually a corollary of *Lemma* 12.1.1): $x \Vdash L_R(\alpha)$ iff $x \in \bigcup\{X : k(X) \vDash \alpha\}$, iff $x \in \bigcup\{X : k(X) \subseteq \phi(\alpha)\}$, iff $x \in \bigcup\{\{X : R(X) \subseteq \phi(\alpha)\}$, iff $\forall y \in U(\langle x, y \rangle \in R \Rightarrow y \in \phi(\alpha))$; $x \Vdash M_R(\alpha)$ iff $x \in \bigcup\{X : \exists X'(g(X') \supseteq X \ \& \ X' \subseteq \phi(\alpha))\}$, iff $x \in \bigcup\{X : \exists X'(R^\smile(X') \supseteq X \ \& \ X' \subseteq \phi(\alpha))\}$, iff $\exists y \in U(\langle x, y \rangle \in R \ \& \ y \in \phi(\alpha))$.   QED

Proposition 12.1.2. *Under the assumptions of Lemma 12.1.2, for any formula $\alpha, \alpha' \in \mathcal{L}$, for all $x \in U$:*

1. $x \Vdash \alpha \wedge \alpha'$ *iff* $x \Vdash \alpha \ \& \ x \Vdash \alpha'$.

2. $x \Vdash \alpha \vee \alpha'$ *iff* $x \Vdash \alpha$ *or* $x \Vdash \alpha'$.

3. $x \Vdash \neg\alpha$ *iff* $x \nVdash \alpha$.

4. $x \Vdash \alpha \rightarrow \alpha'$ *iff* $x \Vdash \neg\alpha$ *or* $x \Vdash \alpha'$.

5. $x \Vdash L_R(\alpha)$ *iff* $\forall y \in U(\langle x, y \rangle \in R \Rightarrow y \Vdash \alpha)$.

6. $x \Vdash M_R(\alpha)$ *iff* $\exists y \in U(\langle x, y \rangle \in R \ \& \ y \Vdash \alpha)$.

7. $x \Vdash L_R(\alpha)$ *iff* $x \Vdash \neg M_R(\neg\alpha)$; $x \Vdash M_R(\alpha)$ *iff* $x \Vdash \neg L_R(\neg\alpha)$.

Proof. From the preceding *Lemma*: (1) $x \Vdash \alpha \wedge \alpha'$ iff $x \in \phi(\alpha) \cap \phi(\alpha')$, iff $x \in \phi(\alpha)$ and $x \in \phi(\alpha')$, iff for some $X, X' \in \mathbf{B}(U)$ such that $X \vDash \alpha$ and $X' \vDash \alpha'$, $x \in X$ and $x \in X'$, iff $x \Vdash \alpha \ \& \ x \Vdash \alpha'$. (2) dually, by substituting $\cup$ for $\cap$ and $\vee$ for $\wedge$. (3) $x \Vdash \neg\alpha$ iff $x \in -\phi(\alpha)$, iff $x \notin \phi(\alpha)$, iff $x \nVdash \alpha$. (4) straightforward from (2) and (3) and the fact that $\phi(\alpha \rightarrow \alpha') = -\phi(\alpha) \cup \phi(\alpha')$. (5) $x \Vdash L(\alpha)$ iff $\forall y \in U(\langle x, y \rangle \in R \Rightarrow y \in \phi(\alpha))$, iff $\forall y \in U(\langle x, y \rangle \in R \Rightarrow y \Vdash \alpha)$. (6) $x \Vdash M_R(\alpha)$ iff

$\exists y \in U(\langle x,y \rangle \in R \ \& \ y \in \phi(\alpha))$, iff $\exists y \in U(\langle x,y \rangle \in R \ \& \ y \Vdash \alpha)$. (7)
The proof is left to the reader [Hints: use the first order equivalences $\forall \equiv \neg\exists\neg$ and $\neg\forall\neg \equiv \exists$]. $\qquad$ QED

Therefore, thanks to the above *Proposition* 12.1.2, we have the following set of forcing clauses over Kripke frames:

Let $\mathcal{L}$ be a propositional modal language and let $\langle U, R \rangle$ be a Kripke frame.

Let ϕ be a set-up: $\hat{\phi} : \mathcal{L} \longmapsto \wp(U)$.

For any point $x \in U$, for any formula $\alpha, \alpha' \in \mathcal{L}$ we set the following forcing clauses:

1. $x \Vdash \alpha$ iff $x \in \hat{\phi}(\alpha)$, for α atomic.
2. $x \Vdash \alpha \wedge \alpha'$ iff $x \Vdash \alpha \ \& \ x \Vdash \alpha'$.
3. $x \Vdash \alpha \vee \alpha'$ iff $x \Vdash \alpha$ or $x \Vdash \alpha'$.
4. $x \Vdash \neg\alpha$ iff $x \not\Vdash \alpha$.
5. $x \Vdash L_R(\alpha)$ iff $\forall y \in U(\langle x,y \rangle \in R \Rightarrow y \Vdash \alpha)$.
6. $x \Vdash M_R(\alpha)$ iff $\exists y \in U(\langle x,y \rangle \in R \ \& \ y \Vdash \alpha)$.

The triple $\langle U, R, \Vdash \rangle$, with the above clauses for $\Vdash$, is called a *Kripke model for modal logic*

Window 12.1. **Forcing over Kripke frames**

From *Lemmata* 12.1.1 and 12.1.2, it follows that once again we can confine our attention to the Boolean set-theoretical operations and define two monadic operators L_R and M_R ranging on subsets of U. In this way, we avoid any reference to the language $\mathcal{L}$ and its formulae.

Otherwise stated, we can associate to Kripke models *Boolean algebras of sets with additional monadic operators*:

Definition 12.1.3. *Let* $\mathbf{B}(U)$ *be the Boolean algebra of* $\wp(U)$. *Let* $R \subseteq U \times U$. *Then* $\langle \mathbf{B}(U), L_R, M_R \rangle$ *is called a Pre-monadic Boolean algebra of sets.*

Strictly speaking, in order to denote a Pre-monadic Boolean algebra of sets, $\langle \mathbf{B}(U), L_R \rangle$ (or $\langle \mathbf{B}(U), M_R \rangle$) suffices, since the two monadic operators are dual *via* the Boolean complementation.

REMARKS. Pay attention that in general L_R (i.e. $[R]$) and M_R (i.e. $\langle R \rangle$) are not adjoint to each other, because $[R]$ is adjoint to $\langle R^{\smile} \rangle$ and $[R^{\smile}]$ is adjoint to $\langle R \rangle$

From the above results we have the following statement, linking algebraic forcing and point-based forcing:

Proposition 12.1.3. *Let $\langle \mathbf{B}(U), L_R \rangle$ be a Pre-monadic Boolean algebra of the powerset of a set U. Let ϕ be an evaluation map from a modal language $\mathcal{L}$ to $\mathbf{B}(U)$. Then for any formula $\alpha \in \mathcal{L}$, $\phi(\alpha) = U$ if and only if $\forall x \in U, x \Vdash \alpha$.*

It is not difficult to derive the abstract (i.e. algebraic) properties of the modal operators, thanks to the following results:

Proposition 12.1.4. *Let $\langle \mathbf{B}(U), L_R \rangle$ be a Pre-monadic Boolean algebra of sets. Then, for any $X, Y \subseteq U$,*

L1. $L_R(U) = U$.

L2. $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$.

L3. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$.

L4. if $X \subseteq Y$ then $L_R(X) \subseteq L_R(Y)$.

M1. $M_R(\emptyset) = \emptyset$.

M2. $M_R(X \cap Y) \subseteq M_R(X) \cap M_R(Y)$.

M3. $M_R(X \cup Y) = M_R(X) \cup M_R(Y)$.

M4. if $X \subseteq Y$ then $M_R(X) \subseteq M_R(Y)$.

Proof. In view of *Corollary* 12.1.1, from the adjunction relations $M_{R^\smile} \dashv L_R$ and $M_R \dashv L_{R^\smile}$ that can be derived from *Proposition* 2.1.1 of Chapter 2.   QED

Therefore, in a more abstract framework we shall set:

Definition 12.1.4. *Let $\mathbf{A}$ be a Boolean algebra. Let L be a monadic operator on $\mathbf{A}$ such that:*

1. $L(1) = 1$ – *L-conormality.*

2. $L(a \wedge b) = L(a) \wedge L(b)$ – *L-cocontinuity (or multiplicativity).*

3. $L(a) \vee L(b) \leq L(a \vee b)$ – *L-discontinuity.*

Then the structure $\langle \mathbf{A}, L \rangle$ is called a Pre-monadic Boolean algebra.

Proposition 12.1.5. *Let* $\langle \mathbf{A}, L \rangle$ *be a Pre-monadic Boolean algebra. Let M be a monadic operator defined, for any $a \in \mathbf{A}$, by $M(a) = \neg L(\neg a)$. Then for any $a, b \in \mathbf{A}$:*

1. $M(0) = 0$ – *M-normality.*

2. $M(a \vee b) = M(a) \vee M(b)$ – *M-continuity (or additivity).*

3. $M(a \wedge b) \leq M(a) \wedge M(b)$ – *M-codiscontinuity.*

4. $a \leq b$ *implies* $M(a) \leq M(b)$ *and* $L(a) \leq L(b)$ – *monotonicity.*

Thus L and M are comodal and, respectively, modal operators in the sense of *Definition* 1.4.3 of Chapter 1.

Definition 12.1.5. *Given a Pre-monadic Boolean algebra $\langle \mathbf{A}, L \rangle$, set $L(A) = \{L(a) : a \in A\}$, $\wedge_L = \wedge \upharpoonright L(A)$ and $\vee_L = \vee \upharpoonright L(A)$. Then we set $\mathbf{L}(\mathbf{A}) = \langle L(A), \wedge_L, \vee_L, 1, 0 \rangle$.*

Now we can notice that the sublattice $\mathbf{L}(\mathbf{A})$ is not necessarily distributive.

At this point we add stronger properties to the monadic operator, obtaining the notion of a *Monadic Boolean algebra*, that will be of central importance in our story:

Definition 12.1.6. *Let* $\langle \mathbf{A}, L \rangle$ *be a Pre-monadic Boolean algebra such that:*

1. $L(a \vee L(b)) = L(a) \vee L(b)$ – *monadic L-continuity.*

2. $L(a) \wedge a = L(a)$ – *L-deflationary property.*

Then $\langle \mathbf{A}, L \rangle$ *is called a* Monadic Boolean algebra.

We have to notice that *Property* 12.1.4.(2) is now derivable from the others.

Let us list other important properties of Monadic Boolean algebras:

Proposition 12.1.6. *Let* $\langle \mathbf{A}, L \rangle$ *be a Monadic Boolean algebra. Define, for all $a \in A$ $M(a)$ as $-L(-a)$. Then:*

1. $M(a \wedge M(b)) = M(a) \wedge M(b)$ – *monadic M cocontinuity.*

2. $a \wedge M(a) = a$ – *M-inflationary property.*

Example 12.1.1. Modal operators induced by R-neighborhoods

As an example of a structure equipped with a binary relation consider the set $U = \{x, y, z\}$ and the following relation $R \subseteq U \times U$:

R	x	y	z
x	0	1	1
y	0	1	0
z	0	0	1

Just by inspecting rows, we can see examples of some definitions and properties:

(a) R-neighborhoods: $R(x) = \{y, z\}, R(\{x, z\}) = \{y, z\}$.
(b) Monotonicity: $\{y\} \subseteq \{x, y\}$. $R(\{y\}) = \{y\} \subseteq \{y, z\} = R(\{x, y\})$.
(c) Continuity: $R(\{x\}) \cup R(\{y\}) = \{y, z\} \cup \{y\} = \{y, z\} = R(\{x, y\}) = R(\{x\} \cup \{y\})$.
(d) 0-preservation: $R(\emptyset) = \emptyset$ (on the contrary, $R(U) = \{y, z\} \neq U$).

Let us now compute some applications of the operators L_R and M_R:

(*)$L_R(\{x, z\}) = \{z\}$. Indeed: $R(\{z\}) = \{z\} \subseteq \{x, z\}, R(\{y\}) = \{y\} \nsubseteq \{x, z\}$, $R(\{x\}) = \{y, z\} \nsubseteq \{x, z\}$.

Notice that thanks to the continuity property, we obtain the result by gathering all the element of U whose R-neighborhoods are included in $\{y, z\}$. A better way in order to compute L_R is based on the duality $L_R(X) = -M_R(-X)$, any X.

(**) $M_R(\{x\}) = \emptyset, M_R(\{z\}) = \{x, z\}, -M_R(-\{x, z\}) = -M_R(\{y\}) = -\{x, y\} = L_R(\{x, z\})$.

Now, consider the following set-up $\widehat{\phi}(A) = \{x, z\}, \widehat{\phi}(B) = \{z, y\}, \widehat{\phi}(C) = \{x, y\}$. From it we have:

- $z, y \models B$, $z, x \models A$, $x, y \models C$;
- $z \models A \wedge B$ (because $z \in \widehat{\phi}(A)$ and $z \in \widehat{\phi}(B)$):
- $y \models L_R(C)$ (because $R(\{y\}) = \{y\}$ and $\{y\} \subseteq \{x, y\} = \hat{\phi}(C)$. Otherwise stated, all the elements R-accessible from y force C. In this case the only element accessible from y is y itself). On the contrary, although $x \models C$, $x \nvDash L_R(C)$ because $R(x) = \{y, z\} \nsubseteq \hat{\phi}(C)$ (indeed, $z \nvDash C$).
- $x \models M_R(A \wedge B)$ (because $\langle x, z \rangle \in R$ and $z \in \phi(A) \cap \phi(B)$, so that $z \models A \wedge B$. Otherwise stated, there is an element R-accessible from x that forces $A \wedge B$).

Exercise 12.1. *Let $\langle U, R \rangle$ be a relational structure. Without using the adjunction properties of $R()$, but pure logical deductions, prove isotonicity, normality, continuity and co-discontinuity of $R()$.*

Example 12.1.2. Example of a Boolean algebra with operator which is not pre-monadic

Consider the Boolean algebra **A** depicted below in the diagram on the left. Let us suppose that we are given the following table for an operator L_0. Then on the right we draw the resulting substructure $\mathbf{L_0} = \mathbf{L(A)}$, embedded in **A**:

x	$L_0(x)$
0	0
a	b
b	0
c	c
d	d
e	f
f	e
1	1
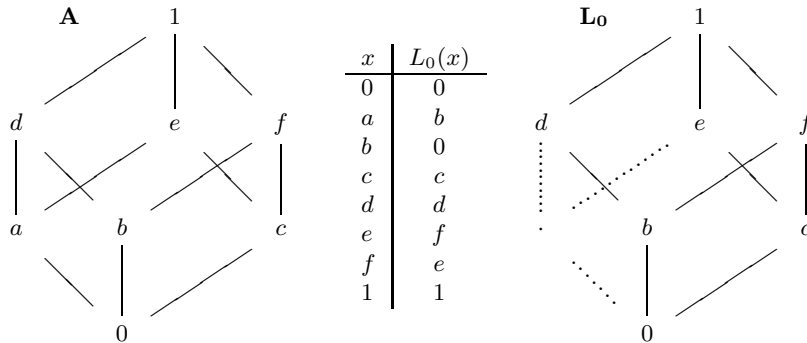
The structure $\langle \mathbf{A}, L_0 \rangle$ is a not a pre-monadic Boolean algebra. In fact, from the above table we can easily verify that $L_0(1) = 1$ and that the monotonicity property holds. But $L-$co-continuity fails: $L_0(d \wedge e) = 0 \neq b = d \wedge f = L_0(d) \wedge L_0(e)$.

Exercise 12.2. *(a) Compute $M_0(x)$ for any $\in \mathbf{A}$. (b) Find an example of $M-$codiscontinuity. (c) Can you find a Boolean algebra of sets $\mathbf{A}'$ such that $\mathbf{L}_R(\mathbf{A}')$ isomorphic to $\mathbf{L}_0$ for some binary relation $R \subseteq A \times A$?*

Example 12.1.3. Example of a pre-monadic Boolean algebra

Consider the Boolean algebra $\mathbf{A}$ depicted in Example 12.1.2. Let us suppose that we are given the following table for an operator L_1. Then on the right we draw the resulting substructure $\mathbf{L_1} = \mathbf{L}(\mathbf{A})$:

x	$L_1(x)$
0	0
a	0
b	a
c	0
d	d
e	c
f	a
1	1
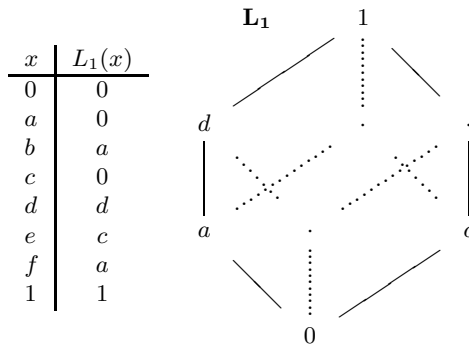
It is evident that $\mathbf{L_1}$ is not distributive. By easy inspection we see that L is monotonic (for instance, $b \leq f$ and $L(b) = L(f) = a$).

Let us verify a case of $L-$cocontinuity and a case of $L-$discontinuity:

$L_1(d \wedge f) = L_1(b) = a = d \wedge a = L_1(d) \wedge L_1(f)$.

$L_1(d \vee f) = L_1(1) = 1 \geq d \vee a = L_1(d) \vee L_1(f)$.

However, $\langle \mathbf{A}, L_1 \rangle$ is not a Monadic Boolean algebra.

- Let us verify that $L(x) \leq x$ is not uniformly valid: $L_1(b) = a \nleq b$.

- Let us verify that the equality $L(x \vee L(y)) = L(x) \vee L(y)$ is not uniformly valid:

$L_1(a \vee L_1(f)) = L_1(a \vee a) = L_1(a) = 0 \neq a = 0 \vee a = L_1(a) \vee L_1(f)$.

Exercise 12.3.

(a) Compute the table of M_1.

(b) Find a case of $M-$co-discontinuity.

(c) Find a case which invalidates the monadic M−co-continuity Property.
(d) Find a Boolean algebra of sets $\mathbf{A}'$ with top element a set U, such that $\mathbf{A}'$ is isomorphic to the above Boolean algebra $\mathbf{A}$ and a relation $R \subseteq U \times U$ such that $\mathbf{L}_R(\mathbf{A}')$ is isomorphic to $\mathbf{L}_1$.
(e) Classify R according to the following properties: reflexivity, transitivity, symmetry.

Example 12.1.4. Example of a pre-monadic operator L inducing a sublattice

Consider the Boolean algebra $\mathbf{A}$ of Example 12.1.2. Consider the following table for L (on the right we draw the resulting sublattice $\mathbf{L_2} = \mathbf{L}(\mathbf{A})$):

x	$L_2(x)$
0	0
a	0
b	b
c	c
d	b
e	c
f	f
1	1
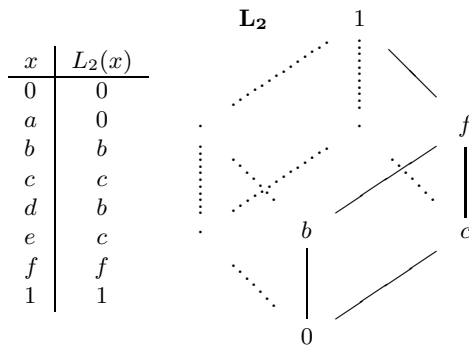
Exercise 12.4.
(a) Verify that $\langle \mathbf{A}, L_2 \rangle$ is a Pre-monadic Boolean algebra.
(b) Verify that $\mathbf{L_2}$ is distributive by computing a representation $\langle \mathbf{L_A}, k^*(\mathbf{L_A}) \rangle$ of $\langle \mathbf{A}, \mathbf{L_2} \rangle$ by means of the Representation Procedure.
(c) Classify the specialization preorder that you find during the Representation Procedure according to the following properties: reflexivity, transitivity, symmetry.
(d) Is $\langle \mathbf{A}, L_2 \rangle$ a Monadic Boolean algebra?

Example 12.1.5. Example of a monadic Boolean algebra

Consider on the Boolean algebra $\mathbf{A}$ the following monadic operator L_m on $\mathbf{A}$ (as usual, on the right we draw the resulting substructure $\mathbf{L_m} = \mathbf{L}(\mathbf{A})$, which is a sublattice, in this case):

x	$L_m(x)$
0	0
a	0
b	0
c	0
d	b
e	e
f	b
1	1
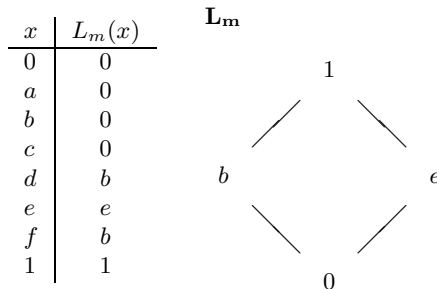
The system $\langle \mathbf{A}, L_m \rangle$ is a monadic Boolean algebra. It is worth noticing that $\mathbf{L_m}$ is a Boolean algebra, too.

If $\langle \mathbf{A}, L_R \rangle$ happens to be a Monadic Boolean algebra, where the operator L_R is induced by a binary relation R, we can ask if R enjoys some particular property. The answer is positive and will be given at the end of the present Section. Indeed we are going to see that the notions of a Pre-monadic Boolean algebra and Monadic Boolean algebra are the two extremes of a path that leads from operators associated with arbitrarily generic relations to operators connected with relations exhibiting the strongest properties, passing through intermediate cases.

For the reader's convenience, let us resume and align the definitions introduced so far in Table 12.1.

Table 12.1: Modalities, relations, forcing and algebraic structures

Operator	Definition	Context
$a \vDash L_k(\alpha)$	$\forall a'(a' \leq k(a) \Rightarrow a' \vDash \alpha)$	Forcing on algebraic structures
$X \subseteq \phi(L_k(\alpha))$	$\forall X'(X' \subseteq R(X) \Rightarrow X' \subseteq \phi(\alpha))$	Lattice of sets with $k(X) = R(X)$
$x \Vdash L_R(\alpha)$	$\forall y \in U(\langle x, y \rangle \in R \Rightarrow y \Vdash \alpha)$	Forcing on Kripke frames
$L_R(X)$	$\{x : \forall y \in U(\langle x, y \rangle \in R \Rightarrow y \in X)\}$	Pre Monadic Boolean algebras of sets
$a \vDash M_k(\alpha)$	$\exists a'(g(a') \geq a \ \& \ a' \vDash \alpha)$	Forcing on algebraic structures
$X \subseteq \phi(M_k(\alpha))$	$\exists X'(R^\smile(X') \supseteq X \ \& \ X' \subseteq \phi(\alpha))$	Lattice of sets with $g(X) = R^\smile(X)$
$x \Vdash M_R(\alpha)$	$\exists y \in U(\langle x, y \rangle \in R \ \& \ y \Vdash \alpha)$	Forcing on Kripke frames
$M_R(X)$	$\{x : \exists y \in U(\langle x, y \rangle \in R \ \& \ y \in X)\}$	Pre Monadic Boolean algebras of sets

We know that, $L_R(X)$ and $M_R(X)$ equal $\{x : R(x) \subseteq X\}$ and, respectively, $\{x : R(x) \cap X \neq \emptyset\}$. Hence, using the distributivity property of R-neighborhoods, we obtain $L_R(X) = \bigcup\{Z : R(Z) \subseteq X\}$ and,

dually, $M_R(X) = \bigcap\{-Z : X \subseteq -R(Z)\}$ (the duality of the two equations will be proved in Frame 15.1). Therefore, if we compare the last definitions with the definitions of lower and, respectively, upper approximations, by substituting $[x]_R$ for $R(x)$, for R an equivalence relation, we observe that they differ slightly but in a significant way. We underline this difference by adding in Table 12.2 the intermediate definition of two hypothetical operators L_R^* and M_R^*.

We can notice that the passage from L_R and M_R to L_R^* and, respectively, M_R^* surely requires some extra features, as well as that from L_R^* and M_R^* to (lR) and, respectively, (uR). In what follows we analyse these extra features and their contexts of application.

<div align="center">Table 12.2: Three degrees of $R-$modal operators</div>

R-modal operators	Necessity	Possibility
normal	$L_R(X) = \bigcup\{Z : R(Z) \subseteq X\}$	$M_R(X) = \bigcap\{-Z : X \subseteq -R(Z)\}$
with extra features	$L_R^*(X) = \bigcup\{R(Z) : R(Z) \subseteq X\}$	$M_R^*(X) = \bigcap\{R^\smile(Z) : X \subseteq R^\smile(Z)\}$
approximation	$(lR)(X) = \bigcup\{[x]_R : [x]_R \subseteq X\}$	$(uR)(X) = \bigcap\{[x]_R : X \subseteq [x]_R\}$

12.2 From Loosely Structured Spaces to Structured Spaces: A Variety of Modal Properties

Now we analyse the properties of the monadic operators L_R and M_R as dependent on the properties of the relation R.

If we do not impose any particular property on R, we cannot predict interesting uniform relationships between X and $L_R(X)$ – or $M_R(X)$ – nor special nice behaviours of the two modal operators.

What we can predict derives just from the fact that our operators happen to be Diodorean modalities, as one can see from *Proposition* 12.1.2.(6) above (*viz* it is valid to assert the possibility of α at point x if there is some state of affair accessible from x in which α is true).

What we can say without adding extra hypothesis is listed in Proposition 12.1.4, and we denote this basic set of properties with the symbol **K** (after the fact that they characterise a modal system usually denoted by this symbol). A modal logic with at least the same properties as **K**, is called *normal*.

Indeed a relation between elements of U without any specific constraints, reflects, in a obvious sense, empirical and variable relationships between pieces of information. But we can impose particular constraints to R according to theoretical intuitions or, as it happens for Approximation Spaces, according to a particular organisation of data. In Table 12.3 one can see how do specific constraints transform the properties of the modelled logic.

Table 12.3: Relational properties and modal properties

Properties of R and derived set-theoretical characteristics	Modelled modal properties on top of **K**	Label
Reflexivity $\forall x(\langle x, x \rangle \in R)$ $X \subseteq R(X)$	$L(\alpha) \to \alpha$ $\alpha \to M(\alpha)$	T
Seriality $\forall x, \exists y(\langle x, y \rangle \in R)$ $X \neq \emptyset$ implies $R(X) \neq \emptyset$	$L(\alpha) \to M(\alpha)$	D
Symmetricity $\forall x, y(\langle x, y \rangle \in R \Rightarrow$ $\langle y, x \rangle \in R); Y \subseteq R(X)$ iff $X \subseteq R(Y); R(X) = R^{\smile}(X)$	$\alpha \to L(M(\alpha))$ $M(L(\alpha)) \to \alpha$	B
Transitivity $\forall x, y, z(\langle x, y \rangle \in R \ \& \ \langle y, z \rangle$ $\in R \Rightarrow \langle x, z \rangle \in R); R(R(X))$ $\subseteq R(X); Y \subseteq R(X)$ implies $R(Y) \subseteq R(X)$	$L(\alpha) \to L(L(\alpha))$ $M(M(\alpha)) \to M(\alpha)$	4
Euclidean property $\forall x, y, z(\langle x, y \rangle \in R \ \& \ \langle x, z \rangle$ $\in R \Rightarrow \langle y, z \rangle \in R); Y \subseteq R(X)$ implies $R(X) \subseteq R(Y)$	$M(L(\alpha)) \to L(\alpha)$ $M(\alpha) \to L(M(\alpha))$	5

Example 12.2.1. Property D implies $L(\alpha) \rightarrow M(\alpha)$; $L(\alpha) \longrightarrow \alpha$ implies T

Suppose that (a) there is at least an element y such that xRy and (b) for all y', if xRy' then $y' \models \alpha$. Then $[\![L(\alpha)]\!] \subseteq [\![M(\alpha)]\!]$, because there is at least an element accessible form x that forces A, so that x forces $[\![M(\alpha)]\!]$ whenever x forces $[\![L(\alpha)]\!]$. So, $L(\alpha) \rightarrow M(\alpha)$. But if we drop hypothesis (a), that is, if we drop seriality, hypothesis (b) is vacuously true if there is no element accessible form x that forces α. In this case $x \models [\![L(\alpha)]\!]$ but $x \not\models \alpha$. Hence either $[\![M(\alpha)]\!] \subset [\![L(\alpha)]\!]$ or $[\![M(\alpha)]\!]$ and $[\![L(\alpha)]\!]$ are incomparable. Example:

$$
\begin{array}{c|cc}
R & x & y \\
\hline
x & 0 & 1 \\
y & 0 & 0 \\
\end{array}
$$
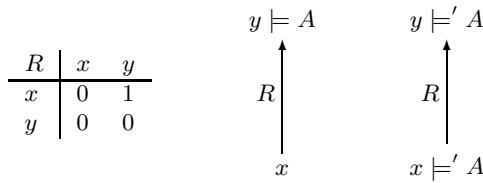
According to the model with forcing $\models$, $[\![M(A)]\!] = \{x\}$, while $[\![L(A)]\!] = \{y\}$ (since no element is accessible from y). Since R is not reflexive (i.e. $x \not\leqslant x$), this prove that if $[\![L(\alpha)]\!] \leq [\![L\alpha]\!]$ then T (reflexivity) must hold.

Example 12.2.2. Example of a non symmetric relation where $\alpha \rightarrow L(M(\alpha))$ fails

According to the model with forcing $\models'$ of Example 12.2.1, we have:

	$\models' A$	$\models' M(A)$	$\models' L(M(A))$
y	yes (set-up)	no (no accessible element forces A)	yes (void precondition "$\forall y'(yRy'\ldots))$"
x	yes (set-up)	yes (because $y \models' A$)	no (because $y \not\models' M(a)$)

Therefore, $[\![A]\!] = \{x, y\}$ and $[\![L(M(A))]\!] = \{y\}$.

Next we verify that adding seriality to a non-symmetric relation does not change the effect:

$$
\begin{array}{c|ccc}
R' & x & y & z \\
\hline
x & 0 & 1 & 1 \\
y & 0 & 1 & 0 \\
z & 0 & 0 & 1 \\
\end{array}
$$
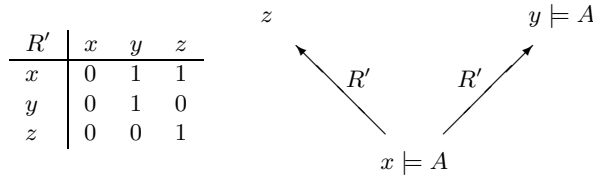
Therefore in this model we have $[\![A]\!] = \{x, y\}$, $[\![M(A)]\!] = \{x, y\}$ and $[\![L(M(A))]\!] = \{y\}$.

Example 12.2.3. $[\![L(A)]\!] \longrightarrow [\![LL(A)]\!]$ implies property 4
Consider the relation

R''	x	y	z
x	1	1	0
y	1	1	1
z	0	1	1

R'' is reflexive and symmetric. However it is not transitive. We leave to the reader the verification of instances of property B. We show that property 4 does not hold. Let $z, y \models A$. Then $[\![L(A)]\!] = \{z\}$ but $[\![LL(A)]\!] = \emptyset$, because $\langle z, y \rangle \in R''$ and $y \not\models L(A)$.

Exercise 12.6.
(a) Prove that property T (reflexivity) implies $L(\alpha) \to \alpha$.
(b) Prove that property 4 (transitivity) implies $L(\alpha) \to L(L(\alpha))$.

Some combinations of the above properties are equivalent. For instance, **K***T5*, **K***TB4*, **K***DB4*, **K***DB5* are equivalent (the reader should try and prove it – in Frame 15.2 it is possible to find some hints).

Indeed, the following result is folklore in Modal Logic:

Proposition 12.2.1. *For any relation $R \subseteq U \times U$, the following are all the possible distinct combinations of the properties D, T, B, 4, 5, on top of* **K***:*
K*,* **K***D,* **K***T,* **K***B,* **K***4,* **K***5,* **K***DB,* **K***D4,* **K***D5,* **K***45,* **K***TB,* **K***T4,* **K***D45,* **K***B4,* **K***T5.*

Some of the above combinations have received a particular attention in modal logic literature, because of their philosophical and/or mathematical importance.[5] As such they are known by means of traditional names: **K***T* = **T**, **K***TB* = **B**, **K***T4* = **S**4, **K***T5* = **S**5   A number

---

[5]Nonetheless, in many cases, properties are adopted not because they reflect specific intuitions about the way states of affairs are organised, but only in view of the formal properties that the modelled Logical system must feature. For instance, if L has to model a doxastic operator (i.e. "subject S believes that . . ."), then since an opinion is not guaranteed to be true, the reflexive property cannot be adopted, otherwise we should have $L(A) \to A$, that is read "If subject S believes A, then A is true". On the contrary, this property is required for modelling epistemic operators, such as "Subject S knows that ..", according to the classical definition advocating that "knowledge" is *true* and justified belief (cf. [Halpern, Moses 1985] for a technical overview. Cf. [Ellis 76] for a philosophical introduction and *Box* "Logico-philosophical remarks. 1" of Section 9.2 of Chapter 9).

of coarser/finer relationships between these systems are well-known in logical literature, as well as some intermediate systems. We address the reader to the *References*, for details.

However, the reader has surely noticed that, as a matter of fact, the properties of system **S**4 have been analysed in Part I, because any IQRS is a Kripke frame with reflexive and transitive accessibility relation.[6] Here we want to mention that Proximity Spaces are models for system **B**: in fact, Proximity Spaces are relational spaces $\langle U, R \rangle$ where R is reflexive and symmetric.[7] And it is clear now, that **S**5 is about to be adopted as the referent modal logic for Approximation Spaces, because **S**5 models are characterised by reflexive, transitive and symmetric relations. Actually, this will be the starting point for understanding the modal features of Rough Set Systems.

For the time being, we shall investigate some further formal properties of relational spaces connected with pre-topological and topological spaces.

12.3   Relations, Pre-Topologies and Topologies

Our interest in studying relations is the fact that the main concern in Rough Set Analysis is the way "perceptions" are connected in order to form conceptually meaningful patterns. Henceforth, a single element of the domain of concern is not interesting by its own ("an sich"), but to the extent it is connected (or not) with other elements. Otherwise stated, we are interested in the geometry that relations impose on a

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[6]More precisely, since IQRS are finite, hence they fulfill the so-called McKinsey condition $\forall x \exists y (\langle x, y \rangle \in R \ \& \ \forall z (\langle y, z \rangle \in R \Rightarrow y = z))$, they are Kripke frames adequate to the system **S**4.1, which is obtained by adding to **S**4 the axiom $L(M(a)) \to M(L(A))$.

[7]The symbol "**B**" is after the name of L. E. J. Brouwer, founder of the Intuitionistic school (cf. Introduction). This traditional use is justified by a translation of the intuitionistic negation $\neg$ as $L \sim$ (here "$\sim$" is the Boolean complementation of the modal system). In accordance with it, the intuitionistically admissible low $a \Longrightarrow \neg\neg a$ becomes $a \to L(\sim L(\sim a))$, i.e. $a \to L(M(a))$ (which characterises modal operators modelled by symmetric relations) which is the characteristic axiom of the "Brouwerian" system. On the contrary, the intuitionistically invalid law $\neg\neg a \Longrightarrow a$ becomes $L(M(a)) \to a$, which is invalidated by models with relations fulfilling *TB*. However, the "real" modal system connected with Intuitionistic Logic is **S**4 + **Grz**, where **Grz** is Segerberg's translation $L(L(p \to L(p)) \to p) \to p$ of the principle introduced by Andrzej Grzegorczyk for a modal interpretation of Heyting's logic (cf. [Grzegorczyk 1967] and [Segerberg 1971]).

universe of possible perceptions/stimulations (or "empirical results", "uninterpreted data" and the like).

Of course, we shall not remain at the abstraction level of a point-like geometry for ever. We are more interested in the general, universal properties of a "perception system". Therefore the abstraction level shall be lifted to a sort of pointless geometry. This more abstract level was discussed at the end of the Introduction and constituted already the playground of our algebraic analysis of Rough Set Systems. Here we are going to reach the same abstraction level for the modal interpretation.

Indeed, in case of the algebraic analysis, first we started noticing that a concrete Approximation Space on U is induced by a subalgebra of the Boolean algebra $\mathbf{B}(U)$, so that it was possible to define the notion of an abstract Approximation System as a pair $\langle \mathbf{B}, \mathbf{B}' \rangle$ made up of a Boolean subalgebra $\mathbf{B}'$ of a given Boolean algebra $\mathbf{B}$. Secondly, $\mathbf{B}$ was transformed into a new algebraic structure (namely, a Rough Set System), embedding the transformation of the elements of $\mathbf{B}$ induced by $\mathbf{B}'$. In the modal analysis, we shall follow the same strategy: the only difference is that we shall transform $\mathbf{B}$ into a modal system in accordance with the way its elements are modalised by means of an operator $L_{\mathbf{B}'}$ (or $M_{\mathbf{B}'}$), which is the abstract companion of L_R (of M_R).

This analysis will not mention the population of the elements of $\mathbf{B}$. However, we shall again reach this abstraction level starting from the intuitive ground of a "concrete" analysis of universes populated by "real" elements connected by "operating" relations.

12.4   Pre-Topological Spaces

We shall approach topological spaces from more general structures, called "pre-topological spaces". This choice is suggested by the fact that pre-topological spaces are widely (although often implicitly) used in Rough Set Theory in order to generalise the basic concepts of lower and upper approximation (as one is able to verify in the Frame section). Intuitively, whereas in Kripke frames any single world is linked with a set of accessible world, in pre-topologies any point x is associated with a family of sets, its neighborhood system $n(x)$. Each element of $n(x)$ may be intended as representing a collection of points that are relevant

to x. Or, from another perspective, $n(x)$ is the family of observable phenomena connected with x. Therefore a formula α is necessarily valid at point x if the set of points validating α is relevant to x, i.e. $x \Vdash L(\alpha)$ *iff* $[\![\alpha]\!] \in n(x)$. This is the basic intuition leading to the definition of a *core map* (see below).

Clearly if $[\![\neg\alpha]\!] \in n(x)$, then α is unnecessary at point x. So, since α is possible at point x if it is not unnecessary at x, we can define $x \Vdash M(\alpha)$ *iff* $[\![\neg\alpha]\!] \notin n(x)$. This leads to the notion of a *vicinity map* dual of the core map.

Obviously, a vicinity map (a core map) is a generalisation of the usual notion of a closure map (interior map). The main difference, intuitively, is that vicinity maps reflect the notion of "x is close to a set X" under one or more possible points of view, while closure operators account for single cumulative points of view, by gluing all the elements of $n(x)$ through the imposition for $n(x)$ to be a filter.

Moreover, neighborhoods of points of U are not required to be subsets of U. Indeed, in a more general setting, we can think of situations in which $n(x) \subseteq \wp(U')$ for $x \in U$ and $U' \neq U$. Hence U' acts as a "medium", via a map $f : U \longmapsto U'$, in the evaluation of a closeness relation between a point x from U and another point y of U. In fact, a certain closeness criterion might not be applicable directly on the elements of U, but can be applicable on their f-images in U' (for instance we cannot understand if professor Smith's and professor Brown's scientific interests are similar by looking at the list of the pure names of the academic body of San Jose University. However, this is possible when we map Smith and Brown onto the set of academic disciplines).

In this case x will belong to the core of a subset $X \subseteq U$ if $f(X)$ belongs to $n(x)$. Below we illustrate this more general situation:

In Figure 12.1, x is related to y, w, w', from the point of view of a criterion α acting between their f-images on U'. On the other hand, x is related to z, q, z', z'' through a different criterion β. The collection of these aggregations forms $n(x)$. It follows that x belongs to the core of the set $\{y, w, w', x\}$ and of the set $\{z, q, z', z''\}$ because both $f(\{y, w, w', x\})$ and $f(\{z, q, z', z''\})$ belong to $n(x)$. Moreover, x belongs, for instance, to the vicinity of $\{y, z\}$ because $-\{y, z\}$ does not belong to $n(x)$. Notice that $f(x)$ does not belong to $f^{\rightarrow}(\{z, q, z', z''\})$.

So, let U, U' be sets. We can consider that the elements of U are connected (classified, characterised, labeled, perceived, ...) by means of
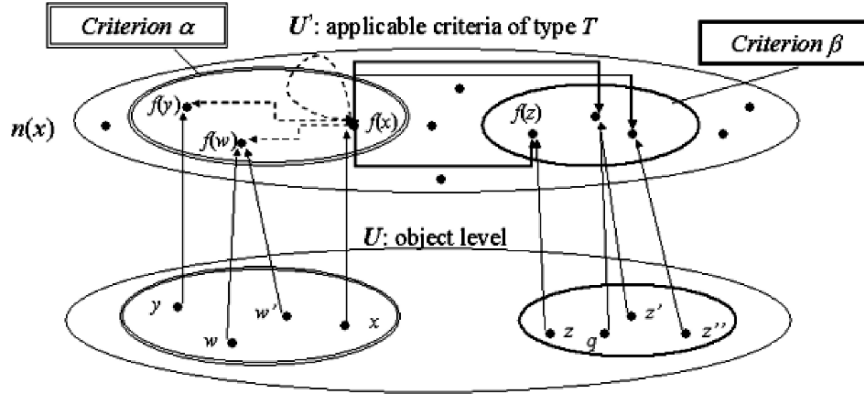
Figure 12.1: Observations and pre-topological spaces

the relationships that occur between the elements of another set U'. Therefore, according to this connection, any element p of U can be associated with one ore more elements of $\wp(U')$, obtaining thereby a family of subsets of U', denoted by $n(p)$, so that each $N \in n(p)$, links p with other elements of U under a specific respect.

We summarize these intuitions in the following definitions:

Definition 12.4.1. *Let U, U' be sets, $X' \subseteq U'$, $u \in U$ and $f : U \longmapsto U'$ a total function. Then,*

1. *A* neighborhood map *is a total function $n : U \longmapsto \wp(\wp(U'))$, such that $f(x) = f(y)$ implies $n(x) = n(y)$.*

2. - *$n(u)$ is called a* concrete neighborhood family *of u.*

   - *If $N \in n(u)$, then N is called a* concrete neighborhood *of u.*

   - *If $u' \in N \in n(u)$, then u' is called a* concrete neighbor *of u.*

   - *The family $\mathcal{N}(U) = \{n(x) : x \in U\}$ is called a* concrete neighborhood system.

   - *The pair $\langle U, \mathcal{N}(U) \rangle$ is called a* concrete neighborhood space.

3. *If $G(X') = \{x : X' \in n(x)\}$, then G is called the* core *map induced by $\mathcal{N}(U)$.*

4. *If $F(X') = -G(-X') = \{x : -X' \notin n(x)\}$, then F is called the* vicinity *map induced by $\mathcal{N}(U)$.*

5. *The set $F(X') \cap -G(X') = \{x : \forall N \in n(x)(N \cap X' \neq \emptyset \neq N \cap -X')\}$ is called the boundary of X', denoted by $\partial(X')$.*

We can notice the reason why the notion of a core map (a vicinity map) is a generalisation of the notion of an interior (closure) operator. Indeed, if $U = U$ and f is the identity map then, as we shall prove in *Lemma* 12.4.1, $x \in G(X)$ if and only if $X \in n(x)$, that is, if and only if X itself is a neighborhood of x, whereas in topological spaces $x \in \mathbb{I}(X)$ (the interior of X) if and only if there is a neighborhood of x included in X. We shall see that the two definitions coincide just under some specific assumptions. Under the same assumptions we shall prove that $x \in F(X)$ if and only if X has no void intersection with all of the neighborhoods of x.

Example 12.4.1. A simple neighborhood system

Let $U = \{x, y, z, w\}, U' = \{a, b, c\}, f(x) = a, f(y) = b, f(z) = f(w) = c$. Consider the following neighborhood system: $n(x) = \{\{a, c\}, \{a, b, c\}\}, n(y) = \{\{b\}, \{a, b\}\},$ $n(z) = n(w) = \{\{c\}\}$. Then, $G(\{b\}) = \{u : \{b\} \in n(u)\} = \{y\}, G(\{b, c\}) = \emptyset$, and so on;
$F(\{b\}) = \{u : -\{b\} \notin n(u)\} = \{u : \{a, c\} \notin n(u)\} = \{y, z, w\}, F(\{a, b\}) = \{x, y\}$ and so on.

Notice that neither G nor F are isotonic.

TERMINOLOGY AND NOTATION. Given $p \in U$, from now on the image $n(p)$ of p along n will be usually denoted by $\mathcal{N}_p$.

Consider the following conditions on $\mathcal{N}(U)$, for any $x \in U$, $A, N, N' \subseteq U'$:

1. $U' \in \mathcal{N}_x$.
0. $\emptyset \notin \mathcal{N}_x$.
Id. if $x \in G(A)$ then $f^{\rightarrow}(G(A)) \in \mathcal{N}_x$.
N1. $f(x) \in N$, for all $N \in \mathcal{N}_x$.
N2. if $N \in \mathcal{N}_x$ and $N \subseteq N'$, then $N' \in \mathcal{N}_x$.
N3. if $N, N' \in \mathcal{N}_x$, then $N \cap N' \in \mathcal{N}_x$.
N4. there is an $N \neq \emptyset$ such that $\mathcal{N}_x = \uparrow_\subseteq N$.

Because function f occurs in the definitions of **Id** and **N1**, the two conditions will be said to be "*point-dependent*".

From a practical point of view the distinction between U and U' is relevant (think, for instance, of the different attributes in relational

databases). However, on a theoretical side, dealing with a single universe is more comfortable and does not cause any information shortcoming, because instead of $x' \in N \in \mathcal{N}_x$ we can consider the inverse image $f^{\leftarrow}(\{x\})$ (this is what we usually do in relational databases when we move from a set of attribute-values V to the entities identified by V).

Definition 12.4.2. *If $U = U'$ and f is the identity map, then the pair $\langle U, \mathcal{N}(U) \rangle$ is called a* Fréchet *space.*

REMARKS. From now on we shall deal only with Fréchet spaces. In a Fréchet space, **N1** reads: "$x \in N$, for all $N \in \mathcal{N}_x$" and **Id** turns into "$\forall A \subseteq U, \forall x \in G(A), G(A) \in \mathcal{N}_x$". A neighborhood system $\mathcal{N}(U)$ will be denoted also by $\mathcal{N}$ if the set U is understood.

Example 12.4.2. A simple Fréchet space
Consider the universe $U = \{a, b, c\}$. The following is a Fréchet neighborhood system:
$\mathcal{N}_a = \{\{b\}, \{a, c\}, U\}, \mathcal{N}_b = \{\{a, b\}, \{b, c\}, U\}, \mathcal{N}_c = \{\{b\}, \{a, c\}, \{a, b\}, U\}$. Clearly in $\mathcal{N}(U)$ **0** and **1** hold. On the contrary, **N1** does not hold because, for instance, $a \notin \{b\} \in n(a)$.

The above conditions carry particular properties that reflect on the operators G and F:

Lemma 12.4.1. *Let $\mathcal{N}(U)$ be a neighborhood system. Then, for any $X, Y \subseteq U$, $x \in U$:*

(**G1**) $x \in G(X)$ *iff* $X \in \mathcal{N}_x$;   (**G2**) $G_x =_{def} \{X : x \in G(X)\} = \mathcal{N}_x$.

Condition	Equivalent properties of G	Equivalent properties of F
1	$G(U) = U$	$F(\emptyset) = \emptyset$
0	$G(\emptyset) = \emptyset$	$F(U) = U$
Id	$G(X) \subseteq G(G(X))$	$F(F(X)) \subseteq F(X)$
N1	$G(X) \subseteq X$	$X \subseteq F(X)$
N2	$X \subseteq Y \Rightarrow G(X) \subseteq G(Y)$ $G(X \cap Y) \subseteq G(X) \cap G(Y)$	$X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$ $F(X \cup Y) \supseteq F(X) \cup F(Y)$
N3	$G(X \cap Y) \supseteq G(X) \cap G(Y)$	$F(X \cup Y) \subseteq F(X) \cup F(Y)$

Proof. (**G2**) $\{X : x \in G(X)\} = \{X : x \in \{y : X \in \mathcal{N}_y\}\} = \{X : X \in \mathcal{N}_x\} = \mathcal{N}_x$. From (G2) we straightforwardly obtain (**G1**). (**1**) Trivial. (**0**) Trivial. (**Id**) Assume **Id** holds and $x \in G(X)$. From **Id**, $G(X) \in \mathcal{N}_x$, so from (G1) $x \in G(G(X))$. Conversely, if $G(X) \subseteq G(G(X))$ then from (G1) $X \in \mathcal{N}_x$ implies $G(X) \in \mathcal{N}_x$, so that **Id** holds. (**N1**) Assume **N1** holds. If $x \in G(X)$, from (G1) $X \in \mathcal{N}_x$ and from **N1** $x \in X$. *Vice-versa*, assume $G(X) \subseteq X$. But $G(X) = \{x : X \in \mathcal{N}_x\}$; thus from (G1) $x \in G(X)$, hence $x \in X$. Henceforth $x \in X$. Thus **N1** holds. (**N2**) (a) Assume **N2**. If $X \in \mathcal{N}_x$ and $X \subseteq Y$ then $Y \in \mathcal{N}_x$. From (G1) we deduce that if $x \in G(X)$ and $X \subseteq Y$ then $x \in G(Y)$, that is, $G(X) \subseteq G(Y)$. Conversely, if $X \subseteq Y$ implies $G(X) \subseteq G(Y)$, then from (G1) $X \in \mathcal{N}_x$ implies $Y \in \mathcal{N}_x$. Hence **N2** holds. (b) Assume **N2**. If $x \in G(X \cap Y)$ then $X \cap Y \in \mathcal{N}_x$. But $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. Thus, from **N2** $X \in \mathcal{N}_x$ and $Y \in \mathcal{N}_x$, so that $x \in G(X)$ and $x \in G(Y)$. Conversely, assume $G(X \cap Y) \subseteq G(X) \cap G(Y)$, $X \in \mathcal{N}_x$ and $X \subseteq Y$. Then $G(X \cap Y) = G(X) \subseteq G(X) \cap G(Y)$. This means that $G(X) \subseteq G(Y)$, from (G1), so that $Y \in \mathcal{N}_x$, and **N2** holds. (**N3**) Assume **N3** and $x \in G(X) \cap G(Y)$. From (G1) we obtain $X \in \mathcal{N}_x$ and $Y \in \mathcal{N}_x$. Therefore in view of **N3**, $X \cap Y \in \mathcal{N}_x$, and again from (G1) $x \in G(X \cap Y)$. Henceforth, $G(X) \cap G(Y) \subseteq G(X \cap Y)$. Conversely, assume $X \in \mathcal{N}_x$, $Y \in \mathcal{N}_x$ and $G(X) \cap G(Y) \subseteq G(X \cap Y)$. From the latter assumption if $x \in G(X) \cap G(Y)$ then $x \in G(X \cap Y)$. Therefore, from (G1), if X and $Y \in \mathcal{N}_x$, then $X \cap Y \in \mathcal{N}_x$, so that **N3** holds. As to F we obtain the results by duality. Here we prove only (i) $G(X \cap Y) \subseteq G(X) \cap G(Y) \Rightarrow F(X) \cup F(Y) \subseteq F(X \cup Y)$ and (ii) $G(X) \subseteq G(G(X)) \Rightarrow F(F(X)) \subseteq F(X)$. (i) Indeed $G(X \cap Y) \subseteq G(X) \cap G(Y)$ iff $-(G(X) \cap G(Y)) \subseteq -G(X \cap Y)$, iff $-(G(-X) \cap G(-Y)) \subseteq -G(-X \cap -Y)$, iff $-G(-X) \cup -G(-Y) \subseteq -G - (X \cup Y)$, iff $F(X) \cup F(Y) \subseteq F(X \cup Y)$. (ii) $G(X) \subseteq G(G(X))$ iff $-G(G(X)) \subseteq -G(X)$ iff $-G(G(-X)) \subseteq -G(-X)$ iff $-G - (-G(-X)) \subseteq -G(-X)$ iff $F(F(X)) \subseteq F(X)$.   QED[8]

REMARKS. One should not confuse **G1** with the principle "$X \in \mathcal{N}_x \Rightarrow x \in X$" which holds if $\mathcal{N}(U)$ fulfills **N1**.

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[8]Note that if $U \neq U'$, then property **1** turns into $G(U') = U$, property **0** turns into $F(U') = U$ and, finally, **N1** turns into $G(X) \subseteq f^{\leftarrow}(X)$ [in the proof of **N1** substitute "$f(x) \in X$" for "$x \in X$" and "$G(X) \subseteq f^{\leftarrow}(X)$" for "$G(X) \subseteq X$", and notice that $f(X) \in X$ iff $x \in f^{\leftarrow}(X)$].

Example 12.4.3. A neighborhood system satisfying **Id** but not **N1**, whose core map G is not idempotent

Consider the universe $U = \{a, b, c\}$ and the neighborhood system $\mathcal{N}(U)$ given by:

x	a	b	c
$\mathcal{N}_x$	$\{\{a\}, \{a,b\}, \{b,c\}, U\}$	$\{\{b\}, \{a,b\}, \{b,c\}, U\}$	$\{\{a,b\}, U\}$

Let us check that in this neighborhood system property **Id** is satisfied. Indeed the core map G is given by:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$G(x)$	$\emptyset$	$\{a\}$	$\{b\}$	$\emptyset$	U	$\emptyset$	$\{a,b\}$	U

It is easy to verify that if $X \in \mathcal{N}_x$ then $G(X) \in \mathcal{N}_x$ (for instance $G(\{b,c\}) = \{a,b\}$ because $\{b,c\}$ belongs to $\mathcal{N}_a$ and $\mathcal{N}_b$. However $G(G(\{b,c\})) = U \neq G(\{b,c\})$). Also, we can observe that $\mathcal{N}(U)$ does not fulfill **N1** ($\{a,b\} \in \mathcal{N}_c$ but $c \notin \{a,b\}$). Actually, had $\mathcal{N}(U)$ fulfilled **N1**, G would have been idempotent (cf. *Proposition* 12.4.5 and Example 12.4.5 below).

Example 12.4.4. A neighborhood system satisfying **Id** but not **N1**, whose core map G is idempotent

Consider the universe $U = \{a, b, c\}$ and the neighborhood system $\mathcal{N}(U)$ given by:

x	a	b	c
$\mathcal{N}_x$	$\{\{a,b\}, \{a,c\}, U\}$	$\{\{a\}, \{b\}, \{a,b\}, \{b,c\}, U\}$	$\{\{c\}, \{a,c\}, U\}$

In this neighborhood system property **Id** is satisfied, but **N1** is not ($\{a\} \in \mathcal{N}_b$ but $b \notin \{a\}$). However the core map G is idempotent:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$G(x)$	$\emptyset$	$\{b\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b\}$	U

Example 12.4.5. A neighborhood system which satisfies **0**, **1**, **N1** and **Id**

Consider the universe $U = \{a, b, c\}$ and the neighborhood system $\mathcal{N}(U)$ given by:

x	a	b	c
$\mathcal{N}_x$	$\{\{a,b\}, \{a,c\}, U\}$	$\{\{b\}, \{a,b\}, \{b,c\}, U\}$	$\{\{c\}, \{a,c\}, U\}$

It is easy to check that in this neighborhood system **0**, **1**, **N1** and **Id** hold. The core map is idempotent:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$G(x)$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b\}$	U

However, **N2** does not hold. In fact, $\mathcal{N}_c$ is not an order filter (for instance, $\{c\} \in \mathcal{N}_c, \{c\} \subseteq \{b,c\}$ but $\{b,c\} \notin \mathcal{N}_c$).

The following is a very simple but useful statement:

Proposition 12.4.1. *Let $\mathcal{N}(U)$ be a neighborhood system. Then $\mathcal{N}(U)$ fulfills* **Id** *if and only if for all $X \subseteq U$ and $x \in U$, $X \in N_x$ implies $G(X) \in N_x$.*

Proof. Suppose **Id** holds and $X \in \mathcal{N}_x$. From **G1** $x \in G(X)$. Hence from **Id**, $G(X)$ must belong to $\mathcal{N}_x$, too. Conversely, suppose $X \in \mathcal{N}_x \Rightarrow G(X) \in \mathcal{N}_x$ and $x \in G(X)$. Again from **G1**, $X \in \mathcal{N}_x$. Therefore $G(X) \in \mathcal{N}_x$. We conclude that **Id** holds.   QED

Notice that if **N2** is assumed, then **N3** is equivalent to the following weaker condition:

$$if \ \ X \in \mathcal{N}_x \ \ and \ \ Y \in \mathcal{N}_x \ \ then \ \ \exists Z \in \mathcal{N}_x \ \ such \ that \ \ Z \subseteq X \cap Y \qquad (\mathbf{N3-})$$

Proposition 12.4.2. *Assume* **N2** *and* **N3−***. Then for any $X, Y \subseteq U$, $G(X \cap Y) \supseteq G(X) \cap G(Y)$.*

Proof. If $x \in G(X) \cap G(Y)$, then $x \in G(X)$ and $x \in G(Y)$. Thus, from **G1** $X, Y \in \mathcal{N}_x$. Therefore from **N3−** there exists $Z \subseteq X \cap Y$ such that $Z \in \mathcal{N}_x$. But from **N2**, $X \cap Y$ must belong to $\mathcal{N}_x$, too.   QED

(Notice that in literature even weaker conditions are studied, such as the so-called "connection condition": if $X \in \mathcal{N}_x$ and $Y \in \mathcal{N}_x$ then $X \cap Y \neq \emptyset$. An example of the use of this condition in modal logic can be found in Frame 15.13.3).

Moreover, if **N2** is assumed then **Id** is equivalent to the following weaker condition:

$$if \ N \in \mathcal{N}_x, \ then \ \exists N' \in \mathcal{N}_x \ such \ that \ for \ any \ y \in N', N \in \mathcal{N}_y \qquad (\tau)$$

This is the familiar topological property usually explained by the sentence: "if X is a neighborhood of a point x, then it is also a neighborhood of all those points that are sufficiently close to x".

Proposition 12.4.3. *Let $\mathcal{N}(U)$ be a neighborhood system. Then if $\mathcal{N}(U)$ satisfies* **Id***, it satisfies (τ), too.*

Proof. Suppose $p \in U$ and $N \in \mathcal{N}_p$. From **Id**, $G(N)$ belongs to $\mathcal{N}_p$. But from definition of $G(N)$, $N \in \mathcal{N}_x$ for any $x \in G(N)$. Hence (τ) holds.   QED

The converse implication does not hold without **N2**, as is illustrated in Example 12.4.6 below.

Proposition 12.4.4. *Let $\mathcal{N}(U)$ be a neighborhood system satisfying* **N2**. *Then an element $\mathcal{N}_x$ of $\mathcal{N}(U)$ satisfies (τ) if and only if it satisfies* **Id**.

Proof. From *Proposition* 12.4.3, **Id** implies (τ). Conversely, let (τ) hold. Consider any neighbor $N \in \mathcal{N}_x$. From (τ) there is an element $N' \in \mathcal{N}_x$ such that for any $y \in N'$, $N \in \mathcal{N}_y$. Clearly the set $G(N) = \{z : N \in \mathcal{N}_z\}$ includes N' because it is the largest collection of elements z such that $N \in \mathcal{N}_z$. Therefore in view of **N2**, $G(N) \in \mathcal{N}_x$. QED

Condition **Id** alone does not guarantee the idempotence of G and F (for a counterexample see Example 12.4.3). We have idempotence by adding **N1** to **Id**:

Proposition 12.4.5. *Let $\mathcal{N}(U)$ be a neighborhood system satisfying* **N1** *and* **Id**. *Then for any $X \subseteq U$, $G(G(X)) = G(X)$ and $F(F(X)) = F(X)$.*

Proof. Immediate from *Lemma* 12.4.1. QED

Proposition 12.4.6. *If G is idempotent, then* **Id** *holds.*

Proof. Suppose **Id** does not hold. Then $\exists x \in U, X \subseteq U$ such that $X \in \mathcal{N}_x$ but $G(X) \notin \mathcal{N}_x$. Therefore, $x \notin G(G(X))$, although $x \in G(X)$. It follows that $G(G(X)) \neq G(X)$. QED

Example 12.4.6. A neighborhood system fulfilling **N1** and (τ) but neither **Id** nor **N2**
Let $U = \{a, b, c, d\}$. Let $\mathcal{N}(U)$ be given by $\mathcal{N}_a = \{\{a\}, \{a, b, c\}, U\}, \mathcal{N}_b = \{\{b\}, \{a, b\}, \{a, b, c\}, U\}$, and $\mathcal{N}_c = \{\{c\}, U\}$. Then property (τ) is fulfilled by all the elements of $\mathcal{N}(U)$. However, $\{a, b, c\} \in \mathcal{N}_a$ but $G(\{a, b, c\}) = \{a, b\} \notin \mathcal{N}_a$. Hence $\mathcal{N}_a$ does not satisfy **Id**. According to *Corollary* 12.4.1 it follows that in the pre-topological space induced by $\mathcal{N}(U)$ the operator G is not idempotent ($G(\{a, b, c\}) = \{a, b\}$, but $G(\{a, b\}) = \{b\}$).
Notice that **N2** does not hold in $\mathcal{N}(U)$ (for instance $\{a, b\} \supseteq \{a\} \in \mathcal{N}_a$, but $\{a, b\} \notin \mathcal{N}_a$). Henceforth (τ) plus **N1** does not imply **N2**.

In general idempotence of G does not imply **N1**. However we have,

Corollary 12.4.1. *In the presence of* **N1**, *G is idempotent if and only if* **Id** *holds.*

In general G and F are not required to be idempotent. Intuitively, the lack of idempotence reflects a sort of flowing situation in which boundaries are not fixed once for ever, so that by adding the boundary $\partial(X)$ to a subset X by means of the vicinity map F we do not gain a stable situation, since a new boundary could appear.

Now we list various combinations of the above properties that shall dealt with in this section. The right column displays their main relational characteristics, that will be proved in this Chapter:[9]

If all the elements of $\mathcal{N}(U)$ satisfy	$\mathcal{N}(U)$ is said to be of type	Elements of $\mathcal{N}(U)$	Relational properties
0, 1	$\mathcal{N}_S$		
0, Id	$\mathcal{N}_{Id}$		
0,1,N1	$\mathcal{N}_1$		
0,1,N1, Id	$\mathcal{N}_{1Id}$		
0,1,N2	$\mathcal{N}_B$	proper order filter w. r. t. $\subseteq$	Induced by systems of serial relations
0, 1, N1, N2	$\mathcal{N}_2$	proper order filter w. r. t. $\subseteq$	Induced by systems of reflexive relations
0, 1, N1, N2, τ	$\mathcal{N}_{2Id}$	proper order filter w. r. t. $\subseteq$	Induced by systems of preorders
0, 1, N1, N2, N3	$\mathcal{N}_3$	proper filter	
0, 1,N1, N2, N3, τ	$\mathcal{N}_{3Id}$	proper filter	
0, 1, N1, N2, N3, N4	$\mathcal{N}_4$	principal filter	Induced by single reflexive relations
0, 1, N1, N2, N3,N4, τ	$\mathcal{N}_{4Id}$	principal filter	Induced by single preorders

[9]In some papers these properties are denoted by different names (for instance in [Stadler & Stadler 2001] we have: $\mathbf{1} = \text{K0}$, $\mathbf{N1} = \text{K2}$, $\mathbf{N2} = \text{K1}$, $\mathbf{N3} = \text{K3}$). Also, the

REMARKS. If $\mathcal{N}(U)$ is finite and of type $\mathcal{N}_3$ then it is also of type $\mathcal{N}_4$.

The resulting picture, that we shall justify throughout this Section, will be the following:
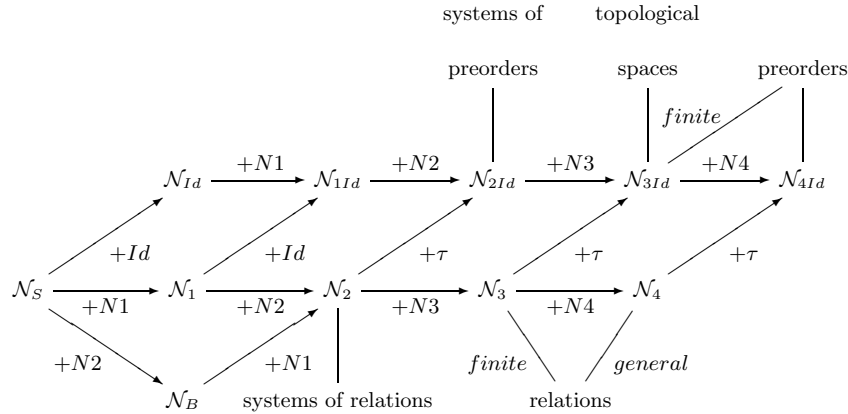
TERMINOLOGY AND NOTATION.

In what follows, we shall deal only with spaces of type at least $\mathcal{N}_1$. Therefore, by abuse of language we shall refer to a neighborhood system at least of type $\mathcal{N}_1$ as a "neighborhood system" tout-court. Moreover, since we shall typically deal with finite spaces, $\mathcal{N}_3$ systems will stand also for $\mathcal{N}_4$ systems. A distinct use will be generally adopted for systematic purposes (introduction of notions, an so on). In the Frame section we shall illustrate some applications of the most general form of neighborhood systems.

Now we shall see that vicinity maps in neighborhood systems of type $\mathcal{N}_1$ reflect, so to say, a process of *extension*. An extension is a process that applied to a set X collects all the elements of X plus those elements that, under some point of view, are connected with them

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terms used to refer to types of pre-topological spaces may vary (in the quoted paper we have $\mathcal{N}_B = Extended\ topology$, $\mathcal{N}_1 = Brissaud\ space$, $\mathcal{N}_2 = Neighborhood\ space$, $\mathcal{N}_3 = pre\text{-}topology$, $\mathcal{N}_{2Id} = Convex\ closure\ space$). Other combinations have been studied. For instance, neighborhood systems satisfying $\mathbf{1} + \mathbf{N2} + \mathbf{N3}$, which induce the so called "*Smith spaces*". Spaces induced by neighborhood systems satisfying $\mathbf{N2} + \mathbf{Id}$ are called "*intersection spaces*", while $\mathcal{N}_{2Id}$ spaces are also called "*topped intersection structures*" or "*closure systems*". Notice that some authors call neighborhood systems satisfying $\mathbf{N1}$ and $\mathbf{N3}-$ "neighborhood basis" and neighborhood systems of type $\mathcal{N}_3$ "neighborhood filters". Neighborhood systems of type $\mathcal{N}_4$ are usually called "binary neighborhood systems", because they are univocally related to binary relations (as we shall widely see in this Chapter).

(if any). Therefore such a process is an increasing map f between subsets of U and we call it an "*expansion process*":

Definition 12.4.3. *Let U be a set. An* expansion process *is any map $f : \wp(U) \longmapsto \wp(U)$ such that for any $X \subseteq U$, $X \subseteq f(X)$.*

Dually, we can think of a process of erosion which cuts down some connections between elements of U, just leaving the elements from a subset X that are strictly connected each others. We call such a process a "*contraction*".

Definition 12.4.4. *Let U be a set. A* contraction process *is any map $g : \wp(U) \longmapsto \wp(U)$ such that for any $X \subseteq U$, $g(X) \subseteq X$.*

Proposition 12.4.7. *Let U be a set and f an expansion process. If for any $X \subseteq U, g(X) = -f(-X)$, then g is a contraction process, called the* dual *of f.*

Proof. For any $X \subseteq U, -X \subseteq f(-X)$. Hence $-f(-X) \subseteq --X = X$.
<div align="right">QED</div>

From now on, by $\langle \varepsilon, \varkappa \rangle$ we shall indicate a pair of duals: expansion and, respective, contraction maps.

Definition 12.4.5. *A* pre-topological space *is a triple $\langle U, \varepsilon, \varkappa \rangle$ such that:* (i) *U is a set,* (ii) *$\varepsilon : \wp(U) \longmapsto \wp(U)$ is an expansion map such that $\varepsilon(\emptyset) = \emptyset$,* (iii) *$\varkappa : \wp(U) \longmapsto \wp(U)$ is a contraction map dual to ε.*

Proposition 12.4.8. *If $\langle U, \varepsilon, \varkappa \rangle$ is a pre-topological space, then $\varkappa(U) = U$.*

The proof is left to the reader.

Now we have to note that the notion of an expansion (contraction) cannot be immediately related with that of a R-neighborhood. In fact, given a generic relation $R \subseteq U \times U$, we do not have either $R(X) \subseteq X$ or $X \subseteq R(X)$, for any $X \subseteq U$ (the same happens for $R^{\smile}$, of course). Indeed, as we have seen in Section 12.2, $X \subseteq R(X)$ is valid only if R is reflexive. Moreover, both ε and $\varkappa$ lack the isotonicity law which, on the contrary, is valid for R-neighborhoods. Finally, differently from R-neighborhoods, neither the definition of ε, nor that of $\varkappa$ make any assumption about the distribution over disjunctions or conjunctions.

Also, notice that neither ε nor $\varkappa$ are required to be idempotent in a pre-topological space (anyway, the same happens for R-neighborhoods). As already pointed out, this reflects a floating situation.

Example 12.4.7. Expansions and contractions
In the following figure we depict an example of a floating boundary:
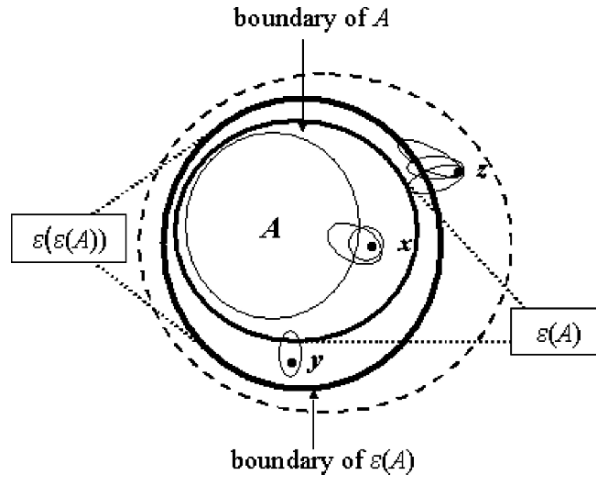
Figure 12.2: Example of a floating boundary

In Figure 12.2, point x belongs to the boundary of A, $\partial(A)$, while $y \notin \partial(A)$. Therefore $x \in \varepsilon(A)$, while $y \notin \varepsilon(A)$. However, $y \in \partial(\varepsilon(A))$. It follows that $\varepsilon(\varepsilon(A)) \supsetneq \varepsilon(A)$, $\varepsilon(\varepsilon(\varepsilon(A))) \supsetneq \varepsilon(\varepsilon(A))$, and so on up to an eventual fix point of the operator ε.

Suppose to process a set $A = \{x, y, z\}$. In A, the elements x and y are tightly linked, while x and z are loosely linked. Moreover, z is connected with the elements a and b, and y with the element c, that lies all outside of A. When we apply the expansion process ε to A, we gather together all the elements of A (x, y and z), plus the elements they are connected with, that is, a, b and c. When we contract A, we keep just the tight connected elements inside A, (x and y) and miss the elements which are loosely connected with these "core" elements of A. Therefore, $\varepsilon(A) = \{x, y, z, a, b, c\}$ and $\varkappa(A) = \{x, y\}$.

Definition 12.4.6. *Let* $\langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space,* $X \subseteq U$. *Then,*

1. *X is said to be "closed" iff $\varepsilon(X) = X$.*

2. *X is said to be "open" iff $\varkappa(X) = X$.*

3. *The intersection of all closed sets containing X, whenever it is a closed set, is called "$\varepsilon-$closure of X" and denoted by $\mathbb{C}_\varepsilon(X)$.*

4. *The union of all open sets contained in X, whenever it is an open set, is called "$\varkappa-$interior of X" and denoted by $\mathbb{I}_\varkappa(X)$.*

Proposition 12.4.9. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then, for any $X, Y \subseteq U$, if $\mathbb{I}_\varkappa(X)$ and $\mathbb{I}_\varkappa(Y)$ exist, then $\mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y) \subseteq X \cap Y$.*

Proof. From the very definition of this operator, $\mathbb{I}_\varkappa(X) \subseteq X$ and $\mathbb{I}_\varkappa(Y) \subseteq Y$. Hence $\mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y) \subseteq X \cap Y$.   QED

In general, the existence of the closure (of the interior) of a set X is not guaranteed, since it is not guaranteed, in a pre-topological space, that the intersection (union) of a family of closed (open) sets is a closed (open) set. In turn, this situation is related to the fact that in a generic pre-topological space, as we have seen, isotonicity fails for both ε and $\varkappa$.

In fact, assume that X and Y are open. By definition of a contraction map, $\varkappa(X \cup Y) \subseteq X \cup Y$, but although $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$ we have neither $X = \varkappa(X) \subseteq \varkappa(X \cup Y)$ nor $Y = \varkappa(Y) \subseteq \varkappa(X \cup Y)$. Therefore we cannot obtain the converse inclusion $X \cup Y = \varkappa(X) \cup \varkappa(Y) \subseteq \varkappa(X \cup Y)$. Hence, $X \cup Y$ may fail to be open since $\varkappa(X \cup Y)$ may be different from $X \cup Y$.[10] By duality we obtain that X and Y closed do not imply that $X \cap Y$ is closed.

Therefore, in pre-topological spaces, neighborhood systems are more important than open set systems.

Now we reveal the obvious fact that $\varkappa$ is the core map induced by a neighborhood system of type (at least) $\mathcal{N}_1$.

Definition 12.4.7. *Given a contraction $\varkappa : \wp(U) \longmapsto \wp(U)$, for any $x \in U$ the family*

$$\varkappa_x = \{Z \subseteq U : x \in \varkappa(Z)\}$$

is called the family of $\varkappa$-neighborhoods of x. We set $\mathcal{N}^\varkappa(U) = \{\varkappa_x\}_{x \in U}$ and call $\mathcal{N}^\varkappa(U)$ a $\varkappa-$neighborhood system.

---

[10]The reader is invited not to confuse the equation $\varkappa(X) \cup \varkappa(Y) = \varkappa(Y \cup Y)$, when both X and Y are open (hence $\varkappa(X) = X$ and $\varkappa(Y) = Y$), which is a situation that does not hold without the isotonicity law, with the same equation when X and Y are generic sets (not necessarily open), which may fail also in the presence of isotonicity.

Intuitively, by means of $\varkappa_x$ we obtain all the subsets Z such that x is strictly connected with some element of Z.

Proposition 12.4.10. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then,*

    1. The family $\mathcal{N}^\varkappa(U)$ is a neighborhood system of type $\mathcal{N}_1$.

    2. $\varkappa$ is the core map induced by $\mathcal{N}^\varkappa(U)$.

Proof. (**1**) If $Z \in \varkappa_x$, then $x \in \varkappa(Z) \subseteq Z$. Hence, $x \in Z$. Thus **N1** holds in $\mathcal{N}^\varkappa(U)$.
(**2**) $G(Z) = \{x : Z \in \varkappa_x\} = \{x : x \in \varkappa(Z)\} = \varkappa(Z)$. **0** and **1** follow from *Definition* 12.4.5.   QED

Conversely, in view of *Lemma* 12.4.1.(N1), we have:

Proposition 12.4.11. *Given a neighborhood system $\mathcal{N}(U)$ of type $\mathcal{N}_1$, the core map G induced by $\mathcal{N}(U)$ is a contraction operator, which is said to be* induced by $\mathcal{N}(U)$.

TERMINOLOGY AND NOTATION. If in a pre-topological space $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ the operator $\varkappa$ is induced by a neighborhood system $\mathcal{N}(U)$, then $\mathbf{P}$ itself is said to be induced by $\mathcal{N}(U)$.

Now we shall prove that a neighborhood system of type (at least) $\mathcal{N}_1$ induces a pre-topological space and, viceversa, that a pre-topological space induces an $\mathcal{N}_1$ neighborhood system.

Corollary 12.4.2. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space induced by a neighborhood system $\mathcal{N}(U)$ of type $\mathcal{N}_1$, then $\mathcal{N}^\varkappa(U) = \mathcal{N}(U)$.*

Proof. Immediate, from *Proposition* 12.4.10, *Definition* 12.4.7 and *Lemma* 12.4.1.(G2).

Proposition 12.4.12. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then, for any $X \subseteq U$, X is open if and only if X belongs to $\varkappa_x$ for any $x \in X$.*

Proof. If X is open, then $X = \varkappa(X)$. Hence for any element $x \in X, x \in \varkappa(X)$. It follows that X belongs to $\varkappa_x$. The converse is trivial in view of **N1**.   QED

Therefore, a set is open if and only if it is a neighborhood for all its own elements. But this is exactly what condition **Id** requires for $G(X)$

(alias $\varkappa(X)$), any $X \subseteq U$. Indeed, it is immediate to verify that a neighborhood system $\mathcal{N}(U)$ is of type $\mathcal{N}_{1Id}$ if and only if for any $X \subseteq U$, $G(X)$ is an open set.

REMARKS. The bi-implication of *Proposition* 12.4.12 holds because in every pre-topological space $\mathcal{N}^{\varkappa}(U)$ is a neighborhood system of at least type $\mathcal{N}_1$.

Example 12.4.8. A sample pre-topology
Consider the universe $U = \{a, b, c\}$. Suppose we are given the following contraction map:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U
$\varkappa(x)$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b\}$	U

If we compute the family $\mathcal{N}^{\varkappa}(U) = \{\varkappa_x\}_{x \in U}$, we obtain that $\mathcal{N}^{\varkappa}(U) = \mathcal{N}(U)$, where $\mathcal{N}(U)$ is the neighborhood system of *Example* 12.4.5 (for instance, $\varkappa_a = \{Z \subseteq U : a \in \varkappa(Z)\} = \{\{a, b\}, \{a, c\}, U\}$).

We know that this neighborhood system is of type $\mathcal{N}_{1Id}$ but not of type $\mathcal{N}_2$. Linked to this fact, we note that the contraction operator $\varkappa$ (i.e. G) is not isotone: $\{c\} \subseteq \{b, c\}$, but $\varkappa(\{c\}) = \{c\} \not\subseteq \{b\} = \varkappa(\{b, c\})$.

We note immediately that $\varkappa$ is a co-discontinuous contraction operator:
$\varkappa(\{a, b\}) \cap \varkappa(\{a, c\}) = \{a\} \neq \emptyset = \varkappa(\{a\}) = \varkappa(\{a, b\} \cap \{a, c\})$.

Given $\mathcal{N}^{\varkappa}(U)$ we can recover the contraction map $\varkappa$ using the equation $\varkappa(X) = \{z : X \in \varkappa_z\}$. Let us compute, for instance, $\varkappa(\{b, c\})$ and $\varkappa(\{a, b\})$:

$\varkappa(\{b, c\}) = \{x : \{b, c\} \in \varkappa_x\} = \{b\}$ (indeed, $\{b, c\}$ belongs only to $\varkappa_b$);

$\varkappa(\{a, b\}) = \{x : \{a, b\} \in \varkappa_x\} = \{a, b\}$ (indeed, $\{a, b\}$ belongs to $\varkappa_b$ and to $\varkappa_a$).

Example 12.4.9. Open sets
In the pre-topology $\langle U, \varkappa, \varepsilon \rangle$ of Example 12.4.8, the sets $\{b\}, \{c\}, \{a, b\}, \{a, c\}, U$ and $\emptyset$ are open, because they are fix points of the contraction map $\varkappa$. On the contrary, $\varkappa(\{b, c\}) = \{b\}$.

We have seen that a set X is open if it is a $\varkappa$-neighborhood of all its points; that is, if for any $x \in X, X \in \varkappa_x$. Therefore, we can verify also in this way that $\{b, c\}$ is not open: indeed, $\{b, c\} \notin \varkappa_c$. Moreover, the set $\{b, c\}$ does not have an interior: the set of all open subsets of $\{b, c\}$ is $\{\emptyset, \{b\}, \{c\}\}$ whose union is $\{b, c\}$ itself, which is not open.[11]

This example shows that in the above pre-topological space not every union of open sets is an open set: $\{b\}$ and $\{c\}$ are open; however, $\{b\} \cup \{c\} = \{b, c\}$ is not open.

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[11]One should not confuse the fact that a set X is a $\varkappa$-neighborhood of all its points, which is always true of open sets, with the existence of a subset Y of X such that X is a $\varkappa$-neighborhood of all the elements of Y, which is related to topological spaces – see further in the text.

Exercise 12.7. *Given a pre-topological space:*
(a) Is the family $\varkappa_x$ closed with respect to intersections (unions), for any x?
(b) Is the family $\varkappa_x$ closed with respect to supersets, for any x?
(c) Compute the table for the dual expansion map ε of the pre-topological space of Example 12.4.8. (i) Is this map continuous?
(ii) Find a subset of U which does not have a closure.

12.4.1 Excursus. Dynamics 1: The Failure of the Isotonicity Law

We want to recall again that the failure of the isotonicity law prevents us from using generic pre-topologies in order to provide the knowledge order embedded in a relation R over a universe U with a pre-topological interpretation, even if R is reflexive. In fact, for any $X \subseteq U$ we cannot coherently set $R(X) = \varepsilon(X)$ or $R(X) = \varkappa(X)$, because if $X \subseteq Y$ we have $R(X) \subseteq R(Y)$, but both $\varkappa(X) \subseteq \varkappa(Y)$ and $\varepsilon(X) \subseteq \varepsilon(Y)$ may fail.

Intuitively this difference reflects the fact that a single relation R on U is a static representation of the relationships between the elements of U, while ε and $\varkappa$ may account for a dynamic evaluation of these relationships. In fact, if $X \subseteq Y$ but $\varepsilon(X) \nsubseteq \varepsilon(Y)$ we can imagine a situation in which an element x of X fulfills a connection with some element x' as far as x is considered just within the set X (which is recorded by the fact $x' \in \varepsilon(X)$), but whenever x is associated, by expanding X to a superset Y by means of ε, with other elements outside of X, then the connection between x and x' is lost, because of a sort of incompatibility between x' and some new element in $Y \cap -X$.

Here is an example.

In Figure 12.3 we have a set $A = \{x, y, z\}$ and a superset of A, $B = \{x, y, z, w\}$. Assume that (i) y is connected with c, (ii) z is connected with b, (iii) w is connected with a, and (iv) that w and b are incompatible. If we expand A, we obtain $\varepsilon(A) = \{x, y, z, b, c\}$. But if we extend A to B, then b, which is incompatible with the new entry w, breaks its alliance with z. Therefore, the expansion of B will be $\varepsilon(B) = \{x, y, z, w, c, a\}$ which is not even comparable with $\varepsilon(A)$. Therefore, expansion is not a monotonic (isotonic) operator, in general.

Moreover, one might have the case in which z and w are incompatible. So that after enlarging A to B, the connection between z and the other elements of A is lost. Therefore, when we expand B to $\varepsilon(B)$ we
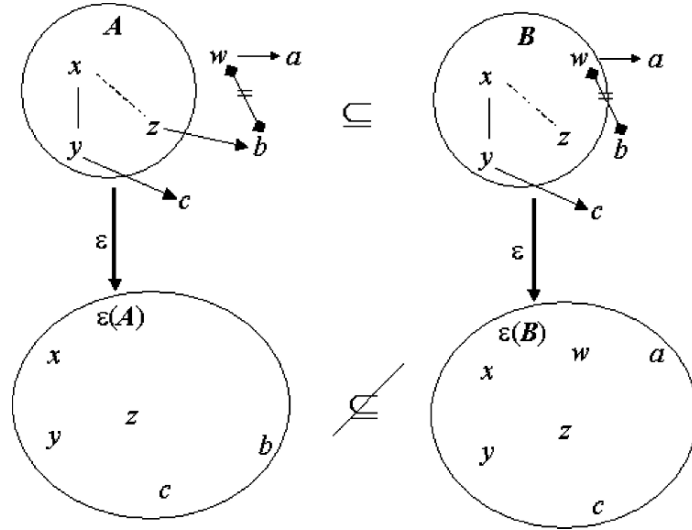
Figure 12.3: A non-isotonic expansion

obtain $\{x, y, z, c, w, a\}$, but if we contract $\varepsilon(B)$ we obtain a set in which z does not appear any longer. It follows that $\varkappa(\varepsilon(B))$ and B are not comparable, in this case. Differently, consider the fact that the relation $\mathbb{IC}(X) \supseteq X$ is always valid in topological spaces. For an extremely simple example of a pre-topology where $\varkappa(\varepsilon(X)) \not\supseteq X$, take $\langle U, \varepsilon, \varkappa \rangle$, where $U = \{a, \ldots\}, \varkappa(\{a\}) = \emptyset$ and $\varepsilon(\{a\}) = \{a\}$. In this pre-topology $\varkappa(\varepsilon(\{a\})) = \emptyset \not\supseteq \{a\}$.

As a concrete simple example, consider the following three binary tables:

R_1	a	b	c		R_2	a	b	c		R_3	a	b	c
a	1	1	0		a	1	0	1		a	1	0	1
b	0	1	0		b	0	1	0		b	1	1	0
c	0	0	1		c	0	0	1		c	0	0	1

Suppose this is the behaviour of the same relation R under different conditions. For instance $R_1(a)$ is the behaviour of R at point a when this element is taken alone and b and c are not considered together; $R_2(a)$ is the behaviour of R at point a when this element is taken alone and b and c are considered together, or when it is joined with b. $R_3(a)$ is the behaviour of R at point a when this element is taken jointly with c. Going on with this interpretation, we can see that for any x and $i \in \{1, 2, 3\}$, $R_i(x)$ is the behaviour of R at x for a certain

context. Here we display a possible context-sensitive set of evaluations, distinguishing "internal contexts" containing the elements to which R applies, and "external contexts" otherwise:

Internal contexts	External contexts	Element	Applicable version of R
$\{a\}$	$\{\{c\}, \{b\}\}$	a	R_1
$\{a\}$	$\{\{b, c\}\}$	a	R_2, R_3
$\{a, b\}$	$\{\{c\}\}$	a, b	R_2
$\{a, c\}$	$\{\{b\}\}$	a, c	R_3
$\{b\}$	$\{\{a\}, \{c\}\}$	b	R_1
$\{b\}$	$\{\{a, c\}\}$	b	R_3
$\{b, c\}$	$\{\{a\}\}$	b, c	R_2
$\{c\}$	$\{\{a\}, \{b\}, \{a, b\}\}$	c	R_1, R_2, R_3
$\{a, b, c\}$	$\emptyset$	a, b, c	R_1

Obviously, the fact that the behaviour of R changes along the contexts, does not make R- neighborhood formation an isotonic process. For instance, although $\{b\} \subseteq \{b, c\}$, if the external context of the evaluation of $R(\{b\})$ is $\{a, c\}$, then we have $R(\{b\}) = R_3(\{b\}) = \{a, b\}$. But the external context of evaluation of $R(\{b, c\})$ is $\{a\}$ so that $R(\{b, c\}) = R_2(\{b, c\}) = \{b, c\}$. Hence $R(\{b\}) \nsubseteq R(\{b, c\})$.

However, these three versions of R may represent other situations. For instance they could be the results of three surveys about the same relation R with respect to three different points in time t_1, t_2 and t_3.

Along this line of interpretation we shall develop interesting dynamic frameworks in information analysis, in which isotonicity is valid, although we have still to renounce other nice properties.

This point will be developed in Excursus 12.6.2 below. First, we have to introduce other kinds of pre-topologies.

To sum up, a dynamic analysis is required by two basic situations and a mixed one.

The first is when we fix the point in time and let the observation process depend on contexts:

$$R \text{ at point in time } t_x$$

$$\textit{Context } C_1 \to R_1 \quad \textit{Context } C_2 \to R_2 \quad \textit{Context } C_n \to R_n$$
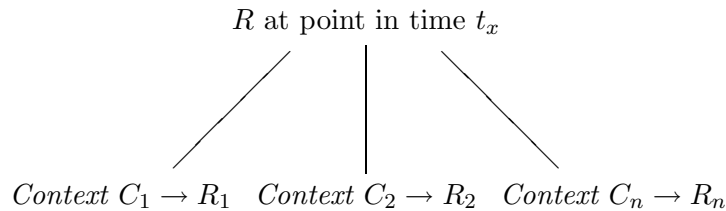
The second happens when we fix the context and let the observation vary over time;

$$R \text{ in context C}$$

$$time\ t_1 \rightarrow R_1 \quad time\ t_2 \rightarrow R_2 \quad time\ t_n \rightarrow R_m$$
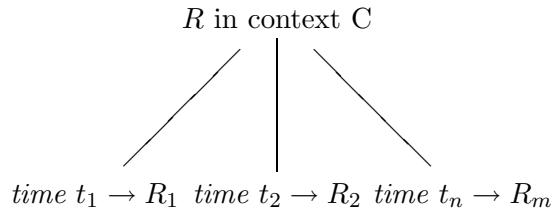
A third situation is given mixing the previous two:

$$R$$

$$point\ in\ time\ t_1 \quad point\ in\ time\ t_2 \quad point\ in\ time\ t_m$$

$$Context\ C_1 \rightarrow R_{1_1}\ Context\ C_2 \rightarrow R_{1_2}\ Context\ t_n \rightarrow R_{1_n}$$
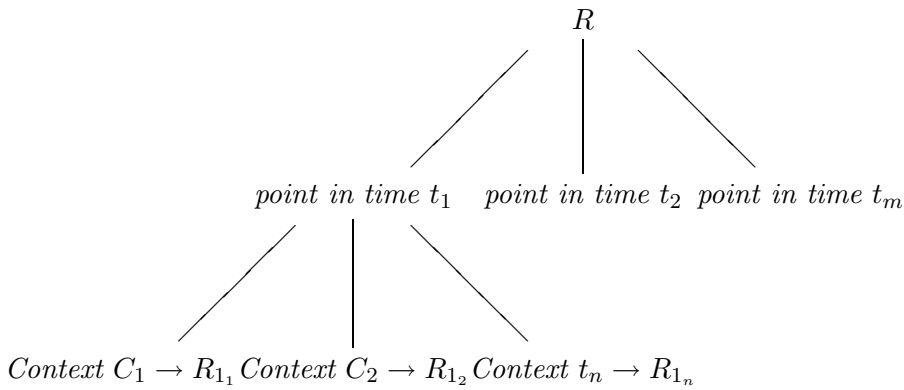
Classical Rough Set Theory does not account for this kind of dynamic phenomena. Indeed, as far as we are confined to a single Information System, we can deal just with a picture taken at a particular point in time and at a particular point in space (meaning that the picture fixes a situation in space and time). In this picture, relations are static and definite.

Dynamics can be taken into account if we consider possible evolutions of Information Systems over time and/or evolutions of these behaviours of the analysed elements.

As we shall see, in all these cases pre-topologies are useful in order to synthesize and represent evolution. For instance, we can think of a collection of Approximation Spaces with operations which are able to synthesise their different information.

12.5   Towards Topology 1

In what follows, we shall progressively impose new properties to a pre-topological space in order to encompass the features required by our analysis, like isotonicity and distribution.

In this journey we still use examples taken from the dynamic approach illustrated above, so that the reader will be able to appreciate where approaches which use topological concepts are positioned in data analysis, like Approximation Space.

Definition 12.5.1. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is said to be of type $\mathfrak{V}_{Id}$ if and only if for all $X \subseteq U$ the operator $\varkappa$ and ε are idempotent.*

It is easy to check that if one of the two conjugate operators is idempotent, so is the other.

Proposition 12.5.1. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is of type $\mathfrak{V}_{Id}$ if $\mathcal{N}^{\varkappa}(U)$ is of type $\mathcal{N}_{1Id}$.*

Proof. Immediate, from *Proposition* 12.4.5 and *Proposition* 12.4.10. QED

Notice that a pre-topological space of type $\mathfrak{V}_{Id}$ is much weaker than a topological space, although $\varkappa$ is idempotent. For instance $\varkappa$ is not required to be isotonic.

Definition 12.5.2. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is said to be of type $\mathfrak{V}_{I}$ if and only if for all $X, Y \subseteq U, X \subseteq Y$ implies $\varepsilon(X) \subseteq \varepsilon(Y)$.*

Proposition 12.5.2. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is of type $\mathfrak{V}_{I}$ if and only if for all $X, Y \subseteq U, X \subseteq Y$ implies $\varkappa(X) \subseteq \varkappa(Y)$.*

Therefore, a pre-topological space is of *type* $\mathfrak{V}_{I}$ if and only if its expansion and contraction operators are isotonic. And this happens if every $\varkappa-$neighborhood system is a proper filter:

Proposition 12.5.3. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then the following statements are equivalent:*

1. *$\mathbf{P}$ is of type $\mathfrak{V}_{I}$.*

2. *The family $\mathcal{N}^{\varkappa}(U)$ is a neighborhood system of type $\mathcal{N}_{2}$.*

3. *$\mathbf{P}$ is induced by a neighborhood system of type $\mathcal{N}_{2}$.*

*Proof.* Immediate, from *Lemma* 12.4.1 and *Proposition* 12.4.10.   QED

Given a neighborhood system of type $\mathcal{N}_{2}$ we can define a pre-topological space of type $\mathfrak{V}_{I}$ in a manner that will be recognised to be very familiar. Let us indeed define two new operators g and f on $\wp(U)$.

Definition 12.5.3. *Let $\mathcal{N}(U)$ be a neighborhood system on U. Let us set:*

1. *$g(X) = \{x \in U : \exists N(N \in \mathcal{N}_x \,\&\, N \subseteq X)\}$.*

2. *$f(X) = \{x \in U : \forall N(N \in \mathcal{N}_x \Rightarrow N \cap X \neq \emptyset)\}$.*

Clearly, the maps g and f are dual (the proof is left to the reader [*hints*: consider the set $-g(-X)$; so, as usual, apply in sequence the first order equivalences $\neg\forall x \equiv \exists x\neg$, $\neg(A \Rightarrow B) \equiv A \,\&\, \neg B$ and, finally, $\neg(Y \cap -X \neq \emptyset) \equiv Y \subseteq X$]). Moreover, these maps are weaker than G and, respectively, F, because, obviously, for all $X \in \wp(U)$, $G(X) \subseteq g(X)$ (since $X \subseteq X$)

On the basis of this definition we have:

Proposition 12.5.4. *Let U be a set. Let $\mathcal{N}(U)$ be a neighborhood system. Then the following are equivalent:*

1. **N2** *holds in $\mathcal{N}(U)$.*

2. *For any subset X of the universe, $g(X) = G(X)$ and $f(X) = F(X)$.*

Proof. Since $G(X) \subseteq g(X)$, $g(X) = G(X)$ if and only if $\forall X \subseteq U$, $\forall x \in U((\exists N \in \mathcal{N}_x \,\&\, N \subseteq X) \Rightarrow X \in \mathcal{N}_x)$, if and only if **N2** holds. Dually for F.   QED

So, in case of neighborhood systems of type $\mathcal{N}_2$ the vicinity map (the expansion operator) is defined in the usual topological way: a point x is close to a set X if and only if all the elements of $\mathcal{N}_x$ have non null intersection with X. A well-known intuitive picture is that displayed by Figure 12.4.
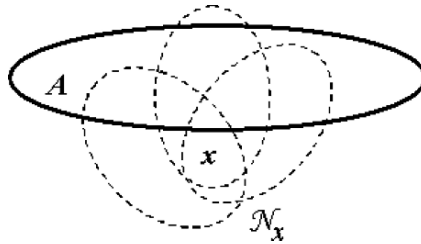
Figure 12.4: Point x is close to the set A because all of its neighborhoods have non empty intersections with A

The above definitions sound familiar to the reader, since it is obvious that as soon as we consider neighborhood systems $\mathcal{N}(U)$ such that for all x, $\mathcal{N}_x$ is a filter with a least element $l(x)$, then (1) and (2) of *Definition* 12.5.3 turn into: (1') $g(X) = \{x : l(x) \subseteq X\}$ and (2') $f(X) = \{x \in U : l(x) \cap X \neq \emptyset\}$.

Thus when this least element is an equivalence class $[x]_{\approx}$, we obtain the definitions of upper and, respectively, lower approximations.

However, from this discussion it follows that a pre-topological space must be equipped with additional structural properties in order to exhibit the characteristics of Approximation Spaces.

Proposition 12.5.5. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space of type $\mathfrak{V}_I$. Then,*

1. *If $\{O_i\}_{i \in I}$ is a family of open sets, then $\bigcup\limits_{i \in I} \{O_i\}$ is open.*

2. *If $\{C_i\}_{i \in I}$ is a family of closed sets, then $\bigcap\limits_{i \in I} \{C_i\}$ is closed.*

3. *U and $\emptyset$ are both closed and open.*

Proof. (**1**) Let $\{O_i\}_{i \in I}$ be a family of open sets. For any element $x \in \bigcup\limits_{i \in I} O_i$ there is a $j \in I$ such that $x \in O_j$. But O_j is open, hence $O_j = x \in \varkappa(O_j)$. Since $O_j \subseteq \bigcup\limits_{i \in I} O_i$, by isotonicity we have $\varkappa(O_j) \subseteq \varkappa(\bigcup\limits_{i \in I} O_i)$ so that $x \in \varkappa(\bigcup\limits_{i \in I} O_i)$. Therefore, for all $x \in \bigcup\limits_{i \in I} O_i$ we have $\bigcup\limits_{i \in I} O_i \in \varkappa_x$ and from *Proposition* 12.4.12 we obtain the result. (**2**) By duality. (**3**) Left to the reader. QED

Corollary 12.5.1. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space of type $\mathfrak{V}_I$. Then for any $X \subseteq U$, the closure $\mathbb{C}_{\varepsilon}(X)$ and the interior $\mathbb{I}_{\varkappa}(X)$ always exist.*

Another way to interpret the above result is that isotonicity implies a sort of fix-point property.

Although pre-topological spaces are spaces endowed with a rather rich structure, nevertheless, they are not able to completely account for the geometrical features of relational spaces. Let us consider again the cause of this limitation.

On the one hand, a pre-topological space provides a decreasing map, $\varkappa$, and an increasing map, ε, while $R-$neighboring is neither, for an arbitrary relation R. It is a decreasing map only if R is reflexive.

On the other hand, $R-$neighboring distributes both on unions and intersection, while this feature is not standard for generic pre-topological spaces.

So, let us now analyse the meaning of reflexivity and distribution.

12.5.1 Excursus: Reflexivity, Distribution and Perception

Since we are interested in how conceptual patterns are formed around the perception of a "point" x (item, object, stimulation, event, ...), we can assume that if a conceptual pattern is induced by a process π that gathers together all the "points" that are related with the given perceived "point" x, then x should belong to the result $\pi(x)$ of this process. Otherwise we should admit, rather metaphysically, that some *phenomena* appear to our consciousness by means of perceptions related with something which still remains a *noumenon* and not a part of the induced phenomena (see Figure 12.5).
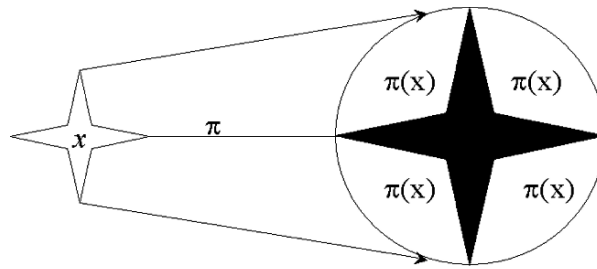
Figure 12.5: A non-reflexive phenomenological process may induce a phenomenon in which it partially or totally disappears

In order to avoid this metaphysical drawback, we can assume reflexivity on a quite intuitive basis.

As for distribution, we can have different attitudes. As a matter of facts, there is no evidence for claiming that if a phenomenon P_1 is the result of an inflationary (i.e. increasing) process π applied to "point" x, $P_1 = \pi(x)$, and P_2 is the result of the same process applied to "point" y, $P_1 = \pi(y)$, then the result of the application of π to both points, $\pi(x+y)$, is $P_1 + P_2$. Indeed, the two points taken together could carry more information than the sum of the two pieces of information carried by the two "points" singularly taken. Proximity Spaces and Concept Lattices are good examples of this situation (see Part I). On the contrary, the classical upper approximation in Rough Set Theory

is additive. Additivity is a symptom of phenomena that fulfill some compositional property, in the sense that our ideal process π is additive: $\pi(x + y) = \pi(x) + \pi(y)$ (or, in a set-theoretical framework, $\pi(\{x\} \cup \{y\}) = \pi(\{x\}) \cup \pi(\{y\})$).

Moreover, one might wonder if it is possible to have an inflationary and distributive map avoiding isotonicity, i.e. monotonicity. We have already seen that this is not possible: we can have inflationary isotonic maps that are not additive. However, if a map is additive, then it is isotonic with respect to the lattice order.

12.6   Towards Topology 2

So, we have done a step towards the direction of relational spaces (Kripke frames) and Rough Sets by means of the concept of a pre-topological space of type $\mathfrak{V}_I$. Now we shall go further ahead, in order to grasp the distribution features.

Definition 12.6.1. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is said to be of type $\mathfrak{V}_D$ if and only if for all $X, Y \subseteq U, \varepsilon(X \cup Y) = \varepsilon(X) \cup \varepsilon(Y)$.*

Proposition 12.6.1. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is of type $\mathfrak{V}_D$ if and only if for all $X, Y \subseteq U, \varkappa(X \cap Y) = \varkappa(X) \cap \varkappa(Y)$.*

We have already seen that these distribution laws implies the isotonicity law. So any pre-topological space of type $\mathfrak{V}_D$ is also of type $\mathfrak{V}_I$. However the converse implication is not valid, as we can see in Example 12.6.2 below.

It is possible to prove that in order for a pre-topology to be of type $\mathfrak{V}_D$, the structure of the $\varkappa$-neighborhoods of any element of U must be a filter and not only an order filter. That is, if X and Y belongs to $\varkappa_x$, any x, then $X \cap Y$ must belong to $\varkappa_x$, too:

Proposition 12.6.2. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then the following statements are equivalent:*

1. $\mathbf{P}$ *is of type $\mathfrak{V}_D$.*

2. *The family $\mathcal{N}^{\varkappa}(U)$ is a neighborhood system of type $\mathcal{N}_3$.*

3. $\mathbf{P}$ *is induced by a neighborhood system of type $\mathcal{N}_3$.*

Proof. Immediate, from *Lemma* 12.4.1, *Proposition* 12.4.10 and *Corollary* 12.4.2.   QED
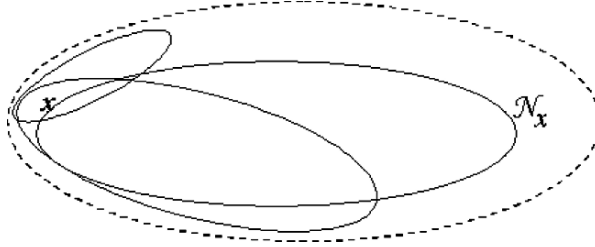
Figure 12.6: In a neighborhood system of type $\mathcal{N}_3$, the elements of the neighborhood family of any point x form a filter with respect to the relation $\subseteq$

Proposition 12.6.3. *Let $\langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space of type $\mathfrak{V}_D$. Then,*

1. *If $\{O_i\}_{i \in I}$ is a finite family of open sets, then $\bigcap_{i \in I} \{O_i\}$ is open.*

2. *If $\{C_i\}_{i \in I}$ is a finite family of closed sets, then $\bigcup_{i \in I} \{C_i\}$ is closed.*

3. *For any $X, Y \subseteq U$: $\mathbb{I}_\varkappa(X \cap Y) = \mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y)$.*

4. *For any $X, Y \subseteq U$: $\mathbb{C}_\varepsilon(X \cup Y) = \mathbb{C}_\varepsilon(X) \cup \mathbb{C}_\varepsilon(Y)$.*

Proof. (**1**) If A and B are open sets, then $\varkappa(A) = A$ and $\varkappa(B) = B$. Since $\langle U, \varepsilon, \varkappa \rangle$ is of type $\mathfrak{V}_D$, $\varkappa(A \cap B) = \varkappa(A) \cap \varkappa(B) = A \cap B$. It follows that $A \cap B$ is an open set. (**2**) Dually. (**3**) From the first statement we have that for any $X, Y \subseteq U$, $\mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y)$ is an open set; moreover, from *Proposition* 12.4.9 we obtain that $\mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y)$ is an open set included in $X \cap Y$. On the other hand, since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we have (i): $\mathbb{I}_\varkappa(X \cap Y) \subseteq \mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y)$. But $\mathbb{I}_\varkappa(X \cap Y)$ is the largest open set included in $X \cap Y$, so that we have (ii): $\mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y) \subseteq \mathbb{I}_\varkappa(X \cap Y)$. We conclude from (i) and (ii) that $\mathbb{I}_\varkappa(X) \cap \mathbb{I}_\varkappa(Y) = \mathbb{I}_\varkappa(X \cap Y)$. (**4**) From duality.   QED

Therefore, a pre-topological space of type $\mathfrak{V}_D$ features properties very close to those that characterise topological spaces. The remaining difference is that in a pre-topological space of type $\mathfrak{V}_D$ the two maps $\varkappa$ and ε are not required to be idempotent. Anyway, before adding the remaining clause and obtaining topological spaces, we have to introduce a new element to the taxonomy of pre-topological spaces, that will make it possible to associate reflexive relations with them.

Definition 12.6.2. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ is said to be of type $\mathfrak{V}_S$, or an* Alexandroff *pre-topological space, if and only if for all $X \subseteq U, \varepsilon(X) = \bigcup_{x \in X} \varepsilon(\{x\})$.*

A pre-topological space is of type $\mathfrak{V}_S$ only if any $\varkappa-$ neighborhood system is a principal filter.

Proposition 12.6.4. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then the following are equivalent:*

1. *$\mathbf{P}$ is of type $\mathfrak{V}_S$.*

2. *$\mathbf{P}$ is of type $\mathfrak{V}_I$ and for any x, for any family $\{X_i\}_{i \in I}$ of elements of $\varkappa_x$, $\bigcap_{i \in I} X_i \in \varkappa_x$.*

3. *The family $\mathcal{N}^\varkappa(U)$ is a neighborhood system of type $\mathcal{N}_4$.*

4. *$\mathbf{P}$ is induced by a neighborhood system of type $\mathcal{N}_4$.*

Proof. Immediate, from *Lemma* 12.4.1, *Proposition* 12.4.10 and *Corollary* 12.4.2.   QED

Therefore, if U is finite, then the notions of $\mathfrak{V}_S$ and $\mathfrak{V}_D$ pre-topological spaces coincide.

Example 12.6.1. A pre-topological space not of type $\mathfrak{V}_I$
We have seen that in the pre-topological space of *Example* 12.4.8, $\varkappa$ is idempotent but not isotonic.

Example 12.6.2. A pre-topological space in which $\varkappa$ is isotonic and idempotent but not multiplicative
We show that $\varkappa-$distributivity is independent of $\varkappa-$isotonicity and $\varkappa-$idempotence.
     Consider the pre-topology $\mathbf{P}_1 = \langle U, \varepsilon, \varkappa \rangle$ such that $U = \{a, b, c\}$ and $\varkappa$ and ε are given by the following table:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varepsilon(x)$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	U	$\{a,c\}$	$\{b,c\}$	U
$\varkappa(x)$	$\emptyset$	$\{a\}$	$\{b\}$	$\emptyset$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U

By easy inspection we can verify that both $\varkappa$ and ε are isotonic. However,

- $\varepsilon(\{a\}) \cup \varepsilon(\{b\}) = \{a\} \cup \{b\} = \{a,b\} \neq U = \varepsilon(\{a,b\}) = \varepsilon(\{a\} \cup \{b\})$.

- $\varkappa(\{a,c\}) \cap \varkappa(\{b,c\}) = \{a,c\} \cap \{b,c\} = \{c\} \neq \emptyset = \varkappa(\{c\}) = \varkappa(\{a,c\} \cap \{b,c\})$.

Indeed, $\varkappa_c = \{\{a,c\}, \{b,c\}, U\}$ is an order filter but not a filter because $\{a,c\} \cap \{b,c\} \notin \varkappa_c$.

It should be noticed, moreover, that the family $\{\varkappa_x\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$. We can conclude that adding **Id** to **N2** does not say anything about **N3**.

Example 12.6.3. Contraction operators and order filters
We have seen that in the pre-topology $\mathbf{P}_1$ above, $\varkappa_c$ is an order filter but not a filter. This is the reason why the cocontinuity law fails when $\varkappa$ is applied to the intersection of $\{a,c\}$ and $\{b,c\}$, and the continuity law fails when ε is applied to their complements, $\{b\}$ and respectively $\{a\}$.

The family of $\varkappa$-neighborhoods is:

x	a	b	c
$\varkappa_x$	$\{\{a\}, \{a,b\}, \{a,c\}, U\}$	$\{\{b\}, \{a,b\}, \{b,c\}, U\}$	$\{\{a,c\}, \{b,c\}, U\}$

Notice that $\varkappa_a$ and $\varkappa_b$, incidentally, are filters. However, $\{\varkappa_x\}_{x \in U}$ is not a neighborhood system of type $\mathcal{N}_3$ because of $\varkappa_c$.

Example 12.6.4. A pre-topological space **P** where $\varkappa$ is idempotent but **P** is not of type $\mathfrak{V}_I$: $\varkappa$-isotonicity is independent of $\varkappa$-idempotence
A simple example is given by the pre-topological space of Example 12.4.8

Example 12.6.5. A pre-topological space **P** where any intersection of two open subsets is open, but not of type $\mathfrak{V}_D$
Consider the neighborhood system $\mathcal{N}(U)$

x	a	b	c
$\mathcal{N}_x$	$\{\{a,b\}, \{a,b,c\}\}$	$\{\{b,c\}, \{a,b\}, \{a,b,c\}\}$	$\{\{a,b,c\}\}$

The open subsets are $\emptyset, \{a,b\}$ and $\{a,b,c\}$, and it is easy to check that they are closed under intersection. However, $\mathcal{N}_b$ is not a filter; thus $\mathcal{N}(U)$ is not of type $\mathfrak{V}_D$.

12.6.1 Bases

Definition 12.6.3. *Given a pre-topological space* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$*, the family* $\Omega_\varkappa(U) = \{\varkappa(A)\}_{A \subseteq U}$ *will be called the* pre-topology *of* U*.*

TERMINOLOGY AND NOTATION. From now on given a pre-topological space $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$, with the symbol **P** we shall mean, whenever convenient and appropriate in the context, also its pre-topology $\Omega_\varkappa$.

Definition 12.6.4. *Let* $\mathbf{P_1} = \langle U, \varepsilon', \varkappa' \rangle$ *and* $\mathbf{P_2} = \langle U, \varepsilon'', \varkappa'' \rangle$ *be two pre-topological spaces on the same universe* U*. Then we say that (the pre-topology of)* $\mathbf{P_1}$ *is finer than (the pre-topology of)* $\mathbf{P_2}$ *(or* $\mathbf{P_2}$ *is coarser than* $\mathbf{P_1}$*), in symbols* $\mathbf{P_2} \trianglelefteq \mathbf{P_1}$*, if for any* $X \subseteq U$*,* $\varkappa''(X) \subseteq \varkappa'(X)$*.*

Proposition 12.6.5. *Given two pre-topological spaces* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *on the same universe* U, $\mathbf{P_2} \trianglelefteq \mathbf{P_1}$ *if and only if* $\varkappa''_x \subseteq \varkappa'_x$, *any* $x \in U$.

Proof. Suppose $\varkappa''_x \subseteq \varkappa'_x$ and $x \in \varkappa''(Z)$. Hence $Z \in \varkappa''_x$ so that $Z \in \varkappa'_x$, too. It follows that $x \in \varkappa'(Z)$. Conversely, if $\varkappa''_x \not\subseteq \varkappa'_x$ there is an $F \in \varkappa''_x$ such that $F \notin \varkappa'_x$. Hence $x \in \varkappa''(F)$ but $x \notin \varkappa'(F)$. It follows that $\varkappa''(X) \not\subseteq \varkappa'(X)$.   QED

TERMINOLOGY AND NOTATIONS. If $\mathcal{X}$ is a family of subsets of a given set U, then by $\Uparrow \mathcal{X}$ we shall denote the set $\{Y \subseteq U : \exists X(X \in \mathcal{X} \ \& \ X \subseteq Y)\}$ (the order filter generated by $\mathcal{X}$ in $\wp(U)$: $\Uparrow \mathcal{X} = \{\uparrow_{\subseteq} X : X \in \mathcal{X}\}$).

The following definition and properties will be useful.

Definition 12.6.5. *Let* U *be a set,* $\mathcal{F}$, $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *order filters or filters of elements of* $\wp(U)$. *Moreover let* $\mathcal{B}$, $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be families of subsets of* U. *Then:*

1. *If* $\mathcal{F} = \Uparrow \mathcal{B}$, *then* $\mathcal{B}$ *is called a* basis *for* $\mathcal{F}$ *and we say that* $\mathcal{B}$ *induces* $\mathcal{F}$. *We call a collection* $\mathfrak{B} = \{\mathcal{B}_i\}_{i \in I}$ *of bases, a* basis system.

2. *If* $\mathcal{F}_1 \subseteq \mathcal{F}_2$, *then* $\mathcal{F}_2$ *is said to be a* finer *filter than* $\mathcal{F}_1$.

Proposition 12.6.6. *Let* $\mathcal{B}_1, \mathcal{B}_2 \subseteq \wp(U)$, $\mathcal{F}_1 = \Uparrow \mathcal{B}_1, \mathcal{F}_2 = \Uparrow \mathcal{B}_2$ *and* $\mathcal{B}_2 \subseteq \mathcal{B}_1$. *Then* $\mathcal{F}_1$ *is finer than* $\mathcal{F}_2$.

The converse of the above *Proposition*, generally does not hold. Consider, indeed, $U = \{a, b, c\}$, $\mathcal{B}_1 = \{\{a\}\}$, $\mathcal{B}_2 = \{\{a, b\}\}$. Then, $\Uparrow \mathcal{B}_2 \subseteq \Uparrow \mathcal{B}_1$ although $\mathcal{B}_1 \not\subseteq \mathcal{B}_2$.

Corollary 12.6.1. *Let* $\mathcal{B} \subseteq \wp(U)$ *and* $\mathcal{F} = \Uparrow \mathcal{B}$. *Then,*
   $\bigcap \mathcal{F} = \bigcap \mathcal{B} \in \mathcal{B}$ *(i.e.* $\bigcap \mathcal{B} \in \mathcal{F}$).

Proof. If $\mathcal{F}$ is a filter and $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. So $\bigcap \mathcal{F} \in \mathcal{F}$. Clearly, for any $X \in \mathcal{F}$, $\bigcap \mathcal{F} \subseteq X$. Therefore, if $\mathcal{F} = \Uparrow \mathcal{B}$, then $\bigcap \mathcal{F} \subseteq B$, for any $B \in \mathcal{B}$. It follows that $\bigcap \mathcal{F} = \bigcap \mathcal{B}$.   QED

In view of *Definition* 12.5.4, if we are given a family of filters induced by a basis system, then in order to compute $\varepsilon(X)$ and $\varkappa(X)$ it is sufficient to consider the bases:

Proposition 12.6.7. *Let* U *be a set. Let* $\mathcal{N}(U)$ *be a neighborhood system of type (at least)* $\mathcal{N}_2$ *and* $\mathcal{N}'(U)$ *a neighborhood system of type*

$\mathcal{N}_4$. *Assume that, for any x, $\mathcal{N}_x = {\Uparrow}\, \mathcal{B}_x$ for some $\mathcal{B}_x \subseteq \wp(U)$ and $\mathcal{N}'_x = {\Uparrow}\, \{Q_x\}$ for some $Q_x \subseteq U$. Then for any $X \subseteq U$ the following equations hold:*

1. $\{x \in U : \exists N(N \in \mathcal{N}_x\ \&\ N \subseteq X)\} =$
   $\{x \in U : \exists A(A \in \mathcal{B}_x\ \&\ A \subseteq X)\}.$

2. $\{x \in U : \forall N(N \in \mathcal{N}_x \Rightarrow N \cap X \neq \emptyset) =$
   $\{x \in U : \forall A(A \in \mathcal{B}_x \Rightarrow A \cap X \neq \emptyset)\}.$

3. $\{x \in U : \forall N'(N' \in \mathcal{N}'_x \Rightarrow N' \cap X \neq \emptyset)\} = \{x \in U : Q_x \cap X \neq \emptyset\}.$

4. $\{x \in U : \exists N'(N' \in \mathcal{N}'_x\ \&\ N' \subseteq X)\} = \{x \in U : Q_x \subseteq X\}.$

Proof. (**1**): Since $\mathscr{F}_x = {\Uparrow}\, \mathcal{B}_x$, if N is such that $N \in \mathcal{N}_x$ and $N \subseteq X$, then there is a $A \in \mathcal{B}_x$ such that $A \subseteq N \subseteq X$. Therefore, since $A \in \mathcal{N}_x$, A itself satisfies the condition of the right term of the equation. The converse is trivial. (**2**) If $X \cap A \neq \emptyset$ for $A \in \mathcal{B}_x$, then $X \cap F \neq \emptyset$ for any $F \supseteq A$. On the other hand, if $X \cap F \neq \emptyset$ for any $F \in \mathcal{N}_x$, then this holds of any $A \in \mathcal{B}_x$. (**3**) Trivially because the left part of the equation reduces to $\{x \in U : \bigcap \mathcal{N}_x \cap X \neq \emptyset)\}$ and $\bigcap \mathcal{N}_x = Q_x$, because $\mathcal{N}'_x = {\Uparrow}\, \{Q_x\}$. (**4**) Trivial, because Q_x is the least element of $\mathcal{N}'_x$.   QED

Definition 12.6.6. *Let $\mathcal{N}(U)$ be a neighborhood system of type at least $\mathcal{N}_2$, $\mathfrak{B} = \{\mathcal{B}_x\}_{x \in U} \subseteq \wp(\wp(I))$ and $\mathcal{N}(U) = \{{\Uparrow}\, \mathcal{B}_x\}_{x \in U}$. If a pre-topological space $\mathbf{P}$ is induced by $\mathcal{N}(U)$, then we say that it is induced by $\mathfrak{B}$, too, and that $\mathfrak{B}$ is a* basis *for $\mathbf{P}$. In this case to define $\varkappa$ and ε we shall also use the right side of the equations (1) and, respectively, (2) of Proposition 12.6.7 above.*

Trivially we have:

Proposition 12.6.8. *In any neighborhood system induced by a basis, 1 and $\mathbf{N2}$ hold.*

Example 12.6.6. From order filters to contraction operators
Given a neighborhood system of type $\mathcal{N}_2$, we can recover the contraction operator of a pre-topological space of type $\mathfrak{V}_I$ by means of the equations of *Proposition* 12.6.7.

Consider the family of neighborhood system $\mathcal{N}(U)$ on $U = \{a, b, c\}$ given by $\mathcal{N}_a = \{\{a\}, \{a, b\}, \{a, c\}, U\}$, $\mathcal{N}_b = \{\{a, b\}, U\}$ and $\mathcal{N}_c = \{\{b, c\}, \{a, c\}, U\}$.

Each neighborhood family is an order filter. Thus $\mathcal{N}(U)$ is of type $\mathcal{N}_2$ and it is induced by the basis $\mathfrak{B} = \{\mathcal{B}_a = \{\{a\}\}, \mathcal{B}_b = \{\{a, b\}\}, \mathcal{B}_c = \{\{a, c\}, \{b, c\}\}\}$.

Let us compute $\varkappa(\{a, b\})$:

(a) $\varkappa(\{a, b\}) = \{x : \exists A(A \in \mathcal{B}_x \ \& \ A \subseteq \{a, b\})\}$:

(a.1) a is OK: $\{a\} \in \mathcal{B}_a$ and $\{a\} \subseteq \{a, b\}$.

(a.2) b is OK: $\{a, b\} \in \mathcal{B}_b$ and $\{a, b\} \subseteq \{a, b\}$.

(a.3) c is not OK: none member of $\mathcal{B}_c$ is included in $\{a, b\}$.

Hence, $\varkappa(\{a, b\}) = \{a, b\}$.

Let us compute $\varkappa(\{c\})$: no element of $\mathcal{B}_a$, $\mathcal{B}_b$ or $\mathcal{B}_c$ is included in $\{c\}$; hence $\varkappa(\{c\}) = \emptyset$.

Exercise 12.8.

(a) Give an example of a pre-topological space not of type $\mathfrak{V}_I$ where $\{x : A \in \varkappa_x\} \neq \{x : \exists X(X \in \varkappa_x \ \& \ X \subseteq A)\}$.

(b) Exploiting Proposition 12.6.7.(ii), compute $\varepsilon(X)$ for any $X \subseteq U$ in the pre-topological space $\mathbf{P}_1$ of Example 12.6.2 above.

(c) Compute a minimal basis for the pre-topology $\mathbf{P}_1$.

(d) Find a minimal binary relation $R \subseteq U \times X$, for some set X, such that the expansion map ε of $\mathbf{P}_1$ coincides with the Galois closure operator on $\wp(U)$, modulo R.

Although bases are enough, in many examples below we shall also show the entire family of filters or order filters inducing a pre-topology.

Proposition 12.6.9. *If a pre-topological space $\mathbf{P}$ is induced by a basis $\mathfrak{B} = \{\mathcal{B}_x\}_{x \in U}$ such that for any $x \in U$, $\mathcal{B}_x$ is a singleton, then $\mathbf{P}$ is of type $\mathfrak{V}_S$.*

Proof. If $\mathcal{B}_x = \{X\}$, then $\Uparrow \mathcal{B}_x = \uparrow X = \{Y \subseteq U : X \subseteq Y)\}$, which is obviously a filter, because $\bigcap \uparrow X = X$ and $X \in \uparrow X$. Therefore, from *Corollary* 12.4.2 we have the result.   QED

12.6.2   Excursus. Dynamics 2: The Failure of the Distributivity Laws

In Excursus 12.4.1, we have seen that dynamics and monotonicity may conflict. Here we shall exhibit examples of dynamic data analysis where monotonicity holds. However, distributivity laws fails to hold because of the intrinsic mechanism of these dynamic analyses.

Suppose we are given a universe U and a system of n binary relations on U, $\mathfrak{R} = \{R_i\}_{1 \leq i \leq n}$.

As we have seen, we can think of $R_1, R_2, \ldots, R_n$ as the results of n surveys about the same relation R with respect to n different points

of time $t_1, t_2, \ldots, t_n$, respectively, or surveys about n different criteria $C_1, C_2, \ldots, C_n$, respectively.

Definition 12.6.7. *Let U be a set and $\{R_i\}_{i \in I}$ a family of binary relations on U. Then the pair $\langle U, \{R_i\}_{i \in I} \rangle$ is called a* Dynamic Relational System.

If each R_i is reflexive, then we can use pre-topology to develop interesting information analyses in which the pre-topological operators are isotonic, although we have still to renounce other nice properties, such as $\varkappa$-cocontinuity and ε-continuity.

Let us list $n \times 2$ basic "use cases" of the above surveys. Given a subset A of U, we have n use cases involving the expansion process, and n use cases involving the contraction process:

1. (*Contraction*): We say that $x \in \varkappa^m(A)$, for $1 \le m \le n$, if every y such that $\langle x, y \rangle \in R_i$ belongs to A, at least in m cases. Otherwise stated: $x \in \varkappa^m(A)$ if $R_{1 \le i \le n}(x) \subseteq A$ for at least m indices. So, for instance, assume $n = 3$, then $x \in \varkappa^2(A)$ if $R_1(x) \subseteq A$ and $R_2(x) \subseteq A$, or if $R_1(x) \subseteq A$ and $R_3(x) \subseteq A$, or if $R_2(x) \subseteq A$ and $R_3(x) \subseteq A$ (i.e. if $R_1(x) \cup R_2(x) \subseteq A$, or $R_1(x) \cup R_3(x) \subseteq A$, or $R_2(x) \cup R_3(x) \subseteq A$).

2. (*Expansion*): We say that $x \in \varepsilon^m(A)$, for $1 \le m \le n$, if A contains at least a y such that $\langle x, y \rangle \in R_i$ in at least $n + 1 - m$ cases. Otherwise stated: $x \in \varepsilon^m(A)$ if $R_{1 \le i \le n}(x) \cap A \ne \emptyset$ for at least $n + 1 - m$ indices. So, for instance, assume $n = 3$, then $x \in \varepsilon^3(A)$ if $R_1(x) \cap A \ne \emptyset$, or $R_2(x) \cap A \ne \emptyset$, or $R_3(x) \cap A \ne \emptyset$ (i.e. if $(R_1(x) \cup R_2(x) \cup R_3(x)) \cap A \ne \emptyset$).

According to these use cases, we can compute the families of expansion and contraction operators, $\varepsilon^{1 \le m \le n}$ and $\varkappa^{1 \le m \le n}$, by transforming the various R_i-neighborhoods into appropriate bases and applying eventually *Proposition* 12.6.7:

Definition 12.6.8. *Let U be a set and let $\mathfrak{R} = \{R_i\}_{1 \le i \le n}$ be a system of n binary reflexive relations on U. For $1 \le m \le n$, let Γ^m be the family of combinations of m elements out of a set of n elements, γ a combination from Γ^m. Then let us set:*

1. $\varepsilon^m : \wp(U) \longmapsto \wp(U); \varepsilon^m(A) = \{x \in U : \forall F(F \in \mathscr{F}_x^m \Rightarrow F \cap A \ne \emptyset)\}$.

2. $\varkappa^m : \wp(U) \longmapsto \wp(U); \varkappa^m(A) = \{x \in U : \exists F(F \in \mathscr{F}_x^m \ \& \ F \subseteq A)\}$,

where: $\mathscr{F}_x^m$ is the (order) filter induced by the basis $\mathcal{B}_x^m$, and $\mathcal{B}_x^m = \{X_\gamma : X_\gamma = \bigcup_{l \in \gamma} R_l(x)\}_{\gamma \in \Gamma^m}$.

Proposition 12.6.10. *Let $\mathfrak{R}$ be a system of n reflexive binary relations on a set U. Then, for each $m, 1 \le m \le n$, $\langle U, \varkappa^m, \varepsilon^m \rangle$ is a pre-topological space of type $\mathfrak{V}_I$.*

The proof is immediate. In fact, from *Proposition* 12.6.8, **1** and **N2** hold in $\mathcal{N}^{\varkappa^m}$. Moreover, **Id** and **0** hold because all relations in $\mathfrak{R}$ are reflexive.

Let us apply all the above definitions to a simple example.

Consider the Dynamic Relational System $\langle U, \{R_1, R_2, R_3\}\rangle$, where $U = \{a, b, c\}$ and R_1, R_2 and R_3 are the relations from the example of Excursus 12.4.1.

In view of the above definitions we have:

m	Γ^m	$\mathcal{B}_x^m$
1	$\{\{1\}, \{2\}, \{3\}\}$	$\{R_1(x), R_2(x), R_3(x)\}$
2	$\{\{1,2\}, \{1,3\}, \{2,3\}\}$	$\{R_1(x) \cup R_2(x), R_1(x)$ $\cup R_3(x), R_2(x) \cup R_3(x)\}$
3	$\{\{1,2,3\}\}$	$\{R_1(x) \cup R_2(x) \cup R_3(x)\}$

In the following tables we show the basis $\mathfrak{B}^m(U) = \{\mathcal{B}_x^m\}_{x \in U}$, the induced neighborhood system $\mathfrak{F}^m(U) = \{\mathscr{F}_x^m\}_{x \in U}$, and, finally the operators ε^m and $\varkappa^m$:

$\mathcal{B}_x^m$	$\mathcal{B}_a^m$	$\mathcal{B}_b^m$	$\mathcal{B}_c^m$
$\mathcal{B}_x^1$	$\{\{a,b\}, \{a,c\}\}$	$\{\{b\}, \{a,b\}\}$	$\{\{c\}\}$
$\mathcal{B}_x^2$	$\{\{a,c\}, U\}$	$\{\{b\}, \{a,b\}\}$	$\{\{c\}\}$
$\mathcal{B}_x^3$	$\{U\}$	$\{\{a,b\}\}$	$\{\{c\}\}$

$\mathscr{F}_x^m$	$\mathscr{F}_a^m$	$\mathscr{F}_b^m$	$\mathscr{F}_c^m$
$\mathscr{F}_x^1$	$\{\{a,b\}, \{a,c\}, U\}$	$\{\{b\}, \{a,b\}, \{b,c\}, U\}$	$\{\{c\}, \{a,c\}, \{b,c\}, U\}$
$\mathscr{F}_x^2$	$\{\{a,c\}, U\}$	$\{\{b\}, \{a,b\}, \{b,c\}, U\}$	$\{\{c\}, \{a,c\}, \{b,c\}, U\}$
$\mathscr{F}_x^3$	$\{U\}$	$\{\{a,b\}, U\}$	$\{\{c\}, \{a,c\}, \{b,c\}, U\}$

ε^m	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
ε^1	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	U	U
ε^2	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,c\}$	$\{a,b\}$	$\{a,c\}$	U	U
ε^3	$\emptyset$	$\{a,b\}$	$\{a,b\}$	$\{a,c\}$	$\{a,b\}$	U	U	U

$\varkappa^m$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varkappa^1$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varkappa^2$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varkappa^3$	$\emptyset$	$\emptyset$	$\emptyset$	$\{c\}$	$\{b\}$	$\{c\}$	$\{c\}$	U

Since each R_i is reflexive, $\langle U, \varkappa^1, \varepsilon^1 \rangle$ is a pre-topological space. However we can notice, for instance, that $\mathscr{F}_a^1$ is not a filter, because $\{a,b\} \cap \{a,c\} \notin \mathscr{F}_a^1$.

Hence the pre-topological space $\langle U, \varepsilon^1, \varkappa^1 \rangle$ is not of type $\mathfrak{V}_D$.

Also, we can directly observe the relationship between ε and $\varkappa$ distributivity and proper filters.

Indeed, since $\mathscr{F}_a^1$ is not a proper filter, there are two minimal distinct elements $A = \{a,b\}$ and $B = \{a,c\}$ of $\mathscr{F}_a^1$ such that $A \cap B \neq \emptyset$ but $A \cap B \notin \mathscr{F}_a^1$. Let us set $Y = B \cap -A = \{b\}$, $Z = A \cap -B = \{c\}$. Therefore, the subset $Y \cup Z = \{b,c\}$ has empty intersection neither with A nor with B; hence $Y \cup Z$ has empty intersections with no members of $\mathscr{F}_a^1$, because A and B are minimal. It follows that a belongs to $\varepsilon^1(Y \cup Z)$. But $a \notin \varepsilon^1(Y)$ and $a \notin \varepsilon^1(Z)$. Henceforth $\varepsilon^1(Y) \cup \varepsilon^1(Z) \subsetneq \varepsilon^1(Y \cup Z)$. Dually for $\varkappa$-codiscontinuity. In fact, $a \in \varkappa^1(A)$ because $A \in \mathscr{F}_a^1$ and $A \subseteq A$. For the same reason $a \in \varkappa^1(B)$. Therefore $a \in \varkappa^1(A) \cap \varkappa^1(B)$. But $A \cap B \subsetneq A$ and $A \cap B \subsetneq B$ (remember that $A \neq B$). Since A and B are minimal in $\mathscr{F}_a^1$, there is not any $F \in \mathscr{F}_a^1$ such that $F \subseteq A \cap B$. Thus $a \notin \varkappa^1(A \cap B)$. Henceforth $\varkappa^1(A \cap B) \subsetneq \varkappa^1(A) \cap \varkappa^1(B)$. As a side consequence, ε^1 is not continuous. In our example:

$$\varkappa^1(\{a,b\} \cap \{a,c\}) = \varkappa^1(\{a\}) = \emptyset \neq \{a\} = \{a,b\} \cap \{a,c\} = \varkappa^1(\{a,b\}) \cap \varkappa^1(\{a,c\}).$$

and

$$\varepsilon^1(\{b\}) \cup \varepsilon^1(\{c\}) = \{b,c\} \subseteq \{a,b,c\} = \varepsilon^1(\{b,c\}) = \varepsilon^1(\{b\} \cup \{c\})$$

[See the Frame section for further details.]

Example 12.6.7. A pre-topological space of type $\mathfrak{V}_D$ which is not topological: $\varkappa$-idempotence is independent of $\varkappa-$distributivity and isotonicity

Consider the pre-topological space $\mathbf{P}_2 = \langle U, \varepsilon, \varkappa \rangle$ such that $U = \{a, b, c\}$ and $\varkappa$ and ε are given by the following table:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varepsilon(x)$	$\emptyset$	$\{a,b\}$	U	$\{c\}$	U	U	U	U
$\varkappa(x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\emptyset$	$\{c\}$	U

By easy inspection we can verify that $\varkappa$ distributes over meets and ε distributes over unions.

However, the two operators are not idempotent: $\varepsilon(\{a\}) = \{a,b\} \neq \{a,b,c\} = \varepsilon(\{a,b\})$; $\varkappa(\{b,c\}) = \{c\} \neq \emptyset = \varkappa(\{c\})$. So this is a case of distributive operators that are not idempotent. Since distributivity implies isotonicity, we have the required example. Therefore property (τ) is not valid.

But (τ) is a typical property of neighborhoods in topological spaces – see further in the text.

The same happens for the structure $\langle U, \varepsilon^3, \varkappa^3 \rangle$ in Excursus Dynamics 2, § 12.6.2. Indeed, consider the neighborhood systems $\mathscr{F}_a^3 = \{\{a,b,c\}\}$, $\mathscr{F}_b^3 = \{\{a,b\}, \{a,b,c\}\}$, $\mathscr{F}_c^3 = \{\{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. Given the element $\{a,b\}$ of $\mathscr{F}_b^3$, there is not any $X \in \mathscr{F}_b^3$ such that $\{a,b\} \in \mathscr{F}_x^3$ for any $x \in X$. In fact, clearly $\{a,b,c\}$ is not such an X. As for the remaining element of $\mathscr{F}_b^3$, $\{a,b\}$ itself, it does not belong to $\mathscr{F}_a^3$. This is the reason for $\varkappa^3(\varkappa^3(\{a,b\})) = \emptyset \neq \{b\} = \varkappa^3(\{a,b\})$. Dually, this is the reason for $\varepsilon^3(\varepsilon^3(\{c\})) = \{a,b,c\} \neq \{a,b\} = \varepsilon^3(\{c\})$.

Exercise 12.9.
(a) Compute the family $\mathscr{F} = \{\varkappa_a, \varkappa_b, \varkappa_c\}$ from $\mathbf{P}_2$.
(b) Check that every member of $\mathscr{F}$ is a filter and not only an order filter.
(c) Compute a minimal basis for the pre-topology $\mathbf{P}_2$.
(d) Verify that property (τ) fails to hold in $\mathbf{P}_2$.

12.7 Pre-Topological Spaces and Binary Relations

Now we are in a good position for understanding how relations and relation neighborhoods are connected with pre-topological spaces.

First of all, let us underline that not every pre-topological space is connected with a binary relation and not every binary relation induces a pre-topology. We know that pre-topologies can be associated with

relations that are *at least reflexive*. Taking into account this proviso, let us formalise in a definition the construction discussed in the above Excursus 12.6.2:

Definition 12.7.1. *Let U be a set and let $\mathfrak{R} = \{R_i\}_{i \in I}$ be a system of reflexive binary relations on U.*

(a) *If $\mathcal{N}(U) = \{R_i(x)\}_{i \in \mathfrak{R}}$ then we say that $\mathcal{N}(U)$ and the pre-topological space $\mathbf{P}(\mathfrak{R}) = \langle U, \varepsilon, \varkappa \rangle$ are connected with $\mathfrak{R}$, where for any $x \in U$, $\varkappa(x) = G(x)$. In this case we shall also write $\mathcal{N}(\mathfrak{R})$.*

(b) *The pre-topological space induced by the basis $\mathfrak{B}^m(U) = \{\mathcal{B}_x^m\}_{x \in U}$ is said to be $m-$associated with the system $\mathfrak{R}$ and denoted by $\mathbf{P}^m(\mathfrak{R}) = \langle U, \varepsilon^m, \varkappa^m \rangle$.*

(c) *In particular, if $\mathfrak{R} = \{R\}$, then the pre-topological space induced by the basis $\{R(x)\}_{x \in U}$ is said to be associated with the relation R and denoted by $\mathbf{P}(R) = \langle U, \varepsilon^R, \varkappa^R \rangle$.*

One should not confuse $\mathbf{P}(\mathfrak{R})$ with $\mathbf{P}(R)$.

Example 12.7.1. Difference between pre-topological spaces using $\{R_i(x)\}_{i \in I, x \in U}$ as a neighborhood system or as a basis for a neighborhood system

Here we show the difference between pre-topological spaces *connected* with a system $\mathfrak{R}$ of reflexive binary relations, and pre-topological spaces *induced* by $\mathfrak{R}$.

Consider the following system of relations $\mathfrak{R} = \{R_1, R_2\}$:

R_1	a	b	c		R_2	a	b	c
a	1	1	0		a	1	0	1
b	0	1	1		b	1	1	0
c	1	0	1		c	0	1	1

If we intend $\{R_1(x), R_2(x)\}_{x \in U}$ as a neighborhood system for a pre-topological space $\mathbf{P}(\mathfrak{R}) = \langle U, \varepsilon, \varkappa \rangle$, then $\varkappa(\{a, b, c\}) = \emptyset$. Actually, $\mathbf{P}(\mathfrak{R})$ is not of type $\mathfrak{V}_I$ because neither $\{R_1(a), R_2(a)\}$ nor $\{R_1(b), R_2(b)\}$ are filters. Therefore we must use the definition $\varkappa(X) = G(X) = \{x : X \in \mathcal{N}_x\}$. But for all $x \in U$, $\{a, b, c\} \notin R_1(x)$ or $R_2(x)$. On the contrary, if we intend $\mathfrak{R}$ as a basis then we obtain $\mathbf{P}^1(\mathfrak{R}) = \langle U, \varepsilon^1, \varkappa^1 \rangle$. In this case $\varkappa^1(\{a, b, c\}) = \{a, b, c\}$. One can observe that in $\mathbf{P}(\mathfrak{R})$, $\varkappa(X) \neq \{x : \exists R_i(R_i(x) \subseteq X)\}$ (indeed for any $i \in \{1, 2\}$, and for any $x \in U$, $R_i(x) \subseteq \{a, b, c\}$). Indeed, $\mathbf{P}^1(\mathfrak{R})$ has type $\mathfrak{V}_{Cl}$ while $\mathbf{P}(\mathfrak{R})$ has type $\mathfrak{V}_{Id}$.

Therefore, in $\mathbf{P}^1(\mathfrak{R})$ we can apply *Proposition* 12.5.4 and *Proposition* 12.6.7, while in $\mathbf{P}(\mathfrak{R})$ we can just set $\varkappa = G$.

However, since for any $x \in U$, $\{a, b, c\} \in \Uparrow (\mathcal{B}_x^1)$, we have that the relation $\mathbf{1} = U \times U$ belongs to the pseudo-uniformity $U(\mathfrak{R})$ see below, because $\mathbf{1} \supseteq R_i$, any $R_i \subseteq U \times U$. It follows that $\mathbf{P}(U(\mathfrak{R})) = \mathbf{P}(\mathfrak{R})$.

Proposition 12.7.1. *Let* $\mathbf{P}^1(\mathfrak{R}) = \langle U, \varepsilon^1, \varkappa^1 \rangle$ *be a pre-topological space 1-associated with a system of reflexive binary relations* $\mathfrak{R} = \{R_i\}_{i \in I}$. *Then;*

1. $\mathbf{P}^1(\mathfrak{R})$ *is of type* $\mathfrak{V}_I$.

2. $\mathfrak{B}^1(U) = \{\mathcal{B}_x^1\}_{x \in U} = \{R_i(x)\}_{i \in I, x \in U}$.

3. $\varkappa^1(X) = \{x : \exists R_i(R_i \in \mathfrak{R} \;\&\; R_i(x) \subseteq X)\}$.

4. $\varepsilon^1(X) = \bigcap\limits_{i \in I} R_i^{\smile}(X)$.

Proof. (**1**) Obvious. (**2**) Obvious. (**3**) In view of (1), $\mathbf{P}^1(\mathfrak{R})$ fulfills **N2**, hence we can apply *Proposition* 12.6.7. Therefore, $\varepsilon^1(X) = \{x : \forall R_i(R_i \in \mathfrak{R} \Rightarrow R_i(x) \cap X \neq \emptyset)\}$. But $R_i(x) \cap X \neq \emptyset$ if and only if $\exists x'(x' \in X \;\&\; \langle x, x' \rangle \in R_i)$. Thus, $\varepsilon^1(X) = \{x : \forall R_i(R_i \in \mathfrak{R} \Rightarrow x \in R^{\smile}(X))\}$. QED

Exercise 12.10.

(a) *Consider the system* $\mathfrak{R}$ *collecting the following two equivalence relations on* $U_4 = \{a, b, c, d\}$:

E_1	a	b	c	d
a	1	1	1	0
b	1	1	1	0
c	1	1	1	0
d	0	0	0	1

E_2	a	b	c	d
a	1	1	0	0
b	1	1	0	0
c	0	0	1	1
d	0	0	1	1

(a.1) *Compute the operators* ε^1, ε^2, $\varkappa^1$ *and* $\varkappa^2$ *starting from the two bases* $\{\mathcal{B}_x^1\}_{x \in U_4}$ *and* $\{\mathcal{B}_x^2\}_{x \in U_4}$.

(a.2) *Consider the pre-topologies* $\mathbf{P}^1(\mathfrak{R}) = \langle U_4, \varepsilon^1, \varkappa^1 \rangle$ *and* $\mathbf{P}^2(\mathfrak{R}) = \langle U_4, \varepsilon^2, \varkappa^2 \rangle$. *Do* $\mathbf{P}^1(\mathfrak{R})$ *or* $\mathbf{P}^2(\mathfrak{R})$ *coincide with the Approximation Space induced by* $\langle U_4, E_1 \cap E_2 \rangle$?

(b) *Consider the following relations on* $U_3 = \{a, b, c, \}$:

R_1	a	b	c
a	1	0	0
b	0	1	0
c	0	1	0

R_2	a	b	c
a	1	1	0
b	1	1	0
c	0	0	1

(b.1) *Compute* ε^1, ε^2, $\varkappa^1$ *and* $\varkappa^2$ *by starting with the two bases* $\{\mathcal{B}_x^1\}_{x \in U_3}$ *and* $\{\mathcal{B}_x^2\}_{x \in U_3}$.

(b.2) *Are* $\langle U_3, \varepsilon^1, \varkappa^1 \rangle$ *and* $\langle U_3, \varepsilon^2, \varkappa^2 \rangle$ *pre-topological structures?*

We must distinguish pre-topological spaces that are *connected* with a system of relations $\mathfrak{R}$ and pre-topological spaces that are *induced* by $\mathfrak{R}$. However, if $\mathbf{P}$ is induced by $\mathfrak{R}$ we can find a system of relations $\mathfrak{R}'$ such that $\mathbf{P}$ is connected with it, in a straightforward way.

Indeed, so far we have discussed spaces generated by arbitrary families of binary reflexive relations. However we can prove that particular types of spaces are generated by families of binary reflexive relations organised in a specific manner.

If $\mathfrak{R} = \{R_i\}_{i \in I}$ is a system of binary reflexive relations and $\mathbf{P}^1(\mathfrak{R}) = \langle U, \varepsilon^1, \varkappa^1 \rangle$ is the pre-topological space 1-associated with $\mathfrak{R}$, then we can regard any R_i as a vicinity (nearness) relation on U. Clearly, if $R_i' \supseteq R_i$, then $R_i'(x) \in \varkappa_x^1$, because by definition $R_i(x) \in \varkappa_x^1$. Therefore we can think of I principal filters of relations ordered by $\subseteq$, generated by $\mathfrak{R} = \{R_i\}_{i \in I}$. The collection of these filters is called a *pseudo-uniformity* generated by $\mathfrak{R}$, and denoted by $U(\mathfrak{R})$ (see Figure 12.7).
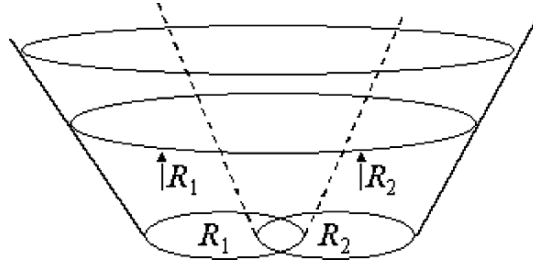
Figure 12.7: A pseudo-uniformity

Definition 12.7.2. *Let $\mathfrak{R} = \{R_i\}_{i \in I}$ be a system of relations, then the family $U(\mathfrak{R}) = \Uparrow \{R_i\}_{i \in I}$ is called a* pseudo-uniformity.

REMARKS. Notice that a pseudo-uniformity is a system of relations, not a system of relation neighborhoods.

Proposition 12.7.2. *Let $U(\mathfrak{R})$ be a pseudo-uniformity such that each $R \in \mathfrak{R}$ is reflexive. Then,*

1. *The pre-topological space $\mathbf{P}(U(\mathfrak{R}))$ connected with $U(\mathfrak{R})$ coincides with the pre-topological space $\mathbf{P}^1(\mathfrak{R})$ 1-associated with $\mathfrak{R}$.*

2. *$\mathbf{P}(U(\mathfrak{R}))$ is a pre-topological space of type $\mathfrak{V}_I$.*

Proof. Directly from *Definitions* 12.7.2 and *Proposition* 12.7.1.   QED

So, pseudo-uniformities provide us with the intuitive concept of a family of vicinity relations, or, under a slightly different point of view, they provide us with a qualitative (non numerical) notion of nearness. In this intuitive context, the requirement that in any pseudo-uniformity $U(\mathfrak{R})$, if $R \in U(\mathfrak{R})$ and $R \subseteq R'$, then $R' \in U(\mathfrak{R})$, rests on the intuition that if two points x and y are estimated to be near with respect to a given point of view (or resolution) then they are near also with respect to a less refined point of view (i.e. with respect to a coarser resolution). The opposite, of course, does not hold, because a better resolution can separate x and y.

Moreover a different scenario is given by the requirement that if x and y are estimated to be near with respect to both the relations R and R', then they must be estimated near also with respect to the relation $R \cap R'$. This is the behaviour of pre-topological spaces of type $\mathfrak{V}_D$, so that the situation in which $U(\mathfrak{R})$ is closed under intersections needs a new name:

Definition 12.7.3. *If $\mathfrak{R} = \{R_i\}_{i \in I}$ is a system of relations, and $U(\mathfrak{R})$ a pseudo-uniformity such that $R, R' \in \mathfrak{R}$ implies $R \cap R' \in \mathfrak{R}$, then $U(\mathfrak{R})$ is called a* pre-uniformity.

Notice that since both R and R' are required to be reflexive, the so called diagonal $\Delta(U) = \{\langle x, x \rangle : x \in U\}$ is always included in $R \cap R'$, so that the intersection of elements of $\mathfrak{R}$ is never empty (see Figure 12.8).
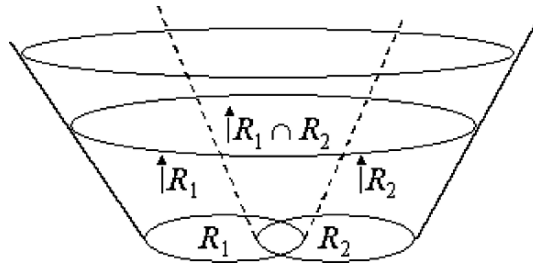
Figure 12.8: A pre-uniformity

Corollary 12.7.1. *Let $U(\mathfrak{R})$ be a pre-uniformity such that each $R \in \mathfrak{R}$ is reflexive and $\mathfrak{R}' = \{R\}$, for R reflexive. Then,*

1. *$\mathbf{P}(U(\mathfrak{R}))$ is a pre-topological space of type $\mathfrak{V}_D$.*

2. *$U(\mathfrak{R}') = \uparrow R$ is a pre-uniformity.*

Proof. From *Definitions* 12.7.1, 12.7.3 and *Proposition* 12.6.2, because if $U(\mathfrak{R})$ is a pre-uniformity then $\mathcal{N}(U(\mathfrak{R}))$ is a filter.   QED

The catalogue of the interesting pre-topologies does not reduces to the above cases. Indeed, it is not complete if we miss the following important case:

Definition 12.7.4. *A pre-topological space* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *is said to be of type* $\mathfrak{V}_{Cl}$ *if the operator* ε *is a* closure operator, *that is, inflationary, isotonic and idempotent and* $\varkappa$ *is an* interior operator, *that is, deflationary, isotonic and idempotent.*

The pre-topological space $\mathbf{P}_1$ of Example 12.6.2 is of type $\mathfrak{V}_{Cl}$ (notice that, however, it is not of type $\mathfrak{V}_D$. Therefore, distributivity might not hold).

Proposition 12.7.3. $\mathcal{N}(U)$ *is a neighborhood system of type* $\mathcal{N}_{2Id}$ *if and only if its induced pre-topological space is of type* $\mathfrak{V}_{Cl}$.

Proof. From *Proposition* 12.4.10 and *Lemma* 12.4.1.(N1) and (N2), G is inflationary and isotonic if and only if **N1** and **N2** hold in $\mathcal{N}(U)$. QED

Moreover,

Proposition 12.7.4. *Let* $\mathfrak{R} = \{R_i\}_{i \in I}$ *be a system of preorder relations on* U. *Then,*

1. *In the pre-topological space* $\mathbf{P}^1(\mathfrak{R}) = \langle U, \varepsilon^1, \varkappa^1 \rangle$ *the operators* ε^1 *and* $\varkappa^1$ *are isotonic and idempotent.*

2. *The family* $\{\Uparrow \mathcal{B}_x^1\}_{x \in U}$ *is a neighborhood system of type* $\mathcal{N}_{2Id}$.

Proof. We prove only (**1**) because $\{\varkappa_x^1\}_{x \in U} = \{\Uparrow \mathcal{B}_x^1\}_{x \in U}$ and (1) implies that $\{\varkappa_x^1\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$.

Isotonicity derives from the construction of $\mathbf{P}^1(\mathfrak{R})$ via $\{\Uparrow \mathcal{B}_x^1\}_{x \in U}$.

Let us prove the assertion about the idempotence of $\varkappa^1$ through its contraposition. If $\varkappa^1$ is not idempotent then there is an $A \subseteq U$ such that $\varkappa^1(A) \nsubseteq \varkappa^1(\varkappa^1(A))$ (indeed, $\varkappa^1(\varkappa^1(A)) \subseteq \varkappa^1(A)$ always holds). In this case there is a $y \in \varkappa^1(A)$ such that $y \notin \varkappa^1(\varkappa^1(A))$. Therefore there is a set $B \in \mathcal{B}_y^1$ such that $B \subseteq A$ (so that $y \in \varkappa^1(A)$), but for every $B' \in \mathcal{B}_y^1$, $B' \nsubseteq \varkappa^1(A)$ (so that $y \notin \varkappa^1(\varkappa^1(A))$). In particular

$B \nsubseteq \varkappa^1(A)$. It follows immediately that there is an element $b \in B$ such that $b \notin \varkappa^1(A)$. This means that for all $B'' \in \mathcal{B}_b^1$, $B'' \nsubseteq A$. But B has the form $R_i(y)$, for some index i, and B'' has the form $R_j(b)$, for every index j. Therefore we can put $i = j$. To sum up, there is a $y \in U$ such that for all $b \in R_i(y), b \in A$ but for some $b \in R_i(y)$ there is a $b' \in R_i(b)$ such that $b' \notin A$. It follows that $b' \notin R_i(y)$. Hence R_i is not transitive, so that it is not true that all the members of $\mathfrak{R}$ are preorders.   QED

The converse of the above *Proposition* does not hold because we can have systems of relations $\mathfrak{R}$ such that none of their components is a preorder but, nonetheless, in $\mathbf{P}^1(\mathfrak{R})$ the operator $\varkappa^1$ (resp. ε^1) is idempotent.

Example 12.7.2. A system $\mathfrak{R}$ of non-preorder relations, which induces a pre-topological space of type $\mathfrak{V}_{Cl}$

Notice: Under the assumptions of *Proposition* 12.6.7, in what follows we shall work on pre-topological bases, instead of induced filters.

Let $\mathfrak{R}$ be the collection of relations of Example 12.7.1.

It is easy to check that neither relation is transitive ($\langle a, c \rangle \in R_2, \langle b, a \rangle \in R_2$ but $\langle b, c \rangle \notin R_2$; $\langle a, b \rangle \in R_1, \langle b, c \rangle \in R_1$ but $\langle a, c \rangle \notin R_1$).

The basis $\mathfrak{B}^1$ is given by

$\mathcal{B}_x^1$	$\mathcal{B}_a^1$	$\mathcal{B}_b^1$	$\mathcal{B}_c^1$
$\mathcal{B}_x^1$	$\{\{a,b\},\{a,c\}\}$	$\{\{b,c\},\{a,b\}\}$	$\{\{a,c\},\{b,c\}\}$

Therefore the operator $\varkappa^1$ is given by:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$\varkappa^1(x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U

Hence $\varkappa^1$ is idempotent. Indeed, the family $\{\varkappa_x\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$.

So it is observed that the important mathematical notion of a closure (interior) operator is connected, in particular contexts, with pre-topological spaces induced by systems of relations featuring specific properties, namely preorders.

We sum-up the above results in Table 12.4.

The last row will be the target of what follows.

We recall that from *Proposition* 12.5.4, if $\langle U, \varepsilon, \varkappa \rangle$ is of type $\mathfrak{V}_I$, then for any $X \subseteq U$, $\varkappa(X) = \{x \in U : \exists N(N \in \mathcal{N}_x \ \& \ N \subseteq X)\}$ and

Table 12.4: Correspondence between pre-topology types and neighborhood system types

$\langle U, \varepsilon, \varkappa \rangle$ is said of type:	Characteristic properties	if $\mathcal{N}^{\varkappa}(U)$ is of type:
$\mathfrak{V}_{Id}$	$\varkappa(\varkappa(X)) = \varkappa(X)\ \varepsilon(\varepsilon(X)) = \varepsilon(X)$	$\mathcal{N}_{1Id}$
$\mathfrak{V}_I$	$X \subseteq Y \Rightarrow \varkappa(X) \subseteq \varkappa(Y)$ [resp. $\varepsilon(X) \subseteq \varepsilon(Y)$]	$\mathcal{N}_2$
$\mathfrak{V}_D$	$\varepsilon(X \cup Y) = \varepsilon(X) \cup \varepsilon(Y)$ [resp. $\varkappa(X \cap Y) = \varkappa(X) \cap \varkappa(Y)$]	$\mathcal{N}_3$
$\mathfrak{V}_{Cl}$	ε [resp. $\varkappa$] is a closure [resp. interior] operator	$\mathcal{N}_{2Id}$
$\mathfrak{V}_S$	$\varepsilon(X) = \bigcup_{x \in X} \varepsilon(\{x\})$	$\mathcal{N}_4$
topological	ε [resp. $\varkappa$] is a topological closure [resp. interior] operator	$\mathcal{N}_{3Id}, \mathcal{N}_{4Id}$

$\varepsilon(X) = \{x \in U : \forall N(N \in \mathcal{N}_x \Rightarrow N \cap X \neq \emptyset)\}$, that is, in $\mathfrak{V}_I$ spaces the contraction operator (the expansion operator) has the same definition as the interior (closure) operator in usual topological spaces. Moreover, notice that if U is finite, then the notions of $\mathfrak{V}_S$ and $\mathfrak{V}_D$ pre-topological spaces coincide.

REMARKS. If we think of a neighborhood system as the image of a relation $R \subseteq U \times \wp(U)$, then we can ask what are the relationships between $\varkappa$, G and the perception operator int introduced in Chapter 2, and the role played by **Id**, **N1**, **N2** and so on in these relationships. This point will be developed il *Lemma* 15.14.4 of Frame 15.14.

In what follows we abandon systems of relations and from now on we shall focus on single relation based pre-topologies. About them we have a first set of results:

Corollary 12.7.2. *Let* $\mathbf{P}(R) = \langle U, \varepsilon^R, \varkappa^R \rangle$ *be a pre-topological space associated with a reflexive binary relation R. Then for any $X \subseteq U$:*

1. $\mathbf{P}(R)$ *is of type* $\mathfrak{V}_S$.

2. $\varkappa^R(X) = \{x : R(x) \subseteq X\}$.

3. $\varkappa^R(X) = \bigcup \{Y : R(Y) \subseteq X\}$.

4. $\varepsilon^R(X) = R^\smile(X)$.

5. $\varkappa_x^R = \Uparrow \{R(x)\}$, for all $x \in U$.

Proof. (**1**) Trivially, from *Proposition* 12.6.9 since $R(x)$ is a single subset of U. (**2**) From *Proposition* 12.7.1. (**3**) From the additivity of R-neighborhoods. (**4**) From *Proposition* 12.7.1. (**5**) From *Proposition* 12.7.1 and the definition of a basis of a filter. QED

Notice that we cannot prove $\varkappa^R(X) = R(X)$, in contrast with $\varepsilon^R(X) = R^\smile(X)$; conversely, we cannot prove $\varepsilon^R(X) = \bigcap\{Y : R^\smile(Y) \supseteq X\}$, in contrast to $\varkappa^R(X) = \bigcup\{Y : R(Y) \subseteq X\}$.

Example 12.7.3. If R is not a preorder, then $R^\smile(X) \neq \bigcap\{R^\smile(Z) : X \subseteq R^\smile(Z)\}$

We show that reflexivity is not enough in order to turn the above inequality into equality. Consider the following reflexive but not transitive relation:

R	a	b	c
a	1	0	1
b	1	1	1
c	0	1	1

$\{R^\smile(Z) : \{b\} \subseteq R^\smile(Z)\} = \{\{a,b\}, \{b,c\}, \{a,b,c\}\}$. Thus $\bigcap\{R^\smile(Z) : \{b\} \subseteq R^\smile(Z)\} = \{b\} \neq R^\smile(\{b\}) = \{b,c\}$. Indeed, the problem is that $b \in R(c)$, $a \in R(b)$ but $a \notin R(c)$. Thus $c \notin R^\smile(\{a\})$ so that $c \notin \bigcap\{R^\smile(Z) : \{b\} \subseteq R^\smile(Z)\}$, although $R^\smile(\{a,b\}) \in \{R^\smile(Z) : \{b\} \subseteq R^\smile(Z)\}$.

In view of *Definition* 12.1.2 we obtain immediately the following

Corollary 12.7.3. *Let* $\mathbf{P}(R) = \langle U, \varepsilon^R, \varkappa^R \rangle$ *be a pre-topological space associated with a reflexive binary relation* R. *Then for any* $X \subseteq U$,

1. $\varkappa^R(X) = L_R(X)$.

2. $\varepsilon^R(X) = M_R(X)$.

12.7.1 Excursus: Pre-topological Spaces and Modal Algebras

Let us set $\Omega_{\varkappa^R}(U) = \{X \subseteq U : \varkappa^R(X) = X\}$. We should now ask if the system $\langle \mathbf{B}(U), \Omega_{\varkappa^R}(U) \rangle$ is a modal system. From the point of view of *Definition* 11.5.4 the answer is negative because in general we

cannot define $\varkappa^R(X)$ (i.e. $L_R(X)$) as $U \overset{\Omega_{\varkappa^R}}{\Longrightarrow} (X)$, that is, $\varkappa^R(X)$ does not coincide with the greatest element of $\Omega_{\varkappa^R}(U)$ below X, because of the trivial reason that this element might not exist. This limitation is due to the fact that without further constraints, generally $\Omega_{\varkappa^R}(U)$ is not a sup-subsemilattice of $\wp(U)$. Moreover notice that this fact is independent of the distributive properties of $\varkappa^R$ (see Example 12.8.1).

To analyse this topic, we shall use the following Lemma:

Lemma 12.7.1. *Let $\langle \mathbf{B}(U), k(\mathbf{B}(U)) \rangle$ be a k-modal system such that the knowledge map k is connected with a relation $R \subseteq U \times U$. Let, for any $X \subseteq U$, $L_R^*(X) = \bigcup \{R(Z) : R(Z) \subseteq X\}$ and $!_R(X) = U \overset{k(\mathbf{B}(U))}{\Longrightarrow} X = \max\{Z \in k(\mathbf{B}(U)) : Z \subseteq X\}$. Then for any $X \subseteq U$,*

  1. *$L_R^*(X) = !_R(X)$.*

  2. *If R is reflexive, then $L_R(X) \subseteq L_R^*(X)$.*

  3. *If R is a preorder, then $L_R(X) = L_R^*(X)$.*

Proof. (**1**) $!_R(X) = \max\{Y \in k(\mathbf{B}(U)) : Y \subseteq X\} = \max\{Y \in \{R(Z)\}_{Z \subseteq U} : Y \subseteq X\} = \max\{R(Z) : R(Z) \subseteq X\}$. But from the additivity of R-neighborhoods, $\max\{R(Z) : R(Z) \subseteq X\} = \bigcup\{R(Z) : R(Z) \subseteq X\} = L_R^*(X)$. (**2**) Suppose $a \in L_R(X)$. Then $R(a) \subseteq X$, so that $R(a) \subseteq \bigcup\{R(Z) : R(Z) \subseteq X\}$. If R is reflexive, $a \in R(a)$ and, therefore, $a \in \bigcup\{R(Z) : R(Z) \subseteq X\} = L_R^*(X)$ (the reverse inclusion does not hold – cf. Example 12.8.1). (**3**) In view of (2) we have only to prove the reverse inclusion. Suppose R is a preorder and $a \in \bigcup\{R(Z) : R(Z) \subseteq X\}$. Then there is a b such that $a \in R(b)$ and $R(b) \subseteq X$. Now, for any c such that $c \in R(a)$, $c \in R(b)$ by transitivity of R. Therefore $c \in X$. It follows that $R(a) \subseteq X$ and we can conclude that $a \in \{x : R(x) \subseteq X\}$.   QED

We show some instances of this point in Example 12.8.1 below.

So, we have partially solved the problem issued at the end of Section 12.1: $L_R(X) = L_R^*(X)$ and $M_R(X) = M_R^*(X)$ if and only if R is a preorder. Otherwise stated:

Corollary 12.7.4. *If $\langle \mathbf{B}(U), k(\mathbf{B}(U)) \rangle$ is a k-modal system such that the knowledge map k is connected with a relation $R \subseteq U \times U$, then $\langle \mathbf{B}(U), k(\mathbf{B}(U)) \rangle$ is a modal system if and only if for any $X \subseteq U$, $X \subseteq k(X)$ and if $X' \subseteq k(X)$, then $k(X') \subseteq k(X)$.*

However, pre-topological spaces are naturally connected with a more general class of modal structures called *modal algebras*:

Definition 12.7.5. *A* modal algebra *is a pair* $\langle \mathbf{B}, \Box \rangle$, *where* $\mathbf{B}$ *is a non degenerate Boolean algebra closed under a unary operation* $\Box$.

We have trivially:

Proposition 12.7.5. *Let* U *be a set and* $\mathcal{N}(U)$ *a neighborhood system over* U. *Then* $\langle \mathbf{B}(U), G \rangle$ *is a modal algebra.*

Conversely,

Proposition 12.7.6. *Let* $\langle \mathbf{B}(U), \Box \rangle$ *be a modal algebra of the subsets of a set* U. *Let us set for all* $X \subseteq U$, $X \in \mathcal{N}_x^{\Box}$ *if and only if* $x \in \Box(X)$. *Then* $\mathcal{N}^{\Box}(U) = \{\mathcal{N}_x^{\Box}\}_{x \in U}$ *is a neighborhood system.*

A more general form of duality between modal algebras and neighborhood system is discussed in Frame 15.15.

Now we continue our analysis of pre-topological spaces.

In order to compare two pre-topological spaces associated with two binary relations it is sufficient to compare the relations themselves:

Corollary 12.7.5. *Let* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *be two pre-topological spaces on the same universe* U, *associated with two reflexive binary relations on* U, R_1 *and* R_2, *respectively. Then* $\mathbf{P_2} \trianglelefteq \mathbf{P_1}$ *if and only if for any* $x \in U, R_1(x) \subseteq R_2(x)$, *if and only if* $R_1 \subseteq R_2$.

Dually, given a pre-topological space, we can define a reflexive binary relation associated with it:

Proposition 12.7.7. *(T-association) Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space. Let us set:*

$$\langle x, y \rangle \in R^T(\mathbf{P}) \text{ iff } y \in \bigcap \{X : X \in \varkappa_x\} \qquad (T)$$

Then $R^T(\mathbf{P})$ *is a reflexive binary relation on* U.

Proof. Trivial: since by definition $x \in \bigcap\{X : X \in \varkappa_x\}$, then $\langle x, x \rangle \in R^T(\mathbf{P})$. $\hfill$ QED

We shall say that $R^T(\mathbf{P})$ is *T-associated* with the pre-topology $\mathbf{P}$ and denote this relation with R^T whenever the pre-topological space $\mathbf{P}$ is understood.

In general, given a pre-topological space $\mathbf{P}$, it is possible to link it to a pre-topological space that is associated with a reflexive binary relation, in a unique way.

Proposition 12.7.8. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space. Then,*

1. $\mathbf{P} \trianglelefteq \mathbf{P}(R^T(\mathbf{P}))$.

2. *If* $\mathbf{P}$ *is of type* $\mathfrak{V}_S$, *then* $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}))$.

3. *If* $\mathbf{P}$ *is associated with a relation* R, *then* R *is T-associated with the pre-topological space* $\mathbf{P}$, *that is,* $R = R^T(\mathbf{P}(R))$.

Proof. (**1**) $\mathbf{P}(R^T(\mathbf{P}))$ $\varkappa_x^{R^T}$ is induced by the family $R^T(x)$, and for any $x \in U$, $R^T(x) = \bigcap\{X : X \in \varkappa_x\}$, so that $R^T(x) \subseteq X$, for any $X \in \varkappa_x$. Therefore, $\varkappa_x \subseteq \varkappa_x^{R^T}$. (**2**) Suppose $\mathbf{P}$ is of type $\mathfrak{V}_S$. Then for any $x \in U$ there is a subset X of U such that $\varkappa_x = \uparrow X$. Since $y \in R^T(x)$ iff $y \in \bigcap\{X : X \in \varkappa_x\} = X$, we obtain $\varkappa_x = \varkappa_x^{R^T}$. (**3**) From *Definition* 12.7.1 if $\mathbf{P}$ is associated with a relation R, then it is induced by the basis $\{R(x)\}_{x \in U}$. Therefore, $\varkappa_x = \Uparrow \{R(x)\} = \{\uparrow R(x)\}$. Hence, from *Proposition* 12.6.4, $\mathbf{P}$ is of type $\mathfrak{V}_S$, so that from point (2) we obtain the result.   QED

REMARKS. The above *Proposition* 12.7.8 guarantees that given a pre-topological space $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ of type $\mathfrak{V}_S$, we can derive the properties of $\mathbf{P}$ from those of $\mathbf{P}(R^T(\mathbf{P}))$.

As to the inequality (1) of *Proposition* 12.7.8, it is possible to show that if $\mathbf{P}$ is of type $\mathfrak{V}_I$, then $\mathbf{P}(R^T(\mathbf{P}))$ is the coarsest pre-topology among those of type $\mathfrak{V}_S$ that are finer than $\mathbf{P}$ (see Frame 15.3 for a proof).

We can also associate a pre-topology to a tolerance (i.e. reflexive and symmetric) relation:

Proposition 12.7.9. *(B-association) Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space. Let us set, for all* $x, y \in U$:

$$\langle x, y \rangle \in R^B(\mathbf{P}) \text{ iff } y \in \bigcap\{X : X \in \varkappa_x\} \Rightarrow x \in \bigcap\{Y : Y \in \varkappa_y\} \quad (B)$$

Then R *is a reflexive and symmetric relation.*

Proof. Trivial.   QED

We shall say, that $R^B(\mathbf{P})$ is *B-associated* with the pre-topology $\mathbf{P}$.

12.7.2 Excursus. Pre-Topological Spaces and Approximation Spaces

From *Proposition* 12.7.8 we know that $\mathbf{P} \trianglelefteq \mathbf{P}(R^T(\mathbf{P}))$ and $R^T(\mathbf{P}(R)) = R$. So we wonder what information $\mathbf{P}(R^B(\mathbf{P}))$ and $R^B(\mathbf{P}(R))$ carry. We shall give the answer as a corollary of the following more general statement about families of binary relations, as treated in Excursus 12.4.1.

Proposition 12.7.10. *Let U be a set and $\{R_j\}_{1 \leq j \leq n}$ a family of n reflexive binary relations on U. Let $1 \leq m \leq n$ and let $\mathbf{P}^m = \langle U, \varepsilon^m, \varkappa^m \rangle$, with the operators ε^m and $\varkappa^m$, as defined by* Definition *12.6.8. Moreover let us set $R^* = \bigcup\limits_{1 \leq j \leq n} R_j$ and $R_* = \bigcap\limits_{1 \leq j \leq n} R_j$. Then,*

1. *$R^T(\mathbf{P}^n) = R^*$.*

2. *$R^T(\mathbf{P}^1) = R_*$.*

3. *$R^B(\mathbf{P}^n)$ is the largest tolerance relation included in R^*.*

4. *$R^B(\mathbf{P}^1)$ is the largest tolerance relation included in R_*.*

The proof is given in Frame 15.3

Corollary 12.7.6. *For any family of n reflexive binary relations, $\mathbf{P}(R^T(\mathbf{P}^n)) \trianglelefteq \mathbf{P}(R^B(\mathbf{P}^n))$.*

Therefore, trivially, if we are given just one reflexive binary relation R, then $R^B(\mathbf{P}(R)) \subseteq R$, because $R^B(\mathbf{P}(R))$ is the largest tolerance relation included in R, while if we are given a pre-topological space $\mathbf{P}$, then $\mathbf{P}(R^B(\mathbf{P}))$ is the pre-topological space associated with the largest tolerance relation included in $R^T(\mathbf{P})$. It follows that $\mathbf{P} \trianglelefteq \mathbf{P}(R^B(\mathbf{P}))$ (the equality is not uniformly valid even if $\mathbf{P}$ is of type $\mathfrak{V}_S$; in fact in this case we have, generally, $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}) \trianglelefteq \mathbf{P}(R^B(\mathbf{P})))$.

Corollary 12.7.7. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space. Then $\mathbf{P}(R^B(\mathbf{P}))$ is the coarsest pre-topology among the pre-topological spaces finer than $\mathbf{P}$ and associated with a tolerance relation.*

A direct proof is in Frame 15.4.

Corollary 12.7.8. *Let U be a set and $\{R_j\}_{1 \leq j \leq n}$ a system of n reflexive binary relations on U, such that $R^* = \bigcup\limits_{1 \leq j \leq n} R_j$ (such that $R_* =$*

$\bigcap_{1 \leq j \leq n} R_j$) is transitive. Then, $\mathbf{P}(R^B(\mathbf{P}^n))$ (respectively $\mathbf{P}(R^B(\mathbf{P}^1))$) is the Approximation Space induced by the largest tolerance relation included in R^* (respectively included in R_*).

Particularly, if $R^T(\mathbf{P})$ is a binary transitive relation on U, then $\mathbf{P}(R^B(\mathbf{P}))$ is the Approximation Space induced by the largest tolerance relation included in $R^T(\mathbf{P})$, so that we can say that $\mathbf{P}(R^B(\mathbf{P}))$ is the coarsest Approximation Space finer than the pre-topological space $\mathbf{P}$.

Definition 12.7.6. *Let $\mathbf{P}$ be a pre-topological space, then:*

1. *If $R^T(\mathbf{P})$ is a tolerance relation, then $\mathbf{P}$ is said to be* weakly symmetric.

2. *If $R^T(\mathbf{P})$ is an equivalence relation, then $\mathbf{P}$ is said to be* strongly symmetric.

Therefore, any pre-topological space of the form $\mathbf{P}(R^B(\mathbf{P}))$, is weakly symmetric and any pre-topological space of the form $\mathbf{P}(R^B(\mathbf{P}))$ such that $R^T(\mathbf{P})$ is a transitive and reflexive, is strongly symmetric.

Corollary 12.7.9. *Let $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ be a pre-topological space, then:*

1. *$\mathbf{P}$ is weakly symmetric if given $x, y \in U, x \in \varepsilon(\{y\})$ implies $y \in \varepsilon(\{x\})$.*

2. *$\mathbf{P}$ is strongly symmetric if given $x, y \in U, x \in \bigcap\{X : X \in \varkappa_y\}$ implies $\varkappa_x = \varkappa_y$.*

3. *$\mathbf{P}$ is strongly symmetric if $\{\bigcap \varkappa_x : x \in U\}$ forms a partition of U.*

Weakly symmetric pre-topological spaces are connected with a particular kind of pre-uniformity structures:

Definition 12.7.7. *Let $U(\mathfrak{R})$ be a pre-uniformity of reflexive relations over a set U, such that for all $R \subseteq U \times U$, $R \in U(\mathfrak{R})$ implies $R^{\smile} \in U(\mathfrak{R})$. Then $U(\mathfrak{R})$ is called a* semi-uniformity.

Proposition 12.7.11. *If $U(\mathfrak{R})$ is a semi-uniformity then its connected pre-topological space $\mathbf{P}(U(\mathfrak{R}))$ is weakly symmetric.*

Example 12.7.4. Associating pre-topologies with reflexive binary relations

Some examples.

T-association: $R^T(\mathbf{P}^1)$, $R^T(\mathbf{P}^n)$

Consider the system of relations $\mathfrak{R}$ worked in Excursus Dynamics 2.

In view of the basis computed in Section 12.6.2, we derive the following table:

$\bigcap \mathcal{B}_x^m$	$\bigcap \mathcal{B}_a^m$	$\bigcap \mathcal{B}_b^m$	$\bigcap \mathcal{B}_c^m$
$\bigcap \mathcal{B}_x^1$	$\{a\}$	$\{b\}$	$\{c\}$
$\bigcap \mathcal{B}_x^2$	$\{a,c\}$	$\{b\}$	$\{c\}$
$\bigcap \mathcal{B}_x^3$	U	$\{a,b\}$	$\{c\}$

Consider $\mathbf{P}^1(\mathfrak{R})$ and call it $\mathbf{P}^1$, for short. Let us compute $R^T(\mathbf{P}^1)$. Since for all $x \in U$, $\varkappa_x^m = \Uparrow \bigcap \mathcal{B}_x^m$, we can work on the generators of the basis.

(i) $a \in \bigcap \mathcal{B}_x^1$, for $x = a$; (ii) $b \in \bigcap \mathcal{B}_x^1$, for $x = b$; (iii) $c \in \bigcap \mathcal{B}_x^1$, for $x = c$.

Therefore we obtain:

$R^T(\mathbf{P}^1)$	a	b	c
a	1	0	0
b	0	1	0
c	0	0	1

We immediately see that $R^T(\mathbf{P}^1)$ and R_* coincide. Notice, anyway, that $R^T(\mathbf{P}^1)$ is a transitive relation by chance.

Incidentally, here we can verify that $\mathbf{P}^1 \trianglelefteq \mathbf{P}(R^T(\mathbf{P}^1))$. Indeed, $\mathbf{P}(R^T(\mathbf{P}^1))$ has the following family of basis: $\mathcal{B}_a = \{\{a\}\}, \mathcal{B}_b = \{\{b\}\}, \mathcal{B}_c = \{\{c\}\}$. Clearly $\mathcal{B}_a$ induces a filter $\mathscr{F}_a = \uparrow \{a\}$ which is finer than $\mathscr{F}_a^1$ (i.e. $\{\{a,b\}, \{a,c\}, \{a,b,c\}\}$ – cf. Section 12.6.2). Indeed, we can recognize that $\mathbf{P}^1$ is not of type $\mathfrak{V}_S$, because $\bigcap \mathcal{B}_a^1 \notin \mathcal{B}_a^1$.

Now let us compute $R^T(\mathbf{P}^3)$:

(i) $a \in \bigcap \mathcal{B}_x^3$, for $x \in \{a,b\}$; (ii) $b \in \bigcap \mathcal{B}_x^2$, for $x \in \{a,b\}$; (iii) $c \in \bigcap \mathcal{B}_x^3$, for $x \in \{a,c\}$.

So, for instance, since $c \in \bigcap \mathcal{B}_a^2$, $\langle a, c \rangle \in R^T(\mathbf{P}^3)$.

Summing up, we obtain:

$R^T(\mathbf{P}^3)$	a	b	c
a	1	1	1
b	1	1	0
c	0	0	1

We immediately verify $R^T(\mathbf{P}^3) = R^*$.

B-association: $R^B(\mathbf{P}^n), R^B(\mathbf{P}^1)$.

Let $U = \{a, b, c, d, e\}$ and R_1, R_2, R^*, R_* be given by

R_1	a	b	c	d	e		R_2	a	b	c	d	e
a	1	1	1	1	1		a	1	1	1	1	1
b	1	1	1	1	1		b	1	1	1	1	1
c	1	1	1	1	1		c	0	0	1	0	0
d	0	0	0	1	1		d	0	0	0	1	1
e	0	0	0	1	1		e	0	0	0	1	1

R_*	a	b	c	d	e		R^*	a	b	c	d	e
a	1	1	1	1	1		a	1	1	1	1	1
b	1	1	1	1	1		b	1	1	1	1	1
c	0	0	1	0	0		c	1	1	1	1	1
d	0	0	0	1	1		d	0	0	0	1	1
e	0	0	0	1	1		e	0	0	0	1	1

By easy computation, applying the two formulas $\mathcal{B}^1_x = \{R_1(x), R_2(x)\}$ and $\mathcal{B}^2_x = \{R_1(x) \cup R_2(x)\}$ we obtain:

$\mathcal{B}^m_x$	$\mathcal{B}^m_a$	$\mathcal{B}^m_b$	$\mathcal{B}^m_c$	$\mathcal{B}^m_d$	$\mathcal{B}^m_e$
$\mathcal{B}^1_x$	$\{U\}$	$\{U\}$	$\{\{c\}, U\}$	$\{\{d, e\}\}$	$\{\{d, e\}\}$
$\mathcal{B}^2_x$	$\{U\}$	$\{U\}$	$\{U\}$	$\{\{d, e\}\}$	$\{\{d, e\}\}$

From the above table we derive the following one:

$\bigcap \mathcal{B}^m_x$	$\bigcap \mathcal{B}^m_a$	$\bigcap \mathcal{B}^m_b$	$\bigcap \mathcal{B}^m_c$	$\bigcap \mathcal{B}^m_d$	$\bigcap \mathcal{B}^m_e$
$\bigcap \mathcal{B}^1_x$	U	U	$\{c\}$	$\{d, e\}$	$\{d, e\}$
$\bigcap \mathcal{B}^2_x$	U	U	U	$\{d, e\}$	$\{d, e\}$

Let us compute $R^B(\mathbf{P}^1)$:

(i) $a \in \bigcap \mathcal{B}^1_x$, for $x \in \{a, b\}$; (ii) $b \in \bigcap \mathcal{B}^1_x$, for $x \in \{a, b\}$; (iii) $c \in \bigcap \mathcal{B}^1_x$, for $x \in \{a, b, c\}$; (iv) $d \in \bigcap \mathcal{B}^1_x$, for $x \in \{a, b, d, e\}$; (v) $e \in \bigcap \mathcal{B}^1_x$, for $x \in \{a, b, d, e\}$.

Therefore, for instance, $\langle a, b \rangle$ and $\langle b, a \rangle \in R^B(\mathbf{P}^1)$, while $\langle a, e \rangle$ and $\langle e, a \rangle \notin R^B(\mathbf{P}^1)$ because although $e \in U = \bigcap \mathcal{B}^1_a$, $a \notin \{d, e\} = \bigcap \mathcal{B}^1_e$ (in order to understand if $\langle x, y \rangle \in R^B(\mathbf{P}^1)$, it is sufficient to compare the ranges of validity of the membership relation for x and y: $\{x, y\}$ is included in both of them, hence $\langle x, y \rangle \in R^B(\mathbf{P}^1)$).

Summing up, we obtain:

$R^B(\mathbf{P}^1)$	a	b	c	d	e
a	1	1	0	0	0
b	1	1	0	0	0
c	0	0	1	0	0
d	0	0	0	1	1
e	0	0	0	1	1

Comparing this relation with R_*, we immediately see that it is the largest tolerance relation included in R_*. Incidentally, since R_* is transitive, $R^B(\mathbf{P}^1)$ is also an equivalence relation.

Now let us compute $R^B(\mathbf{P}^2)$:

(i) $a \in \bigcap \mathcal{B}_x^2$, for $x \in \{a, b, c\}$; (ii) $b \in \bigcap \mathcal{B}_x^2$, for $x \in \{a, b, c\}$; (iii) $c \in \bigcap \mathcal{B}_x^2$, for $x \in \{a, b, c\}$; (iv) $d \in \bigcap \mathcal{B}_x^2$, for $x \in \{a, b, d, e\}$; (v) $e \in \bigcap \mathcal{B}_x^1$, for $x \in \{a, b, d, e\}$.

Therefore, for instance, now $\langle a, c \rangle$, $\langle b, c \rangle$, $\langle c, a \rangle$ and $\langle c, b \rangle \in R^B(\mathbf{P}^2)$.

We obtain:

$R^B(\mathbf{P}^1)$	a	b	c	d	e
a	1	1	1	0	0
b	1	1	1	0	0
c	1	1	1	0	0
d	0	0	0	1	1
e	0	0	0	1	1

Comparing this relation with R^*, we immediately see that it is the largest tolerance relation included in R^*. Also in this case, since R^* is transitive, $R^B(\mathbf{P}^2)$ is an equivalence relation.

We conclude noticing that if the pre-topological space $\mathbf{P}$ is of type $\mathfrak{V}_S$, then it can be associated with a pre-topology which is the finest among the pre-topology T-associated with a tolerance relation and that are coarser than $\mathbf{P}$. Suffices it to consider for any $x \in U$ the basis $\mathcal{B}_x = \bigcap \varkappa_x \cup \{y : x \in \varkappa_y\}$.

Exercise 12.11.
Draw directed graphs representing $R_1, R_2, R^B(\mathbf{P}^1)$ and $R^B(\mathbf{P}^2)$.

A relation R such that $R^B(\mathbf{P}(R))$ is a tolerance but not an equivalence relation

Consider the following relation R:

R	a	b	c	d
a	1	1	0	0
b	1	1	1	0
c	0	1	1	1
d	0	0	0	1

By easy inspection we can observe that R is not transitive. For instance $\langle a, b \rangle \in R$, $\langle b, c \rangle \in R$, but $\langle a, c \rangle \notin R$.

If we transform it into the relation $R^B(\mathbf{P}(R))$, then we obtain a tolerance relation and not an equivalence (because of the lack of transitivity). The basis for $\mathbf{P}(R)$ is:

x	a	b	c	d
$\mathcal{B}_x$	$\{\{a, b\}\}$	$\{\{a, b, c\}\}$	$\{\{b, c, d\}\}$	$\{\{d\}\}$

(i) $a \in \bigcap \mathcal{B}_x$, for $x \in \{a, b\}$; (ii) $b \in \bigcap \mathcal{B}_x$, for $x \in \{a, b, c\}$; (iii) $c \in \bigcap \mathcal{B}_x$, for $x \in \{b, c\}$; (iv) $d \in \bigcap \mathcal{B}_x$, for $x \in \{c, d\}$. Therefore, for instance, $a \in \bigcap \mathcal{B}_b$ and $b \in \bigcap \mathcal{B}_a$, so that $\langle a, b \rangle \in R^B(\mathbf{P}(R))$; $d \in \bigcap \mathcal{B}_c$ but $c \notin \bigcap \mathcal{B}_d$, so that $\langle c, d \rangle \notin R^B(\mathbf{P}(R))$.

Summing up:

$R^B(\mathbf{P}(R))$	a	b	c	d
a	1	1	0	0
b	1	1	1	0
c	0	1	1	0
d	0	0	0	1

and we can easily notice that $R^B(\mathbf{P}(R)) \subseteq R$ (in fact, $\langle c, d \rangle \in R$, while $\langle c, d \rangle \notin R^B(\mathbf{P}(R))$).

Exercise 12.12.
Draw two directed graphs representing R and $R^B(\mathbf{P}(R))$.

12.8   Topological Spaces and Binary Relations

We are eventually at one step from our main goal in the Subsection: linking topologies and relations.

Until now, we have considered pre-topological spaces and specifically, pre-topological spaces associated with reflexive or tolerance relations.

Now we have to understand what happens in case of a topological space.

First of all, let us define topological spaces and understand the basic differences between them and their closest relatives: pre-topological spaces of type $\mathfrak{V}_{Id}$, $\mathfrak{V}_{Cl}$ and $\mathfrak{V}_S$.

Definition 12.8.1. *A pre-topological space $\langle U, \varepsilon, \varkappa \rangle$ of type $\mathfrak{V}_D$ is a topological space if for any x, $\varkappa_x$ satisfies property (τ).*

For the reader's convenience, we recall here this property: for any $x \in U, X \subseteq U$,

if $X \in \varkappa_x$, then there is a $Y \in \varkappa_x$ such that for any $y \in Y, X \in \varkappa_y$

Usually, pre-topological spaces do not fulfill this property.

A pre-uniformity connected with a pre-topological space fulfilling (τ), satisfies the property described in the following definition:

Definition 12.8.2. *Let $U(\mathfrak{R})$ be a pre-uniformity such that for each $R \in U(\mathfrak{R})$ there is an $R' \in U(\mathfrak{R})$ such that $R' \otimes R' \subseteq R$. Then $U(\mathfrak{R})$ is called a* quasi-uniformity*.*

We remind that given two relations R and R' on a set U, $R \otimes R'$ is the concatenation $\{ \langle x, y \rangle : \exists z (\langle x, z \rangle \in R \ \& \ \langle z, y \rangle \in R') \}$.

Intuitively, quasi-uniformities provide us with a notion of "non-discontinuity": if $\langle a, c \rangle \in R$, then there is a b in between a and c, that is, a b such that $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$. Conversely if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$, too.

Proposition 12.8.1. *Let $U(\mathfrak{R})$ be a pre-uniformity of reflexive relations. Then, if $U(\mathfrak{R})$ is a quasi-uniformity, its connected pre-topological space $\mathbf{P}(U(\mathfrak{R}))$ fulfills (τ).*

We have seen at the very beginning of this story that isotonicity plus property τ give idempotence. Therefore, if a neighborhood system fulfills **0**, **1**, **N1**, **N2** and (τ), then the operators $\varkappa$ and ε are idempotent in the induced pre-topological space.

In this Section we want to analyse the connections between topological properties and binary relations. More precisely, let $\langle U, \varepsilon^R, \varkappa^R \rangle$ be a pre-topological space associated with a binary relation R. We wonder whether R has to enjoy specific properties whenever $\langle U, \varepsilon^R, \varkappa^R \rangle$ is a topological space. The answer is positive: there is a strict connection between topological spaces and preorders, i.e. binary reflexive and transitive relations.

Proposition 12.8.2. *Let $\mathbf{P} = \langle U, \varepsilon^R, \varkappa^R \rangle$ be a pre-topological space associated with a reflexive binary relation $R \subseteq U \times U$. Then $\mathbf{P}$ is a topological space if and only if R is transitive.*

Proof. (A) $\Rightarrow$: Assume that (τ) holds. In case of pre-topological spaces induced by a reflexive binary relation R, property (τ) reads:

$$(*) \quad \forall x \in U, \forall X \subseteq U(X \in \varkappa_x^R \Rightarrow \exists Y(Y \in \varkappa_x^R \& \forall y \in Y(X \in \varkappa_y^R))).$$

So, take $X = R(x)$. Assume $(*)$ holds for some Y. Then, (i) X is the least element of $\varkappa_x^R$. (ii) $\forall y \in Y, X \supseteq R(y)$ (from the assumption and *Corollary* 12.7.2.(5)). (iii) $Y \subseteq X$, because R is reflexive. Hence, (iv) $X = Y$. Therefore, (v) for all $x' \in R(x)$, $R(x') \subseteq R(x)$, that is, $R(x) \supseteq R(x')$ for all $x' \in X$. Hence, (vi) $R(x) \supseteq R(y)$. But $X = Y$. Therefore $R(x) \supseteq R(R(x))$, which is the axiom for transitivity.

(B) $\Leftarrow$: Suppose $(*)$ does not hold. Therefore we have:

$$(**) \quad \exists x, \exists X(X \in \varkappa_x^R \& \forall Y(Y \in \varkappa_x^R \Rightarrow \exists y(y \in Y \& X \notin \varkappa_y^R))).$$

So, choose $Y = R(x)$. We have elements x and y such that: (i) $\langle x, y \rangle \in R$ and $X \notin \varkappa_y^R$; hence (ii) $\langle x, y \rangle \in R$ and $X \notin\, \uparrow R(y)$; (iii) $\langle x, y \rangle \in R$ and $R(y) \nsubseteq X$; (iv) $\langle x, y \rangle \in R$ and there exists a z such that $\langle y, z \rangle \in R$ and $z \notin X$. But $X \in \varkappa_x^R$, so that $R(x) \subseteq X$. We obtain: (v) $\langle x, x' \rangle \in R$ implies $x' \in X$. From (iv) and (v) we conclude that $\langle x, z \rangle \notin R$. Hence R is not transitive. QED

Proposition 12.8.3. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a topological space induced by a basis* $\mathfrak{B} = \{\mathcal{B}_x\}_{x \in U}$. *Then,*

1. *For any* $x \in U$: (a) $\bigcap \varkappa_x$ *is open;* (b) $\bigcap \mathcal{B}_x$ *is open:* (c) $\bigcap \varkappa_x$ *is the least open set containing* x.

2. *For any* $X \subseteq U$, $\varkappa(X) = \bigcup \{\varkappa(Z) : \varkappa(Z) \subseteq X\}$.

Proof. (**1**) (a) For any $x \in U$, $\varkappa_x = \Uparrow \mathcal{B}_x$ and from *Proposition* 12.6.1 we have $\bigcap \Uparrow \mathcal{B}_x \in \mathcal{B}_x$. Thus we obtain $\bigcap \varkappa_x \in \varkappa_x$. So, from the topological property (τ), there is a $Y \in \varkappa_x$ such that $\bigcap \varkappa_x \in \varkappa_y$ for any $y \in Y$. But this means that any y belonging to Y belongs to $\bigcap \varkappa_x$, too (because, by definition, if $A \in \varkappa_y$, then $y \in A$). Henceforth, $Y \subseteq \bigcap \varkappa_x$. But $\bigcap \varkappa_x$ is the least element of $\varkappa_x$ and Y belongs to $\varkappa_x$. It follows that $Y = \bigcap \varkappa_x$ and we can conclude that $\bigcap \varkappa_x$ is a neighborhood of all its own elements. Hence it is open. (b) is straightforward from the equality $\varkappa_x = \Uparrow \mathcal{B}_x$. (c) is obvious, because if $x \in \varkappa(A) = A$, then $A \in \varkappa_x$, so that $A \supseteq \bigcap \varkappa_x$. (**2**) If $\mathbf{P}$ is a topological space, then $\varkappa$ is isotonic and idempotent. Therefore, if $\varkappa(Z) \subseteq X$, then, for isotonicity, $\varkappa(\varkappa(Z)) \subseteq \varkappa(X)$. That is to say, for idempotence, $\varkappa(Z) \subseteq \varkappa(X)$. Conversely, if $\varkappa(Z) \subseteq \varkappa(X)$, since $\varkappa$ is deflationary $\varkappa(Z) \subseteq \varkappa(X) \subseteq X$. It follows, immediately, $\varkappa(X) = \bigcup \{\varkappa(Z) : \varkappa(Z) \subseteq X\}$.   QED

Notice that *Proposition* 12.8.3.(2) is not that trivial when we frame it in topological spaces connected with binary relations, as we are going to see in the next corollary. Indeed, if isotonicity or idempotence fails, then this result does not hold any longer.

Example 12.8.1. Idempotence, preorders and contractions

The pre-topology $\mathbf{P}_2$ of Example 12.6.7 provides us with an example of the fact that the lack of idempotence makes the equivalence $\varkappa(X) = \bigcup \{\varkappa(Y) : \varkappa(Y) \subseteq X\}$ fail. In fact, $\varkappa(\{c\}) = \emptyset$, while $\bigcup \{\varkappa(Y) : \varkappa(Y) \subseteq X\} = \{c\}$.

It must be noticed that, a fortiori, $\varkappa(X) \neq \bigcup \{Y : \varkappa(Y) \subseteq X\}$. So do not confuse the set $\{x : \varkappa^R(x) \subseteq X\}$ (which has no meaning) with the set $\{x : R(x) \subseteq X\}$ (which gives $\varkappa^R(X)$ in case of pre-topological spaces of type $\mathfrak{V}_S$) and the set $\{R(Y) : R(Y) \subseteq X\}$ (whose union gives $\varkappa^R(X)$ in case of a preorder R).

For another example consider the following non-transitive relation:

R_3	a	b	c
a	1	1	0
b	0	1	1
c	0	0	1

It gives the basis $\mathfrak{B}^R$:

x	a	b	c
$\mathcal{B}_x^{R_3}$	$\{\{a,b\}\}$	$\{\{b,c\}\}$	$\{\{c\}\}$

By applying in $\mathbf{P}(R_3)$ the formula $L_{R_3}(X) = \varkappa^{R_3}(X) = \{x : \exists B (B \in \mathcal{B}_x^{R_3}$ & $B \subseteq X)\}$ we obtain:

X	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\emptyset$	U
$L_{R_3}(X)$	$\emptyset$	$\emptyset$	$\{c\}$	$\{a\}$	$\{c\}$	$\{b,c\}$	$\emptyset$	U

Thus, $L_{R_3}(\{a,b\}) = \{a\} \neq \bigcup\{R_3(x) : R_3(x) \subseteq \{a,b\}\} = \bigcup\{R_3(\{a\})\} = R_3(\{a\}) = \{a,b\}$. This is due to the fact that although $b \in R_3(\{a\})$ and $R_3(\{a\}) \subseteq \{a,b\}$, nonetheless $R_3(\{b\}) = \{b,c\} \not\subseteq \{a,b\} = R_3(\{a\})$, because of the failure of transitivity. Hence, $b \notin \{x : R_3(x) \subseteq \{a,b\}\}$. Therefore $\bigcup\{R_3(Z) : R_3(Z) \subseteq \{a,b\}\}$ contains elements (*viz.* b) that do not fulfill the universal proviso of the necessity operator (i.e. "$L_{R_3}(X) = \{x : \forall y, \langle x,y \rangle \in R_3 \Rightarrow y \in X\}$"). It follows that $\bigcup\{R_3(Z) : R_3(Z) \subseteq X\}$ is not a suitable formula for $L(X)$, in this case.

Moreover notice that in case of lack of transitivity, the set of R-neighborhoods, $\{\emptyset, \{a,b\}, \{b,c\}, \{c\}, U\}$, and the set of necessitations $\{\emptyset, \{a\}, \{c\}, \{b,c\}, U\}$ do not coincide. Henceforth we can observe that in this case it is not true that for any $A \subseteq U$, if $X = L_{R_3}(A)$, then $X = R_3(Y)$ for some $Y \subseteq U$.

Also, observe that $\Omega_{R_3}(U) = \{L_{R_3}(X) : X \subseteq U\}$ is not a distributive sublattice of $\mathbf{B}(U)$. Due to this fact we can see that the element $\max\{X \in \Omega_{R_3}(U) : X \subseteq Z\}$ might not exist for some subset Z of U. Indeed this is the case of $\{a,c\}$. Actually, $\{X \in \Omega_{R_3}(U) : X \subseteq \{a,c\}\} = \{\{a\}, \{c\}\}$ which does not have a greatest element. It can be noticed that $\bigvee\{X \in \Omega_{R_3}(U) : X \subseteq \{a,b\}\} = U$, but $U \not\subseteq \{a,c\}$.

To sum up, if a relation R is not a preorder, then $\mathfrak{B}^R$ is not a basis for $\Omega_R(U)$ in the topological sense. That is, there can be $X, Y \in \mathfrak{B}^R$ such that $X \cup Y \neq L_R(Z)$ for all $Z \subseteq U$.

Finally, it is worth noticing that if R lacks reflexivity, then $\{x : R(x) \subseteq X\}$ is not included in $\bigcup\{R(Z) : R(Z) \subseteq X\}$. For instance, consider the (non-reflexive) relation R of Example 12.1.1.

Then, $\{x : R(x) \subseteq \{b,c\}\} = \{a,b,c\}$, while $\bigcup\{R(Z) : R(Z) \subseteq \{b,c\}\} = \{b,c\}$.

Indeed, the inclusion here failed was proved in *Lemma* 12.7.1 by exploiting reflexivity. On the contrary, the reverse inclusion $\bigcup\{R(Z) : R(Z) \subseteq X\} \subseteq \{x : R(x) \subseteq X\}$ requires just transitivity. For instance, to prove that if $b \in \bigcup\{R(Z) : R(Z) \subseteq \{b,c\}\}$ then $b \in \{x : R(x) \subseteq \{b,c\}\}$, first we need to notice that the antecedent is valid because $b \in R(\{a\})$ and $R(\{a\}) \subseteq \{b,c\}$; second, we apply the transitivity of R to show that $R(\{b\}) \subseteq R(\{a\})$. So we conclude $R(\{b\}) \subseteq \{b,c\}$ and can derive $b \in \{x : R(x) \subseteq \{b,c\}\}$.

We can restate the property (τ) of *Definition* 12.8.1, in terms of idempotence of ε^R and $\varkappa^R$:

Corollary 12.8.1. *Let $\mathbf{P} = \langle U, \varepsilon^R, \varkappa^R \rangle$ be a pre-topological space associated with a reflexive binary relation $R \subseteq U \times U$. Then ε^R and $\varkappa^R$ are idempotent if and only if R is transitive.*

Proof. (A) $\Rightarrow$: Suppose $y \in R(x)$ and $z \in R(y)$. It follows, from *Proposition* 12.7.2, that $y \in \varepsilon^R(\{z\})$ and $x \in \varepsilon^R(\{y\})$. Since **P** is of type $\mathfrak{V}_S$ (again from *Proposition* 12.7.2), the operator ε^R is isotonic. Hence $x \in \varepsilon^R(\varepsilon^R(\{z\}))$. But if ε^R is idempotent, $x \in \varepsilon^R(\{z\})$ and, as a consequence, $z \in R(x)$, which proves that R is transitive.

(B) $\Leftarrow$: Suppose R is transitive. Then, since by default R is also reflexive, we have $R(R(x)) = R(x)$. Hence $R^{\smile}(R^{\smile}(x)) = R^{\smile}(x)$. So, from *Proposition* 12.7.2 $\varepsilon^R(\varepsilon^R(x)) = \varepsilon^R(x)$. For $\varkappa^R$ the proof is by duality.   QED

Therefore, reflexive and transitive relations, i.e. preorders, are tightly linked with topological spaces.

Example 12.8.2. A pre-topological space associated with a reflexive and transitive relation R

Let $U = \{a, b, c, d, e\}$ and let R be the following preorder:

R	a	b	c	d	e
a	1	1	1	1	1
b	0	1	0	1	1
c	0	0	1	0	0
d	0	0	0	1	1
e	0	0	0	1	1

Consider the family $\mathfrak{B}^R = \{R(x)\}_{x \in U} = \{\{c\}, \{d, e\}, \{b, d, e\}, U\}$. Let us compute the family $\{\mathcal{B}_x^R\}_{x \in U}$, where for any $x \in U$, $\mathcal{B}_x^R = \{X : X \in \mathfrak{B}^R \ \& \ x \in X\}$:

x	a	b	c	d	e
$\mathcal{B}_x^R$	$\{U\}$	$\{\{b, d, e\}, U\}$	$\{\{c\}, U\}$	$\{\{d, e\}, \{b, d, e\}, U\}$	$\{\{d, e\}, \{b, d, e\}, U\}$

We can observe what follows

$$for \ any \ x \in U, \ if \ X \in \mathcal{B}_x^R \ then \ there \ is \ a \ Y \in \mathcal{B}_x^R$$
$$such \ that \ for \ any \ y \in Y, X \in \mathcal{B}_y^R \qquad\qquad (12.8.1)$$

Indeed we can chose $Y = X$. Moreover for any x, $\mathscr{F}_x =\Uparrow \{R(x)\} =\Uparrow \mathcal{B}_x^R$ (this is proved in the following way: Let $a \in R(b)$ for $b \neq a$. Suppose $R(a) \nsubseteq R(b)$. Then there is an x such that $x \in R(a)$ and $x \notin R(b)$. But $a \in R(b)$; it follows that R is not transitive). Obviously, property (τ) is inherited by $\mathscr{F}_x$ from $\mathcal{B}_x^R$ and this makes the topological property (τ) hold. In fact take any $x \in U$ and any $F \in \mathscr{F}_x$. Let us look for an $X \in \mathscr{F}_x$ such that for any $y \in X$, $F \in \mathscr{F}_y$. It is sufficient to take any member Y of $\mathcal{B}^R$, such that $Y \subseteq F$ (and it exists, because $\mathscr{F}_x =\Uparrow \{R(x)\}$ and $R(x) \in \mathfrak{B}^R$). In fact, since Y is open, it is a neighborhood of all its points. But since $Y \subseteq F$, then F is a neighborhood of all the points of Y, too.

For instance, let $x = d$ and take $F = \{b, c, d, e\}$ which is a member of $\mathscr{F}_d$. Take $Y = \{b, d, e\}$. Y belongs to $\Uparrow \{R(b)\}, \Uparrow \{R(d)\}$ and $\Uparrow \{R(e)\}$. But since $Y \subseteq F$, we obtain $R(b) = \{b, d, e\} = Y \subseteq \{b, c, d, e\} = F, R(d) = \{d, e\} \subseteq Y \subseteq F$

and $R(e) = \{d, e\} \subseteq Y \subseteq F$. That is to say, $F \in \Uparrow \{R(b)\}, F \in \Uparrow \{R(d)\}$ and $F \in \Uparrow \{R(e)\}$. Which is the same thing as saying that F is a neighborhood of b, d and e.

Therefore $\mathscr{F} = \{\mathscr{F}_x\}_{x \in U}$ is a neighborhood system for a topology $\Omega(U)$ with subbasis $\mathfrak{B}^R$. But $\mathscr{F}$ is a neighborhood system for $\mathbf{P}(R)$, too. So we conclude that $\langle U, \Omega(U) \rangle = \mathbf{P}(R)$. Hence $\mathbf{P}(R)$ is a topological space and the interior operator $\mathbb{I}_R$ coincides with the contraction operator $\varkappa^R$.

Let us compute, for instance, $\varkappa^R(\{a, c, d, e\})$ and $\mathbb{I}_R(\{a, c, d, e\})$:

$\varkappa^R(\{a, c, d, e\}) = \{x : \exists X(X \in \Uparrow \{R(x)\} \ \& \ X \subseteq \{a, c, d, e\})\} = \{x : R(x) \subseteq \{a, c, d, e\}\} = \{c, d, e\}$;

$\mathbb{I}_R(\{a, c, d, e\}) = \bigcup\{X : X \in \mathfrak{B}^R \ \& \ X \subseteq \{a, c, d, e\}\} = \bigcup\{\{c\}, \{d, e\}\} = \{c, d, e\}$.

It is immediate that $\Omega_{\varkappa^R}(U) = \{\emptyset, \{c\}, \{d, e\}, \{c, d, e\}, \{b, d, e\}, \{b, c, d, e\}, U\}$.

Conversely, suppose we are given a topological space $\langle U, \Omega(U) \rangle$, such that $\Omega(U) = \{\emptyset, \{c\}, \{d, e\}, \{c, d, e\}, \{b, d, e\}, \{b, c, d, e\}, U\}$.

Let us set $\mathcal{O}_x = \{O : O \in \Omega(U) \ \& \ x \in O\}$. Then we obtain a reflexive and transitive binary relation S in the following way:

$$\langle x, y \rangle \in S \ \textit{iff} \ y \in \bigcap \mathcal{O}_x \qquad (12.8.2)$$

x	a	b	c	d	e
$\bigcap \mathcal{O}_x$	U	$\{b, d, e\}$	$\{c\}$	$\{d, e\}$	$\{d, e\}$

Clearly, $\bigcap \mathcal{O}_x = \bigcap \mathcal{B}_x^R$. Summing up, we found $S = R$.

Now we have a list of results connected with the fact that preorder is the relational counterpart of the topological property "A set X is open if and only if it is a neighborhood of all its own points". A sentence which, in turn, reflects, as we know, the intuitive reading "If a set X is close to a point x, then it is close to all the points that are sufficiently close to x".

Proposition 12.8.4. *Let* $\mathbf{P} = \langle U, \varepsilon, \varkappa \rangle$ *be a pre-topological space of type* $\mathfrak{V}_S$. *Then* $R^T(\mathbf{P})$ *is a preorder if and only if* $\mathbf{P}$ *is a topological space.*

Proof. In any pre-topology $\mathbf{P}$ of type $\mathfrak{V}_S$, $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}))$ (*Proposition* 12.7.8). Hence from *Proposition* 12.8.2 we obtain the result. $\qquad$ QED

Example 12.8.3. A pre-topological space **P** which is not a topological space, even if $R^T(\mathbf{P})$ is a preorder

This example seems to question *Proposition* 12.8.4. But there is a trick. The topological space $\mathbf{P}^1$ of Example 12.7.4 is such an example. Indeed, it is not difficult to see that $\mathbf{P}^1$ is not a topological space (we have seen that $\varkappa^1(\{a,b\}) \cap \varkappa^1(\{a,c\}) = \{a\} \neq \emptyset = \varkappa(\{a,b\} \cap \{a,c\})$; however, in Example 12.7.4 we have shown that $R^T(\mathbf{P}^1)$ is the identity relation, hence reflexive and transitive).

Notice, anyway, that $\mathbf{P}^1$ is not of type $\mathfrak{V}_S$. Indeed, there are not pre-topological spaces **P** of type $\mathfrak{V}_S$ such that $R^T(\mathbf{P})$ is a preorder but **P** is not a topological space [for a related example, see Frame 15.7].

Exercise 12.13. *Give a direct proof of Proposition 12.8.4.*

Moreover, it is important to point out that there are pre-topologies **P** such that $\Omega_\varkappa(U)$ is a distributive lattice, so that $\langle U, \Omega_\varkappa(U) \rangle$ is a topological space, but **P** is not topological (see a counter example below). This may happen because the interior operator induced by $\Omega_\varkappa(U)$ as a topology on U and $\varkappa$ may fail to coincide. However, this cannot happen if **P** is topological.

Example 12.8.4. A pre-topological space **P** which is not topological such that $\Omega_\varkappa(U)$ is a lattice of sets

Consider the pre-topological space $\mathbf{P}_2$ of Example 12.6.7.

Let us compute the family $\{\varkappa_x\}_{x \in U}$:

x	a	b	c
$\varkappa_x$	$\{\{a,b\},U\}$	$\{\{a,b\},U\}$	$\{\{b,c\},U\}$

P is of type $\mathfrak{V}_s$ because each $\varkappa_x$ is a principal filter. However **P** is not topological because $\varkappa_c$ does not satisfy τ. Indeed, given the member $\{b,c\}$ of $\varkappa_c$ there is no $X \in \varkappa_c$ such that $\{b,c\}$ belongs to $\varkappa_x$ for every $x \in X$. In fact if we hope to have some chance we must consider the least element $\{b,c\}$ of $\varkappa_c$. But $\{b,c\} \notin \varkappa_b$ (indeed, $\varkappa(\varkappa(\{b,c\})) = \emptyset \neq \{c\} = \varkappa(\{b,c\})$), so that idempotence fails.

However, $\Omega_\varkappa(U)$ is clearly a lattice of sets, hence a distributive lattice:
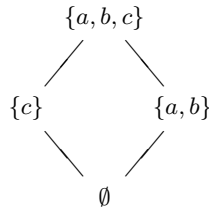
Thus, $\langle U, \Omega_\varkappa(U) \rangle$ is a topological space. Let $\mathbb{I}_{\Omega_\varkappa}$ and $\preceq_{\Omega_\varkappa}$ denote the interior operator and, respectively, the specialization preorder induced by $\Omega_\varkappa(U)$. Since **P**

is not topological, $\varkappa$ and $\mathbb{I}_{\Omega_\varkappa}$ cannot coincide. Indeed here is a counterexample: $\mathbb{I}_{\Omega_\varkappa}(\{c\}) = \{c\} \neq \emptyset = \varkappa(\{c\})$.

Finally, neither $R^T(\mathbf{P})$ and $\preceq_{\Omega_\varkappa}$ coincide. Indeed $\langle b, c \rangle \in R^T(\mathbf{P})$ because $b \in \bigcap \varkappa_c = \{b, c\}$, while $\langle b, c \rangle \notin \preceq_{\Omega_\varkappa}$ because, $c \in \{c\}$ but $b \notin \{c\}$.

The proof of this fact makes it possible to have a brief tour through some of the results so far achieved.

Proposition 12.8.5. *Let* $\mathbf{P} = \langle U, \varkappa, \varepsilon \rangle$ *be a pre-topology. If* $\mathbf{P}$ *is topological then* $\varkappa$ *and the interior operator induced by* $\Omega_\varkappa(U)$ *as a topology on* U *coincide.*

Proof.

1. $R^T(\mathbf{P}) = \preceq_{\Omega_\varkappa}$, where $\preceq_{\Omega_\varkappa}$ is the specialization preorder induced by $\Omega_\varkappa(U)$ *qua* topology on U. In fact, by definition $\langle x, y \rangle \in R^T(\mathbf{P})$ if and only if $y \in \bigcap \varkappa_x$. But from *Proposition* 12.8.3.(1) $\bigcap \varkappa_x$ is the least open set containing x. It follows that $\langle x, y \rangle \in R^T(\mathbf{P})$ if and only if $x \in \bigcap \varkappa_x \Rightarrow y \in \bigcap \varkappa_x$ if and only if $x \preceq_{\Omega_\varkappa} y$ (remember that $x \in \bigcap \varkappa_x$ always holds). This means that the specialization preorder $\preceq$ and $R^T(\mathbf{P})$ coincide.

2. Hence, $R^T(\mathbf{P})$ is a preorder. Recalling that $\mathbf{P}$ is finite and topological, thus of type $\mathfrak{V}_S$, it follows that:

   (a) $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}))$ (from *Proposition* 12.7.8.(2)), so that $\varkappa = \varkappa^{R^T(\mathbf{P})}$.

   (b) $\varkappa^{R^T(\mathbf{P})} = L_{R^T}(\mathbf{P})$ (see *Corollary* 12.7.3).

   (c) $\mathbf{F}(\langle U, \preceq \rangle) = \Omega_\varkappa(U)$.

   (d) $L_{R^T}(\mathbf{P}) = !_{\preceq}$ (suffice it to substitute $\preceq$ for R in *Lemma* 12.7.1). But $!_{\preceq}$ is indeed the interior operator induced by $\mathbf{F}(\langle U, \preceq \rangle)$.

   Hence

3. $!_\varkappa$ is the interior operator induced by $\Omega_\varkappa(U)$.

From 1 and 3 we obtain the result.   QED

REMARKS. Moreover, notice that we can have neighborhood systems $\mathcal{N}(U)$ with a related core map G such that $\{G(X)\}_{X \subseteq U}$ is a lattice of sets but such that G is not a contraction operator. Obviously, in this case G does not coincide with the interior operator induced by $\{G(X)\}_{X \subseteq U}$

qua frame of the open subsets of a topological space. Obviously, in view of *Proposition* 12.4.11, $\mathcal{N}(U)$ cannot be of type $\mathcal{N}_1$. The reader is referred to Example 12.8.5.

Example 12.8.5. A neighborhood system whose core map G induces a topological space but such that G is neither a contraction operator, nor coincides with the interior operator

In Example 12.6.7 we have shown a pre-topological space which is not topological but such that $\Omega_\varkappa(U)$ is a topology. Now we exhibit a neighborhood system such that G is not an interior operator but such that $\Omega_G(U)$ is a topology.

Let $U = \{a, b, c\}$ and let the neighborhood system $\mathcal{N}(U)$ be given by

x	a	b	c
$\mathcal{N}_x$	$\{\{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, U\}$	$\{\{b\}, \{a,b\}, \{b,c\}, U\}$	$\{U\}$

Therefore the core map G is:

X	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\emptyset$	U
$G(X)$	$\{a\}$	$\{b\}$	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,b\}$	$\emptyset$	U

It is easy to check that $\mathcal{N}(U)$ fulfills **N2**, **N3** and **Id**, so that G is idempotent. Moreover, $\{G(X)\}_{X \subseteq U}$ is a distributive lattice. It follows that $\mathbf{T} = \langle U, \{G(X)\}_{X \subseteq U}\rangle$ is a topological space. However G does not coincide with the interior operator of $\mathbf{T}$. In fact, $G(\{b, c\}) = \{a, b\}$ while $\mathbb{I}_\mathbf{T}(\{b, c\}) = \bigvee\{A \in \{G(X)\}_{X \subseteq U} : A \subseteq \{b, c\}\} = \{b\}$. Indeed, $\mathcal{N}(U)$ does not fulfill **N1**, so that G cannot be a contraction operator (cf. $G(\{b, c\})$).

Therefore, the lack of **N1** is the reason why $\mathcal{N}(U)$ does not induce a topological interior operator.

Exercise 12.14.
(a) Draw a graph for $\Omega_\varkappa(U)$ where $\mathbf{P}$ is the pre-topological space of Example 12.4.8.
(b) Is $\Omega_\varkappa(U)$ a lattice of sets?
(c) Is $\Omega_\varkappa(U)$ distributive?
(d) What known properties does $\Omega_\varkappa(U)$ fulfill?

Corollary 12.8.2. *Let $\mathbf{P}(R) = \langle U, \varepsilon^R, \varkappa^R\rangle$ be a topological space associated with a preorder $R \subseteq U \times U$. Then:*

1. *For any $x \in U$, $R(x)$ is the least open set containing x.*

2. *For any open set O, $O = \bigcup_{x \in O} R(x)$ (any open set is a union of minimal R-neighborhoods). We record this fact by saying that*

$\{R(x)\}_{x \in U}$ is a basis of open subsets *for the topological space* $\mathbf{P}(R)$.

3. *For any $X \subseteq U$, $R(X)$ is open.*

4. *X is open if and only if $X = R(X)$.*

5. *For any $x \in U$, $R^{\smile}(x)$ is the least closed set containing x.*

6. *For any $X \subseteq U$, $R^{\smile}(X)$ is closed.*

7. *X is closed if and only if $X = R^{\smile}(X)$.*

8. *$R^T(\mathbf{P}(R)) = \preceq$ (where $\preceq$ is the specialization preorder induced by $\mathbf{P}(R)$).*

Proof. (**1**) From *Proposition* 12.8.3, for any $x \in U$, $\bigcap \varkappa_x^R$ is open. But $\bigcap \varkappa_x^R = R(x)$ (alternative proofs are reported in Frame 15.5).

(**2**) Since O is open, from *Proposition* 12.7.2. (2) we have $O = \varkappa^R(O) = \{x : R(x) \subseteq O\}$. This means that for any $x \in O$, $R(x)$ is included in O. Thus $\bigcup_{x \in O} R(x)$ is included in O. Moreover, if $x \in O$, then $\langle x, x \rangle \in R(x)$, for reflexivity. Hence $x \in \bigcup_{x \in O} R(x)$, so that O, in turn, is included in $\bigcup_{x \in O} R(x)$. (**3**) For any $x \in X$, $R(x)$ is open; therefore, $R(X) = \bigcup_{x \in X} R(x)$ is open (because unions of open sets are open and R-neighboring is additive). (**4**) If $X = R(X)$, then from (3) X is open. Conversely, suppose X is open. Then, from point (2), $X = \bigcup_{x \in X} R(x) = R(X)$. (**5**) From *Proposition* 12.7.2.(4), $\varepsilon^R(\{x\}) = R^{\smile}(x)$. But $\varepsilon^R(\{x\})$ is closed and contains x. (**6**) From (5), using R-neighborhood additivity. (**7**) X is closed if and only if $X = \varepsilon^R(X)$ if and only if $X = R^{\smile}(X)$. (**8**) from Proposition 12.7.8.(3), $R^T(\mathbf{P}(R)) = R$. Obviously, $\mathbf{P}(R)$ is induced by the basis $\{R(x)\}_{x \in U}$. Thus we have to prove: $a \preceq b$ if and only if $\langle a, b \rangle \in R$. But $a \preceq b$ if and only if $a \in R(x) \Rightarrow b \in R(x)$, for all x. Therefore if $a \preceq b$ then $a \in R(a) \Rightarrow b \in R(a)$. But $a \in R(a)$, since R is reflexive. Thus $b \in R(a)$, that is, $\langle a, b \rangle \in R$. Conversely suppose $b \in R(a)$. Therefore if $a \in R(x)$ then, by transitivity, $b \in R(x)$. Hence $a \preceq b$. QED

From *Corollary* 12.8.2.(1) and (2) we obtain a well-known fact about topological spaces, namely that any topological space is induced by a basis of open sets by means of union formation.

Corollary 12.8.3. *Let R be a preorder. Then $\mathfrak{B}^R = \{R(x)\}_{x \in U}$ is a topological basis of a topological space $\langle U, \Omega(U) \rangle$ such that for any $X \subseteq U$, $\mathbb{I}(X) = \bigcup \{Y \in \mathfrak{B}^R : Y \subseteq X\} = \varkappa^R(X)$.*

This is specific of preorder relations. For counterexamples see Example 12.7.4.

TERMINOLOGY AND NOTATION. From now on, given a topological space $\langle U, \varepsilon^R, \varkappa^R \rangle$ associated with a preorder R, the frame of open subsets of U (i.e. the family of open subsets of U equipped with the operations $\cap, \cup, -, U, \emptyset$) will be denoted by $\Omega_R(U)$ and, consequently, the topological space will be also denoted by $\langle U, \Omega_R(U) \rangle$. By $\Gamma_R(U)$ we shall mean the family of closed subsets of U. The interior and the closure operators induced by $\Omega_R(U)$, will be denoted by $\mathbb{I}_R$ and, respectively, $\mathbb{C}_R$ (we recall that $\mathbb{I}_R(X) = \bigcup \{Y \in \Omega_R(U) : Y \subseteq X\}$; $\mathbb{C}_R$ is defined dually). Moreover, remember that $\Omega_{\varkappa^R}(U)$ denotes the set $\{\varkappa^R(X) : X \subseteq U\}$.

Corollary 12.8.4. *Let $\langle U, \varepsilon^R, \varkappa^R \rangle$ be a topological space associated with a preorder $R \subseteq U \times U$. Then for any $X \subseteq U$:*

1. $\Omega_R(U) = \Omega_{\varkappa^R}(U)$; $\Gamma_R(U) = \{\varepsilon^R(X) : X \subseteq U\}$.

2. $\mathbb{I}_{\varkappa^R}(X) = \varkappa^R(X) = \mathbb{I}_R(X)$; $\mathbb{C}_{\varepsilon^R}(X) = \varepsilon^R(X) = \mathbb{C}_R(X)$.

3. *For any $X \subseteq U$, $\mathbb{I}_R(X) = \bigcup \{R(Z) : R(Z) \subseteq X\}$.*

4. *For any $X \subseteq U$, $\mathbb{C}_R(X) = \bigcap \{R^{\smile}(Z) : X \subseteq R^{\smile}(Z)\}$.*

5. $\mathbb{I}_R(X) = \bigcup \{Z : Z \in \Omega_R(U) \ \& \ Z \subseteq X\} = U \overset{\Omega_R}{\Longrightarrow} X$.

6. $\langle \mathbf{B}(U), \Omega_R(U) \rangle$ *is a modal system.*

Proof. (**1**) $\Omega_R(U) = \{X : X = \varkappa^R(X)\}$. So, since $\varkappa^R(X) = \varkappa^R(\varkappa^R(X))$ the result is obvious. (**2**) Immediately from (1) and *Definition* 12.4.6. (**3**) Directly from *Lemma* 12.7.1. (**4**) Let $\mathcal{I} = \{R^{\smile}(Z) : X \subseteq R^{\smile}(Z)\}$. Clearly, since R is reflexive, so is $R^{\smile}$. Thus $X \subseteq R^{\smile}(X)$, so that $R^{\smile}(X) \in \mathcal{I}$. Moreover, because R is transitive, so is $R^{\smile}$. Thus if $X \subseteq R^{\smile}(Y)$, for some $Y \subseteq U$, then $R^{\smile}(X) \subseteq R^{\smile}(Y)$. It follows that $R^{\smile}(X)$ is the least element of $\mathcal{I}$. We conclude that $\bigcap \mathcal{I} = R^{\smile}(X)$. (**5**) Since $\Omega_R(U) = \{R(X) : X \subseteq U\}$, the thesis is just a translation of (3). (**6**) Directly from (5)   QED.

Corollary 12.8.4.(3) tells us that an open set O is a fixpoint of the process of formation of R-neighborhoods limited by some set X. We can restate this image, saying that an open set O is the result of a process π of approximation of a phenomenon X by means of some basic pieces of information: $O = \pi(X)$. As such it is stable: $\pi(\pi(X)) = \pi(X) = \pi(O) = O$. Otherwise stated, it is the core of a "phenomenon", modulo a perception process π. This stability is precisely the nice property we can derive from the topological property (τ) discussed above, which, in turn is strictly connected with transitivity. Indeed, transitivity makes it possible to drill down until the limit, or to collect everything that immediately or mediately is connected with a given perception point x.

We have seen that in order to obtain this nice property we have to renounce some dynamic features. Classical Rough Set Theory is within this choice. And in this framework we can review the story of the modal operators we have suspended at the end of the last paragraph.

Let us continue it.

In view of *Proposition* 12.8.2.(2) we have that if O is open, $x \in O$ and $\langle x, y \rangle \in R$, then $y \in O$. From this, one can easily understand why open sets are images of the necessity operator L.

In fact, compare the last property with the definitions of L as shown in the table at the end of Section 12.1. In view of those definitions, in set-theoretical terms we have: $x \in [\![L(\alpha)]\!]$ iff $\forall y, \langle x, y \rangle \in R \Rightarrow y \in [\![\alpha]\!]$. That is to say, $x \in [\![L(\alpha)]\!]$ iff $R(x) \subseteq [\![A]\!]$. Therefore if we are given a topological space $\langle U, \varepsilon^R, \varkappa^R \rangle$ and for any formula α, $[\![\alpha]\!]$ is a subset of U, then $[\![L(\alpha)]\!]$ is to be interpreted as the largest open subset included in $[\![\alpha]\!]$.

Indeed, we have:

Corollary 12.8.5. *Let* $\langle \mathbf{B}(U), L_R \rangle$ *be a pre-monadic Boolean algebra of the powersets of a set* U. *Let* R *be a preorder and* $X \subseteq U$. *Then,*

1. $L_R(X) = \bigcup \{R(Z) : R(Z) \subseteq X\}$.

2. $M_R(X) = \bigcap \{R^{\smile}(Z) : R^{\smile}(Z) \supseteq X\}$.

Proof. Straightforwardly, from *Corollaries* 12.7.3, 12.7.1 and 12.8.4. QED

In Frame 15.6 we give a direct proof of the second equation. Therefore, we have accomplished almost all the moves listed in the last table of Section 12.1.

Chapter 13

Modalities, Topologies and Algebras

13.1 Topological Boolean Algebras

We pack the above properties in the following definition:

Definition 13.1.1. *Let $\langle U, \Omega(U) \rangle$ be a topological space. Then the pair $\langle \mathbf{B}(U), \mathbb{I} \rangle$ is said to be a* topological Boolean algebra of sets.

Corollary 13.1.1. *Let $\langle \mathbf{B}(U), L_R \rangle$ be a pre-monadic Boolean algebra of the powersets of a set U, such that R is a preorder. Then $\langle \mathbf{B}(U), L_R \rangle$ is a topological Boolean algebra of sets.*

In a more abstract setting, we can easily observe that taking into account the formal properties of the operators $\mathbb{I}$ and $\mathbb{C}$ in a topological space $\langle U, \Omega(U) \rangle$, we obtain the following definition:

Definition 13.1.2. *Let $\mathbf{B}$ be a Boolean algebra and $\mathbb{I}$ a monadic operator on $\mathbf{B}$ such that, for any $a, b \in \mathbf{B}$*

1. *$\mathbb{I}(1) = 1$.*

2. *$\mathbb{I}(a) \leq a$.*

3. *$\mathbb{I}(\mathbb{I}(a)) = \mathbb{I}(a)$.*

4. *$\mathbb{I}(a \wedge b) = \mathbb{I}(a) \wedge \mathbb{I}(b)$.*

then the pair $\langle \mathbf{B}, \mathbb{I} \rangle$ is called a "topological Boolean algebra, tBa".

From *Definition* 13.1.2, it follows immediately that $\mathbb{I}(a \vee b) \geq \mathbb{I}(a) \vee \mathbb{I}(b)$. Therefore, any *tBa* is a pre-monadic Boolean algebra with additional features (namely (2) and (3) – cf. *Definition* 12.1.3).

Proposition 13.1.1. *Let* $\langle \mathbf{B}, \mathbb{I} \rangle$ *be a* tBa *and* $\mathbb{C}$ *a monadic operator such that for any* $a \in \mathbf{B}, \mathbb{C}(a) = \neg \mathbb{I}(\neg a)$. *Then, for any* $a, b \in \mathbf{B}$,

1. $\mathbb{C}(0) = 0$.

2. $\mathbb{C}(a) \geq a$.

3. $\mathbb{C}(\mathbb{C}(a)) = \mathbb{C}(a)$.

4. $\mathbb{C}(a \vee b) = \mathbb{C}(a) \vee \mathbb{C}(b)$.

The above abstraction is adequate in that the following proposition holds:

Proposition 13.1.2. *Let* $\langle U, \Omega(U) \rangle$ *be a topological space, then* $\langle \mathbf{B}(U), \mathbb{I} \rangle$ *is a* tBa.

Proof. straightforward.   QED

Proposition 13.1.3. *Any* tBa *is a model for the modal system* **S***4.*

For the complete proof see, for instance, Rasiowa [1974], Chapter XIII, where **S**4 is called $\mathcal{S}_{\lambda 4}$. By going back from *Corollary* 12.8.4 through our preceding discussion, we can easily obtain that **S**4 modal systems are characterised by reflexive and transitive binary relations, i.e. preorders (see Section 12.2).[1]

13.2   Monadic Topological Boolean Algebras

The equations stated in *Corollary* 12.8.5 give a partial answer to the problem risen at the end of Section 12.1. To completely solve it we have to understand when $M_R(X) = \bigcap \{R(Z) : R(Z) \supseteq X\}$.

Immediately we observe that the second equation holds whenever $R = R^{\smile}$. So, let us specialize the above results for the case when the binary relation at hand is an equivalence relation.

---

[1]However, if we are confined to finite partial orders we characterise the logic S4**GRZ** – see above.

Proposition 13.2.1. *Let $\langle U, \Omega_R(U) \rangle$ be a topological space associated with a relation $R \subseteq U \times U$. Then $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) = \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$ if and only if R is an equivalence relation.*

Proof.

(A) $\Rightarrow$: Let R be an equivalence relation. Since $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) \supseteq \mathbb{I}_R(X) \cup \mathbb{I}_R(\mathbb{I}_R(Y))$, from idempotence of $\mathbb{I}_R$ we have $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) \supseteq \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$. Therefore we have to prove the reverse inclusion. So, let $a \in \mathbb{I}_R(X \cup \mathbb{I}_R(Y))$; we prove that $a \in \mathbb{I}_R(X)$ or $a \in \mathbb{I}_R(Y)$. We recall that $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) = \{x : R(x) \subseteq X \cup \mathbb{I}_R(Y)\} = \{x : R(x) \subseteq X \cup \{y : R(y) \subseteq Y\}\}$. Thus, $R(a) \subseteq X \cup \{y : R(y) \subseteq Y\}$, so that for any $a' \in R(a)$, $a' \in X \cup \{y : R(y) \subseteq Y\}$. Therefore, let $a' \in \{y : R(y) \subseteq Y\}$, then $R(a') \subseteq Y$. But $R(a') = R(a)$, because R is an equivalence relation. Hence, in this case, $a \in \mathbb{I}_R(Y)$. Otherwise $R(a') \cap Y = \emptyset$. But in this case we must have $R(a) \subseteq X$ and $a \in \mathbb{I}_R(X)$.

(B) $\Leftarrow$: Assume now that $\mathbb{I}_R(X \cup \mathbb{I}_R(Y)) \subseteq \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$. We have to prove that R is an equivalence relation. Suppose $\mathbb{I}_R(X \cup \mathbb{I}_R(Y))$ is not included in $\mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$. We show that in this case R cannot be an equivalence relation. So, assume (i) $a \in \mathbb{I}_R(X \cup \mathbb{I}_R(Y))$ and (ii) $a \notin \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$. Therefore, $a \in \{x : R(x) \subseteq X \cup \mathbb{I}_R(Y)\}$. However, $R(a)$ cannot be included in X, otherwise $a \in \mathbb{I}_R(X)$. It follows that there is an $a' \in R(a)$ such that $a' \in \mathbb{I}_R(Y)$. This means that $R(a') \subseteq Y$. Suppose $R(a') = R(a)$. In this case $R(a) \subseteq Y$ and $a \in \mathbb{I}_R(Y)$, which contradicts our assumption (ii). Henceforth, $R(a) \neq R(a')$ (if R is transitive, $R(a') \not\subseteq R(a)$). It follows immediately that R is not an equivalence relation. QED

Proposition 13.2.2. *Let $\langle U, \Omega_E(U) \rangle$ be a topological space associated with an equivalence relation $E \subseteq U \times U$. Then,*

1. *$\Omega_E(U) = \Gamma_E(U)$.*

2. *$\Omega_E(U)$, is a Boolean algebra.*

3. *$\langle U, \Omega_E \rangle = \langle U, \mathbf{AS}(U/E) \rangle$.*

Proof. (**1**) $X \in \Omega_E(U)$ if and only if $X = \mathbb{I}_E(X) = \varkappa^E(X)$ if and only if $X = E(X)$, if and only if $X = E^{\smile}(X)$, if and only if $X = \varepsilon^E(X) = \mathbb{C}_E(X)$, if and only if $X \in \Gamma_E(U)$. (**2**) Since $\mathbb{I}_E$ and $\mathbb{C}_E$ are dual, from point (1) we have that if $X \in \Omega_E(U)$, then $X = \mathbb{I}_E(X)$, so that $-X = -\mathbb{I}_E(X) = \mathbb{C}_E(-X)$. Therefore, $-X \in \Gamma_E(U) = \Omega_E(U)$. So,

$\Omega_E(U)$ is closed under complementation. Moreover, from *Proposition* 12.5.5 and point (1), if $X, Y \in \Omega_E(U)$, then both $X \cup Y$ and $X \cap Y$ belong to $\Omega_E(U)$. Moreover, $\varkappa^E(U) = U$ and $\varkappa^E(\emptyset) = \emptyset$. Therefore $\Omega_E(U)$ is a Boolean algebra of sets. (**3**) From the definitions of Ω_E and **AS**(U/E).   QED

So far, we have distilled the topological features of Approximation Spaces. Now we have enough material in order to understand why the pair $\langle \mathbf{B}(U), L_E \rangle$, where L_E is induced by an Approximation Space $\langle U, \Omega_E \rangle = \langle U, \mathbf{AS}(U/E) \rangle$, is a particular kind of *topological Boolean algebra of sets*.

As we have seen, this term applies, more in general, to any pair $\langle \mathbf{B}(U), L \rangle$ where L is the interior operator of any topology on U. In particular, it applies to the pair $\langle \mathbf{B}(U), L_R \rangle$ where L_R is induced by the topology $\{\varkappa^R(X) : X \subseteq U\}$ for some transitive and reflexive relation R. The distinguishing properties of Approximation Spaces, *qua* topological Boolean algebra of sets, are consequences of the fact that Approximation Spaces are induced not just by generic preorders, but by equivalence relations:

Corollary 13.2.1. *Let $\langle \mathbf{B}(U), L_R \rangle$ be a pre-monadic Boolean algebra of the powersets of a set U. Let R be an equivalence relation and $X \subseteq U$. Then,*

    1. $L_R(X) = \bigcup \{R(Z) : R(Z) \subseteq X\}$.

    2. $M_R(X) = \bigcap \{R(Z) : R(Z) \supseteq X\}$.

Proof. Straightforward, from *Corollary* 12.8.5.   QED

The above result completes the answer to the problem risen at the end of Section 12.1.

We summarize the properties of pre-monadic Boolean algebras induced by topological spaces associated with equivalence relations, in the following corollary:

Corollary 13.2.2. *Let $\langle \mathbf{B}(U), L_R \rangle$ be a pre-monadic Boolean algebra of the powersets of a set U, such that R is an equivalence relation. Then $\langle \mathbf{B}(U), L_R \rangle$ is a monadic topological Boolean algebra of sets.*

Proof. Straightforwardly from *Definition* 12.1.6 and *Proposition* 13.2.1.
<div align="right">QED</div>

In abstraction we set:

Definition 13.2.1. *If* $\langle \mathbf{B}, \mathbb{I} \rangle$ *is a* tBa *such that, for any* $a, b \in \mathbf{B}$, $\mathbb{I}(a \vee \mathbb{I}(b)) = \mathbb{I}(a) \vee \mathbb{I}(b)$, *then it is called a* "monadic topological Boolean algebra (mtBa)".

Corollary 13.2.3. *Any monadic Boolean algebra of sets* $\langle \mathbf{B}(U), L_R \rangle$ *is a mtBa of sets.*

Proof. From *Definition* 12.1.6.(2) and *Proposition* 13.2.1 the proof follows.
<div align="right">QED</div>

Proposition 13.2.3. *For all tBa* $\langle \mathbf{A}, L \rangle$, $\mathbf{L}(\mathbf{A})$ *is a sublattice of* $\mathbf{A}$.

Proof. Let $x, y \in \mathbf{L}(\mathbf{A})$. Then $x \wedge y = L(x) \wedge L(y)$ (from idempotence of L. Thus $x \wedge y = L(x \wedge y)$. Since for all $a, b \in \mathbf{A}$, $L(a) \vee L(b) \le L(a \vee b)$, if $x, y \in \mathbf{L}(\mathbf{A})$, $x \vee y \le L(x \vee y)$. But $L(x \vee y) \le x \vee y$ (because L is deflationary). It follows that $x \vee y = L(x \vee y)$.
<div align="right">QED</div>

Proposition 13.2.4. *For all mtBa* $\langle \mathbf{A}, L \rangle$, $\mathbf{L}(\mathbf{A})$ *is a Boolean algebra.*

Proof. We have to prove that $\mathbf{L}(\mathbf{A})$ is closed under complementation. Let $x \in \mathbf{L}(\mathbf{A})$. Since $x \wedge -x = 0$ we have $L(x \wedge -x) = 0$ (from the deflationary property). Thus, $L(x \wedge -x) = L(x) \wedge L(-x) = 0$. Moreover, $x \vee -x = 1$ so that $L(L(x) \vee -x) = 1$ (because $x = L(x)$). From monadicity, $L(x) \vee L(-x) = x \vee L(-x) = 1$. Thus $L(-x)$ is the complement of x, because $\mathbf{L}(\mathbf{A})$ is a sublattice of $\mathbf{A}$. We conclude that $L(-x) = x$ and $\mathbf{L}(\mathbf{A})$ is closed under complementation.
<div align="right">QED</div>

Corollary 13.2.4. *In any mtBa* $\langle \mathbf{A}, L \rangle$, $\mathbf{M}(\mathbf{A}) = \{M(x) : x \in \mathbf{A}\}$ *coincides with* $\mathbf{L}(\mathbf{A})$.

Proof. For all $x \in \mathbf{A}$, $M(x) \in \mathbf{L}(\mathbf{A})$. In fact, $M(x) = -L(-x)$. But from *Proposition* 13.2.4 $-L(-x) \in \mathbf{L}(\mathbf{A})$. For all $y \in \mathbf{A}$, $L(x) \in \mathbf{L}(\mathbf{A})$: dually.
<div align="right">QED</div>

Corollary 13.2.5. *In any mtBa* $\langle \mathbf{A}, L \rangle$, *(a)* $ML(x) = L(x)$; *(b)* $LM(x) = M(x)$, *any* $x \in \mathbf{A}$.

Proof. (a) From *Proposition* 13.2.4, for all $x \in \mathbf{A}$, $-L(x) \in \mathbf{L}(\mathbf{A})$. Therefore, $-L(x) = L - L(x) = -ML(x)$. It follows that $L(x) = ML(x)$. (b) By duality.
<div align="right">QED</div>

Proposition 13.2.5. *Any* mtBa *is a model for the modal system* **S***5.*

As to the proof see, for instance, Rasiowa [1974], Chapter XIII, where **S**5 is called $\mathcal{S}_{\lambda 5}$. This result confirms what we have discussed in Section 12.2: **S**5 modal systems are characterised by symmetric, reflexive and transitive binary relations, i.e., equivalence relations.

The following results link the definitions of lower and upper approximations to the fact that $\langle \mathbf{B}(U), \Omega_E(U) \rangle$ is a modal system:

Corollary 13.2.6. *Let $\langle U, \Omega_E(U) \rangle$ be a topological space associated with an equivalence relation $E \subseteq U \times U$ and let $\langle U, E \rangle$ be the Indiscernibility Space based on E. Then for any $X \in \mathbf{B}(U)$,*

1. $L_E(X) = \mathbb{I}_E(X) = \bigcup\{Y : Y \in \Omega_E \ \& \ Y \subseteq X\} = \bigcup\{[x]_E : [x]_E \subseteq X\} = (lE)(X).$

2. $M_E(X) = \mathbb{C}_E(X) = \bigcap\{Y : Y \in \Omega_E \ \& \ Y \supseteq X\} = \bigcap\{[x]_E : X \subseteq [x]_E\} = (uE)(X).$

3. (i) $L_E(M_E(X)) = M_E(X)$; (ii) $M_E(L_E(X)) = L_E(X).$

Proof. (**1**) and (**2**) come straightforward from the above results. (**3**) $L_E(X)$ is an open set. Thus it has the form $E(Y)$ for some $Y \subseteq U$ (namely $Y = \bigcup\{E(x) : E(x) \subseteq X\}$). Therefore, $M_E(L_E(X)) = M_E(E(Y)) = E^{\smile}(E(Y)) = E(E(Y)) = E(Y) = L_E(X)$. Dually for the first equation.   QED

Corollary 13.2.7. *Let $\langle U, \mathbf{AS}(U) \rangle$ be an Approximation Space. Then $\langle \mathbf{B}(U), (lE) \rangle$ (or $\langle \mathbf{B}(U), (uE) \rangle$) is a* mtBa *of sets.*

The two axioms which characterise **S**5, that is $L(M(\alpha)) \longleftrightarrow M(\alpha)$ and $M(L(\alpha)) \longleftrightarrow L(\alpha)$ say, in logical terms, that any string $(m_1, ..., m_n)$ of nested modal operators $m_i \in \{[R], \langle R \rangle\}$, collapses into the one-term string m_n. In our Rough Set reading, this collapse says that any single approximation of a subset of the universe of discourse provides an *exact set*, that is a set invariant under further approximations.

At this point we can list a series of connections between some fundamental results we have proved so far.

- From a topological point of view, in any mtBa, for all $x \in \mathbf{L}(\mathbf{A})$, $x = \mathbb{I}C(x)$ (from *Corollary* 13.2.5). Hence any $x \in \mathbf{L}(\mathbf{A})$ is a regular element [cf. Subsection 7.3.1 of Chapter 7].

- Any Approximation Space $\mathbf{AS}(U/E)$ is a Boolean subalattice of the Boolean algebra of $\wp(U)$.

- In $\mathbf{S}5$ modal systems, $ML(\alpha) \longleftrightarrow L(a)$ and $LM(\alpha) \longleftrightarrow M(a)$, any formula α) [cf. Table 12.3 of Section 12.2].

- In any Approximation Space $\mathbf{AS}(U/E)$, for any $X \subseteq U, (uE)(X)$ and $(lE)(X)$ are exact elements. Hence, $(lE)(uE)(X) = (uR)(X)$ and $(uR)(lE)(X) = (lE)(X)$.

- $M_R \dashv L_R$ (because $R = R^{\smallsmile}$, from *Proposition* 13.2.1 and *Proposition* 13.2.1) [cf. *Corollary* 8.2.1 of Chapter 8].

Example 13.2.1. A topological Boolean algebra and a monadic topological Boolean algebra

The pre-monadic Boolean algebra $\langle \mathbf{A}, L_2 \rangle$ of Example 12.1.4 is a topological Boolean algebra. The structure $\langle \mathbf{A}, L' \rangle$ with the operator L' below, is a monadic topological Boolean algebra:

x	0	a	b	c	d	e	f	1
$L'(x)$	0	0	b	0	b	e	b	1

The sublattices $\mathbf{L}'(\mathbf{A})$ of the images of the operator L' coincides with that of the monadic Boolean algebra $\langle \mathbf{A}, L_m \rangle$ of Example 12.1.5. However, $\langle \mathbf{A}, L_m \rangle$ is not topological because, for instance, $L_m L_m(d) = 0 \neq b = L_m(d)$ (i.e. L_m is not idempotent).

Chapter 14

The Propositional Modal Logic of Rough Sets

14.1 Introduction

In this Section we introduce the notion of a *monadic topological quasi Boolean algebra*. It turns out that monadic topological quasi-Boolean algebras are rough set systems construed after starting with topological Boolean algebras, that is, algebras exhibiting an explicit modal operator L (intuitively, lower approximation) and its dual M. Actually, monadic topological quasi-Boolean algebras can be made into Rough Sets Systems by translating L with $\lrcorner\lrcorner$. And *vice-versa*.

It will be proved that monadic topological quasi-Boolean algebras are the Lindenbaum algebras of a logic that, therefore, we can call a "logic of Rough Sets. In what follows we need to set up a bridge between syntax and semantics. This bridge is provided by a method of defining a family $\mathcal{LB}$ of sets of formulas in a suitable way, so that $\mathcal{LB}$ may be the carrier of an algebraic structure.

Let $\mathcal{L}$ be a propositional language with logical constants $\wedge, \vee, \neg \Longrightarrow$. Given a logical system $\mathfrak{L}$ on $\mathcal{L}$ we define a relation $\sim$ over formulas of $\mathcal{L}$ as follows:

$$\varphi \sim \psi \ \textit{iff} \ \vdash \varphi \Longleftrightarrow \psi$$

provided that $\Longrightarrow$ has the property: if $\models_{\mathfrak{L}} \alpha \Longrightarrow \beta$ and $\models_{\mathfrak{L}} \alpha$ then $\models_{\mathfrak{L}} \beta$.

With suitable axioms of the logical system $\sim$ turns out to be an equivalence relation.

Let $LB = \mathcal{L}/\sim$ be the set of $\sim$ equivalence classes. We define the operations $\oplus$ and $\cdot$, $\ni$, $\overline{[\alpha]}$ on LB by:

$$
\begin{array}{rcl}
[\alpha] \oplus [\beta] & = & [\alpha \vee \beta] \\
[\alpha] \cdot [\beta] & = & [\alpha \wedge \beta] \\
\overline{[\alpha]} & = & [\neg \alpha] \\
[\alpha] \ni [\beta] & = & [\alpha \Longrightarrow \beta]
\end{array}
$$

Due to choice of appropriate axioms and rules, above definitions remain valid, the set of all theorems (tautologies) forms a class, 1, and the set of all antitheorems (contradictions) forms a class, 0.

The structure $\mathbf{LB}(\mathfrak{L}) = (LB, \oplus, \cdot, \ni, ^{-}, 0, 1)$ is called the *Lindenbaum algebra* of $\mathfrak{L}$.

One can prove that if $\mathfrak{L}$ is a system for propositional Classical Logic, then $\sim$ is a congruence relation and $\mathbf{LB}(\mathfrak{L})$ is a Boolean algebra [to prove this one can turn $\mathbf{LB}(\mathfrak{L})$ into a lattice by setting a partial order $[\alpha] \leq [\beta]$ iff $\alpha \vdash_{\mathfrak{L}} \beta$, i.e. – via the Deduction Theorem – iff $\vdash_{\mathfrak{L}} \alpha \Longrightarrow \beta$; then one shows that $\oplus$ is *sup* (or *lub*) and $\cdot$ is *inf* (or *glb*) with respect to $\leq$ and $[\alpha] \oplus \overline{[\alpha]} = 1$, $[\alpha] \cdot \overline{[\alpha]} = 0$, by exploiting axioms and theorems of Classical Logic. For instance, $\vdash_{\mathfrak{L}} \alpha \Longrightarrow (\beta \Longrightarrow \alpha)$ and $\vdash_{\mathfrak{L}} \beta \Longrightarrow (\alpha \Longrightarrow \beta)$ and the Deduction Theorem shows that $\cdot$ is *inf*].

At this point, by means of Stone's representation theorem (see Subsection 7.2.1 of Chapter 7 and the cross-references therein), one can prove that $\mathbf{LB}(\mathfrak{L})$ is isomorphic to a Boolean algebra of sets (a variant of this fact, namely Sikorski's representation theorem, will be exploited in *Theorem* 14.3.2).

One can apply the method to other systems of logic, as we are going to do.

14.2   From Syntax to Semantics

We have seen that **S**5 and *mtBa* capture the propositional aspects of concrete Approximation Spaces, as *mBa* captures abstract Approximation Spaces (see Section 13.2).

In order to lift from Approximation Spaces to a modal logic of Rough Sets we need to adequately account for the syntactic counterpart of the concept of rough equality.

Let us consider the modal system **S**5. The language of **S**5 is the usual classical propositional language with monadic modalities L (necessity) and M (possibility) such that one is definable in terms of the other e.g. $M = \neg L \neg$.

Axioms of **S**5 are the usual propositional logic axioms along with the following additional axioms for L and M.

1. $L(\alpha \implies \beta) \implies (L(\alpha) \implies L(\beta))$.

2. $L(\alpha) \implies \alpha$.

3. $M(\alpha) \implies LM(\alpha)$.

The rules of inference are

$$\frac{\alpha, \alpha \implies \beta}{\beta} \text{ (modus ponens)}$$

and

$$\frac{\alpha}{L(\alpha)} \text{ (necessitation)}$$

As above, in the algebra $\mathfrak{F}$ of well formed formulas an equivalence relation R_1 is defined by $\langle \alpha, \beta \rangle \in R_1$ if and only if $\vdash_{S5} \alpha \iff \beta$.

In the quotient space $\mathfrak{F}/R_1$, all the operators in $\mathfrak{F}$ may be extended. That this can be done is due to the theorems obtained in **S**5 which reduces the equivalence relation R_1 into a congruence relation. The quotient algebra $\mathfrak{F}/R_1$ is the Lindenbaum algebra for the logic system **S**5.

Exercise 14.1. *To check that $\mathfrak{F}/R_1$, is a monadic tBa.*

Exercise 14.2. *To show that all theorems of **S**5 (i.e. α such that $\vdash_{S5} \alpha$) constitute the top element 1 and all anti theorems (i.e. α such that $\vdash_{S5} \neg\alpha$) constitute the least element 0 of $\mathfrak{F}/R_1$.*

An evaluation v maps any wff α to a subset $v(\alpha)$ of U of an approximation system $\langle U, AS(U/E) \rangle$ such that

1. $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$.

2. $v(\alpha \vee \beta) = v(\alpha) \cup v(\beta)$.

3. $v(\neg\alpha) = -v(\alpha)$.

4. $v(L(\alpha)) = (lE)(v(\alpha))$.

5. $v(M(\alpha)) = (uE)(v(\alpha))$.

Let $\models \alpha$ be defined as $v(\alpha) = U$ for all evaluations in all approximation systems $\langle U, AS(U/E) \rangle$. Then the standard **S**5 soundness – completeness (meta)theorems state that

$$\vdash_{\mathbf{S5}} \alpha \text{ iff } \models \alpha.$$

The proof hinges upon the above mentioned Lindenbaum construction and the fact that the canonical mapping $\alpha \longmapsto [\alpha]_{R_1}$ is indeed an evaluation in the defined sense and also that $\vdash_{\mathbf{S5}} \alpha$ iff $[\alpha]_{R_1} = 1$ in the Lindenbaum algebra. This is, however a standard text book exercise.

It should, however, be noted that the Lindenbaum construction is nothing special of the modal system **S**5.

If there is an inference procedure $\vdash_{\mathcal{L}}$ in a logic $\mathcal{L}$ satisfying the conditions that $\alpha \vdash_{\mathcal{L}} \alpha$ and $\alpha \vdash_{\mathcal{L}} \beta$ and $\beta \vdash_{\mathcal{L}} \gamma$ implies $\alpha \vdash_{\mathcal{L}} \gamma$, then an equivalence relation R can always be defined by $\langle \alpha, \beta \rangle \in R$ iff $\alpha \vdash_{\mathcal{L}} \beta$ and $\beta \vdash_{\mathcal{L}} \alpha$ hold.

Also let it be assumed that the logical operations preserve substitution of equivalents and that the set of theorems *viz.* $\{\alpha : \vdash_{\mathcal{L}} \alpha\}$ is closed under substitution by equivalents. Then the quotient algebra $\mathcal{L}/R$ may be constructed and it is possible to extend the basic logical operations to this algebra that will construe the Lindenbaum algebra for the logic. The canonical map viz. $\alpha \longmapsto [\alpha]_R$ will make an evaluation in a natural way. This quotient algebra shall be instrumental to proving soundness and completeness of the logic $\mathcal{L}$ relative to set theoretic semantics.

So, the system **S**5 and tBa appear to capture the propositional aspects of rough set theory provided that sets with the same lower and upper approximations are considered to be identical in rough context in all respects and logical operations are classical. We have, under this consideration that $\langle \alpha, \beta \rangle \in R_1$ if and only if $\vdash_{\mathbf{S5}} \alpha \iff \beta$ i.e. $v(\alpha)$ is the same set as $v(\beta)$ whatever interpretations be given to the constituent atomic formulae in α and β and whatever approximation space is taken as the domain of interpretation.

But this interpretation falls short of the real import of rough set theory that would intend to make no distinction between $v(\alpha)$ and $v(\beta)$ when these are roughly equal, that is, $v(\alpha) \approx v(\beta)$.

In other words we would require syntactic ways for the interpretations of wffs α and β to be roughly equal in all models.

To express that, let us introduce a new binary connective in the language of **S**5. By abuse of notation we shall borrow the symbol $\approx$ from

semantics. Thus, $\alpha \approx \beta$ stands for $(L(\alpha) \iff L(\beta)) \wedge (M(\alpha) \iff M(\beta))$, where α, β are any well-formed formulae.

Then, the syntactic counterpart of the above demand is reflected in $\vdash_{\mathbf{S5}} \alpha \approx \beta$.

Hence a binary relation R in $\mathfrak{F}$ may be defined by $\langle \alpha, \beta \rangle \in R$ if and only if $\vdash_{\mathbf{S5}} \alpha \approx \beta$ and R is an equivalence relation partitioning $\mathfrak{F}$ giving rise to the quotient set $\mathfrak{F}/R$. A few interesting observations are the following

- The set $\mathfrak{F}/R$ with $\leq$ defined by $[\alpha] \leq [\beta]$ iff $\vdash_{\mathbf{S5}} (L(\alpha) \implies L(\beta)) \wedge (M(\alpha) \implies M(\beta))$, is a partially ordered set. So, $[\alpha] \leq [\beta]$ if and only if $v(\alpha)$ is roughly included in $v(\beta)$ for any evaluation in any approximation space. The zero, 0, and the unit, 1, of this bounded poset are, as before, the equivalence classes of antitheorems and theorems of $\mathbf{S5}$ respectively.

- R is not a congruence relation, so, Lindenbaum construction is not possible in this situation.

But a Lindenbaum-like construction can be carried out.

In fact, In the poset $\langle \mathfrak{F}/R, \leq \rangle$, we are able to define the following operations, for any classes $[\alpha], [\beta] \in \mathfrak{F}/R$,

F1 $\neg[\alpha] \equiv [\neg\alpha]$.

F2 $[\alpha] \sqcap [\beta] \equiv [(\alpha \wedge \beta) \vee (\alpha \wedge M(\beta) \wedge \neg M(\alpha \wedge \beta))]$.

F3 $[\alpha] \sqcup [\beta] \equiv [(\alpha \vee \beta) \wedge (\alpha \vee L(\beta) \vee \neg L(\alpha \vee \beta))]$.

F4 $L[\alpha] \equiv [L(\alpha)]$.

F5 The zero (0) and unit (1) of the poset are respectively the equivalence classes of all antitheorems (α is an antitheorem if and only if $\vdash_{\mathbf{S5}} \neg\alpha$) and theorems of $\mathbf{S5}$.

where $\neg, \wedge$ and $\vee$ are the operations of $\mathbf{S5}$.

The structure $\mathbf{A} = \langle \mathfrak{F}/R, \leq, \sqcap, \sqcup, \sim, L, 0, 1 \rangle$, turns out to be short of a monadic topological Boolean algebra, in that the laws of contradiction and excluded middle (viz. $x \sqcap \sim x = 0$ and $x \sqcup \sim x = 1$, respectively) fail to hold in it generally, in contrast with monadic Boolean algebras (because of their Boolean basis). It is, indeed, a quasi-Boolean algebra (see Part II) that has, in addition, a monadic topological operation

(i.e. L). Thus we arrive at the notion of a *monadic topological quasi-Boolean algebra*:

Definition 14.2.1. *An algebra* $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, 0, 1 \rangle$ *is a* monadic topological quasi-Boolean algebra *(mtqBa) if:*

A1. $\langle A, \leq, \wedge, \vee \rangle$ *is a distributive lattice.*

A2. $\sim\sim a = a.$

A3. $\sim (a \vee b) = \sim a \wedge \sim b.$

A4. $L(a) \leq a.$

A5. $L(a \wedge b) = L(a) \wedge L(b).$

A6. $L(L(a)) = L(a).$

A7. $L(1) = 1.$

A8. $M(L(a)) = L(a),$ *where* $M(a) \equiv \sim L(\sim a),$ $a, b \in A.$

A question naturally arises at this juncture: what is the logic that corresponds to monadic topological quasi-Boolean algebras?

Moreover, given an Indiscernibility Space $\langle U, E \rangle$, the Rough Set-counterpart of $\mathbf{A}$ will be the mtqBa $\langle \wp(U)/\approx, \overline{\sqsubseteq}, \sqcap, \sqcup, \sim, L, [\emptyset], [U] \rangle$, where $\approx$ is the relation of rough equality (an equivalence on $\wp(U)$) induced by the equivalence relation E. The binary relation $\overline{\sqsubseteq}$ and the other operations on the quotient set $\wp(X)/\approx$ are defined as follows. Let $[S], [T]$ be members of $\wp(U)/\approx$.

R1 $[S]\overline{\sqsubseteq}[T]$ if and only if $(lE)(S) \subseteq (lE)(T)$ and $(uE)(S) \subseteq (uE)(T)$ (that is, S is roughly included in T).

R2 $[S] \sqcap [T] = [S \sqcap T].$

R3 $[S] \sqcup [T] = [S \sqcup T].$

R4 $\sim [S] = [-S].$

R5 $L([S]) = [(lE)(S)].$

where

(i) $S \sqcap T = (S \cap T) \cup (S \cap (uE)(T) \cap -(uE)(S \cap T))$ and
(ii) $S \sqcup T \equiv (S \cup T) \cap (S \cup (lE)(T) \cup -(lE)(S \cup T)),$

$\cap, \cup, -$ denoting the operations of intersection, union, and complementation in $\wp(U)$ respectively.

Recalling the discussion in Part II, it is possible to show that the above algebra can be embedded in the mtqBa that is obtained by equipping the semi-simple Nelson algebra $\langle RS(U), \leq, \wedge, \vee, \sim, 0, 1 \rangle$ with a monadic topological operator L such that for any $\langle X_1, X_2 \rangle \in RS(U)$, $L(\langle X_1, X_2 \rangle) = \langle X_2, X_2 \rangle$.[1] Thus we obtain the mqBa $\mathbf{TRS}(U) = \langle RS$ $(U) \leq, \wedge, \vee, \sim, L, 0, 1 \rangle$, where the operations on $RS(U)$ are those described in Subsection 7.1.1 of Chapter 7 and for any $\langle X_1, X_2 \rangle \in RS(U)$, $L(\langle X_1, X_2 \rangle) = \langle X_2, X_2 \rangle$.

The above construction is a particular case of the following:

Lemma 14.2.1. *Let* $\mathbf{B} = \langle B, \vee, \wedge, \neg, 0, 1 \rangle$ *be a Boolean algebra. Let* $\mathbf{L} = \langle \mathbf{B}, L \rangle$ *be a mBa. Let us consider the families of ordered pairs* $B^{[3]} = \{\langle a, b \rangle : b \leq a\}_{a,b \in B}$ *and* $ML(\mathbf{L}) = \{\langle M(a), L(a) \rangle\}_{a \in \mathbf{B}}$. *Then,*

1. *The algebra* $\mathbf{ML}(\mathbf{L}) = \langle ML(\mathbf{L}), \leq, \wedge, \vee, \sim, L, 0, 1 \rangle$ *is a mtqBa, called the mtqBa* associated with *the mBa* $\mathbf{L}$.

2. *The algebra* $\mathbf{TQ}(\mathbf{B}) = \langle B^{[3]}, \leq, \wedge, \vee, \sim, L, 0, 1 \rangle$ *is a mtqBa, called the* monadic topological quasi-Boolean extension *of* $\mathbf{B}$,

where for any $\langle a_1, a_2 \rangle$ *and* $\langle b_1, b_2 \rangle$ *belonging to* $B^{[3]}$ *or* $ML(\mathbf{L})$,

1. $1 = \langle 1, 1 \rangle$, $0 = \langle 0, 0 \rangle$.

2. $\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle = \langle a_1 \wedge b_1, a_2 \wedge b_2 \rangle$.

3. $\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle = \langle a_1 \vee b_1, a_2 \vee b_2 \rangle$.

4. $\sim \langle a_1, a_2 \rangle = \langle \neg a_2, \neg a_1 \rangle$.

5. $L(\langle a_1, a_2 \rangle) = \langle a_2, a_2 \rangle$.

The easy proof is left as an exercise. It must only be shown that the operator $\sim$ enjoys properties A2–A3 and the operator L enjoys properties A4–A8 above, where the operator M is defined as usual, viz. $M(\langle a_1, a_2 \rangle) = \sim L(\sim \langle a_1, a_2 \rangle)$, for any $\langle a_1, a_2 \rangle \in B^{[3]}$ (whence, it turns out that $M(\langle a_1, a_2 \rangle) = \langle a_1, a_1 \rangle$) (however, all this machinery was developed in Part II).

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[1] We recall that $RS(U) = \{\langle X_1, X_2 \rangle \in \mathbf{AS}(U) \times \mathbf{AS}(U) : X_2 \subseteq X_1 \ \& \ X_2 \cap B = B \cap X_1\}$, where B is the union of all singleton equivalence classes of U/E.

Further, it is not difficult to prove:

Proposition 14.2.1. *Let* $\mathbf{A} = \langle \mathbf{B}, L \rangle$ *be a mBa and let* $\mathbf{L}(\mathbf{A})$ *be the Boolean sub-algebra of the set* $L(\mathbf{A}) = \{L(a) : a \in \mathbf{B}\}$*. Then* $\mathbf{ML}(\mathbf{A})$ *is a sublattice of* $\mathbf{TQ}(\mathbf{L}(\mathbf{A}))$*.*

In the present context, given any Indiscernibility Space $\langle U, E \rangle$, we give these algebras a special status, calling $\mathbf{ML}(\mathbf{L})$ an *abstract Rough Set algebra* (whenever $\mathbf{L} = \langle \mathbf{B}(U), (lE) \rangle$ and $\mathbf{TQ}(\mathbf{AS}(U))$ a *Rough Set algebra (of sets)*. Moreover, because of the results of the discussion about singleton equivalence classes in an Approximation Space that was developed in Part II, we can notice that $\langle \wp(U)/\approx, \overline{\sqsubseteq}, \sqcap, \sqcup, \sim, L, [\emptyset], [X] \rangle$ is isomorphic to the monadic topological quasi-Boolean extension of $\mathbf{AS}(U)$, $\mathbf{TQ}(\mathbf{AS}(U))$, if and only if no equivalence class is a singleton. Actually we can notice that $\langle \wp(U)/\approx, \overline{\sqsubseteq}, \sqcap, \sqcup, \sim, L, [\emptyset], [X] \rangle$ is isomorphic to $\mathbf{TRS}(U)$, which, in turns, coincides with $\mathbf{ML}(\langle \mathbf{B}(U), (lE) \rangle)$. In other words, $\mathbf{TRS}(U)$ is the mtqBa associated with the mBa (Approximation System) $\langle \mathbf{B}(U), \mathbf{AS}(U) \rangle$. In fact, as we know, its domain is built using the modal operators provided by the mtBa $\langle \mathbf{B}(U), (lE) \rangle$, so that only particular ordered pairs of decreasing elements of $\mathbf{AS}(U)$ are eligible in its domain (viz, those ordered pairs $\langle X_1, X_2 \rangle$ fulfilling besides the property $X_2 \subseteq X_1$ also the property $X_1 \cap C = C \cap X_2$ for all elements C of $\mathbf{AS}(U)$ that are atoms in $\wp(U)$). However, it suffices for our present purpose to consider Rough Set algebras in general, as it will become apparent later.

Example 14.2.1. The smallest non-trivial pre-Rough algebra
The following is the smallest non-trivial pre-Rough algebra:
$\mathbf{T} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$, where $A = \{1, \delta, 0\}$:

$$1$$
$$|$$
$$\delta$$
$$|$$
$$0$$

with $\sim 0 = 1$, $\sim 1 = 0$, $\sim a = a$, $L(1) = 1$, $L(0) = L(a) = 0$.

Example 14.2.2. A topological quasi-Boolean algebra

$$L(a) = L(c) = L(0) = 0$$
$$L(e) = L(b) = b; L(d) = d$$
$$L(f) = f; L(1) = 1$$
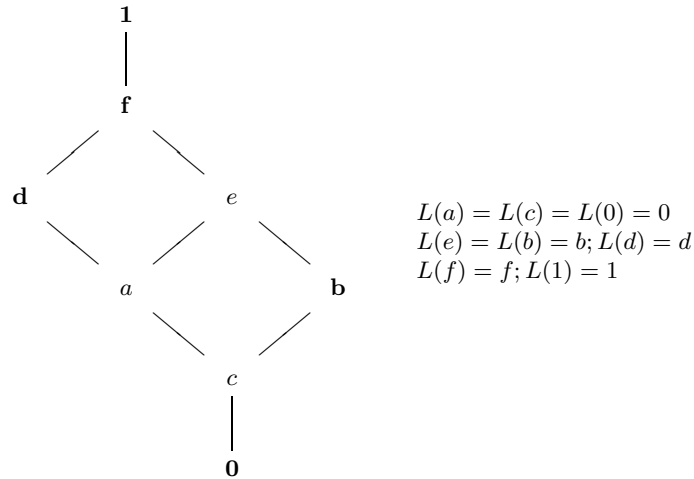
Example 14.2.3. A rough algebra

To show an example of rough algebra, let us draw the monadic topological quasi-Boolean algebra $\mathbf{ML}(\mathbf{L}'(\mathbf{A}))$ associated with the monadic Boolean algebra $\mathbf{L}'(\mathbf{A})$ of Example 13.2.1:
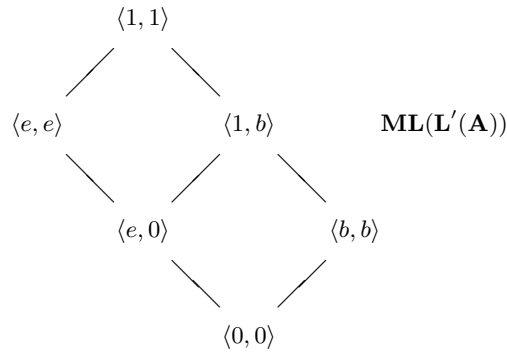
$$\mathbf{ML}(\mathbf{L}'(\mathbf{A}))$$

We can observe that it is isomorphic to the lattice $\mathbf{L_3}$ of Example of Frame 10.4 of Part II. And this does not happen by chance.

Exercise 14.3.

(a) Explain why $\mathbf{ML}(\mathbf{L}'(\mathbf{A}))$ of Example 14.2.3 and the lattice $\mathbf{L_3}$ of the Example of Frame 10.4 of Part II are isomorphic.

(b) Draw the monadic topological quasi-Boolean extension $\mathbf{TQ}(\mathbf{A})$ of $\mathbf{A}$. In view of the discussion about singleton definable sets developed in Part II, explain why $\mathbf{TQ}(\mathbf{A})$ and $\mathbf{ML}(\mathbf{L}'(\mathbf{A}))$ do not coincide.

(c) Exhibit a necessary and sufficient condition for **TQ**(**A**) *and* **ML**(**L**) *to coincide.*

A deeper look into a Rough Set algebra reveals some more properties. On abstraction, it is a mtqBa with some additional axioms. As such, it will be referred to as a *Rough algebra*.

So the earlier question gives way to the following:

- What is the logic corresponding to the class of Rough algebras?

- Could a Rough Set semantics be given to such a logic?

The first issue includes the task of finding an appropriate implication operator $\Rightarrow$, one of its essential properties being

$$a \Rightarrow b = 1 \ iff \ a \leq b \qquad (14.2.1)$$

By a Rough Set semantics we shall mean that a model for this logic is an Approximation Space equipped with a meaning function, that is well formed formulas of the language are interpreted as rough sets. Conjunction and disjunction are not assigned ordinary set intersection ($\cap$) and union ($\cup$), but operations which reduce to $\cap$ and $\cup$ respectively only when working on definable sets. Negation is interpreted as complementation and the necessity operator L as lower approximation. The other salient point of such a model is that implication is interpreted as rough inclusion, and bi-implication as rough equality.

In the following Subsection, we present the complete set of axioms for a Rough algebra, but as an intermediate step, a *pre-Rough algebra* is defined. Two representation theorems, crucial for this work are also proved.

The logics $\mathcal{L}_1, \mathcal{L}_2$ corresponding to pre-Rough and Rough algebras respectively, are proposed in Section 14.4. Subsection 14.4.3 brings forth a smaller class of Rough algebras, that are able to educe soundness and completeness of $\mathcal{L}_2$ as well. This result is used to impart Rough Set semantics to $\mathcal{L}_2$.

14.3   Rough Algebras

We have already observed that any Approximation Space **AS**(U) is a Boolean algebra so that we can consider its monadic topological quasi-Boolean extension **TQ**(**AS**(U)).

In fact, up to isomorphism, one can claim the following.

Lemma 14.3.1. *Let* **B** *be a Boolean algebra, then* **TQ**(**B**) *is a monadic topological quasi-Boolean algebra such that* **B** *is its largest Boolean sub-algebra.*

Proof. Indeed, let us consider the family $L(B^{[3]}) = \{L(\langle a, b\rangle)\}_{\langle a,b\rangle \in B^{[3]}}$. Then it is easy to see that $\mathbf{L}(\mathbf{TQ}(\mathbf{B})) = \langle L(B^{[3]}), \leq, \wedge, \vee, \sim, 0, 1\rangle$ is the largest Boolean sub-algebra of **TQ**(**B**). Also, **B** is clearly isomorphic to $\mathbf{L}(\mathbf{TQ}(\mathbf{B}))$. Indeed, $L(B^{[3]}) = \{\langle X, X\rangle : X \in \mathbf{B}\} = \mathcal{B}(\mathbf{B})$, but $\mathcal{B}(\mathbf{B})$ is isomorphic to **B** (cf. Part II). QED

So, let us look at some more properties of **TQ**(**B**).

Proposition 14.3.1. *In* $\mathbf{TQ}(\mathbf{B}) = \langle B^{[3]}, \leq, \wedge, \vee, \sim, L, 0, 1\rangle$, *for any* $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle$ *in* $B^{[3]}$,

1. $L(\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle) = L(\langle a_1, a_2\rangle) \vee L(\langle b_1, b_2\rangle)$.

2. *If* $L(\langle a_1, a_2\rangle) \leq L(\langle b_1, b_2\rangle)$ *and* $M(\langle a_1, a_2\rangle) \leq M(\langle b_1, b_2\rangle)$, *then* $\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle$.

Proof. By easy verification [*Hints*: $L(\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle) = \langle a_2 \vee b_2, a_2 \vee b_2\rangle = \langle a_2, a_2\rangle \vee \langle b_2, b_2\rangle = L(\langle a_1, a_2\rangle) \vee L(\langle b_1, b_2\rangle)$]. QED

Let us define another binary operation $\Rightarrow$ on **TQ**(**B**) as follows.

Definition 14.3.1. *Let* **TQ**(**B**) *be the monadic topological quasi-Boolean extension of the Boolean algebra* **B**. *For any* $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle$ *in* $B^{[3]}$,

$$\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle \;=\; (\sim L(\langle a_1, a_2\rangle) \vee L(\langle b_1, b_2\rangle))$$
$$\wedge(\sim M(\langle a_1, a_2\rangle) \vee M(\langle b_1, b_2\rangle)).$$

Proposition 14.3.2. *Let* **TQ**(**B**) *be the monadic topological quasi-Boolean extension of the Boolean algebra* **B**. *Then, for any* $\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle$ *in* $B^{[3]}$,

1. $\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle \in L(B^{[3]})$.

2. $\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle = 1$ *if and only if* $\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle$.

3. *If* "$\Longrightarrow$" *is the implication operation in the Boolean algebra* **B**, *i.e.* $a \Longrightarrow b = \neg a \vee b$, *then* $\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle = \langle a_1 \Longrightarrow b_1, a_1 \Longrightarrow b_1\rangle \wedge \langle a_2 \Longrightarrow b_2, a_2 \Longrightarrow b_2\rangle$.

Proof. By easy inspection.   QED

REMARKS. Notice that $a \Rightarrow b \leq a \rightarrowtail b$ (where $\rightarrowtail$ is the implication defined in *Definition* 9.6.1). However, by easy calculation one can verify that while $a \Rightarrow b = \langle a_1 \Longrightarrow b_1 \wedge a_2 \Longrightarrow b_2, a_1 \Longrightarrow b_1 \wedge a_2 \Longrightarrow b_2 \rangle$, $a \rightarrowtail b = \langle a_2 \Longrightarrow b_1, a_1 \Longrightarrow b_1 \wedge a_2 \Longrightarrow b_2 \rangle$. Therefore, if $a \rightarrowtail b = 1$, we must have $a_1 \Longrightarrow b_1 \wedge a_2 \Longrightarrow b_2 = 1$, hence $a_1 \leq b_1$ and $a_2 \leq b_2$. It follows that $a \Rightarrow b = 1$ if and only if $a \rightarrowtail b = 1$.

We can pack the above properties in a new abstract concept.

Definition 14.3.2. *An algebra* $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ *is a pre-Rough algebra, if and only if* $\langle A, \leq, \wedge, \vee, \sim, L, 0, 1 \rangle$ *is a monadic topological quasi-Boolean algebra and, in addition, the following hold for any* $a, b \in A$.

  A9. $\sim L(a) \vee L(a) = 1$.

A10. $L(a \vee b) = L(a) \vee L(b)$.

A11. $L(a) \leq L(b)$ *and* $M(a) \leq M(b)$ *imply* $a \leq b$.

A12. $a \Rightarrow b = (\sim L(a) \vee L(b)) \wedge (\sim M(a) \vee M(b))$.

Proposition 14.3.3. *In any pre-Rough algebra* $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$, $\mathbf{L}(\mathbf{A}) = \langle L(\mathbf{A}), \leq, \wedge, \vee, \sim, 0, 1 \rangle$ *is a Boolean algebra.*

Proof. Left to the reader (*Hints*: use A9).   QED

It must also be recalled that $L(\mathbf{A}) = M(\mathbf{A})$.

Proposition 14.3.4. *Let* $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ *be a monadic topological quasi-Boolean algebra such that* $\mathbf{L}(\mathbf{A}) = \langle L(A), \leq, \wedge, \vee, \sim, 0, 1 \rangle$ *is a Boolean algebra.*
Then $\mathbf{A}$ *is epimorphic to the sub-algebra generated by the set* $\{\langle M(a), L(a) \rangle : a \in A \ \& \ L(a \vee b) = L(a) \vee L(b), \text{ for any } b \in A\}$.

Now we give a representation theorem that allows us to view any pre-Rough algebra as an algebra of pairs of Boolean elements.

Theorem 14.3.1. *(Representation) If* $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ *is a pre-Rough algebra, it is isomorphic to the sub-algebra of the monadic topological quasi-Boolean extension* $\mathbf{TQ}(\mathbf{L}(\mathbf{A}))$ *generated by the set* $\{\langle M(a), L(a) \rangle : a \in A\}$.

Proof. The proof is obtained using *Proposition* 14.3.4 and A11.   QED

The monadic topological quasi-Boolean extension **TQ**(**B**) of any Boolean algebra **B** is a pre-Rough algebra. Moreover, $\langle \mathfrak{F}/R, \leq, \sqcap, \sqcup, \sim, L, \Rightarrow, 0, 1 \rangle$ is a pre-Rough algebra; it may also be noticed that $L(\mathfrak{F}/R) = \mathfrak{F}/R_1$ (cf. the above introduction).
If **AS**(U) is an Approximation System, $\langle \wp(U)/ \approx, \overline{\sqsubseteq}, \sqcap, \sqcup, \sim, L, \Rightarrow, [\emptyset], [U] \rangle$ and the monadic topological quasi-Boolean extension **TQ**(**AS**(U)) are pre-Rough algebras too.
(In each of the preceding examples, $\Rightarrow$ is defined in terms of $\sim, \sqcap$ (or $\wedge$), $\sqcup$ (or $\vee$), L and M as in A12).
In fact, $\langle \wp(U)/\approx, \overline{\sqsubseteq}, \sqcap, \sqcup, \sim, L, \Rightarrow, [\emptyset], [U] \rangle$ and **TQ**(**AS**(U)) enjoy some additional properties that lead to the notion of a *Rough algebra*.

Definition 14.3.3. *An algebra* **A** $= \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ *is a* Rough algebra*, if and only if* **A** *is a pre-Rough algebra such that* **L**(**A**) $= \langle L(A), \leq, \wedge, \vee, \sim, 0, 1 \rangle$ *is a subalgebra of* **A** *that is*

A13. Complete.

A14. Completely distributive, i.e. $\bigvee_{i \in I} \bigwedge_{j \in J} a_{i,j} = \bigwedge_{f \in J^I} \bigvee_{i \in I} a_{i,f(i)}$,
for any index sets I, J *and elements* $a_{i,j}, i \in I, j \in J$, *of* $L(A)$, *J^I being the set of maps of I into J.*

It follows that for any index set I and $a_i \in L(A), i \in I$, $\bigvee_{i \in I} L(a_i) = L(\bigvee_{i \in I} a_i)$.

Proposition 14.3.5. *Given an Approximation Space* **AS**(U), *the monadic topological quasi Boolean extension* **TQ**(**AS**(U)) *is a Rough Algebra.*

Definition 14.3.4. *The Rough algebra* **TQ**(**AS**(U)) *is called the* Rough Set algebra *corresponding to the Approximation Space* **AS**(U).

Theorem 14.3.2. *[Representation] Any Rough algebra is isomorphic to a subalgebra of the Rough Set algebra corresponding to some Approximation Space* **AS**(U).

Proof. Let **A** $= \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ be a Rough algebra. Then **L**(**A**) $= \langle L(A), \leq, \wedge, \vee, \sim, 0, 1 \rangle$ is a complete Boolean subalgebra of **A** that is completely distributive. Hence it is isomorphic to a complete field of sets Sikorski [1969], **C** $= \langle C, \subseteq, \cap, \cup, -, \emptyset, 1 \rangle$, say. **C** is atomic

Sikorski [1969], so let U denote the union of all its atoms. The atoms induce a partition E, of U. Thus we have an Approximation Space $\mathbf{AS}(U)$. It may then be noticed that $\mathbf{C} = \mathbf{AS}(U)$, that is, $\mathbf{C}$ coincides with the collection of all definable sets. So the isomorphism of $\mathbf{L}(\mathbf{A})$ and $\mathbf{C}$ implies the isomorphism of $\mathbf{TQ}(\mathbf{L}(\mathbf{A}))$ and the Rough Set algebra $\mathbf{TQ}(\mathbf{AS}(U))$. Now there is an isomorphic copy of $\mathbf{A}$ in $\mathbf{TQ}(\mathbf{L}(\mathbf{A}))$ (cf. *Theorem* 14.3.1), and hence in $\mathbf{TQ}(\mathbf{AS}(U))$. This is the required subalgebra of $\mathbf{TQ}(\mathbf{AS}(U))$.   QED

We had earlier remarked that Approximation Spaces with no singleton definable set are sufficient for our purpose. *Proposition* 14.3.7 substantiates this remark to an extent. It is obtained as a consequence of the following general result concerning Boolean algebras of sets.

Proposition 14.3.6. *Any atomic Boolean algebra of sets is isomorphic to an atomic Boolean algebra of sets of which no atom is a singleton.*

Proposition 14.3.7. *Given any Indiscernibility Space $\langle U, E \rangle$, there is an Indiscernibility Space $\langle U', E' \rangle$ such that the following hold:*

1. *The Boolean algebras $\mathbf{AS}(U)$ and $\mathbf{AS}(U')$ of definable sets in $\langle U, E \rangle$ and $\langle U', E' \rangle$, respectively, are isomorphic.*

2. *The Rough Set algebras $\mathbf{TQ}(\mathbf{AS}(U))$ and $\mathbf{TQ}(\mathbf{AS}(U'))$ corresponding to $\mathbf{AS}(U)$ and $\mathbf{AS}(U')$, respectively, are isomorphic.*

3. *For any element $\langle D_1', D_2' \rangle$ of $\mathbf{TQ}(\mathbf{AS}(U'))$, there is a rough set S in the Approximation System $\langle \mathbf{B}(U'), \mathbf{AS}(U') \rangle$ such that $(lE')(S) = D_2'$ and $(uE')(S) = D_1'$.*

14.4   The Systems $\mathcal{L}_1, \mathcal{L}_2$

Let us now look at the formal systems $\mathcal{L}_1$ and $\mathcal{L}_2$. It will be shown that these are sound and complete relative to the class of all pre-rough algebras and rough algebras respectively.

14.4.1   The System $\mathcal{L}_1$

The language of $\mathcal{L}_1$ consists of propositional variables $p, q, r, ...$, logical symbols $\sim, \sqcap, L$ and parentheses. The formation rules are as usual.

$\sqcup, M$ and $\Rightarrow$ are definable connectives:

(i) $\alpha \sqcup \beta \equiv \sim (\sim \alpha \sqcap \sim \beta)$
(ii) $M(\alpha) \equiv \neg L(\neg \alpha)$ and
(iii) $\alpha \Rightarrow \beta \equiv (\sim L(\alpha) \sqcup L(\beta)) \sqcap (\sim M(\alpha) \sqcup M(\beta))$

for any wffs α, β of $\mathcal{L}_1$.

Axiom schemata:

1. $\alpha \Rightarrow \alpha$
2a. $\sim\sim \alpha \Rightarrow \alpha$ 2b. $\alpha \Rightarrow \sim\sim \alpha$.
3. $\alpha \sqcap \beta \Rightarrow \alpha$ 4. $\alpha \sqcap \beta \Rightarrow \beta \sqcap \alpha$.
5a. $\alpha \sqcap (\beta \sqcup \gamma) \Rightarrow (\alpha \sqcap \beta) \sqcup$ 5b. $(\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma) \Rightarrow$
$(\alpha \sqcap \gamma)$ $\alpha \sqcap (\beta \sqcup \gamma)$
6. $L(\alpha) \Rightarrow \alpha$
7a. $L(\alpha \sqcap \beta) \Rightarrow L(\alpha) \sqcap L(\beta)$ 7b. $L(\alpha) \sqcap L(\beta) \Rightarrow L(\alpha \sqcap \beta)$
8. $L(\alpha) \Rightarrow L(L(\alpha))$ 9. $M(L(\alpha)) \Rightarrow L(\alpha)$
10a. $L(\alpha \sqcup \beta) \Rightarrow L(\alpha) \sqcup L(\beta)$ 10b. $L(\alpha) \sqcup L(\beta) \Rightarrow L(\alpha \sqcup \beta)$

Rules of inference:

1. $$\frac{\begin{array}{c}\alpha \\ \alpha \Rightarrow \beta\end{array}}{\beta}$$
modus ponens

2. $$\frac{\begin{array}{c}\alpha \Rightarrow \beta \\ \beta \Rightarrow \gamma\end{array}}{\alpha \Rightarrow \gamma}$$
hypothetical syllogism

3. $$\frac{\alpha}{\beta \Rightarrow \alpha}$$

4. $$\frac{\alpha \Rightarrow \beta}{\sim \beta \Rightarrow \sim \alpha}$$

5. $$\frac{\begin{array}{c}\alpha \Rightarrow \beta \\ \alpha \Rightarrow \gamma\end{array}}{\alpha \Rightarrow \beta \sqcap \gamma}$$

6. $$\frac{\begin{array}{c}\alpha \Rightarrow \beta, \beta \Rightarrow \alpha \\ \gamma \Rightarrow \delta, \delta \Rightarrow \gamma\end{array}}{(\alpha \Rightarrow \gamma) \Rightarrow (\beta \Rightarrow \delta)}$$

7. $$\frac{\alpha \Rightarrow \beta}{L(\alpha) \Rightarrow L(\beta)}$$

8. $$\frac{\alpha}{L\alpha}$$

9. $$\frac{\begin{array}{c}L(\alpha) \Rightarrow L(\beta) \\ M(\alpha) \Rightarrow M(\beta)\end{array}}{\alpha \Rightarrow \beta}$$

Let $\mathfrak{F}_1$ denote the set of all wffs of $\mathcal{L}_1$. Following the standard notation, $\vdash_{\mathcal{L}_1} \alpha$ will denote that α is a theorem of $\mathcal{L}_1$, and $\Gamma \vdash_{\mathcal{L}_1} \alpha$, that α is a syntactic consequence of Γ, Γ being any set of wffs in $\mathcal{L}_1$.

Definition 14.4.1. *An evaluation v of the wffs of $\mathcal{L}_1$ in a pre-rough algebra $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ is a map from the set of propositional variables of $\mathcal{L}_1$ to A. It can be uniquely extended to $\mathfrak{F}_1$ (the extension is also denoted as v, and called an evaluation as well) as : $v(\alpha \sqcap \beta) \equiv v(\alpha) \wedge v(\beta), v(\sim \alpha) \equiv \sim v(\alpha), v(L(\alpha)) \equiv L(v(\alpha))$.*

Let **PRA** denote the class of all pre-rough algebras.

Definition 14.4.2. *Let Γ be a set of wffs and α any wff of $\mathcal{L}_1$. If in any pre-rough algebra $\mathbf{A}$, for any evaluation v in $\mathbf{A}$, $v(\Gamma) = \{1\}$ implies that $v(\alpha) = 1$, then we write $\Gamma \models_{\mathbf{PRA}} \alpha$. In particular (when $\Gamma = \emptyset$), α is valid ($\models_{\mathbf{PRA}} \alpha$) provided in any pre-rough algebra $\mathbf{A}$, for each evaluation v, $v(\alpha) = 1$.*

If an evaluation v in a pre-rough algebra $\mathbf{A}$ is such that $v(\Gamma) = \{1\}$, then we say that v is a *model* of Γ in $\mathcal{P}$. In particular, if $v(\alpha) = 1$ ($\Gamma = \{\alpha\}$), v is said to be a model of α in $\mathbf{A}$.

Lemma 14.4.1. *Let v be an evaluation in a pre-rough algebra $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$. For any wffs β, γ of $\mathcal{L}_1$, $v(\beta \Rightarrow \gamma) = 1$ if and only if $v(\beta) \leq v(\gamma)$ in $\mathbf{A}$.*

Proof. $1 = v(\beta \Rightarrow \gamma) = v((\sim L(\beta) \sqcup L(\gamma)) \wedge (\sim M(\beta) \sqcup M(\gamma)))$ if and only if $v(\sim L(\beta) \sqcup L(\gamma)) = 1 = v(\sim M(\beta) \sqcup M(\gamma))$, as $\mathbf{A}$ is a lattice. Now $1 = v(\sim L(\beta) \sqcup L(\gamma)) = \sim L(v(\beta)) \vee L(v(\gamma))$ if and only if $L(v(\beta)) \leq L(v(\gamma))$, as $\langle L(A), \leq, \wedge, \vee, \sim, 0, 1 \rangle$ is a Boolean algebra. For the same reason, $M(v(\beta)) \leq M(v(\gamma))$. So, by property $A11$ in the definition of a pre-rough algebra, $v(\beta) \leq v(\gamma)$ in $\mathbf{A}$. The other direction is obtained using the fact that L, M both distribute over $\wedge, \vee$. QED

Let Γ be a set of wffs and α any wff of $\mathcal{L}_1$. We prove that $\mathcal{L}_1$ is sound and complete relative to the class of all pre-rough algebras.

Theorem 14.4.1. (Soundness) *If $\Gamma \vdash_{\mathcal{L}_1} \alpha$, $\Gamma \models_{\mathbf{PRA}} \alpha$.*

Proof. The proof is straightforward, and involves easy verification using *Lemma* 14.4.1. One can, in fact, show that in an arbitrary pre-rough algebra, say $\mathbf{A}$, (i) any evaluation v is a model for each axiom of $\mathcal{L}_1$

(i.e., each axiom is valid), and (ii) if v is a model for the premise(s) of any rule of inference of $\mathcal{L}_1$, it is also a model of the conclusion of that rule.

Indeed, v is a model of axioms 1–5b, since $\mathbf{A}$ is a quasi-Boolean algebra. It is a model of axioms 6–9, as $\mathbf{A}$ is a topological quasi-Boolean algebra. And finally, it is a model of axioms 10a-b, due to property $A10$ in the definition of a pre-rough algebra.

Now suppose that v is a model of the premise(s) of any of the rules of inference 1–6 of $\mathcal{L}_1$. That v is a model of the conclusion of the respective rules follows as $\mathbf{A}$ is a quasi-Boolean algebra. In case of rules 7–8 and rule 9, this follows from the properties of a tqBa and the defining property $A11$ of a pre-rough algebra respectively.   QED

We also have the converse of the above theorem (the completeness theorem), the proof of which – as we shall see – follows the routine technique in algebraic logic. One can define a binary relation R on $\mathfrak{F}_1$ as: $\langle \alpha, \beta \rangle \in R$ if and only if $\Gamma \vdash_{\mathcal{L}_1} \alpha \Leftrightarrow \beta$. This can be easily proved to be an equivalence relation on $\mathfrak{F}_1$. Thus the quotient set $\mathfrak{F}_1/R$ is obtained. The following may be established.

Lemma 14.4.2. *The Lindenbaum algebra* $\mathcal{LA}_1 \equiv \langle \mathfrak{F}_1/R, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ *is a pre-rough algebra, where for any* $[\alpha][\beta] \in \mathfrak{F}_1/R$, *the operations* $\leq, \wedge, \sim, L, 1$ *are defined on* $\mathfrak{F}_1/R$ *as:*

1. *$[\alpha] \leq [\beta]$ if and only if $\Gamma \vdash_{\mathcal{L}_1} \alpha \Leftrightarrow \beta$,*

2. *$[\alpha] \wedge [\beta] \equiv [\alpha \sqcap \beta]$,*

3. *$\sim [\alpha] \equiv [\sim \alpha]$, and*

4. *$L([\alpha]) \equiv [L(\alpha)]$.*

5. *The unit element 1 of the algebra is just the equivalence class $[\alpha]$, with $\Gamma \vdash_{\mathcal{L}_1} \alpha$.*

6. *The other operators are defined in terms of the above as usual.*

Proof. The operators $\leq, \wedge, \sim, L$ are well-defined:
$\leq$ is so by rule of inference 6; $\wedge$ by axiom 3, rules 2 and 5; $\neg$ by rule 4; and L by rule 7.

That $[\alpha] \wedge [\beta]$ is the greatest lower bound of $[\alpha]$ and $[\beta]$, follows from axiom 3 and rule 5.

$\Gamma \models_{\mathcal{L}_1} \alpha$ and $\Gamma \models_{\mathcal{L}_1} \beta$, then $\Gamma \models_{\mathcal{L}_1} \alpha \Leftrightarrow \beta$, by rule 3. On the other hand, if $\Gamma \vdash_{\mathcal{L}_1} \alpha$ and $\langle \alpha, \beta \rangle \in R$, then by rule 1, $\Gamma \vdash_{\mathcal{L}_1} \beta$. So if $\Gamma \vdash_{\mathcal{L}_1} \alpha$, then $[\alpha]$ consists precisely of those wffs β for which $\Gamma \vdash_{\mathcal{L}_1} \beta$. Further, $\Gamma \models_{\mathcal{L}_1} \alpha$ implies, for any $[\beta]$, $\Gamma \models_{\mathcal{L}_1} \beta \Rightarrow \alpha$ by rule 3. So $[\beta] \leq [\alpha]$. Therefore, $[\alpha]$ such $\Gamma \vdash_{\mathcal{L}_1} \alpha$, is the unit element (1) of the lattice $\langle \mathfrak{F}_1/R, \leq, \wedge \rangle$.

The properties that make the structure $\langle \mathfrak{F}_1/R, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ a pre-rough algebra, are simple to verify.   QED

Next we prove the completeness theorem.

Theorem 14.4.2. (Completeness) *If* $\Gamma \models_{\overline{\mathbf{PRA}}} \alpha$, *then* $\Gamma \vdash_{\mathcal{L}_1} \alpha$.

Proof. Let $\Gamma \models_{\overline{\mathbf{PRA}}} \alpha$. In particular, as $\mathcal{L}\mathcal{A}_1$ is a pre-rough algebra (by Lemma 14.4.2), any evaluation v in $\mathcal{L}\mathcal{A}_1$ which is a model for Γ, would be a model for α. We define an evaluation v_0 from the set of propositional variables of $\mathcal{L}_1$ to $\mathfrak{F}_1/R$ as follows.

$v_0(p) \equiv [p]$, for any propositional variable p. v_0 is extended to $\mathfrak{F}_1$ in the usual manner. Then, for any wff β of $\mathcal{L}_1$, it can be shown that $v_0(\beta) = [\beta]$. The proof is by induction on the number of connectives in β. Now $v_0(\gamma) = 1$, for each $\gamma \in \Gamma$ (as $\Gamma \vdash_{\mathcal{L}_1} \gamma$) – i.e. v_0 is a model for Γ. So v_0 is a model for α, i.e. $1 = v_0(\alpha) = [\alpha]$. Hence (by *Lemma* 14.4.2), $\Gamma \vdash_{\mathcal{L}_1} \alpha$.   QED

In view of the properties listed in the Table of Section 12.2, one should notice the following correspondences between the syntactic description of the system $\mathcal{L}_1$ and the concrete intuitions behind Rough Set algebras (where, we recall, the modal operator L is induced by the rough equality relation $\approx$):

Axiom 6: reflexivity of $\approx$.
Axiom 8: transitivity of $\approx$.
Axiom 9: Euclidean property of $\approx$ (and the derived symmetry property).

Axioms 7a-b: L is an interior operator, since $\approx$ gives rise to a topological space (see above Section 12.8).
Axioms 10a-b: we have already seen the proof (cf. *Proposition* 14.3.1) based on the fact that any rough set is representable by a pair $\langle X_1, X_2 \rangle$ of elements of some Approximation System $\mathbf{AS}(U)$. Speaking more conceptually, we can notice that any element of a rough set $\langle X_1, X_2 \rangle$

is a fix point of the operator M, respectively, L induced by $\mathbf{AS}(U)$. As such $a_2 \in L(U) = \{L(X)\}_{X \subseteq U}$ which, as we already know, is the carrier of the Boolean algebra $\mathbf{AS}(U)$ that, in turns, is isomorphic to $\mathbf{L}(\mathbf{TQ}(\mathbf{AS}(U)))$. Hence, applying first the operator L and then the operation $\vee$ or applying the operation $\vee$ first and then the operator L yield to the same result, since both sequences reduce to a union of elements of the same algebra, namely $X_2 \cup X_2$.

14.4.2 The System $\mathcal{L}_2$

The language of $\mathcal{L}_2$ is the same as that of $\mathcal{L}_1$, only enhanced by the presence of the logical symbol $\bigsqcup$, standing for infinite disjunction and giving rise to the following formation rule:
for any index set I, $\bigsqcup_{i \in I} \alpha_i$ is a wff in $\mathcal{L}_2$ if and only if α_i is of the form $L(\beta_i)$, for some $\beta_i, i \in I$.
$\prod$ (infinite conjunction) stands for $\sim \bigsqcup \sim$.
Let $\mathfrak{F}_2$ denote the set of all wffs of $\mathcal{L}_2$.
   Axiom schemata for $\mathcal{L}_2$.

> *Axioms $1 - 5b$ of $\mathcal{L}_1$*
> 11. $L(\alpha_j) \Rightarrow \bigsqcup_{i \in I} L(\alpha_i)$,
>   *for each* $\alpha_j, j \in I$,
> 12a. $\bigsqcup_{i \in I} \alpha_i \Rightarrow L(\bigsqcup_{i \in I} \alpha_i)$ 12b. $L(\bigsqcup_{i \in I} \alpha_i) \Rightarrow \bigsqcup_{i \in I} \alpha_i$
> 13a. $\bigsqcup_{i \in I} \prod_{j \in J} L(\alpha_{i,j})$ 13b. $\prod_{f \in J^I} \bigsqcup_{i \in I} L(\alpha_{i,f(i)})$
> $\Rightarrow \prod_{f \in J^I} \bigsqcup_{i \in I} L(\alpha_{i,f(i)})$ $\Rightarrow \bigsqcup_{i \in I} \prod_{j \in J} L(\alpha_{i,j})$

where I, J are index sets and J^I is the set of maps of I into J.
   Rules of inference of $\mathcal{L}_2$:

> 1.   *Rules $1 - 9$ of $\mathcal{L}_1$*
> 10. $\dfrac{L(\alpha_i) \Rightarrow L(\beta)}{\bigsqcup_{i \in I} L(\alpha_i) \Rightarrow L(\beta)}$ *for each $i \in I$*

Now we prove that $\mathcal{L}_2$ is sound and complete relative to the class of all rough algebras.

   $\vdash_{\mathcal{L}_2} \alpha$ will denote that α is a theorem of $\mathcal{L}_2$, and $\Gamma \vdash_{\mathcal{L}_2} \alpha$, that α is a syntactic consequence of Γ, Γ being any set of wffs in $\mathcal{L}_2$.

   An *evaluation* v of the wffs of $\mathcal{L}_2$ in a rough algebra $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ is defined as in case of $\mathcal{L}_1$ and pre-rough algebras. It is a map from the set of propositional variables of $\mathcal{L}_2$ to A. It can be

uniquely extended to $\mathfrak{F}_2$ (keeping the same notation and name for the extension) for the connectives $\sqcap, \sim$ and L as before, while $v(\bigsqcup_{i \in I} \alpha_i) \equiv \bigvee_{i \in I} v(\alpha_i)$, where α_i is of the form $L(\beta_i)$, for some β_i, $i \in I$. Let **RA** denote the class of all rough algebras. The definitions of $\Gamma \models_{\overline{\mathbf{RA}}} \alpha$, $\models_{\overline{\mathbf{RA}}} \alpha$ and a *model* of Γ (or α) in a rough algebra **A**, Γ being a set of wffs and α any wff of $\mathcal{L}_2$, are all as before.

Theorem 14.4.3. (Soundness) *If $\Gamma \vdash_{\overline{\mathcal{L}_2}} \alpha$, then $\Gamma \models_{\overline{\mathbf{RA}}} \alpha$.*

Proof. We note that Lemma 14.4.1 still applies, and the proof of *Theorem* 14.4.1 can be carried out here while verifying that, in an arbitrary rough algebra, say **A**, (i) any evaluation v is a model for axioms 1–10b, and (ii) if v is a model for the premise(s) of any of the rules 1–9, it is also a model of the conclusion of that rule.

v is a model of axiom 11, as $\bigvee_{i \in I} v(\alpha_i)$ is an upper bound of $\{v(\alpha_i)\}_{i \in I}$ in **A**. The proof in case of axioms 12a-b is based on the observation that $\bigvee_{i \in I} L(b_i) = L(\bigvee_{i \in I} L(b_i))$, the subalgebra $\mathbf{L(A)} \equiv \langle L(A), \leq, \wedge, \vee, \sim, 0, 1\rangle$ of **A** being complete by definition. Complete distributivity of the same subalgebra results in the proof in case of axioms 13a-b.

Finally, if v is a model for the premises of rule 10, it is a model of the conclusion as well, because $\bigvee_{i \in I} v(\alpha_i)$ is the least upper bound of $\{v(\alpha_i)\}_{i \in I}$ in **A**. $\hfill$ QED

The completeness theorem can be proved in this case also. *Lemma* 14.4.2 has to be extended for the purpose. The binary relation R on $\mathfrak{F}_2$ is defined as before: $\langle \alpha, \beta \rangle \in R$ if and only if $\Gamma \vdash_{\overline{\mathcal{L}_2}} \alpha \Leftrightarrow \beta$. This is an equivalence relation on $\mathfrak{F}_2$, and we get the quotient set $\mathfrak{F}_2/R$.

Lemma 14.4.3. *The Lindenbaum algebra $\mathcal{LA}_2 \equiv \langle \mathfrak{F}_2/R, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1\rangle$ is a rough algebra, where for any $[\alpha], [\beta] \in \mathfrak{F}_1/R \leq, \wedge, \sim, L, 1$ are defined on $\mathfrak{F}_2/R$ as:*

1. $[\alpha] \leq [\beta]$ if and only if $\Gamma \vdash_{\overline{\mathcal{L}_2}} \alpha \Rightarrow \beta$.

2. $[\alpha] \wedge [\beta] \equiv [\alpha \sqcap \beta]$.

3. $\sim [\alpha] \equiv [\sim \alpha]$.

4. $L([\alpha]) \equiv [L(\alpha)]$.

5. $\bigvee_{i \in I} [\gamma_i] \equiv [\bigsqcup_{i \in I} L(\beta_i)]$, where $\Gamma \mid_{\overline{\mathcal{L}_2}} \gamma_i \Leftrightarrow L(\beta_i)$, for some β_i, $i \in I$.

6. *The unit element 1 of the algebra is just the equivalence class $[\alpha]$, with $\Gamma \vdash_{\mathcal{L}_2} \alpha$.*

7. *The other operators are defined in terms of the above as usual.*

Proof. That the operators $\leq, \wedge, \sim, L$ are well-defined, is proved just as in *Lemma* 14.4.2. In fact, a similar proof establishes that $\langle \mathfrak{F}_2/R, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ is also a pre-rough algebra. We show that the structure is a rough algebra. $\vee$ is well-defined:

Let $[\delta_i] = [\gamma_i]$, $\Gamma \vdash_{\mathcal{L}_2} \gamma_i \Leftrightarrow L(\beta_i)$ and $\Gamma \vdash_{\mathcal{L}_2} \delta_i \Leftrightarrow L(\zeta_i)$, for some β_i, ζ_i, $i \in I$. $\Gamma \vdash_{\mathcal{L}_2} \delta_i \Leftrightarrow \gamma_i$. By rule 2, $\Gamma \vdash_{\mathcal{L}_2} L(\zeta_i) \Leftrightarrow L(\beta_i)$ for each $i \in I$, so that, using rule 10, one obtains $\Gamma \vdash_{\mathcal{L}_2} \bigsqcup_{i \in I} L(\beta_i) \Leftrightarrow \bigsqcup_{i \in I} L(\zeta_i)$. Thus, $[\bigsqcup_{i \in I} L(\beta_i)] = [\bigsqcup_{i \in I} L(\zeta_i)]$.

Further, $\bigvee_{i \in I} [\gamma_i] = [\bigsqcup_{i \in I} L(\beta_i)]$ is an upper bound for $[\gamma_i]_{i \in I}$ in $\mathfrak{F}_2/R$ by axiom 11 and rule 2. Now suppose $[\gamma_i] \leq [\gamma]$, for each $i \in I$. So $\Gamma \vdash_{\mathcal{L}_2} \gamma_i \Rightarrow \gamma$ implies $\Gamma \vdash_{\mathcal{L}_2} L(\beta_i) \Rightarrow \gamma$, using rule 2. Then applying rule 7, $\Gamma \vdash_{\mathcal{L}_2} L(L(\beta_i)) \Rightarrow L(\gamma)$. By axiom 8 and rule 2, $\Gamma \vdash_{\mathcal{L}_2} L(\beta_i) \Rightarrow L(\gamma)$, and this holds for each $i \in I$. By rule 10 and finally by axiom 6, $\Gamma \vdash_{\mathcal{L}_2} \bigsqcup_{i \in I} L(\beta_i) \Rightarrow \gamma$, i.e. $[\bigsqcup_{i \in I} L(\beta_i)] \leq [\gamma]$.

$\mathcal{L}(\mathfrak{F}_2/R)$ is thus complete. Its complete distributivity follows from axioms 13a-b. $\langle \mathfrak{F}_2/R, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ is thus a rough algebra. QED

The proof of *Proposition* 14.4.2 is now modified to give general completeness.

Theorem 14.4.4. (Completeness) *If $\Gamma \models_{\mathbf{RA}} \alpha$, then $\Gamma \vdash_{\mathcal{L}_2} \alpha$.*

Proof. We define an evaluation v_0 from the set of propositional variables of $\mathcal{L}_2$ to $\mathfrak{F}_2/R$ as before, i.e. $v_0(p) \equiv [p]$, for any propositional variable p. v_0 is then extended to $\mathfrak{F}_2$, as mentioned earlier. Again, for any wff β of $\mathcal{L}_2$, it can be shown by induction on the number of connectives in β, that $v_0(\beta) = [\beta]$. The rest of the proof follows exactly the same arguments as presented for *Proposition* 14.4.2. QED

14.4.3 Rough Set Semantics for $\mathcal{L}_2$

Let **RE** denote the class of all those Rough Set algebras of the form $\mathbf{TQ(AS}(U))$, corresponding to an Approximation Space $\mathbf{AS}(U)$ induced by an Indiscernibility Space $\langle U, E \rangle$, such that for any element $\langle X_1, X_2 \rangle$ of $\mathbf{TQ(AS}(U))$, there is a rough set S with $(lE)(S) = X_2$ and $(uE)(S) = X_1$.

Our contention is that $\mathcal{L}_2$ is sound and complete with respect to this smaller class of rough algebras as well. Let Γ be a set of wffs and α any well formed formulas of $\mathcal{L}_2$. $\Gamma \models_{\overline{\textbf{RE}}} \alpha$ has the same meaning as before.

Theorem 14.4.5. (Soundness) *If $\Gamma \vdash_{\mathcal{L}_2} \alpha$. Then $\Gamma \models_{\overline{\textbf{RE}}} \alpha$.*

Proof. This follows directly from *Theorem* 14.4.3.   QED

Theorem 14.4.6. (Completeness) *Let $\Gamma \models_{\overline{\textbf{RE}}} \alpha$. Then $\Gamma \vdash_{\mathcal{L}_2} \alpha$.*

Proof. Let $\mathbf{A} = \langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$ be a Rough Algebra. By the representation theorem and *Proposition* 14.3.7 it follows that there is a monomorphism from $\mathbf{A}$ into a member $\mathbf{TQ}(\mathbf{AS}(U))$ of $\mathbf{RE}$, corresponding to some Approximation Space. Let i denote such a monomorphism and let v be any evaluation. It can be proved that v is an evaluation from $\mathfrak{F}_2$ to $\mathbf{TQ}(\mathbf{AS}(U))$. As $\Gamma \models_A \alpha$, $v(\alpha) = \langle X, X \rangle$, so that we must have $v(\alpha) = 1$. Thus, using completeness of $\mathcal{L}_2$ (*Theorem* 14.4.6), $\Gamma \vdash_{\mathcal{L}_2} \alpha$.   QED

We now give a Kripke-style semantics for $\mathcal{L}_2$.

Definition 14.4.3. *A model of $\mathcal{L}_2$ is a triple of the form $\langle U, E, \phi \rangle$, where $\langle U, E \rangle$ is an Indiscernibility Space and ϕ is a map from the set of propositional variables of $\mathcal{L}_2$ to $\wp(U)$ ϕ is extended to the set $\mathfrak{F}_2$ of all wffs of $\mathcal{L}_2$ as follows:*

1. $\phi(\sim \alpha) = -\phi(\alpha)$.

2. $\phi(L(\alpha)) = (lE)(\phi(\alpha))$.

3. $\phi(\alpha \sqcap \beta) = \phi(\alpha) \sqcap \phi(\beta)$.

4. $\phi(\alpha \sqcup \beta) = \phi(\alpha) \sqcup \phi(\beta)$.

5. $\phi(\alpha \Rightarrow \beta) = \phi(\alpha) \Rightarrow \phi(\beta)$.

6. $\phi(\bigsqcup_{i \in I} \alpha_i) = \bigcup_{i \in I} \phi(\alpha_i)$.

where, for any $S, T \in \wp(X)$, $S \sqcap T = (S \cap T) \cup (S \cap (uE)(T \cap -(uE)(S \cap T))$, $S \sqcup T \equiv (S \cup T) \cap (S \cup (lE)(T) \cup (lE)(S \cup -T))$ and
$S \Rightarrow T \equiv (-(lE)(S) \cup (lE)(T)) \cap (-(uE)(S) \cup (uE)(T))$,
$\bigcup$ denoting arbitrary union in $\wp(X)$.

Let Γ be a set of wffs and α any wff of $\mathcal{L}_2$.

Definition 14.4.4.

1. α *is true in a model* $\langle U, E, \phi \rangle$ *if and only if* $\phi(\alpha) = U$.

2. α *is valid (written* $\models \alpha$*) if and only if* α *is true in all models.*

3. α *is a semantic consequence of* Γ *(*$\Gamma \models \alpha$*), if and only if, when-ever each member of* Γ *is true in a model,* α *is true in that model as well. In particular, when* $\Gamma = \emptyset$*, we say that* α *is valid (*$\models \alpha$*).*

Remarks 14.4.1.

(i) $\phi(\alpha)$ is a rough set, where α is any wff in $\mathfrak{F}_2$.

(ii) $\alpha \Rightarrow \beta$ is true in $\langle U, E, \phi \rangle$ if and only if $\phi(\alpha)$ is roughly included in $\phi(\beta)$.

(iii) $(\alpha \Rightarrow \beta) \sqcap (\beta \Rightarrow \alpha)$ is true in $\langle U, E, \phi \rangle$ if and only if $\phi(\alpha)$ is roughly equal to $\phi(\beta)$.

Theorem 14.4.7. (Soundness) *Let* $\Gamma \vdash_{\mathcal{L}_2} \alpha$*. Then* $\Gamma \models \alpha$*.*

Proof. Let $\Gamma \vdash_{\mathcal{L}_2} \alpha$. Then $\Gamma \models \alpha$. We consider an arbitrary model $\langle U, E, \phi \rangle$ in which all the members of Γ are true and prove that α is true in this model. Actually, it can be proved that each axiom of $\mathcal{L}_2$ is valid, and all the rules of $\mathcal{L}_2$ preserve validity. It is straightfor-ward to establish the validity of the axioms, using *Remarks* 14.4.1(ii), and facts such as: for any subsets S, T of U in the Indiscernibility Space $\langle U, E \rangle$, $(lE)(S \sqcup T) = (lE)(S) \sqcup (lE)(T) = (lE)(S) \cup (lE)(T)$, or $(lE)((lE)(S) \cup (lE)(T)) = (lE)(S) \cup (lE)(T)$ (or the dual results with respect to the upper approximation operator). Axioms 11-13b follow from the properties of arbitrary union/intersection in $\wp(U)$.

That rules 1–9 preserve validity can be shown easily, again using the results just mentioned. In case of rule 10, denoting $\phi(\alpha_i)$ and $\phi(\beta)$ by A_i and B respectively, and assuming that $(lE)(A_i) \subseteq (lE)(B)$,

$i \in I$, a simple set-theoretic argument shows that $(lE)(\bigcup\limits_{i \in I}(lE)(A_i)) \subseteq$ $(lE)(B)$, $(uE)(\bigcup\limits_{i \in I}(lE)(A_i)) \subseteq (lE)(B)$.   QED

Finally, we come to the completeness of $\mathcal{L}_2$ with respect to the preceding semantics.

Theorem 14.4.8. (Completeness) *Let* $\Gamma \models \alpha$. *Then* $\Gamma \vdash_{\mathcal{L}_2} \alpha$.

Proof. Let $\mathbf{TQ}(\mathbf{AS}(U)) \in \mathbb{A}$, corresponding to an Indiscernibility Space $\langle U, E \rangle$ and v be any evaluation from $\mathfrak{F}_2$ to $\mathbf{TQ}(\mathbf{AS}(U))$ that is a model for Γ. So $v(\gamma) = \langle U, U \rangle$, for each $\gamma \in \Gamma$. We define an evaluation ϕ from $\mathfrak{F}_2$ to $\wp(U)$ as follows. Let p be any propositional variable in $\mathfrak{F}_2$. As $v(p) \in \mathbf{TQ}(\mathbf{AS}(U))$, $v(p) = \langle X_1, X_2 \rangle$ and there is $S \in \wp(U)$ with $(lE)(S) = X_2$ and $(uE)(S) = X_1$. We choose and fix such a set S, and set $\phi(p) = S$. ϕ is then extended to all well formed formulas in $\mathfrak{F}_2$ as in *Definition* 14.4.3. Then $\langle U, E, \phi \rangle$ is a model and for any wff α, it can be proved that $v(\alpha) = \langle (lE)(\phi(\alpha)), (uE)(\phi(\alpha)) \rangle$. In particular, for each $\gamma \in \Gamma$, $\langle U, U \rangle = v(\gamma) = \langle (lE)(\phi(\gamma)), (uE)(\phi(\gamma)) \rangle$, so that $\phi(\gamma) = U$ that is, each member of Γ is true in the model $\langle U, E, \phi \rangle$. As $\Gamma \models \alpha$, $\phi(\alpha) = U$. So $v(\alpha) = \langle (lE)(\phi(\alpha), (uE)(\phi(\alpha)) \rangle = \langle U, U \rangle$ and hence $\Gamma \models_{\mathbb{A}} \alpha$. Using *Theorem* 14.4.6, $\Gamma \vdash_{\mathcal{L}_2} \alpha$.   QED

14.5   Algebraic Interpretation and Modal Interpretation of Rough Set Systems

We conclude this Chapter with a short list of the main relationships between the above modal interpretations of Rough Set Systems and the algebraic interpretations that were introduced in Part II.

First of all, let us resume the algebraic constructions we have seen so far.

In Tables 14.1 and 14.2, $\mathbf{B}$ is any Boolean algebra, $\mathbf{B}'$ a Boolean sub-algebra of $\mathbf{B}$ and $\mathbf{L}$ a distributive sub-lattice of $\mathbf{B}$. $\mathbf{D}$ is any distributive lattice, $\mathbf{D}'$ a sub-lattice of $\mathbf{D}$, $\mathbf{S}$ any lattice and $\mathbf{S}'$ any sub-lattice of $\mathbf{S}$; E is an equivalence relation on the set U, R is a preorder or partial order relation on U, x is an element of U, a is an element of $\mathbf{B}$, $\mathbf{S}$ or $\mathbf{D}$. Moreover, let us compare the resulting plethora of modal notations.

Table 14.1: Algebraic operators and modal systems

Operators	Modal system	Modal logic	Algebraic system
$\varkappa^E(x) = \mathbb{I}_E(x) = L_E(x)$	$\langle B(U), \Omega_E(U) \rangle$	$\mathbf{S}5$	mtBa
$(lE)(X)$	$\langle B(U), \mathbf{AS}(U) \rangle$	$\mathbf{S}5$	mtBa
$(l\mathbf{B}')(a) = 1 \xrightarrow{\mathbf{B}'} a$	$\langle \mathbf{B}, \mathbf{B}' \rangle$	$\mathbf{S}5$	mtBa
$\varkappa^R(x) = \mathbb{I}_R(x) = L_R(x)$	$\langle B(U), \Omega_R(U) \rangle$	$\mathbf{S}4$	tBa
$(l\mathbf{L})(a) = 1 \xrightarrow{\mathbf{L}} a$	$\langle \mathbf{B}, \mathbf{L} \rangle$	$\mathbf{S}4$	tBa
$(l\mathbf{D}')(a) = 1 \xrightarrow{\mathbf{D}'} a$	$\langle \mathbf{D}, \mathbf{D}' \rangle$	$\mathbf{R}4$	
$\mathbf{L_k}(a) = \{a' : f(a') \leq a\}$	$\langle \mathbf{S}, \mathbf{S}' \rangle$, for $\mathbf{S}' = f(\mathbf{S})$		

Table 14.2: Modal operators and algebraic systems

Necessity operators	Possibility operators	Context
L_E	M_E	$\mathbf{S}5$ modal systems
$\mathbb{I}_E$	$\mathbb{C}_E$	0-dimensional Topological Spaces
(lE)	(uE)	Approximation Spaces
$(l\mathbf{B}')$	$(u\mathbf{B}')$	abstract Approximation Spaces
$\sim \lrcorner \equiv \lrcorner\lrcorner$	$\lrcorner \sim \equiv \neg\neg$	semi-simple Nelson algebras
ϕ_2	ϕ_1	3-valued Łukasiewicz algebras
D_2	D_1	Post algebras of order 3
!	¡	P_2-algebras

Chapter 15

Frames (Part III)

15.1 Frame – Proof of the Duality Between $L_R(X) = \bigcup\{Z : R(Z) \subseteq X\}$ and $M_R(X) = \bigcap\{-Z : R(Z) \subseteq -X\}$
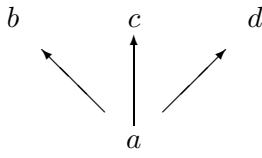
Given a function $f(x)$, its dual function is $-(f(-x))$. Therefore:

$$-L_R(-X) = -\bigcup\{Z : R(Z) \subseteq -X\} = \bigcap\{-Z : R(Z) \subseteq -X\}.$$

The reader is invited to distinguish between $\{-Z : \zeta(Z)\}$ and $-\{Z : \zeta(Z)\}$ ($\zeta(Z)$ any first order formula with parameter Z). The latter expression is equivalent to $\{Z : \neg\zeta(Z)\}$. So, if $A \in \{-Z : \zeta(Z)\}$ then A is the complement of a set fulfilling ζ, while $A \in \{Z : \neg\zeta(Z)\}$ is a set that does not fulfill ζ.

For example consider the following relation R:

R	a	b	c	d
a	1	1	1	1
b	0	1	0	0
c	0	0	1	0
d	0	0	0	1

Let $A = \{b\}$. So $-A = \{a, c, d\}$. Let $\zeta(X, Y) \equiv R(X) \subseteq Y$ then $\{X : \zeta(X, -A)\} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$; thus $\{-X : \zeta(X, -A)\} = \{\{a, b, c, d\}, \{a, b, d\}, \{a, b, c\}, \{a, b\}\}$. It follows that $\bigcap\{-Z : R(Z) \subseteq -A\} = \{a, b\} = R^{\smile}(A)$.

Therefore, we have also verified that $M_R(X) = R^{\smile}(X)$, as stated in *Corollary* 12.1.1.

On the contrary, $\{X : \neg\zeta(X, -A)\} = \{X : R(X) \nsubseteq \{a, c, d\}\} = \{X : a \in X \lor b \in X\}$, i.e. X is any superset of $\{a\}$ or any superset of $\{b\}$. Hence both $\{a\}$ and $\{b\}$ belong to $\{X : \neg\zeta(X, -A)\}$. It follows that $\bigcap\{X : \neg\zeta(X, -A)\} = \emptyset$, because both $\{a\}$ and $\{b\}$ belong to it.

15.2   Frame – Relational Properties and Logical Characteristics

In what follows, given a relation R we shall denote $\langle x, y \rangle \in R$ by xRy, too.

15.2.1   Proof of KT5 = KTB4

Suppose $U = \{w, z, k\}$ and that we are given a Reflexive and Euclidean binary relation $R \subseteq U \times U$, such that wRz and zRk:

$$w$$

$$z \longrightarrow k$$
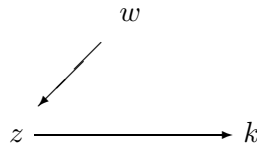
First of all, let us verify that zRw:

$$\frac{1: \ wRz \ (hypothesis) \ 2: \ wRw \ (reflexivity)}{zRw \ (1, 2: \ from \ the \ Euclidean \ property)}$$

Similarly, we obtain kRz. That is, *Reflexive + Euclidean $\Rightarrow$ Symmetric*:

$$w$$

$$z \longleftarrow k$$
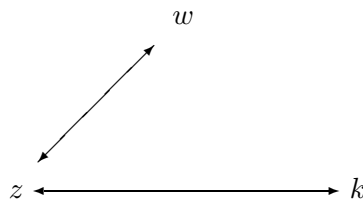
Now, let us prove that R is transitive, too:

$$\frac{1: \ wRz \ (hypothesis) \ \ 2: \ zRk \ (hypothesis) \ \ 3: \ zRw \ (just \ proved)}{wRk \ (1, 2 \ : \ from \ the \ Euclidean \ property)}$$

Similarly we obtain that kRz and zRw imply kRw.
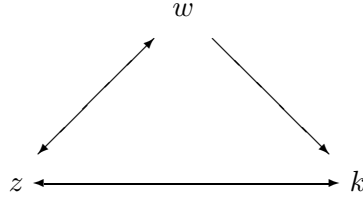
Hence, we deduce that *Reflexive + Euclidean = Equivalence.*

15.2.2   Proof of KDB4 = KTB4

Let $U = \{w, z, k\}$ and $R \subseteq U \times U$. Suppose R is Serial, Symmetric and Transitive. If neither wRz nor zRw, then since R is serial we must have both wRw and zRz, and we are done (R is also reflexive). Suppose now wRz. Then:

$$\frac{1 : \ wRz \ (hypothesis) \ 2 : \ zRw \ (symmetry)}{zRz, wRw \ (1, 2 : \ by \ transitivity)}$$

The same applies to k. Therefore, R is also reflexive.

However, if R is not serial then the deduction fails. Indeed suppose that w is not connected with any element. Obviously we cannot apply Symmetricity and Transitivity in order to obtain wRw.

Exercise 15.1. *Prove* **K**DB*5* = **K**TB*4*.

15.3   Frame – Proof of Proposition 12.7.10

Let $R^* = \bigcup \{R_j\}_{1 \leq j \leq n}$ and $R_* = \bigcap \{R_j\}_{1 \leq j \leq n}$. Then,

(1) Consider the bases for $\mathbf{P}^n$: for any x, $\mathcal{B}_x^n = \{R_1(x) \cup, \ldots, \cup R_n(x)\}$. Obviously $R_1(x) \cup, \ldots, \cup R_n(x) = (R_1 \cup, \ldots, \cup R_n)(x) = R^*(x)$ (in fact, $R_1(x) \cup R_2(x) = \{\langle x, y \rangle : \langle x, y \rangle \in R_1 \text{ or } \langle x, y \rangle \in R_2\} = \{\langle x, y \rangle : \langle x, y \rangle \in R_1 \cup R_2\} = (R_1 \cup R_2)(x)$. Therefore, since $\mathscr{F}_x^n = \Uparrow \mathcal{B}_x^n$, we have, for any x, y: $\langle x, y \rangle \in R^T(\mathbf{P}^n)$ if and only if $y \in \bigcap \mathscr{F}_x^n$ if and only if $y \in \mathcal{B}_x^n$ if and only if $\langle x, y \rangle \in R^*(x)$. It follows that $R^T(\mathbf{P}^n) = R^*$.

(2) Consider the bases for $\mathbf{P}^1$: for any x, $\mathcal{B}_x^1 = \{R_1(x), \ldots, R_n(x)\}$. Since $\mathscr{F}_x^1 = \Uparrow \mathcal{B}_x^1$, $\langle x, y \rangle \in R^T(\mathbf{P}^1)$ if and only if $y \in \bigcap \mathscr{F}_x^1$ if and only if $y \in \bigcap \mathcal{B}_x^1$ if and only if $y \in \bigcap \{R_j(x)\}_{1 \leq j \leq n}$ if and only if $y \in R_*(x)$. It follows that $R^T(\mathbf{P}^1) = R_*$.

(3) For any x, y, $\langle x, y \rangle \in R^B(\mathbf{P}^n)$ if and only if $y \in \bigcap \mathscr{F}_x^n$ and $x \in \bigcap \mathscr{F}_y^n$. Therefore, if $\langle x, y \rangle \in R^B(\mathbf{P}^n)$ then $y \in \bigcap \mathscr{F}_x^n$, so that $\langle x, y \rangle \in$

$R^T(\mathbf{P}^n)$. So, $R^B(\mathbf{P}^n) \subseteq R^T(\mathbf{P}^n)$. Moreover, $R^B(\mathbf{P})$ is symmetric by definition, for any pre-topology $\mathbf{P}$. Hence it is a tolerance relation, because reflexivity is inherited by the pre-topological structure. Finally, because of the bi-implication in the definition of $R^B(\mathbf{P}^n)$, it is the largest tolerance relation included in R^*.

(4) From a similar argument we can prove that $R^B(\mathbf{P}^1)$ is the largest tolerance relation included in R_*.

(5) Since $R^B(\mathbf{P}^n)$ is included in $R^T(\mathbf{P}^n)$, we obtain straightforwardly $\mathbf{P}(R^T(\mathbf{P}^n)) \unlhd \mathbf{P}(R^B(\mathbf{P}^n))$.

15.4   Frame – Proofs of the Propositions about the Uniqueness of $\mathbf{P}(R^T(\mathbf{P}))$ and $\mathbf{P}(R^B(\mathbf{P}))$

Let $\mathbf{P} = \langle U, \varkappa, \varepsilon \rangle$ be a pre-topological space of type $\mathfrak{V}_I$. Let $\mathbf{P}' = \langle U, \varkappa', \varepsilon' \rangle$ be a pre-topological space of type $\mathfrak{V}_S$ finer than $\mathbf{P}$. Thus for all $x \in U$, $\varkappa_x \subseteq \varkappa'_x$, so that $\bigcap \varkappa'_x \subseteq \bigcap \varkappa_x = \bigcap \varkappa_x^{R^T}$. It follows that $\varkappa_x \subseteq \varkappa$. Thus, $\mathbf{P}(R^T(\mathbf{P}))$ is the coarsest among the pre-topologies of type $\mathfrak{V}_S$ finer than $\mathbf{P}$.   QED

15.5   Frame – Alternative Proofs of Corollary 12.8.2.(1)

Proof 1. We have to prove $\varkappa^R(R(x)) = R(x)$. Notice that since $\varkappa^R(R(x)) = \{y : \exists F (F \in \varkappa_y^R \ \& \ F \subseteq R(x)\}$, $\varkappa^R(R(x)) = \{y : R(y) \subseteq R(x)\}$ (because $R(y)$ is the least element of $\varkappa_y^R$). Clearly, if $y' \in R(y)$, then $R(y') \subseteq R(y)$ because R is transitive. Moreover R is reflexive. It follows that for all $y' \in R(y)$, $R(y') \subseteq \{y : R(y) \subseteq R(x)\}$. Therefore, $\{y : R(y) \subseteq R(x)\} = \bigcup \{R(y) : R(y) \subseteq R(x)\} = R(x)$.   QED

Proof 2. We know that for any $x \in U$, given $R(x)$ there is a $Y \in \varkappa_x^R$, such that $R(x) \in \varkappa_y^R$ for all $x \in Y$. But if this is true of some Y, then it is true for every $Y' \subseteq Y$. But $R(x)$ is minimal in $\varkappa_x^R$ (because of *Proposition* 12.6.9). It follows that $R(x)$ is a neighborhood of all its own points. Hence $R(x)$ is open.   QED

15.6   Frame – Direct Proof of $M_R(X) = M_R^*(X)$, for R a Preorder Relation

A proof is given in *Corollary* 12.8.5.

We remind that we have to prove in a direct way that $M_R(X) = \bigcap\{-Z : R(Z) \subseteq -X\}$ equals $M_R^*(X) = \bigcap\{R^\smile(Z) : X \subseteq R^\smile(Z)\}$. Clearly $\bigcap\{R^\smile(Z) : X \subseteq R^\smile(Z)\} = R^\smile(X)$, because R is a preorder. Moreover, $\bigcap\{-Z : R(Z) \subseteq -X\} = -\bigcup\{Z : R(Z) \subseteq -X\}$. Therefore we have to prove that $-R^\smile(X) = \bigcup\{Z : R(Z) \subseteq -X\}$.

So, let us set $A = \bigcup\{Z : R(Z) \subseteq -X\}$ and let $a \in A$. Thus $R(a) \subseteq -X$ and there is not $x \in X$ such that $\langle a, x \rangle \in R$, otherwise $R(a) \cap X \neq \emptyset$. Therefore if $a \in A$, then $a \in -R^\smile(X)$, i.e. $A \subseteq -R^\smile(X)$.

Conversely, $-R^\smile(X)$ is an open set, because $R^\smile(X)$ is closed, since R is a preorder. Hence $R(-R^\smile(X)) = -R^\smile(X)$. Moreover, $X \subseteq R^\smile(X)$, because R is reflexive. It follows that $-R^\smile(X) \subseteq -X$. We can deduce that $-R^\smile(X) \in \{Z : R(Z) \subseteq -X\}$ and conclude $-R^\smile(X) \subseteq A$.

15.7   Frame – Transforming a Pre-Topological Space of Type $\mathfrak{V}_S$ into a Topological Space

Consider the pre-topological space $\mathbf{P}(R_3)$ of Example 12.8.1. First of all, observe that $\mathbf{P}(R_3)$ is of type $\mathfrak{V}_S$ and $\Omega_{R_3}(U)$ is a non-distributive lattice:
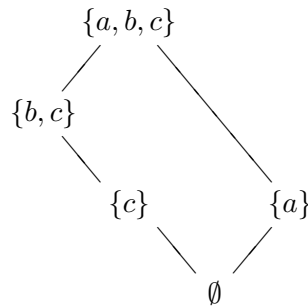
Indeed, $\Omega_{R_3}(U)$ is not a lattice of sets: $L_R(\{c\}) \vee L_3(\{a, b\}) = \{c\} \vee \{a\} = \{a, b, c\} \neq \{c, a\} = \{c\} \cup \{a\}$.

Let us show that $R^T(\mathbf{P}(R_3)) = R_3$: indeed, since $\mathbf{P}(R_3)$ is of type $\mathfrak{V}_S$, $\langle x, y \rangle \in R^T(\mathbf{P}(R_3))$ if and only if $y \in \bigcap \mathcal{B}_x^{R_3}$. But since $\bigcap \mathcal{B}_x^{R_3} = R(x)$, we have $y \in \bigcap \mathcal{B}_x^{R_3}$ if and only if $\langle x, y \rangle \in R_3$.

Moreover notice that $R^T(\mathbf{P}(R_3))$ is different from the specialization preorder $\preceq$ induced by $\Omega_{R_3}(U)$. In fact, $\preceq$ is given by:

$\preceq$	a	b	c
a	1	0	0
b	0	1	1
c	0	0	1

Let us construe $\mathbf{F}(\langle U, \preceq \rangle)$:
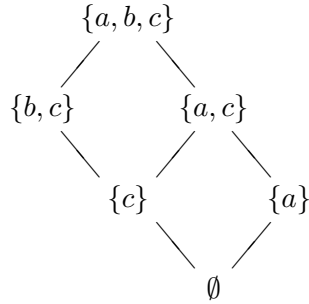
We can observe that $\mathbf{F}(\langle U, \preceq \rangle)$ equals the lattice of sets that we obtain by taking $\Omega_{R_3}(U)$ as a topological basis, that is, by considering all the unions of elements of $\Omega_{R_3}(U)$ (for instance, in $\mathbf{F}(\langle U, \preceq \rangle)$ we have the new element $\{a, c\} = \{c\} \cup \{a\}$, that does not appear in $\Omega_{R_3}(U)$).

The structure $\mathbf{P}_\preceq = \langle U, \mathbf{F}(\langle U, \preceq \rangle) \rangle$ is a topological space. Let us compute the operator $\varkappa$ using the formula $\varkappa(X) = \bigcup \{Y \in \mathbf{F}(\langle U, \preceq \rangle) : Y \subseteq X\}$:

X	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\emptyset$	U
$\varkappa(X)$	$\{a\}$	$\emptyset$	$\{c\}$	$\{a\}$	$\{a, c\}$	$\{b, c\}$	$\emptyset$	U

Therefore, we obtain the neighborhood system:

x	a	b	c
$\varkappa_x$	$\{\{a\}, \{a, b\}, \{a, c\}, U\}$	$\{\{b, c\}, U\}$	$\{\{c\}, \{a, c\}, \{b, c\}, U\}$

Here, $R^T(\mathbf{P}_\preceq)$ is a maximal preorder included in R_3, but only by chance, as the following example witnesses.

Consider the following relation R_4:

R_4	a	b	c	d
a	1	1	0	0
b	0	1	1	0
c	0	0	1	1
d	0	0	0	1

The elements of $\Omega_{R_4}(U)$ are:

Element	Generated by
$\{a\}$	$\varkappa(\{a,b\})$
$\{b\}$	$\varkappa(\{b,c\})$
$\{d\}$	$\varkappa(\{d\}), \varkappa(\{b,d\}), \varkappa(\{a,d\})$
$\{a,b\}$	$\varkappa(\{a,b,c\})$
$\{a,d\}$	$\varkappa(\{a,b,d\})$
$\{c,d\}$	$\varkappa(\{c,d\}), \varkappa(\{a,c,d\})$
$\{b,c,d\}$	$\varkappa(\{b,c,d\})$
$\emptyset$	remaining cases $(\varkappa(\{a\}), \ldots)$
U	$\varkappa(U)$

Therefore, the specialization preorder of $\Omega_{R_4}(U)$ is:

$\preceq$	a	b	c	d
a	1	0	0	0
b	0	1	0	0
c	0	0	1	1
d	0	0	0	1

which is not a maximal preorder included in R_4 (indeed, it lacks $\langle a,b \rangle$, which is an admissible pair for a preorder included in R_4).

15.8 Frame – Modal Interpretations of Approximation Spaces and Rough Sets

Approximation Spaces were interpreted as modal spaces as soon as they came to the stage of logical researches. Nonetheless, the first researchers who systematically dealt with this topic was probably Ewa Orłowska, in a number of pioneering works, such as [Orłowska, 1984, 1988a, 1989] and [Orłowska, 1990a].

We have also to mention, at least, [Nakamura, 1993]. Other authors, such as Dimiter Vakarelov, Ivo Düntsch or Günter Gediga, who made some important advances in the topic, are largely quoted in this Book.

Of course, this list is far from being exhaustive.

A variant of the modal interpretation of Rough Set Theory was exploited to account for a logic for knowledge and learning by [Pawlak, 1985; Orłowska, 1987, 1989; Ras & Zemankova, 1986] and others.

This interpretation is close to the perception-oriented interpretation.

15.9   Frame – Kripke-Joyal Models

As a part of the development of Sheaf Theory, it was realised around 1965 that Kripke semantics was intimately related to the treatment of quantification in topos theory, especially existential quantification. That is, the 'local' aspect of existence for sections of a sheaf was a kind of logic of the 'possible'.

Actually, at present this is not a surprise (however, it was an exciting achievement in 1965). Indeed, in view of the definition of the basic "perception operators" in Chapter 2, we know that $\langle R \rangle$ and $\langle R^{\smile} \rangle$ are connected with existential quantification, while $[R]$ and $[R^{\smile}]$ are connected with universal quantification. Moreover, we know from *Proposition* 2.1.1.(3) that $\mathbf{A} \dashv^{\langle \hat{f}^{\smile} \rangle, \langle \hat{f} \rangle} \mathbf{B}$ for any function $f : A \longmapsto B$, where $\mathbf{A} = \langle \wp(A) \subseteq \rangle$ and $\mathbf{B} = \langle \wp(B) \subseteq \rangle$. In other terms, $\mathbf{A} \dashv^{f^{\rightarrow}, f^{\leftarrow}} \mathbf{B}$. This is a specialization of $\mathbf{B} \dashv^{\langle R^{\smile} \rangle, [R]} \mathbf{A}$. Thus if we define:

$$_{\leftarrow}f : \wp(A) \longmapsto \wp(B); _{\leftarrow}f(X) = \{y \in B : \forall x \in A(f(x) = y \Rightarrow x \in X)\},$$

then we can specialize $\mathbf{B} \dashv^{\langle R \rangle, [R^{\smile}]} \mathbf{A}$ and obtain $\mathbf{B} \dashv^{f^{\leftarrow}, _{\leftarrow}f} \mathbf{A}$. Particularly, if $f : A \times B \longmapsto B$, then $f^{\rightarrow}$ maps a relation R (a subset of $A \times B$) onto a property P (a subset of B). Therefore, recalling that $\leq$ stands for $\Longrightarrow$, the adjunction properties above can be restated as follows:

$$
\begin{array}{cc}
\langle x, y \rangle \in R & P(y) \\
\vdots & \vdots \\
\dfrac{P(y)}{\exists x(\langle x, y \rangle \in R) \Longrightarrow P(y)} & \dfrac{\forall x(\langle x, y \rangle \in R)}{P(y) \Longrightarrow \langle x, y \rangle \in R}
\end{array}
$$

Therefore, we can call $_{\leftarrow}f$ and $f^{\rightarrow}$ the "universal quantifier $\forall_f$ of f" and, respectively, the "existential quantifier of f", $\exists_f$.

If we apply this machinery to presheaves of functions, f is a transformation between two presheaves X and Y, so that the universal quantifier $\forall_f$ (the existential quantifier $\exists_f$) associates a subobject $X'_{\forall_f} \subseteq Y$ ($X'_{\exists_f} \subseteq Y$) with each subobject $X' \subseteq Y$, such that $X'_{\forall_f}$ ($X'_{\exists_f}$) is a sub-presheaf of Y fulfilling the adjunction properties.

This topic was developed by different researchers (for instance, W. Lawvere). However the name Kripke-Joyal semantics is used in this connection in [Mac Lane & Moerdijk, 1992].

From this discussion, it is clear that $\exists$ is a lower adjoint and $\forall$ an upper adjoint. This justifies that $\exists$ distributes over $\vee$ and $\forall$ distributes over $\wedge$, as like as $\diamond$ and, respectively, $\square$ (which, in turn, are defined by means of $\exists$, respectively $\forall$.

15.10 Frame – Quantum Logic and Internal Modalities

In [Bell, 1983] a semantics for Quantum Logic (see, for instance, [Jammer, 1974]) is approached by means of "proximity spaces". From Part I, Frame 4.6, we know that a proximity space is a set equipped with a tolerance relation, $\mathbf{T} = \langle U, T \rangle$. In this paper J. B. Bell argues that the failure of the distributivity law in Quantum Logic is a consequence of more fundamental causes, namely the way in which phenomena are perceived. In this approach phenomena, or "events" in Bell's terminology, are "quanta at a location". Let $p \in U$, then a quantum at location $p \in U$ is p along with all the other points that are similar to p. Thus a quantum at a location has the form $T(p)$. A quantum is intended as the "minimum perceptibilium" at a certain location. Unions of quanta ("assemblage of quanta") give rise to the space of events $Q(\mathbf{T})$. As we have seen in Part I, a quantum assemblage is an ortholattice.

In a quantum assemblage $Q(\mathbf{T})$, the following "localization property" fails:

(LOC) if a and b cover U, that is, if $a \vee b = U$, then for each element z of $Q(\mathbf{T})$, the set $\{a, b\}$ "localizes" to a cover $\{a \wedge z, b \wedge z\}$.

Otherwise stated, the distributivity law $(a \wedge z) \vee (b \wedge z) = (a \vee b) \wedge z$ fails in quantum assemblages. Therefore, the Persistence Property introduced in *Window* 11.1 of this Part, fails too, since it is a property stating that phenomena are valid also in subspaces.

The lack of the Persistence Property is able to model the fact that in *Quantum Theory* we can have properties that hold globally, but do not hold locally. Indeed, we can have properties P_1 and P_2 such that the disjunction $P_1 \vee P_2$ holds in the whole event space, but does not hold in sub-parts of the space. Actually in such a sub-part S we may have a superposition of P_1 and P_2 but S cannot be split into two sub-parts, one fulfilling P_1 and the other P_2. In other terms, we cannot have S' and S'' such that $S' \cup S'' = S, S' \models P_1, S' \models P_2$, that which is explicitly required by Kripke-Joyal semantic in order for S to force $P_1 \vee P_2$. In this case $1 \models P_1 \vee P_2$ but $P_1 \vee P_2$ does not localise to all the sub-parts of 1 (see also Example 11.5.2).

Here is an example from [Bell, 1983]: Let C be a closed disc in the complex plane and for any $x, y \in C$ define $\langle x, y \rangle \in T$ if and only if the angular distance between x and y is $\leq \frac{\pi}{4}$. Then $\mathbf{C} = \langle C, T \rangle$ is a proximity space and for any $x \in C$, the quantum at x is the quadrant $T(x) = \{y \in C : \arg x - \frac{\pi}{4} \leq \arg y \leq \arg x + \frac{\pi}{4}\}$.

Let $Q(\mathbf{C})$ be a complete quantum assemblage. Suppose we are given two atomic attributes W ("white") and B ("black"). Let us assign a forcing relation between points and attributes such that $[\![W]\!] = 1^{st}$ *quadrant* $\cup \, 3^{rd}$ *quadrant* and $[\![B]\!] = 2^{nd}$ *quadrant* $\cup \, 4^{th}$ *quadrant*:

Obviously $[\![W \vee B]\!] = C$, or, otherwise stated, the disjunction of the two attributes is manifested (forced) over the entire universe C. But if S is the quantum $[3\frac{\pi}{4}, \frac{\pi}{4}]$, with respect to the positive x-axis, then $S \nsubseteq [\![W \vee B]\!]$ (i.e. $W \vee B$ is not manifested over S) since there is not a disjunction of S, $S' \cup S'' = S$, such that $S' \subseteq [\![W]\!]$ and $S'' \subseteq [\![B]\!]$ (see Figure 15.1). Therefore $S \nvDash W \vee B$ because it does not satisfy what is required by the Kripke-Joyal forcing condition for $\vee$ (see *Window* 11.1).
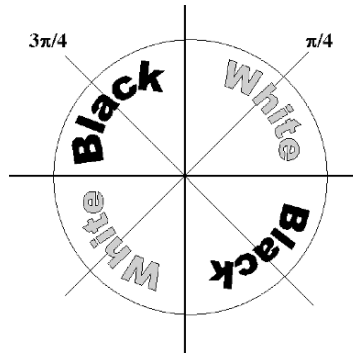
Figure 15.1: Superposition and non-persistence

If we define a modality L by means of the internal forcing condition $x \models L(\alpha)$ if and only if $\forall y(y \leq x \Rightarrow y \models \alpha)$, then L exactly models persistent events, in that $[\![L(\alpha)]\!] = 1$ if and only if α is persistent.

L induces a dual modality M defined by $M(\alpha) = \neg(L(\neg\alpha))$, where $\neg$ is the orthocomplementation:

$x \models M(\alpha)$ if and only if $\forall y(x \models L(\neg\alpha) \Rightarrow y \leq \neg x)$,

if and only if $\forall y(\forall z \leq y(\forall v(v \models \alpha \Rightarrow v \leq \neg z))) \Rightarrow y \leq \neg x$,

if and only if $\forall y(\forall v(v \models \alpha \Rightarrow v \leq y)) \Rightarrow x \leq y$,

if and only if $\exists y(y \geq x \ \& \ y \models A)$.

Therefore, $S \subseteq M([\![\alpha \vee \beta]\!])$ asserts that there is a superpart of S manifesting a superposition of α and β.

In the quoted paper, an example about the manifestation of incompatible attributes is given, too.

15.11 Frame – Persistence of Modalised Formulas

In the approach developed in the present Part, we have imposed on the knowledge map k a particular condition: *continuity* (*viz.* $k(A) \vee k(B) = k(A \vee B)$). As a consequence k is isotonic (if $A \leq B$, then $k(A) \leq k(B)$). This makes any evaluation of modalised formulas satisfy the Persistence Property. Indeed, suppose we are given a k-modal system $\langle \mathbf{L}, k(\mathbf{L}) \rangle$ and p is an element of $\mathbf{L}$. By definition, if $p \models L_k(\alpha)$, then for any $p' \leq k(p)$, $p' \models \alpha$. Suppose $p'' \leq p$. By isotonicity $k(p'') \leq k(p)$. Hence from the preceding clause, $k(p'') \models \alpha$. Given any $p''' \leq k(p'')$, by transitivity $p''' \leq k(p)$, so that $p''' \models \alpha$, too. But since $p''' \leq k(p'')$, it follows that $p'' \models L_k(\alpha)$, as well.[1]

Trivially, for any formula α, $M_g(\alpha)$ is persistent too. In fact if $p \models M_R(\alpha)$, then there exists a p' such that $p \leq g(p')$ and $p' \models \alpha$. Suppose $p'' \leq p$. Then by transitivity $p'' \leq g(p')$, so that the forcing clause for M_R is satisfied at point p'', too.

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[1] If the *Persistence Property* is satisfied by non-modalised formulas, the proof is much shorter. Indeed in this case $p \models L_k(\alpha)$ if and only if $k(p) \models \alpha$. Now suppose $p' \leq p$. By isotonicity $k(p') \leq k(p)$ and from persistence $k(p') \models \alpha$. Henceforth, $p' \models L_k(\alpha)$.

However, at a deeper insight, the above mechanism works because k is, from its very definition, an endomorphism, that is, k maps elements of $\mathbf{L}$ onto elements of $\mathbf{L}$. So, when we start with a k-modal system, things run by definition. We have also seen that if we start with a generic finite distributive modal system $\langle \mathbf{S}, \mathbf{S}' \rangle$ we have a method for transforming it into an isomorphic k-modal system $\langle \mathbf{L_S}, k^*(\mathbf{L_S}) \rangle$ (called the "representation" of $\langle \mathbf{S}, \mathbf{S}' \rangle$) where k^* is connected with a binary relation R on the carrier of $\mathbf{L_S}$. We have also seen that R is not arbitrary but, instead, is the preorder $\preceq$ induced by $\mathbf{L_{S'}}$. The passage from $\mathbf{S}$ to $\mathbf{L_S}$ produces exactly the turning point in which *persistence with respect to the lattice* $\mathbf{S}$ (*viz.* if $p \models \alpha$ and $p' \leq p$, then $p' \models \alpha$) turns into *persistence with respect to the Kripke frame* $\langle \mathcal{J}(\mathbf{S}), \leq \rangle$ (if $x \Vdash \alpha$ and $x' \leq x$, then $x' \Vdash \alpha$) where $\leq$ is the partial order induced by $\mathbf{S}$ on the set of co-prime elements $\mathcal{J}(\mathbf{S})$. But modalised formulas are evaluated using $\preceq$, not $\leq$ ($x \Vdash L(\alpha)$ iff for any x' such that $x \preceq x'$, $x' \Vdash \alpha$). How persistency of modalised formulas is guaranteed with respect to $\leq$? The answer is: since $\mathbf{S}'$ is a sub-lattice of $\mathbf{S}$, $\preceq$ is coherent with $\leq$, because $x \preceq y$ if and only if $\uparrow_\leq x \subseteq \uparrow_\leq y$ if and only if $y \leq x$. Moreover, this coherence is guaranteed *a priori* if we start with a k-modal system $\langle \mathbf{S}, k(\mathbf{S}) \rangle$ where $\mathbf{S}$ is a lattice of subsets of a set U and k is connected with a binary relation $R \subseteq U \times U$. In fact, in this case $k(\mathbf{S})$ is as above and R coincides with the specialization preorder of the sublattice $k(\mathbf{S})$.

Persistence of modalised formulas becomes a critical topic when we want to combine $\mathbf{S}$ with a generic binary relation $R \subseteq U \times U$. Or, which is the same, when we evaluate non modalised formulas over a Kripke frame $\langle U, R_1 \rangle$ and modalised formulas on a different Kripke frame $\langle U, R_2 \rangle$.[2]

This was a main topic in [Humberstone, 1981]. In this well-known paper a modal frame is a tuple $\langle W, \leq, R, V \rangle$, where W is a set, $\leq$ is a partial order on W and R is a binary relation on W. V is a set-up map such that for any atomic formula γ, for any $x, x' \in W$, if $V(\gamma, x)$ is defined (i.e. $V(\gamma, x) = T$ or $V(\gamma, x) = F$) and $x \leq x'$, then $V(\gamma, x) =$

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[2]This is not a problem if we deal, as happened in the majority of the cases in this Part, with modal systems where the non modal part is based on a Boolean algebra $\mathbf{B}$ (in particular $\mathbf{B}(U)$, the Boolean algebra of $\wp(U)$). Indeed, in this case the specialization preorder of (the representation of) $\mathbf{B}$ is immaterial since it reduces to the identity relation. Therefore non modalised formulas are persistent for free as like as modalised formulas, even if they are evaluated by means of a completely arbitrary relation R.

$V(\gamma, x')$, which constitutes the persistency requirement for non modalised formulas (an additional "Refinability" requirement is added, that we do not discuss here). On the basis of this set-up, forcing clauses are defined for non modal operations. It turns out that any non modal formula α is $\leq$-persistent. Modalised formulas are evaluated by means of the relation R:

$$x \Vdash \Box(\alpha) \ \mathit{iff\ for\ all\ } y \in W, \ \mathit{if\ } xRy \ \mathit{then\ } y \Vdash \alpha \qquad (15.11.1)$$

The use of two relations (instead of just one dedicated to modalised formulas, while non modalised formulas are evaluated on a "flat" Boolean space, as usual) was suggested by previous arguments about Tense Logic developed in [Humberstone, 1979], where it was shown the necessity to distinguish between the relation "x is a sub-interval of the interval of time y" and the relation "the interval x wholly precedes the interval y".

Humberstone explicitly examines the relationships that $\leq$ and R must satisfy in order to make modalised formulas $\leq$-persistent. He finds that the following sort of transitivity constraint is fundamental:

$$\mathit{for\ all\ } x, x' \ \mathit{and\ } x'', \ \mathit{if\ } x \leq x' \ \mathit{and\ } x'Rx'' \ \mathit{then\ } xRx'' \qquad (15.11.2)$$

(a dual constraint of (15.11.2) and a third constraint connected with the notion of "Refinability" above cited are added, as well).

It is shown that this suffices to guarantee that if $x \Vdash \Box(\alpha)$ and $x \leq x'$, then $x' \Vdash \Box(\alpha)$.

As to possibility, it is shown that given the usual definition $\Diamond(\alpha) =_{def} \sim \Box(\sim \alpha)$, the correct forcing clause is the following:

$$x \Vdash \Diamond(\alpha) \ \mathit{iff\ } \forall x' \geq x, \exists x'' \ \mathit{such\ that\ } x'Rx''$$
$$\mathit{and\ } \exists x''' \geq x'' \ \mathit{such\ that\ } x''' \Vdash \alpha \qquad (15.11.3)$$

This clause makes $\Diamond(\alpha)$ persistent, any α. It is shown, on the contrary, that if an operator M is defined by the usual clause

$$x \Vdash M(\alpha) \ \mathit{iff\ } \exists x' \ \mathit{such\ that\ } xRx' \ \mathit{and\ } x' \Vdash \alpha \qquad (15.11.4)$$

then $M(\alpha)$ is not $\leq$-persistent and $\Diamond(\alpha) \vdash M(\alpha)$, but not conversely.

In what follows we add some examples.

First example: Consider the lattices $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ and $\mathbf{L}_{\mathbf{L}'}$ of Example 11.5.3. Let $\lesssim$ be the specialization preorder of $\mathbf{F}(\mathbf{J}(\mathbf{L}))$ and $\preceq$ the specialization preorder of $\mathbf{L}_{\mathbf{L}'}$. It is easy to verify that if we put $\lesssim = \leq$ and $\preceq = R$, then (15.11.2) is satisfied.

Second example: Consider the following relations on $W = \{a, b, c, d\}$:

R_1	a	b	c	d
a	1	1	1	1
b	0	1	0	1
c	0	0	1	0
d	0	0	0	1

R_2	a	b	c	d
a	1	1	0	0
b	0	1	0	0
c	1	1	1	0
d	1	1	0	1

R_3	a	b	c	d
a	1	1	1	1
b	0	1	1	1
c	0	0	1	0
d	0	0	0	1

We can observe that (15.11.2) is not satisfied by the pair $\langle R_1, R_2 \rangle$. Indeed, bR_1d, dR_2a but not bR_2a. In particular we can observe that the set $R'' = \{R_2(X) : X \subseteq W\}$ is not a subset of $R' = \{R_1(X) : X \subseteq W\}$ (for instance, $R_2(\{a\}) = \{a, b\}$ is not an element of R'). On the contrary, (15.11.2) is satisfied by the pair $\langle R_1, R_3 \rangle$. And we can observe that $R''' = \{R_3(X) : X \subseteq W\}$ is a subset of R' and $\langle R''', \subseteq \rangle$ is a sublattice of $\langle R', \subseteq \rangle$. Consider now a formula α such that $[\![\alpha]\!] = \{b, d\}$. If we evaluate $\square(\alpha)$ on the basis of R_2, then $[\![\square(\alpha)]\!] = \{b\}$ (i.e. $[\![\square(\alpha)]\!] = L_{R_2}([\![\alpha]\!])$). But $\{b\}$ is not an open set in R_1 because $\geq b$ but $d \notin \{b\}$. Hence b forces $\square(\alpha)$, $d \geq b$ but d does not force $\square(\alpha)$.

A third example is about the operator $\diamond$. Consider a modal frame (in the sense of Humberstone) $\langle W, R_1, R_3, V \rangle$ with $d \models \alpha$. We have $\{x : x \Vdash M(\alpha)\} = \{a, b, d\}$ and $\{x : x \Vdash \diamond(\alpha)\} = \{b, d\}$. Let us check why $a \nVdash \diamond(\alpha) : c \geq a$, c is the only element such that cR_3c, c is the only element such that $c \leq c$ but $c \nVdash \alpha$. Therefore, a does not satisfy clause (15.11.3).

Finally it must be noticed that in case R coincides with the preorder $\leq$, the forcing clause for $\diamond$ becomes a clause for $\square M(\_)$. That is, $x \Vdash \diamond(\alpha)$ iff $x \Vdash \square(M(\alpha))$. Otherwise stated, $\diamond$ in this case, coincides with Lawvere's local operator (see Part II).

15.12   Frame – Coherence Between Information and Knowledge

In Subsection 11.4.1 given a lattice **L** we required a certain sort of coherence between the information order $\leq$ given by the lattice order and the new order induced by a knowledge map k on **L**. Namely, we

required that k must be a $\vee$-endomorphism. Coherence, in this case, is embedded in the persistence property for modalised formulas (cf. the discussion at the beginning of Frame 15.11).

We anticipated that there are other approaches to the problem concerning the coherence between a knowledge order and an information order. A first example is Humberstone's constraint (15.11.2) that we have seen above.

Another interesting approach was proposed by [Ginsberg, 1986] and developed in [Ginsberg, 1988] and [Ginsberg, 1990], upon the following intuition. One can order values on the basis of truth and falsity, or on the basis of the completeness of the information they represent. Thus, instead of a lattice structure equipped with a single order relation, Matthew Ginsberg proposes using a "bi-lattice", that is, a set equipped with two order relations: a truth order $\leq_t$ and a knowledge order $\leq_k$. If $a \leq_k b$ then we say that the evidence underlying the assignment of the truth-value a is subsumed by the evidence underlying the assignment of the truth-value b. The two orders have two distinct top and bottom elements. For $\leq_t$ we have, as usual, 0 and 1. For $\leq_k$ the bottom element is u, meaning "unknown", while the top element is $\perp$, meaning "overdefined" or "contradictory".[3] Clearly $\leq_k$ and $\leq_t$ are intimately connected with each other and this is expressed by a set of constraints about the operations of inf and sup that they induce.

Therefore, a bi-lattice is a structure $\mathbf{B} = \langle B, \wedge, \vee, \cdot, +, 1, 0, u, \perp \rangle$ such that:

(1) $\langle B, \wedge, \vee, 1, 0 \rangle$ and $\langle B, \cdot, +, u, \perp \rangle$ are lattices; (2) $a \wedge b = a$ iff $a \leq_t b$ (i.e. $\leq_t$ is the order relation of the first lattice); (3) $a \cdot b = a$ iff $a \leq_k b$ (i.e. $\leq_k$ is the order relation of the second lattice); (4) each operation respects the order relations in the alternate lattice (for instance, if $a \leq_k b$ and $a' \leq_k b'$, then $a \wedge a' \leq_k b \wedge b'$.

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[3]This notion of "overdefined value" derives from the very origin of this business: Default Logic. More precisely, we can deduce a statement S using default pieces of information. If the default is later overridden by a fresh information, then S might be retracted (a typical example in Artificial Intelligence is Tweety, that can fly because she is a bird and, by default, birds can fly. But later we learn that Tweety is a penguin. Then the default does not work any longer and we have to conclude that Tweety cannot fly). Therefore we can admit that we can prove a statement S using one method and $\neg S$ using another.

The following diagram depicts the smallest non trivial bilattice:
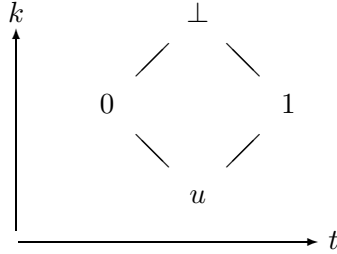
The smallest non trivial bilattice

It is possible to note, for instance, that $u \wedge \perp + u \wedge 1 = 0 + u = 0 = u \wedge (\perp + 1)$. More generally, we have that $x\wedge$ and $x\vee$ are lattice endomorphisms of $\langle B, \cdot, +, u, \perp \rangle$, while $x\cdot$ and $x+$ are lattice endomomorphisms of $\langle B, \wedge, \vee, 1, 0 \rangle$, which corresponds to condition (4) above.

As for negation, the underlying intuition suggests that $\sim$ must be such that $a \leq_t b$ implies $\sim b \leq_t \sim a$, while $a \leq_k b$ implies $\sim a \leq_k \sim b$. Indeed, if we know less about a than about b, we know less about $\sim a$ than $\sim b$. So, for instance, in the above bi-lattice we have $\sim 1 = 0$, $\sim 0 = 1$, $\sim u = u$ and $\sim \perp = \perp$.

Developing this approach, Melvin Fitting in [Fitting, 1988] extends the evaluations to topological spaces dual to Kripke models, that is, by building bi-lattices over intuitionistic semantics. In this approach given a topological space $\langle U, \Omega(U) \rangle$ dual of a Kripke frame $\langle U, R \rangle$, elements of a bi-lattice are ordered pairs $\langle O, C \rangle$, such that O is an open set (meaning "belief") and C is a closed set (meaning "disbelief"), possibly, but not necessarily, the complement of O. Since the case "overdefined" is admitted, no particular relationship between O and C is imposed. This liberality induces the following cases: a pair $\langle O, C \rangle$ will be said (a) *overdefined* if $O \cap C \neq \emptyset$, (b) *consistent* if $O \cap C = \emptyset$, (d) *exact* if $O \cap C = \emptyset$ and $O \cup C = U$.

The two orders are defined as follows: (ti) $\langle O_1, C_1 \rangle \leq_t \langle O_2, C_2 \rangle$ if $O_1 \subseteq O_2$ and $C_1 \subseteq C_2$. (tii) $\langle O_1, C_1 \rangle \leq_k \langle O_2, C_2 \rangle$ if $O_1 \subseteq O_2$ and $C_1 \supseteq C_2$. The operations are defined in a familiar way. For the truth order: $\bigvee S = \langle \bigcup \{O : \langle O, C \rangle \in S\}, \bigcap \{C : \langle O, C \rangle \in S\} \rangle$, $\bigwedge S = \langle \mathbb{I}(\bigcap \{O : \langle O, C \rangle \in S\}), \mathbb{C}(\bigcup \{C : \langle O, C \rangle \in S\}) \rangle$ (the interior and closure operators $\mathbb{I}$ and $\mathbb{C}$ are not required if S is finite). For the knowledge order: $\sum S = \langle \bigcup \{O : \langle O, C \rangle \in S\}, \mathbb{C}(\bigcup \{C : \langle O, C \rangle \in S\}) \rangle$, $\prod S = \langle \mathbb{I}(\bigcap \{O : \langle O, C \rangle \in S\}), \bigcap \{C : \langle O, C \rangle \in S\} \rangle$ (again, $\mathbb{I}$ and $\mathbb{C}$ are not required if S is finite). The interconnections between the two orders,

are guaranteed by 4) above. It must be noticed that the operations $\wedge$ and $\vee$ distribute over $+$ and, respectively, $\cdot$, and viceversa.

As for the negation we have (tiii) $\sim \langle O, C \rangle = \langle \mathbb{I}(C), \mathbb{C}(O) \rangle$. In subsequent papers (cf. [Fitting, 1991]), a dual form of negation, $-$, called a "*conflation*", was introduced such that $a \leq_t b$ implies $-a \leq_t -b$, while $a \leq_k b$ implies $-b \leq_k -a$ (a further negation definable on bilattices was shown in Part II, Frame 10.12.2, Figure 10.4).

Below, on the left the topological version of the above mentioned bi-lattice is reproduced. On the right a topological bi-lattice is built from the dual space of the Kripke frame $\{a \leq b\}$:
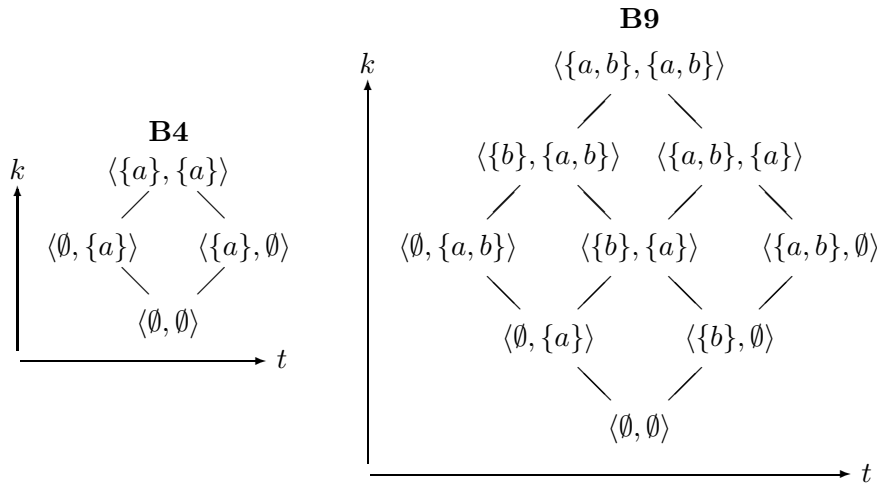
Observe that the right side lattice looks like a Post algebra of order three. Nonetheless it is not a Post algebra. In fact, both $\langle \{a,b\}, \{a,b\} \rangle$ and $\langle \emptyset, \emptyset \rangle$ are fix-points of the negation $\sim$. It follows that the inequality $a \wedge \sim a \leq b \vee \sim b$ (which is valid in Post algebras) is not satisfied.

Finally, notice that the operation $- \langle O, C \rangle = \langle -C, -O \rangle$ is a conflation. Neither in this case we obtain a Post algebra of order three because there are two fix points of the operator "$-$", namely $\langle \emptyset, \{a,b\} \rangle$ and $\langle \{a,b\}, \emptyset \rangle$.

Recently bi-lattices has been used to define extensional interpretations for the "logic of questions" (cf. [Nelken & Francez]). In such logic we have indicative and interrogative sentences. Clearly it is not wise to interpret interrogative sentences over the set $\{0, 1\}$, because a question cannot be true or false. Instead, a question can be "resolved", r, or "unresolved", ur. A question $\text{S}(\alpha)$ is resolved or unresolved depending on the knowledge status of the underlying indicative sentence α.

In turn this may assume the values 1 (meaning "α is known to be true"), 0 (meaning "α is known to be false") and δ (meaning " α is unknown to be true or false"). More precisely, the semantic for indicative sentences is assumed to be the Kleene Strong Semantic (see Frame 10.20). This leads to symmetric notions of *truth* and *answerability conditions.*

In this semantic setting we can evaluate sentences modalised by the interrogative operator $\text{\textcircled{S}}$:

$$v(\text{\textcircled{S}}(\alpha)) = \left\{ \begin{array}{ll} r & if\ v(\alpha) \in \{1,0\} \\ ur & if\ v(\alpha) = \delta \end{array} \right.$$

The relationship between indicative evaluations and interrogative evaluations is guided by an "answerwhood principle": an indicative sentence answers an interrogative one, if whenever the former is assigned 1, the latter is assigned r.[4]

This approach yields to a bi-latice with five elements structured by a truth-order $\leq_t$ and an answerability order $\leq_r$:
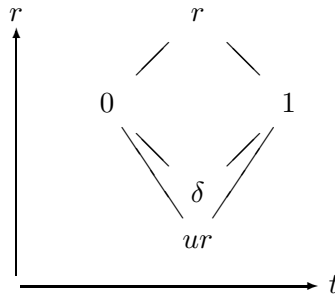
The designed values are 1 and r. Notice that the bi-lattice negation $\sim$ does not affect, correctly, the amount of resolvedness as, in the previous bi-lattices, it does not affect the knowledge degree.[5]

15.13   Frame – Neighborhood Systems

15.13.1   Some History and Recent Applications

Čech in his book [Čech, 1966] showed that spaces in which just **1**, **N1**, **N2**, and **N3** are assumed, are sufficient to derive most of the

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[4]Actually, one should distinguish between "yes/no questions" and so-called "wh-questions" (what, which, when, . . .). The latter are infinitary versions of the former, since the structure of *wh-questions* is similar to that of quantified sentences.

[5]However, this is technically wise but it is unlikely from the point of view of Natural Language (what is the negation of a question?). More difficulties in interpretation, of course, arise if we assume also a conflation operation.

fundamental results of point set topology, thus making advances in the streamline of Fréchet's pioneering ideas on pre-topologies (see [Fréchet, 1928]).

Pre-topologies have been exploited in a limited, albeit significant, number of fields.

Recently, interesting applications have been found in the study of search spaces in Combinatorial Chemistry as well as in sequence spaces underlying molecular evolution. Indeed these spaces are conventionally represented by graphs. However, recombination implies combinatorial search spaces structures, which are not graphical and pre-topological spaces revealed to be a promising formal tool (see [Stadler et al., 2000] and [Stadler & Stadler, 2001] and the references quoted therein).

The formal setting of pre-topologies in which the expansion operator is a closure operator has been used in Computer Science, for instance to model data flow between a client and a server. Indeed, it is worthwhile reproducing in Figure 15.2 a schema from [Hancock & Hyvernat, 2002] because it is quite similar to the schema we presented in Figure 2.6 of Chapter 2:
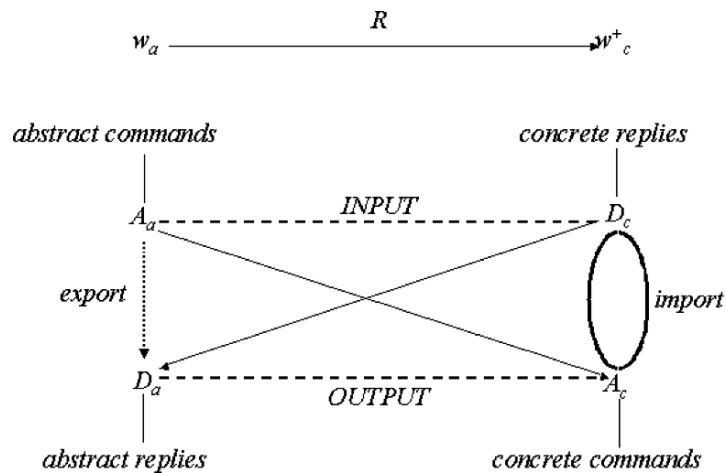
Figure 15.2: The programmer's predicament

In the last decade of the XX century some French scholars began applying pre-topologies to data analysis. We have to mention especially [Belmandt, 1993] (cf. [Langeron & Bonnevay, 1999], too):

Particularly, in Belmandt's work, instead of the functions ε^m and $\varkappa^m$ discussed in Excursus 12.6.2, the following functions where introduced:

(a) $a^m : \wp(U) \longmapsto \wp(U); a^m(A) = \bigcap_{\gamma \in \Gamma^m} \left(\bigcup_{l \in \gamma} \mathbf{b}_l^{\varepsilon}(A) \right);$

(b) $i^m : \wp(U) \longmapsto \wp(U); i^m(A) = \bigcup_{\gamma \in \Gamma^m} (\bigcap_{l \in \gamma} \mathbf{b}_l^{\nsucc}(A)),$

where Γ^m is defined as in Excursus 12.6.2 and for $l \in \gamma$, $\mathbf{b}_l^{\varepsilon}(A) = \{x : R_l(x) \cap A \neq \emptyset\}$ and $\mathbf{b}_l^{\nsucc}(A) = \{x : R_l(x) \subseteq A\}$.

We give an example based on the family of relations of Excursus 12.4.1. Let us compute $a^m(x)$ and $i^m(x)$ for some cases of m and x.

First of all, we have to apply the operators $\mathbf{b}_l^{\varepsilon}$ and $\mathbf{b}_l^{\nsucc}$:

	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\mathbf{b}_1^{\varepsilon}$	$\emptyset$	$\{a\}$	$\{a,b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\mathbf{b}_2^{\varepsilon}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,c\}$	$\{a,b\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\mathbf{b}_3^{\varepsilon}$	$\emptyset$	$\{a,b\}$	$\{b\}$	$\{a,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\mathbf{b}_1^{\nsucc}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{c\}$	$\{b,c\}$	$\{a,b,c\}$
$\mathbf{b}_2^{\nsucc}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{c\}$	$\{b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\mathbf{b}_3^{\nsucc}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{c\}$	$\{b\}$	$\{a,c\}$	$\{c\}$	$\{a,b,c\}$

For instance, in order to obtain $\mathbf{b}_2^{\nsucc}(\{a,b\})$ we reason in the following way: (i) $R_2(a) = \{a,c\} \nsubseteq \{a,b\}$, hence a is discharged; (ii) $R_2(b) = \{b\} \subseteq \{a,b\}$, hence b is admitted; (iii) $R_2(c) = \{c\} \nsubseteq \{a,b\}$, hence c is discharged. Thus we obtain $\mathbf{b}_2^{\nsucc}(\{a,b\}) = \{b\}$. For $\mathbf{b}_2^{\varepsilon}(\{c\})$ we compute: (i) $R_2(a) = \{a,c\} \cap \{c\} \neq \emptyset$, hence a is admitted; (ii) $R_2(b) = \{b\} \cap \{c\} = \emptyset$, hence b is discharged; (iii) $R_2(c) = \{c\} \cap \{c\} \neq \emptyset$, hence c is admitted. Thus we obtain $\mathbf{b}_2^{\varepsilon}(\{c\}) = \{a,c\}$.

The developed formulas for a^m and i^m are:

m	$a^m(A)$
1	$\mathbf{b}_1^{\varepsilon}(A) \cap \mathbf{b}_2^{\varepsilon}(A) \cap \mathbf{b}_3^{\varepsilon}(A)$
2	$(\mathbf{b}_1^{\varepsilon}(A) \cup \mathbf{b}_2^{\varepsilon}(A)) \cap (\mathbf{b}_1^{\varepsilon}(A) \cup \mathbf{b}_3^{\varepsilon}(A)) \cap (\mathbf{b}_2^{\varepsilon}(A) \cup \mathbf{b}_3^{\varepsilon}(A))$
3	$\mathbf{b}_1^{\varepsilon}(A) \cup \mathbf{b}_2^{\varepsilon}(A) \cup \mathbf{b}_3^{\varepsilon}(A)$

m	$i^m(A)$
1	$\mathbf{b}_1^{\nsucc}(A) \cup \mathbf{b}_2^{\nsucc}(A) \cup \mathbf{b}_3^{\nsucc}(A)$
2	$(\mathbf{b}_1^{\nsucc}(A) \cap \mathbf{b}_2^{\nsucc}(A)) \cup (\mathbf{b}_1^{\nsucc}(A) \cap \mathbf{b}_3^{\nsucc}(A)) \cup (\mathbf{b}_2^{\nsucc}(A) \cap \mathbf{b}_3^{\nsucc}(A))$
3	$\mathbf{b}_1^{\nsucc}(A) \cap \mathbf{b}_2^{\nsucc}(A) \cap \mathbf{b}_3^{\nsucc}(A)$

Then, let us compute, for instance, $i^2(\{a,b\})$, $i^1(\{a,b\})$, $i^1(\{a,c\})$ and $a^3(\{a\})$:

(i) $i^2(\{a,b\}) = (\mathbf{b}_1^{\varkappa}(\{a,b\}) \cap \mathbf{b}_2^{\varkappa}(\{a,b\})) \cup (\mathbf{b}_1^{\varkappa}(\{a,b\}) \cap \mathbf{b}_3^{\varkappa}(\{a,b\})) \cup (\mathbf{b}_2^{\varkappa}(\{a,b\}) \cap \mathbf{b}_3^{\varkappa}(\{a,b\})) = (\{a,b\} \cap \{b\}) \cup (\{a,b\} \cap \{b\}) \cup (\{b\} \cap \{b\}) = \{b\}$.

(ii) $i^1(\{a,b\}) = \mathbf{b}_1^{\varkappa}(\{a,b\}) \cup \mathbf{b}_2^{\varkappa}(\{a,b\}) \cup \mathbf{b}_3^{\varkappa}(\{a,b\}) = \{a,b\} \cup \{b\} \cup \{b\} = \{a,b\}$.

(iii) $i^1(\{a,c\}) = \mathbf{b}_1^{\varkappa}(\{a,c\}) \cup \mathbf{b}_2^{\varkappa}(\{a,c\}) \cup \mathbf{b}_3^{\varkappa}(\{a,c\}) = \{c\} \cup \{a,c\} \cup \{a,c\} = \{a,c\}$.

(iv) $a^3(\{a\}) = \mathbf{b}_1^{\varepsilon}(\{a\}) \cup \mathbf{b}_2^{\varepsilon}(\{a\}) \cup \mathbf{b}_3^{\varepsilon}(\{a\}) = \{a\} \cup \{a\} \cup \{a,b\} = \{a,b\}$.

The reader should continue this exercise and verify that the operators i^m and a^m correspond to $\varkappa^m$ and ε^m, respectively.

15.13.2 Neighborhood Systems and Approximation of Information

As we have already seen in Frame 4.10 of Part I, in a series of papers, neighborhood systems have been suggested by T. Y. Lin as a framework to generalise Approximation Spaces and Rough Set Theory (see, for instance [Lin, 1998]).

In order to give an example of the use of neighborhood systems to approximate information, let us use an example from the quoted paper. Consider the following restaurant database:

RESTAURANT	TYPE	LOCATION	PRICE
Wendy	American	West wood	inexpensive
Le Chef	French	West LA	moderate
Great Wall	Chinese	St Monica	moderate
Kiku	Japanese	Hollywood	moderate
South Sea	Chinese	Los Angeles	expensive

If Q is the query "*Select RESTAURANT where TYPE= 'Japanese' and LOCATION= 'West wood' and PRICE= 'Moderate*'", then a traditional database returns a null answer.

But suppose we are given the relations:

(i) *close_location*, so that, for instance $\langle West\ wood,\ St.\ Monica \rangle \in$ *close_location*;

(ii) *close_type*, so that, for instance $\langle Japanese, \quad Chinese \rangle \quad \in$ *close_type*;

(iii) *close_price*, so that, for instance $\langle moderate, \quad inexpansive \rangle \quad \in$ *close_price*.

(iv) *very_close_location* . . . . . .

....

Thus, the machine can relax Q and ask for something which fits with Q or is "similar"(close) to Q, so to obtain an answer on the basis of the neighborhoods of the above relations, such as "Great Wall" which is close to "West wood" as for *location* and to "Japanese" as for *type*.

This reasoning is conceptually, if not technically, analogous to that illustrated in Excursus 12.6.2.

In order to obtain results like this, Lin proposes extensions of Rough Set Theory based on various types of neighborhood systems, for instance binary neighborhood systems $\{R_i\}_{i \in I}$ for binary relations $R_i \subseteq U \times U'$, or fuzzy binary systems, where R_i is a fuzzy relation $U \times U' \longmapsto [0,1]$.

15.13.3   Neighborhood Systems and Modal Systems

Neighborhood systems have been used in order to model non-normal and normal modal logics. Semantics of this kind are therefore called "neighborhood semantics" or "Montague-Scott semantics" (see [Montague, 1968] and [Scott, 1970]). Given a neighborhood system $\mathcal{N}(U)$, the necessity operator L is modeled as follows, for any formula α of a given modal language:

$$x \models L(\alpha) \ iff \ \{y : y \models \alpha\} \in \mathcal{N}_x$$

From this definition we immediately obtain:

$$[\![L(\alpha)]\!] = \{x : [\![\alpha]\!] \in \mathcal{N}_x\} = \varkappa([\![\alpha]\!])$$

where $\varkappa$ is the contraction operator of the induced pre-topological space (the latter equivalence occurs only if $\mathcal{N}(U)$ is at least of type $\mathcal{N}_1$).

In the following table we give a taste of this modeling technique, by listing some axioms along with the properties of its neighborhood models:

	Axioms	System	Properties of $\mathcal{N}(U)$
1	$\vdash \alpha \Longleftrightarrow \beta$ implies $\vdash L(\alpha) \Longleftrightarrow L(\beta)$	E	
2	$1 +$ $L(\alpha \Longrightarrow \beta) \Longrightarrow (L(\alpha) \Longrightarrow L(\beta))$	EK	$(-X \cup Y \in \mathcal{N}_x$ & $X \in \mathcal{N}_x) \Rrightarrow Y \in \mathcal{N}_x$
3	$1 + 2 + \vdash L(\top)$	K	$U \in \mathcal{N}_x$, any $x \in U$
4 5	$1 + 3 +$ $\alpha \Longrightarrow L(\alpha) +$ $L(\alpha) \wedge L(\beta) \Longrightarrow \neg L(\neg(\alpha \wedge \beta))$	ES	$\mathbf{0} + \mathbf{N2} +$ $N, N' \in \mathcal{N}_x \Rrightarrow N \cap N' \neq \emptyset$

A note about the logical system ES is in order (see [Ben David et al., 2001]).

ES is a logic in which the modal operator L has the meaning "expectation", so that $L(\alpha)$ means that "α is expected to happen" (possibly from some assumed defaults in a default logic). Obviously, if event α and β are singularly expected to happen, nothing induces to argue that they must be expected to happen at the same time. It follows that the law

$$(L(\alpha) \wedge L(\beta)) \to L(\alpha \wedge \beta)$$

is not accepted in ES. This is the reason why for any $x \in U$, $\mathcal{N}_x$ is not required to be a filter, but only a so called *semi-filter*. Indeed we have seen in this Part that if $\mathcal{N}(U)$ is of type $\mathcal{N}_3$, then L (i.e. $\varkappa$) distributes over meets.

It is possible to show that some modal logics have neighborhood models but do not have Kripke models (modal logics weaker than **K** provide an example). Moreover, there are normal modal logics which are neither neighborhood complete nor Kripke complete (see – [Gerson, 1975a,b]).

An interesting example of such a logic is given by system G^* obtained in the following way.

If L is interpreted as $Bew(\ulcorner p \urcorner)$, where $\ulcorner p \urcorner$ denotes the Gödel number of the arithmetic sentence p, while $Bew(\alpha)$ (after "Beweisbar") denotes the predicate "α is provable in the first order system of Peano Arithmetic, PA", then the following facts are provable:

- $PA \vdash p$ if and only if $Bew(\ulcorner p \urcorner)$ is true in the standard PA model $\langle \omega, \times, +, 0, 1 \rangle$. Since all PA-provable sentences are true, it follows that for any p the sentence $Bew(\ulcorner p \urcorner) \to p$ is true (in other words, principle T, viz. $L(\alpha) \to \alpha$, is true). However, in view of Gödel's first incompleteness theorem this principle is not always provable. Otherwise $Bew(\ulcorner 0 \neq 0 \urcorner) \to 0 \neq 0$ would be provable in PA.

Hence also the contraposition $\neg(0 \neq 0) \to \neg Bew(\ulcorner 0 \neq 0 \urcorner)$ would be provable and since $\neg(0 \neq 0)$ is indeed provable we would obtain the provability of $\neg Bew(\ulcorner 0 \neq 0 \urcorner)$ in PA. But $\neg Bew(\ulcorner 0 \neq 0 \urcorner)$ exactly states that PA is consistent, so that we would contradict Gödel's second incompleteness theorem.

- In [Löb, 1955], Martin Löb showed that the following hold: (i) $PA \vdash Bew(\ulcorner p \to q \urcorner) \to (Bew(\ulcorner p \urcorner) \to Bew(\ulcorner q \urcorner))$ (i.e. the "normality" principle $L(\alpha \to \beta) \to (L(\alpha) \to L(\beta))$ holds in PA; (ii) if $PA \vdash p$ then $PA \vdash Bew(\ulcorner p \urcorner)$ (i.e. the necessitation rule holds in PA); (iii) if $PA \vdash Bew(\ulcorner p \urcorner) \to p$, then $PA \vdash p$.

- Point (iii) above amounts to say that if $Bew(\ulcorner Bew(\ulcorner p \urcorner) \to p \urcorner)$ is true, then so is $Bew(\ulcorner p \urcorner)$. Otherwise stated $Bew(\ulcorner Bew(\ulcorner p \urcorner) \to p \urcorner) \to Bew(\ulcorner p \urcorner)$ is true. The modal version of this formula is $L(L(A) \to A) \to L(A)$. This modal principle is known as Segerberg's formula W or as "Löb's formula",[6] so that if we denote by G the set of all PA-valid formulas then G is a normal modal logic equivalent to $\mathbf{K}4W$ (cf. [Löb, 1955] and [Solovay, 1976]).[7]

$\mathbf{K}4W$ (or G), is characterised by the class of finite strict modal orderings (transitive and irreflexive relations) (cf. [Segerberg, 1971]). But Solovay, in the quoted paper, also shows that if one adds the set of all ω-valid sentences (that is, sentences that are true for all assignment of the propositional variables to some PA-sentence), then one obtains a logical system G^* which, of course, includes G, but $Bew(\ulcorner p \urcorner) \to p$, too (i.e. $L(\alpha) \to \alpha$). However G^* it is not closed under necessitation.

Differently from G, G^* does not have a Kripke semantics (because it is not a normal logic). But it does not have a neighborhood semantics, either.

We conclude this Frame by emphasizing that the point of contact between pre-topological systems and relational modal systems, stated in *Proposition* 12.7.8.(2), is proved in [Chellas, 1980] as follows: a neighborhood system is called *augmented* if it satisfies **N2** and **N4** (more precisely, for all neighborhood family: $N \in \mathcal{N}_x$ & $N \subseteq N' \Rightarrow N' \in \mathcal{N}_x$ and $\bigcap \mathcal{N}_x \in \mathcal{N}_x$).

---

[6] Actually, axiom 4 of ES is derivable from W.

[7] W (in Löb's version) states that a formula asserting its own provability is necessarily true and provable.

Then, for every Kripke model $\langle W, R, \models \rangle$ there is a pointwise equivalent augmented neighborhood model and vice-versa.

15.13.4   The "Omniscience Problem" in Epistemic and Doxastic Logics

Neighborhood systems are exploited in other fields, such as Epistemic Logic and Doxastic Logic (see the introduction of this Part). In his excellent survey [Liau, 2000], Churn-Jung Liau recalls two main drawbacks of the attitude of modeling reasoning about knowledge and belief on the basis of the interpretation "*The epistemic (doxastic) subject knows (believes) α*" $=_{def} \Box\alpha$. The first is the logical omniscience problem: from $\Box\alpha$ and $\alpha \to \beta$, we deduce $\Box\beta$. Otherwise stated, an epistemic (or doxastic) subject knows (believes) all the logical consequences of his own knowledge (beliefs). Various modifications have been proposed to circumvent this problem (cf. [Halpern, Moses, 1985]). The second problem, connected with this, is that if an epistemic (or doxastic) subject has inconsistent knowledge (beliefs) then she/he knows (believes) everything (from the well-known principle "*Ex falso sequitur quodlibet*"). To limit the consequences, a "local reasoning" approach has been proposed, that conceives a subject as a society of minds, each with its own knowledge (or beliefs). Therefore a local reasoning model is a triple $\langle U, \mathcal{N}(U), \phi \rangle$ where U is a set, ϕ is an evaluation $\phi : \mathcal{L} \longmapsto \wp(U)$ and $\mathcal{N}(U)$ is a neighborhood system. Each neighborhood $\mathcal{N}_p$ represents a frame of the subject's mind in p, so that the evaluation of the modalised formulas turns into:

(i) $\phi(x, \Box(\alpha)) = \bigvee_{S \in \mathcal{N}_x} \bigwedge_{s \in S} \phi(s, \alpha)$; (ii) $\phi(x, \Diamond(\alpha)) = \phi(x, \neg\Box(\neg\alpha))$;

Therefore $x \Vdash \Box(\alpha)$ if there is at least a neighborhood S of x such that all the elements of x force α. Dually, $x \Vdash \Diamond(\alpha)$ if in all its neighborhoods there is at least an element which forces α.

It is shown that if $\mathcal{N}(U)$ is of type $\mathcal{N}_B$, then $[\![\Box(\alpha)]\!] = \varkappa([\![\alpha]\!])$ and $[\![\Diamond(\alpha)]\!] = \varepsilon[\![\alpha]\!]$.

Incidentally, notice that a historical approach for localizing (circumscribing) fallacious reasoning was developed in [Belnap, 1977] exploiting the smallest non-trivial bi-lattice depicted in the preceding Frame 15.12.

15.14   Frame – Pre-Topologies
and Intuitionistic Formal Spaces

In Part I we have considered a *P-system* or a basic pair $\langle G, M, \Vdash \rangle$ and derived a family of "intensional" and "extensional" operators from it. Now let us cast a deeper insight into that construction.

Basic pairs induce particular sorts of pre-topologies, called *Formal Pre-topologies*. In what follows we want to analyse the relationships between concrete pre-topologies (i.e. the pre-topologies we dealt with in this Part) and formal pre-topological properties.

Otherwise stated, we want to understand the extent to which the construction of concrete topologies from neighborhood systems runs in parallel with the construction of formal topologies. To this end we introduce some intermediate notions, such as those of a *semi-topological formal system*, *quasi-topological formal system*, *pseudo-topological formal system* and a *formal neighborhood system*, eventually with some additional features.

Formal neighborhood systems constitute a bridge between concrete neighborhood systems and quasi-topological formal systems. In turn, quasi-topological formal systems are extensions of semi-topological formal systems which, finally, are the structures immediately induced by basic pairs (i.e. *P-systems*) and their formal (pre)-topological operators. Indeed, the latter is the starting point of the construction of pointfree pre-topologies and topologies by the *Padua School* (see [Sambin, 1987, 1989, 1999, 2001]). In order to relieve a basic pair of too much meaning, let us avoid the symbols G, M and $\models$.

Thus, consider any basic pair $\langle A, B, R \rangle$, where A and B are sets and R is a binary relation between A and B (i.e. $R \subseteq A \times B$). A is thought of as a set of points and B as a set of formal neighborhoods. Therefore, for any $a \in A$, $R(a)$ is to be thought of as a neighborhood family and $\{R(a)\}_{a \in A}$ as a neighborhood system. For the definition of the basic derived operators induced by basic pairs, the reader is addressed to Chapter 2, *Definitions* 2.2.1 and 2.3.1.

The main goal of this Frame is to understand the relationship between formal pre-topological spaces and "concrete" neighborhood systems.

We shall show the principal elements about this topics along with a limited set of proofs, in order to make the reader understand the

mathematical approach. For a complete study, we address the interested reader to [Pagliani, 2002].

Let us enrich the machinery developed in Part I by admitting that the members of B can be combinable (intuitively, observables should be combinable). Therefore, B is equipped with a binary operation "$\cdot$" If $b, b' \in B$, then $b \cdot b'$ is called the *combination*, or the *fusion*, of b and b' and it is an abstraction of intersection between concrete neighborhoods.

Also, we can define a formal meet between the elements of $\mathbf{Sat}_{\mathcal{A}}(B)$, inherited by $\mathcal{A}$−saturated sets from the monadic operator $\cdot$ just introduced. First of all, observe that $\cdot$ is lifted from B to $\wp(B)$ by means of the following definition

$$X \cdot Y = \{x \cdot y : x \in X \ \& \ y \in Y\} \qquad (\textit{point-meet})$$

Notice that even if X and Y are saturated, $X \cdot Y$ is not necessarily saturated. In order to obtain a suitable operation on $\mathbf{Sat}_{\mathcal{A}}(B)$ we must close $X \cdot Y$ under $\mathcal{A}$, obtaining the following operator $\bullet$:

$$X \bullet Y = \mathcal{A}(X \cdot Y) \qquad\qquad (15.14.5)$$

However, $\mathbf{Sat}_{\mathcal{A}}(B)$ and $\mathbf{Sat}_{\mathcal{C}}(B)$ do not give information about the distributivity behaviour of $\wedge$, $\vee$, $\cap$, $\cup$ and $\bullet$. This happens of their "concrete" counterparts $\mathbf{Sat}_{int}(B)$ and $\mathbf{Sat}_{cl}(B)$, too. In order to make some distributive law hold, we must add some additional structure.

15.14.1   Formal Covering Relations

Given a basic pair $\langle A, B, R \rangle$, the operator $\mathcal{A}$ makes it possible to synthesize the formal properties of formal pre-topologies in logical (or, better, type theoretical) terms.

The starting point is the definition, by means of the operator $\mathcal{A}$, of a relation, denoted by the symbol $\blacktriangleleft$, which connects (subsets of) formal neighborhoods with subsets of formal neighborhoods. We can interpret $\blacktriangleleft$ as a formal semi-covering relation, because, in view of the symmetry of $\mathcal{A}$ and cl, $\blacktriangleleft$ is the formal counterpart of the "concrete" concept of an "adherence" without any commitment about its behaviour with respect to $\cdot$, by now. Alternatively $\blacktriangleleft$ may be thought of as a sort of type-theoretic sequent relation between sets of propositions (or, more precisely, multisets of propositions, as we shall see at the end).

Definition 15.14.1. *Let $\langle A, B, R \rangle$ be a basic pair. Then for any $b \in B$ and $Y, Y' \subseteq B$, the following relation is called a* formal semi-cover *or, shortly, a* semi-cover:

$$(basis)\ b \blacktriangleleft Y \text{ iff } b \in \mathcal{A}(Y), \qquad (step)\ Y \blacktriangleleft Y' \text{ iff } \forall y \in Y, y \blacktriangleleft Y'.$$

TERMINOLOGY AND NOTATION. Instead of $b \blacktriangleleft \{b'\}$ we shall write $b \blacktriangleleft b'$. Remember that $\langle R \rangle(b)$ is short for $\langle R \rangle(\{b\})$. A semi-cover is called a "basic-cover" in Formal Topology.

Proposition 15.14.1. *Let $\langle A, B, R \rangle$ be a basic pair. Then for any $b, b' \in B$ and $Y, Y' \subseteq B$,*

1. *$b \blacktriangleleft Y$ iff $\langle R \rangle(b) \subseteq \langle R \rangle(Y)$, iff $\forall a \in A(b \in R(a) \Rightarrow \exists y \in Y(y \in R(a)))$.*

2. *$Y \blacktriangleleft Y'$ iff $\langle R \rangle(Y) \subseteq \langle R \rangle(Y')$, iff $Y \subseteq \mathcal{A}(Y')$, iff $\mathcal{A}(Y) \subseteq \mathcal{A}(Y')$.*

3. *(i) $Y \subseteq Y'$ implies $Y \blacktriangleleft Y'$; (ii) $\mathcal{A}(Y) \blacktriangleleft Y$.*

4. *$\mathcal{A}(Y) \blacktriangleleft Y'$ iff $Y \blacktriangleleft \mathcal{A}(Y')$.*

5. *if $Y' \in \Omega_{\mathcal{A}}(B)$, then (i) $Y \blacktriangleleft Y'$ iff $Y \subseteq Y'$, (ii) $b \blacktriangleleft Y$ iff $b \in Y$.*

6. *$\dfrac{b \in Y}{b \blacktriangleleft Y}$ (reflexivity).*

7. *(i) $\dfrac{b \blacktriangleleft Y \quad Y \blacktriangleleft Y'}{b \blacktriangleleft Y'}$; (ii) $\dfrac{b \blacktriangleleft b' \quad b' \blacktriangleleft Y}{b \blacktriangleleft Y}$ (transitivity).*

8. *(i) $Y \blacktriangleleft Y$; (ii) $b \blacktriangleleft b$ (identity).*

9. *$b \blacktriangleleft Y$ iff $\{b\} \blacktriangleleft Y$ (lifting).*

Proof. (**1**) Immediately from *Proposition* 2.3.1.(1). (**2**) By definition, $Y \blacktriangleleft Y'$ iff for all $y \in Y$, $\langle R \rangle(y) \subseteq \langle R \rangle(Y')$, iff $\{y\} \subseteq [R^{\smile}]\langle R \rangle(Y')$ (by *Corollary* 2.1.1) iff $\{y\} \subseteq \mathcal{A}(Y')$ iff $Y \subseteq \mathcal{A}(X)$. By monotonicity and idempotence of $\mathcal{A}$ we obtain $Y \blacktriangleleft Y'$ iff $\mathcal{A}(Y) \subseteq \mathcal{A}(Y')$. (**3**) (i) Trivially from (2) and monotonicity of $\mathcal{A}$. Note, however, that the converse is trivially not true: we can have $b \blacktriangleleft b'$ (i.e. $b \blacktriangleleft \{b'\}$) even if $b \neq b'$; (ii) from $\mathcal{A}(Y) \subseteq \mathcal{A}(Y)$ and (2). (**4**) $\mathcal{A}(Y) \blacktriangleleft Y'$ iff $\mathcal{A}(\mathcal{A}(Y)) \subseteq \mathcal{A}(Y')$, iff $\mathcal{A}(Y) \subseteq \mathcal{A}(Y')$ iff $\mathcal{A}(Y) \subseteq \mathcal{A}(\mathcal{A}(Y'))$ iff $Y \blacktriangleleft \mathcal{A}(Y')$. (**5**) (i) From (2) immediately because $\mathcal{A}(Y') = Y'$. As for (ii), observe that $b \blacktriangleleft Y$ iff $b \in \mathcal{A}(Y) = Y$. (**6**) From (*basis*) because $b \in Y \subseteq \mathcal{A}(Y)$. (**7**) (i) Directly

from (1) and (2) and transitivity of the relation $\subseteq$. (ii) From (7).(i) and (9) below. (**8**) (i) Immediately from (3) and $Y \subseteq Y$. (ii) From $b \in \{b\}$ and (6). (**9**) Trivially from (2), because $b \blacktriangleleft Y$ iff $b \in \mathcal{A}(Y)$ iff $\{b\} \subseteq \mathcal{A}(Y)$.   QED

REMARKS. (Lifting) is one of the simplest but most useful property, because it allows us to move freely from b to $\{b\}$ and vice-versa on the left of the symbol $\blacktriangleleft$, any $b \in B$.

Notice that *Proposition* 15.14.1.(2) and (*identity*) tell us that the relation $\blacktriangleleft$ is a preordering on $\wp(B)$. On the contrary *Proposition* 15.14.1.(5) shows that $\blacktriangleleft$ is a partial ordering relation in $\mathbf{Sat}_{\mathcal{A}}(B)$. Indeed $\mathcal{A}$-saturated elements are representatives of the equivalence classes on $\wp(B)$ modulo the equality of $\mathcal{A}$ closures or, which is the same, modulo the relation $\blacktriangleleft\blacktriangleright$ defined by "$X \blacktriangleleft\blacktriangleright Y$ iff $X \blacktriangleleft Y$ and $Y \blacktriangleleft X$". It follows that $\mathbf{Sat}_{\mathcal{A}}(B)$ is isomorphic to the quotient $\wp(B)/_{\blacktriangleleft\blacktriangleright}$ and in this quotient structure the relation induced by the preorder $\blacktriangleleft$ is clearly a partial order. Finally, (4) says that $\mathcal{A}$ is self-adjoint with respect to $\blacktriangleleft$.

The above proof of transitivity and reflexivity of the relation $\blacktriangleleft$ might hide the role played by the properties of the operator $\mathcal{A}$. These properties are revealed if we generalise the above results to an entire class of operators:

Lemma 15.14.1. *Let K be an operator on $\wp(X)$ for some set X. For any $x \in X, Z, Z' \subseteq X$ define $x \blacktriangleleft Z \equiv_{def} x \in K(Z)$ and $Z \blacktriangleleft Z'$ iff $z \blacktriangleleft Z'$ for all $z \in Z$. Then,*

1. *If K is increasing, then $\blacktriangleleft$ is reflexive.*

2. *If K is monotone, then $\blacktriangleleft$ is weakly transitive, viz.* $\dfrac{x \blacktriangleleft Z \quad Z \blacktriangleleft Z'}{x \blacktriangleleft K(Z')}$

   (weak transitivity).

3. *If K is monotone and idempotent, then $\blacktriangleleft$ is transitive.*

Proof. (**1**) If for all $Z \subseteq X, Z \subseteq K(Z)$, then $x \in K(Z)$ whenever $x \in Z$. (**2**) Suppose (i) $Z \subseteq Z' \Rightarrow K(Z) \subseteq K(Z')$, (ii) $x \blacktriangleleft Z$ and (iii) $Z \blacktriangleleft Z'$. Then, by definition $x \in K(Z)$ because $x \blacktriangleleft Z$. Moreover, $Z \blacktriangleleft Z'$ implies $Z \subseteq K(Z')$, trivially. Thus by monotonicity we obtain $K(Z) \subseteq K(K(Z'))$. It follows that $x \in K(K(Z'))$, that is, $x \blacktriangleleft K(Z')$. (**3**) From (2), if $K(K(Z')) = K(Z')$ we obtain transitivity.   QED

Conversely, if we are given a relation $\blacktriangleleft \subseteq X \times \wp(X)$ fulfilling transitivity and reflexivity and for all $Z \subseteq X$ we put $K(Z) \equiv_{def} \{x : x \blacktriangleleft Z\}$, then K is a monotonic, increasing and idempotent operator, i.e. a closure operator (the easy proof is left as an exercise).

Therefore, since $\mathcal{A}$ is a closure operator *Proposition* 15.14.1.(6) and 15.14.1. (7) are corollary of the above *Lemma* and the relationships between the closure properties of $\mathcal{A}$ and those of $\blacktriangleleft$ are revealed.

Up to now we have spoken of "semicovering" and not of "covering" relations because until now we have not assumed any commitment to the relationships between the operator $\mathcal{A}$ and the "meet" operations $\cdot$, $\cap$ or $\bullet$. In particular we have not information as whether some condition makes $\bullet$ and $\cap$ coincide. Indeed we can notice that in general the following equation:

$$\mathcal{A}(Y \cdot Y') = \mathcal{A}(Y) \cap \mathcal{A}(Y') \qquad (A\text{-}distributivity)$$

does not hold, even in the case $\cdot$ is $\cap$ itself.

Indeed, this principle requires some additional properties to be fulfilled by R.

TERMINOLOGY AND NOTATION.
Given a basic pair $\langle A, B, R \rangle$ the operation $\cdot$ is intended to be defined according to (*point-meet*) whenever it applies to subsets of B. We recall that the expression $b \cdot Y$ will be intended as $\{b\} \cdot Y$, the expressions $[R](b)$ and $\langle R \rangle(b)$ will be intended as $[R](\{b\})$ and, respectively, $\langle R \rangle(\{b\})$, as usual. It follows that $\langle R \rangle(b)$ and $R^{\smile}(b)$ will denote the same object and we shall use either expression in dependence on the context. However, properties of the operator $\cdot$ (such as idempotence, commutativity and the like) refer to the application of $\cdot$ to elements of B, if not otherwise stated.

We recall that a *commutative monoid* is a set U equipped with a binary associative operation $\cdot$ and a unity 1 such that for any $a, b \in U$, $a \cdot b = b \cdot a$ and $1 \cdot a = a \cdot 1 = a$.

Definition 15.14.2. *Let $\langle B, \cdot, 1 \rangle$ be a commutative monoid, let $\perp$ be a subset of B and let $\blacktriangleleft$ be a semi-cover relation. Then the structure $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$, is called a* semi-topological formal system.[8]

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[8]We adopt this name instead of "pre-topological formal system" because the term "pre-topological formal space" will be used for a slightly different notion according with the traditional usage in the theory of Formal Spaces.

Corollary 15.14.1. *Let $\langle A, B, R \rangle$ be a basic pair, $\langle B, \cdot, 1 \rangle$ a commutative monoid and put $\perp = \{b \in B : \langle R \rangle(b) = \emptyset\}$. Then the structure $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ induced by the operator $\mathcal{A}$ is a* semi-topological formal system *called the semi-topological formal system induced by the basic pair $\langle A, B, R \rangle$.*

In semi-topological formal systems, commutativity of $\cdot$ induces just a few properties connecting $\cdot$ and $\blacktriangleleft$, such as $\quad \dfrac{b \cdot b' \blacktriangleleft Y \cdot Y'}{b' \cdot b \blacktriangleleft Y' \cdot Y}.\quad$ Things change if we add further properties to the monoidal operator "$\cdot$".

Definition 15.14.3. *Let $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ be a semi-topological formal system such that $\cdot$ is idempotent. Then $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ is called a* quasi-topological formal system *and $\blacktriangleleft$ is called a* quasi-cover *relation.*

Proposition 15.14.2. *In any quasi-topological formal system $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$, $b \blacktriangleleft b \cdot b$ and $b \cdot b \blacktriangleleft b$ hold, any $b \in B$.*

Proof. Trivially, since the two relations reduce to the identity $b \blacktriangleleft b$.
<div align="right">QED</div>

Definition 15.14.4. *Let U, U' be sets and let f be a total function $U \longmapsto U'$. Let $R \subseteq U \times \wp(U')$ such that $f(x) = f(y)$ implies $R(x) = R(y)$, and let $\perp = \{X \in \wp(U') : \langle R \rangle(X) = \emptyset\}$. Then $\langle U, \wp(U'), R, f \rangle$ will be called a* basic neighborhood pair *and the induced structure $\langle \wp(U'), \cap, U', \blacktriangleleft, \perp, f \rangle$ will be called a* formal neighborhood system. *If $U = U'$ and f is the identity function, then $\langle U, \wp(U), R \rangle$ and $\langle \wp(U), \cap, U, \blacktriangleleft, \perp \rangle$ will be called a* Fréchet basic neighborhood pair *(on U) and, respectively, a* Fréchet formal neighborhood system.

Proposition 15.14.3. *Any formal neighborhood system is a quasi-topological formal system.*

Proof. Trivial: actually the underlying monoid is the semilattice $\langle \wp(U'), \cap, U' \rangle$, so that $\cap$ is commutative and idempotent.
<div align="right">QED</div>

Quasi-topological formal systems are abstraction of formal neighborhood systems, where the element $\perp$ represents the set of elements of $\wp(U')$ that do not belong to any neighborhood family (remember that $\langle R \rangle(X) = R^{\smile}(X)$). Finally, given a neighborhood system $\{R(x)\}_{x \in U}$, we shall liberally use $\mathcal{N}_x$ or $R(x)$ to denote the neighborhood family of x, i.e. $R(x)$.

Remarks.

(A) The difference between Fréchet formal neighborhood systems and generic formal neighborhood systems is immaterial from the formal point of view, because the difference relays on function f which maps point onto points and makes properties **Id** and **N1** work. Indeed, we shall see that these two properties have no formal counterparts.

(B) While dealing with formal neighborhood systems, pay attention that relation R is defined between elements of a set U and subsets of U. Therefore one have to distinguish between X and $\{X\}$, for $X \subseteq U$. X is a subset of the domain of R, but also a single member of the codomain of R. Therefore X is an argument of $R(-)$ while $\{X\}$ is an argument of $R^{\smile}(-)$. Particularly, pay attention to distinguish between $\langle R \rangle(\emptyset)$ and $\langle R \rangle(\{\emptyset\})$, $[R](\emptyset)$ and $[R](\{\emptyset\})$ because both $\emptyset$ and $\{\emptyset\}$ are elements of $\wp(\wp(U))$. Clearly, if **0** holds, that is, $\emptyset \notin R(x)$ any $x \in U$, then $\langle R \rangle(\emptyset) = \langle R \rangle(\{\emptyset\}) = \emptyset$ and if R is onto, then $[R](\emptyset) = [R](\{\emptyset\}) = \emptyset$. On the contrary, if **0** does not hold both $\langle R \rangle(\{\emptyset\})$ and $[R](\{\emptyset\})$ might be non-empty.

(C) For the above reasons, from now on we shall only deal with Fréchet basic neighborhood pairs and with Fréchet neighborhood formal systems, that, thereafter, will be referred to as "formal neighborhood systems" without qualification, if there is not risk of ambiguity. Moreover, given a formal neighborhood system $\langle \wp(U), \cap, U, \blacktriangleleft, \bot \rangle$, the basic neighborhood pair $\langle U, \wp(U), R \rangle$ will be implicitly understood.

(D) Formal neighborhood systems are a bridge between concrete neighborhood systems and quasi-topological formal systems because in formal neighborhood systems for any $a \in U$ and $X \subseteq U$ we can represent the membership relation $X \in \mathcal{N}_a$ as $\langle a, X \rangle \in R$ and $\cap$ is idempotent. In a basic neighborhood pair, *qua* a basic pair, we can define the operators $int, cl, \mathcal{A}$ and $\mathcal{C}$, while as a concrete neighborhood system, we can define a core map G and a vicinity map F. However, the reader is strongly invited to notice that in general neither int corresponds to G, nor cl corresponds to F. This fact is easily verified after *Lemma* 15.14.1 by considering that int is symmetric-dual of $\mathcal{A}$. Therefore, the core map (the vicinity map) induced by a concrete neighborhood system cannot coincide with the operator int (with cl), unless it is an interior operator (a closure operator). In Section 12.5 we have seen the "concrete" conditions on $\{R(a)\}_{a \in U}$ to make G into an interior operator. However, this is just a necessary condition. In what follows we shall zoom in on concrete neighborhood systems to analyse how formal properties are connected to concrete conditions, in order to discover if they make int and G coincide when G is an interior operator.

Terminology and Notation.

When dealing with formal neighborhood systems we shall continue using our notation-conventions: U, X, Z and so on, denote sets; $x, w, b, c, \ldots$ denote elements of sets; $\mathcal{Y}, \mathcal{Y}', \mathcal{Z}, \ldots$ denote sets of sets (hence, for instance, $x \in Y \in \mathcal{Z} \subseteq \wp(U)$). Therefore, when we refer to results about general formal systems, pay attention to translate symbols according to their meaning.

As like as in formal systems, given a formal neighborhood system we write $X \blacktriangleleft X$ instead of $X \blacktriangleleft \{X\}$, unless there is risk of confusion (however, note that $X \blacktriangleleft X$ iff $\{X\} \blacktriangleleft X$, by (*lifting*)).

With this proviso, we see that, for instance, given a formal neighborhood system on U, *Proposition* 15.14.1.(3) by no means allows us to conclude, for two sets $X, Y \in \wp(U)$, that $X \subseteq Y$ implies $X \blacktriangleleft Y$. In fact, in this formal neighborhood system X and Y are elements of the codomain of the relation R. Therefore to have $X \blacktriangleleft Y$ (meaning $X \blacktriangleleft \{Y\}$) we should have $\{X\} \subseteq \mathcal{A}(\{Y\})$. But the latter relation is implied by $\{X\} \subseteq \{Y\}$, which is not implied by $X \subseteq Y$ (example: $\{x\} \subseteq \{x, y\}$ but $\{\{x\}\} \nsubseteq \{\{x, y\}\}$). For the same reason, in formal neighborhood systems, in general we do not have $X \cdot Y \blacktriangleleft Y$, although $X \cdot Y$ (i.e. $X \cap Y$) is included in Y, (in the previous example, $\{x\} \cdot \{x, y\} = \{x\} \cap \{x, y\} = \{x\} \subseteq \{x, y\}$, but, again, $\{\{x\}\} \nsubseteq \{\{x, y\}\}$). On the contrary, $\mathcal{Y} \subseteq \mathcal{Y}'$ really implies $\mathcal{Y} \blacktriangleleft \mathcal{Y}'$ and we have $\mathcal{Y} \blacktriangleleft \mathcal{Y}$, $\mathcal{Y} \cdot \mathcal{Z} \blacktriangleleft \mathcal{Z}$, and so on. Therefore, the reader must carefully distinguish when we are dealing with abstract formal systems (where a set X is a set of elements of the codomain of R) or, on the contrary, with formal neighborhood systems (where a set X is an element of the codomain of R).

Having in mind the above remarks, let us discover the particular features of semi-topological and quasi-topological formal systems.

First of all, notice that although in quasi-topological formal systems the operator $\cdot$ is idempotent, nevertheless it might fail to be idempotent on subsets of B, in view of (*point-meet*):

Proposition 15.14.4. *Let $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ be a quasi-topological formal system. Then for any $Y \subseteq B$, $Y \subseteq Y \cdot Y$.*

Proof. $Y \cdot Y = \{y \cdot y' : y, y' \in Y\}$. Thus, since $\cdot$ is idempotent, $Y \cdot Y = Y \cup \{y \cdot y' : y, y' \in Y \ \& \ y \neq y' \ \& \ y' \neq y \cdot y' \neq y\}$.   QED

The above relation may be restated using the operator $\mathcal{A}$. Since by definition $x \in \mathcal{A}(Y)$ if and only if $a \blacktriangleleft Y$, from *Proposition* 15.14.4 we

obtain, for quasi-topological formal systems, $\mathcal{A}(Y) \cap \mathcal{A}(Y) = \mathcal{A}(Y) \subseteq \mathcal{A}(Y \cdot Y)$.

This is not sufficient in order to obtain (A-*distributivity*). Therefore, we must add particular properties.

In particular three principles require to be carefully analysed:

$$(\textit{left}) \quad \frac{b \blacktriangleleft Y}{b \cdot b' \blacktriangleleft Y}; \quad (\textit{right}) \quad \frac{b \blacktriangleleft Y \quad b \blacktriangleleft Y'}{b \blacktriangleleft Y \cdot Y'}; \quad (\textit{stability}) \quad \frac{b \blacktriangleleft Y \quad b' \blacktriangleleft Y'}{b \cdot b' \blacktriangleleft Y \cdot Y'}.$$

All these principles fail to hold even in quasi-topological formal systems:

– *Counterexample* for (*left*): suppose $R^{\smile}(b \cdot b') \not\subseteq R^{\smile}(b)$. Let $Y = \{b\}$. Then $b \blacktriangleleft Y$ – by definition – but $b \cdot b' \blacktriangleleft Y$ fails to hold (for instance take $A = \{a, a'\}$, $B = \{b, b'\}$, $b \cdot b' = b'$ and $R = \{\langle a, b \rangle, \langle a', b' \rangle\}$).

– *Counterexample* for (*right*): suppose $R^{\smile}(b) = R^{\smile}(b')$ and $R^{\smile}(b) \not\subseteq R^{\smile}(b \cdot b')$. Put $Y = \{b\}$ and $Y' = \{b'\}$. Then $b \blacktriangleleft Y$ – trivially – and $b \blacktriangleleft Y'$ because $R^{\smile}(b) = R^{\smile}(b')$. However, $R^{\smile}(Y \cdot Y') = R^{\smile}(b \cdot b') \not\supseteq R^{\smile}(b)$, Therefore (*right*) fails.

– *Counterexample* for (*stability*): suppose $A = \{a\}$, $B = \{b, b', b'', b''', \ldots\}$, $b \cdot b' = b'$, $b' \cdot b'' = b'''$ and $R = \{\langle a, b \rangle, \langle a, b' \rangle, \langle a, b'' \rangle\}$. Let $Y = \{b'\}$ and $Y' = \{b''\}$. It is easy to verify that $b \blacktriangleleft Y$ and $b' \blacktriangleleft Y'$, while $R^{\smile}(b \cdot b') = R^{\smile}(b') = \{a\} \not\subseteq \emptyset = R^{\smile}(b''') = R^{\smile}(b' \cdot b'') = R^{\smile}(Y \cdot Y')$. Thus (*stability*) fails.

Since the semilattice properties do not add anything about these three rules, if singularly taken (see below), then they fail also in quasi-topological formal systems.

Lemma 15.14.2 (restricted stability)**.** *Let* $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ *be a semi-topological formal system. Then* (*stability*) *holds for all* $b, b' \in B$ *and* $Y, Y' \in \Omega_{\mathcal{A}}(B)$.

Proof. Since for all $Y \in \Omega_{\mathcal{A}}$, $\mathcal{A}(Y) = Y$, in view of *Proposition* 15.14.1 (5), from $b \blacktriangleleft Y$ and $b' \blacktriangleleft Y'$ we derive $b \in Y$ and $b' \in Y'$. It follows that $b \cdot b' \in Y \cdot Y'$. Hence, $b \cdot b' \blacktriangleleft Y \cdot Y'$.   QED

Lemma 15.14.3 (restricted right)**.** *Let* $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ *be a quasi-topological formal system. Then* (*right*) *holds for all* $b \in B$ *and* $Y, Y' \in \Omega_{\mathcal{A}}(B)$.

Proof. Immediate by *Corollary* 15.14.2, and idempotence of $\cdot$ (or *Corollary* 15.14.2, (*lifting*), *Proposition* 15.14.2 and transitivity of $\blacktriangleleft$).   QED

Definition 15.14.5. *A semi-topological formal system in which (stability) holds, is called a* pre-topological formal system *and ◄ is called a* pre-cover*.*

Definition 15.14.6. *A semi-topological (pre-topological, quasi-topological) formal system in which (left) holds, is called a* left semi-topological (left pre-topological, left quasi-topological) *formal system. The relation ◄ is then called a* regular semi-cover (*a* regular pre-cover, *a* regular quasi-cover)*.*

Definition 15.14.7. *A semi-topological (pre-topological, quasi-topological) formal system in which (right) holds, is called a* right semi-topological (*right pre-topological, right quasi-topological*) *formal system. The relation ◄ is then called a* distributive semi-cover (*a* distributive pre-cover, *a* distributive quasi-cover)*.*

Definition 15.14.8. *A semi-topological (pre-topological, quasi-topological) formal system in which (left) and (right) hold, is called a* topological formal system*. The relation ◄ is called a* cover*.*

We have now an obvious duty: to prove the extent to which the above notions do not collapse and, moreover, that *Definition* 15.14.8 is appropriate.

The following equivalence may be proved:

Proposition 15.14.5. *Let $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ be any semi-topological formal system. Then for any $b, b' \in B$ and $Y, Y' \subseteq B$, the following are equivalent*:

1. $b \cdot b' \blacktriangleleft b, \qquad b \cdot b' \blacktriangleleft b'$ *(l-3)*

2. $Y \cdot Y' \blacktriangleleft Y, \quad Y \cdot Y' \blacktriangleleft Y'$ *(l-2)*

3. *(Left).*

From this we obtain:

Proposition 15.14.6. *Let $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ be any left semi-topological formal system. Then for any $b, b' \in B$ and $Y, Y' \subseteq B$:*

1. *(l-3) and (l-2) hold identically.*

2. *(i) $b \blacktriangleleft 1$ (l-1a); (ii) $Y \blacktriangleleft 1$ (l-1b) (we denote the two rules collectively as (l-1a,1b).*

3. $Y \cdot Y \blacktriangleleft Y$ *(abs).*

Proposition 15.14.7. *Let* $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ *be any semi-topological formal system. Then for any* $b, b' \in B$ *and* $Y, Y' \subseteq B$ *the following are equivalent:*

1. $Y \blacktriangleleft Y \cdot Y$ *(r-1),*

2. $\dfrac{b \blacktriangleleft Y}{b \blacktriangleleft Y \cdot Y}$ *(r-2)* ,

3. $b \blacktriangleleft b \cdot b$ *(r-3).*

Moreover, one can prove that (*right*) implies all the (*r-*) rules.

Corollary 15.14.2.

1. *In any pre-topological formal system (left) and all the (l-) principles are equivalent; (right) and all the (r-principles) are equivalent.*

2. *In any quasi-topological formal system, in the presence of (stability), (right) and all the (r-) principles are equivalent and derivable.*

3. *Any quasi-topological formal system in which (stability) and (left) hold is topological.*

4. *Any formal neighborhood system in which (stability) and (left) hold is topological.*

It is not difficult to exhibit an example supporting the following sentence:

Proposition 15.14.8. *In a semi-topological formal system (stability) implies neither (right) nor (left).*

To sum up, if the operator $\cdot$ is not idempotent on B, then all the relationships marked by (*r-*) are not for free, but their validity requires (*right*), while all the relationships marked by (*l-*) require (*left*), independently of the properties of $\cdot$.

In the presence of (*stability*) any relationship marked (*r-*) is equivalent to (*right*) and all (*l-*)-marked principles are equivalent to (*left*). But in the presence of (*stability*) all of the relationships marked (*r-*) are derivable from idempotence of $\cdot$ in B and we can conclude that in quasi-topological formal systems (*stability*) alone provides us with a

valuable set of equivalent properties acting on the right of the quasi-covering relation ◄. On the contrary, in order to obtain (*left*) from (*stability*), idempotence of · is not enough. Actually we need some (*l-*) marked principle, but (*l-*) marked principle are not derivable from simple idempotence of ·. In fact, the only left-side principle obtained from idempotence of ·, is $b \cdot b \blacktriangleleft b$ which is insufficient to obtain (*left*) even jointly with (*stability*).

Finally, observe that even without (*stability*) all (*r-*)-marked principles are equivalent and all (*l-*)-marked principles are equivalent, except (*l-1a*) and (*1-lb*).

This suggests that (*stability*), (*right*), (*left*), (*l-1a*) and *(l-1b)* correspond to different properties of neighborhood systems.

15.14.2   Semi-Topological Formal Systems and Neighborhood Systems

When comparing semi-topological formal systems and neighborhood systems of concrete points, we find that, often, concrete properties have just one-way relationships with those of formal systems or, in some cases, no relationships at all. For instance, neither **N1** has a formal counterpart (because in formal terms points are transparent) nor **Id** (because **Id** would take the form $b \blacktriangleleft \langle R \rangle(b)$, that cannot be formally interpreted).[9] Indeed these principles are *point-dependent*, that is, they are closely related to the individuality of points. This means that **N1** and **Id** have nothing to say about formal principles like (*left*), (*right*) or (*stability*). Conversely, (*left*), (*right*) and (*stability*) have nothing to say about **N1**, **Id** and even about **N3**. In fact, we have the following one-way implication.

Remember that in a formal neighborhood system on a set U, $R \subseteq U \times \wp(U)$ and · is $\cap$. Moreover, remember that if $X \subseteq U$ then $\langle R \rangle(X)$ means $\langle R \rangle(\{X\})$, because X is a member of the codomain of R.

Proposition 15.14.9. *Let $\langle \wp(U), \cap, U, \blacktriangleleft, \perp \rangle$ be a formal neighborhood system. If $\{R(x)\}_{x \in U}$ fulfills* **N3***, then (right) holds.*

Proof. If **N3** holds, then for all $X, X' \subseteq \wp(U)$, $(X, X' \in R(a)) \Rightarrow (X \cap X' \in R(a))$. In view of *Proposition* 15.14.1, $X \blacktriangleleft \mathcal{Y}$ and $X \blacktriangleleft \mathcal{Y}'$

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[9]$b \in \langle R \rangle(b)$ means $R^{\smile}(b) \subseteq R^{\smile}(R^{\smile}(b))$ which, incidentally, makes sense only if $A = B$. Moreover, this relation reduces to the triviality $R^{\smile}(b) \subseteq R^{\smile}(b)$ when R is a preorder.

are equivalent to $\forall a \in U(X \in R(a) \Rightarrow \exists Y \in \mathcal{Y}\ \&\ Y \in R(a))$ and, respectively, $\forall a \in U(X \in R(a) \Rightarrow \exists Y' \in \mathcal{Y}'\ \&\ Y' \in R(a))$. By (*point-meet*) $X \cap Y \in \mathcal{Y} \cdot \mathcal{Y}'$ and by **N3**, $X \cap Y \in R(a)$, so that $X \blacktriangleleft \mathcal{X} \cdot \mathcal{Y}$. QED

But the converse does not hold (think of a formal neighborhood system in which, for any $X \neq \emptyset$, $X \blacktriangleleft \mathcal{Y}$ if and only if $X \in \mathcal{Y}$ or $U \in \mathcal{Y}$. In this case (*right*) holds trivially, independently of **N3** because either $U \in \mathcal{X} \cdot \mathcal{Y}$ or $X \in \mathcal{X} \cdot \mathcal{Y}$ (since $X \cap U = X \cap X = X$ – see an example in Frame 15.14.7).

On the contrary **N2** and (*left*) are equivalent:

Proposition 15.14.10. *Let $\langle \wp(U), \cap, U, \blacktriangleleft, \bot \rangle$ be a formal neighborhood system. Then, $\{R(x)\}_{x \in U}$ is a neighborhood system fulfilling **N2** if and only if (left) holds.*

Proof. (A)$\Rightarrow$: Suppose (*left*) does not hold. Then (*l-3*) does not hold. It follows that $X \cap X' \blacktriangleleft X$ does not hold. Therefore, there exists $a \in U$ such that $X \cap X' \in R(a)$ and $X \notin R(a)$. But $X \cap X' \subseteq X$. Hence **N2** does not hold.

(B)$\Leftarrow$: Conversely, suppose **N2** does not hold. Then there are $X, X' \in \wp(U)$ and $x \in U$ such that $X' \subseteq X$, $X' \in R(x)$ and $X \notin R(x)$. Clearly $X \blacktriangleleft X$ and, by definition, $X \cdot X' = X \cap X' = X'$. But $x \in \langle R \rangle(X') = \langle R \rangle(X \cap X')$ while $x \notin \langle R \rangle(X)$, and hence $\langle R \rangle(X \cap X')$ is not included in $\langle R \rangle(X)$. It follows that $X \cdot X' \blacktriangleleft X$ is not implied by $X \blacktriangleleft X$ and hence (*left*) fails.   QED

REMARKS.

(A) To be more precise, (*left*) implies both $X \cdot X' \blacktriangleleft X$, and $X \cdot X' \blacktriangleleft X'$. Therefore (*left*) implies $X \cap X' \in R(a) \Rightarrow (X \in R(a)\ \&\ X' \in R(a))$, which is the opposite condition of **N3**, and is obtained by reading (*left*) upside-down.

(B) We have seen that $X \subseteq X'$ does not imply $X \blacktriangleleft X'$. Therefore, although in formal neighborhood systems $X \cdot X' = X \cap X' \subseteq X$ holds for any $X, X' \in \wp(U)$, nevertheless $X \cdot X' \blacktriangleleft X$ does not hold without (*left*) (without **N2**).

Finally, the condition corresponding to (*stability*) is the following: for all $a, a', a'' \in U, X, X' \in \wp(U), \mathcal{Y}, \mathcal{Y}' \subseteq \wp(U)$, if $X \in R(a)$ implies $\exists Y \in \mathcal{Y}(Y \in R(a))$ and if $X' \in R(a')$ implies $\exists Y' \in \mathcal{Y}'(Y' \in R(a'))$, then $X \cap X' \in R(a'')$ implies $\exists Y'' \in \mathcal{Y}, \exists Y''' \in \mathcal{Y}'(Y'' \cap Y''' \in R(a''))$.

So far we have analysed how and to what extent formal properties of the relation ◄ parallel "concrete" properties of neighborhood systems.

We can notice, as was anticipated, that not only (*stability*), (*right*) and (*left*) are independent of **N1** and **Id**, but, more generally, that there are no formal property representing **N1** and **Id**.

However, in a sense **Id** and **N1** are embedded in ◄, *via* the closure properties of $\mathcal{A}$.

Clearly this was not a mystery by any means. In fact, since *int* is the "concrete" counterpart of $\mathcal{A}$, the above remark is a part of the obvious answer to the question, related to formal neighborhood systems, "*under what circumstances an operator $Op : \wp(U) \longmapsto \wp(U)$ has the same properties as a given operator "int"?*". The answer, of course, is "*when Op is an interior operator*", since *int* has such properties. Thus, if we consider a basic neighborhood pair $\langle U, \wp(U), R \rangle$ and we specialize this question to $Op = G$, where G is the vicinity map induced by the neighborhood system $\mathcal{N}(U) = \{R(x)\}_{x \in U}$ and if we ask "*what properties must be satisfied by the neighborhood system $\mathcal{N}(U)$ in order to make G fulfill the same properties as int?*", then, trivially again, the answer is: "$\mathcal{N}(U)$ *must be of type* $\mathcal{N}_{2Id}$", because G is an interior operator if and only if $\mathcal{N}(U)$ fulfills **N1**, **N2** and **Id** (see *Proposition 12.7.3*).

Anyway, we want to know if the above three properties, **N1**, **N2** and **Id**, co-operate in order to make G **equal** *int*.

According to (2.3.4) of Chapter 2, $int(X) = \{a \in U : \exists X'(X' \in R(a) \ \& \ R^{\smile}(\{X'\}) \subseteq X)\}$.

REMARKS. In formal neighborhood systems do not confuse $R^{\smile}$ (that is, $\langle R \rangle$) and G. In fact, $R^{\smile} : \wp(\wp(U)) \longmapsto \wp(U)$, while $G : \wp(U) \longmapsto \wp(U)$. Notice, however, that $G(X) = \{x : X \in R(x)\} = R^{\smile}(\{X\})$. Therefore, in what follows we liberally use $G(X)$ of $R^{\smile}(X)$ as equivalent alternatives.

Therefore our question turns into the following: does the recursive equation

$$G(X) = \{a : \exists X'(X' \in R(a) \ \& \ G(X') \subseteq X)\} \quad (G - int)$$

have a solution when $\{R(x)\}_{x \in U}$ is of type $\mathcal{N}_{2Id}$?

The answer is affirmative.

Lemma 15.14.4. *Let $\langle U, \wp(U), R \rangle$ be a basic neighborhood pair and $\mathcal{N}(U) = \{R(x)\}_{x \in U}$. Then, for all $X \in \wp(U)$, $G(X) = int(X)$ if and only if $\mathcal{N}(U)$ is of type $\mathcal{N}_{2Id}$.*

Proof. (A) $\Rightarrow$: [**N1** implies $G(X) \subseteq int(X)$] For all $a \in U$, $a \in G(X)$ implies $X \in R(a)$, by definition. From **N1** and *Lemma* 12.4.1 we have $G(X) \subseteq X$. Therefore so that $a \in int(X)$.. [**N2** + **Id** implies $int(X) \subseteq G(X)$] $a \in int(X)$ if $\exists X' \in \wp(U)$ such that $X' \in R(a)$ and $G(X') \subseteq X$. By **Id**, $G(X') \in R(a)$ which in turn, by means of **N2**, gives $X \in R(a)$, so that $a \in G(X)$.

To sum up, **N1**, **N2** and **Id** together imply $G(X) = int(X)$, all X.

(B) $\Leftarrow$: [$\neg$**N1** implies $G(X) \nsubseteq int(X)$] Suppose **N1** fails to hold in $\mathcal{N}(U)$. Then for some $X \in \wp(U)$, $G(X) \nsubseteq X$. Thus there is $a \in U$ such that $a \in G(X)$ and $a \notin X$. Clearly $X \in R(a)$ and for all the other X' such that $X' \in R(a)$, $G(X')$ is not included in X, because $a \in G(X')$ but $a \notin X$. Therefore, $a \notin int(X)$.

So, assume **N1**. [$\neg$**Id** implies $int(X) \nsubseteq G(X)$] If **Id** does not hold, then there are $a \in U$ and $X \in \wp(U)$, such that $X \in R(a)$ but $G(X) \notin R(a)$. Clearly $a \in int(G(X))$, because by **N1** $G(X) \subseteq X$ and $X \in R(a)$. On the contrary, $a \notin G(G(X))$ because $G(X) \notin R(a)$. Hence $int(G(X)) \nsubseteq G(X)$.

Therefore, assume **Id**, too. [$\neg$**N2** implies $int(X) \nsubseteq G(X)$] Let $X \in R(a)$, $X \subseteq X'$ and $X' \notin R(a)$. We can deduce $a \in int(X')$, because $X \in R(a)$ and, by **N1**, $G(X) \subseteq X$, so that $G(X) \subseteq X'$. But $a \notin G(X')$, because $X' \notin R(a)$.

To sum up, $\forall X(G(X) = int(X))$ implies **N1**, **N2** and **Id**.   QED

Corollary 15.14.3. *Let $\langle U, \wp(U), R \rangle$ be a basic neighborhood pair. Then, $\Omega_{int}(U) = \Omega_G(U)$ if and only if $\{R(x)\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$.*

REMARKS. Let $\mathbf{N} = \langle U, \wp(U), R \rangle$ be a basic neighborhood pair such that $\{R(x)\}_{x \in U}$ is a neighborhood system induced by a relation $Z \subseteq U \times U$. This means that for any $X \in U$, $R(x) = \uparrow Z(x)$. Let $\mathbf{Z} = \langle U, U, Z \rangle$. Then, pay attention that $L_Z(X) = G(X) = int^{\mathbf{N}}(X)$ but, on the contrary, as we know from *Corollary* 3.3.1, $L_Z(X) = G(X) = [Z](X) \neq int^{\mathbf{Z}}(X)$ (the last inequality turns into an equality only if Z is an equivalence relation).

To be sure, the above equation (*G-int*) has also a naive, but trivial, solution: $G(X) = X$, all $X \subseteq U$.

Definition 15.14.9. *Let $\langle U, \wp(U), R \rangle$ be a basic neighborhood pair. If for all $X \in \wp(U)$, $G(X) = X$, then $\langle U, \wp(U), R \rangle$ and its induced formal neighborhood system will be called* extensional.

Proposition 15.14.11. *Let $\langle U, \wp(U), R \rangle$ be an extensional basic neighborhood pair. Then $\{R(x)\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{3Id}$.*

The principal intermediate conclusion of the above discussion is that if $\{R(x)\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$, then *int* and G coincide. And vice-versa. We reserve formal neighborhood systems with this property a particular name:

Definition 15.14.10. *Let $\langle U, \wp(U), R \rangle$ be a formal neighborhood system such that for all $X \in \wp(U)$, $int(X) = G(X)$. Then the induced quasi-topological formal system $\langle \wp(U), \cap, U, \blacktriangleleft, \bot \rangle$ is called a* pseudo-topological formal system.

Proposition 15.14.12. *Every pseudo-topological formal system is a left quasi-topological formal system.*

Proof. The proof is immediate, since **N2** implies (*left*).   QED

Therefore, in pseudo-topological formal systems G is an interior operator coinciding with *int* and, dually, the vicinity map F is a closure operator coinciding with *cl*. However, this is not completely a surprise because we have already proved that $\blacktriangleleft$ is a precovering relation (reflexive and transitive) if and only if it is induced by a closure operator.

The novelty is that now we know that when G and *int* coincide then (*left*) holds automatically because of **N2**. Further, we know that if **N3** is added on top an $\mathcal{N}_{2Id}$ neighborhood system, $\mathcal{N}(U)$, then $\mathcal{N}(U)$ induces a topological space. Indeed, **N3** implies (*right*) and this is connected and consistent with the fact that semi-topological formal systems in which both (*left*) and (*right*) hold are topological, as stated in *Definition* 15.14.8 which we are going to justify immediately.

Indeed, we can interpret this move under different perspectives. From the point of view of $\mathcal{N}_{2Id}$ neighborhood systems, the addition of **N3** makes the induced vicinity map F into a topological closure operator by making F distribute over $\cup$. From the point of view of pseudo-topological formal systems (and, more generally, of left quasi-topological formal systems) the same goal is achieved by adding (*stability*) (cf. *Corollary* 15.14.2.(3)), while in generic semi-topological formal systems we have to add explicitly both (*left*) and (*right*).

To see how these connections work, let us start from the vicinity map F induced by a neighborhood system of type $\mathcal{N}_{2Id}$. We know that it coincides with cl. From *Lemma* 12.4.1, if we add **N3** then cl distributes over $\cup$. Since the operator $\mathcal{A}$, which acts on the formal side of formal neighborhood systems, is symmetric to cl, we expect that $\mathcal{A}$ distributes over $\cup$ in $\wp(\wp(U))$. That is, we expect that the following hold:

$$\mathcal{A}(\mathcal{X} \cup \mathcal{Y}) = \mathcal{A}(\mathcal{X}) \cup \mathcal{A}(\mathcal{Y}) \qquad (A\text{-}sup\text{-}distr)$$

Since in $\mathbf{Sat}_{\mathcal{A}}(\wp(U))$, $\mathcal{A}(\mathcal{Y} \cup \mathcal{Y}) = \mathcal{X} \cup \mathcal{Y}$, the above equation is equivalent to the following:

$$\mathcal{X} \vee \mathcal{Y} = \mathcal{X} \cup \mathcal{Y} \qquad\qquad (15.14.6)$$

Since $\mathbf{Sat}_{\mathcal{A}}(\wp(U))$ is isomorphic to $\mathbf{Sat}_{int}(U)$, i.e. $\mathbf{Sat}_{G}(U)$, we should also obtain that if (A-sup-$distr$) holds, then $\cap$ and $\vee$ distribute according to the distributive laws of the frames of open subsets of a topological space. Moreover, in view of the fact that **N2** implies (*left*) and **N3** implies (*right*), we expect that these two principles are likely to play a fundamental role on the formal side, in order to make (A-sup-$distr$) hold. Finally, notice that in the metalanguage "&" distributes over "or", so that $\cdot$ distributes over $\cup$ (by the very definition (*point-meet*)). Thus we presume, also, that $\cdot$ and $\cap$ coincide in $\mathbf{Sat}_{\mathcal{A}}(\wp(U))$ for any formal neighborhood systems $\{R(x)\}_{x \in U}$ of type $\mathcal{N}_{3Id}$. This is what actually happens, also at a more abstract level.

So let us consider an arbitrary semi-topological formal system $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$.

Let us first note that in $\mathbf{Sat}_{\mathcal{A}}(B)$, $\cap$ does not distribute over $\vee$ (see examples in Frame 15.14.5). What about the operation $\bullet$? If we add (*stability*) then $\bullet$ distributes over $\vee$:

Proposition 15.14.13. *Let* $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ *be a pre-topological formal system and let* $\mathbf{Sat}_{\mathcal{A}}(B)$ *be the equipped with the additional operation* $\bullet$. *Then for all* $X \in \mathbf{Sat}_{\mathcal{A}}(B)$, *for all* $\{Y_i\}_{i \in I} \subseteq \mathbf{Sat}_{\mathcal{A}}(B)$, $X \bullet \bigvee_{i \in I} Y_i = \bigvee_{i \in I}(X \bullet Y_i)$.

REMARKS. We have seen in *Lemmata* 15.14.2 and 15.14.3 that in any semi-topological formal system, (*right*) and (*stability*) hold for saturated sets. However, this is not sufficient in order to obtain distributivity of $\bullet$ over $\vee$, because to prove *Proposition* 15.14.13 we need to apply (*stability*) also to terms involving arbitrary unions, and arbitrary unions of saturated

sets are not necessarily saturated. Therefore we need the general form of (*stability*).

Moreover, we can prove that in any pre-topological formal system the following hold:

$$\mathcal{A}(\mathcal{A}(X) \cdot \mathcal{A}(Y)) = \mathcal{A}(X \cdot Y) \ (A\text{-stability})$$

$$X \cdot \bigcup_{i \in I} Y_i = \bigcup_{i \in I}(X \cdot Y_i).$$

Now let us come back to quasi-topological formal systems and pseudo-topological formal systems.

We have seen that in quasi-topological formal systems (and, more in general, in semi-topological formal systems), $\cdot$ distributes over $\cup$, by the very definition (*point-meet*). But in general, in quasi-topological formal systems $\cdot$ does not coincide with $\cap$. If we add (*left*) things change:

Proposition 15.14.14. *Let $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ be a left quasi-topological formal system. Then for all $X, Y \in \Omega_{\mathcal{A}}(B)$, $X \bullet Y = X \cdot Y = X \cap Y$.*

Proof. Since (*left*) holds, $X \cdot Y \blacktriangleleft X$ and $X \cdot Y \blacktriangleleft Y$. Therefore, $X \cdot Y$ is a lower bound of $\{X, Y\}$ with respect to the partial order $\blacktriangleleft$. Suppose Z is a lower bound of $\{X, Y\}$, then $Z \blacktriangleleft X$ and $Z \blacktriangleleft Y$. But from *Lemma* 15.14.3, $Z \blacktriangleleft X \cdot Y$. It follows that $X \cdot Y$ is the greatest lower bound of $\{X, Y\}$ with respect to the partial order $\blacktriangleleft$. But in view of *Proposition* 15.14.1.(5), in $\Omega_{\mathcal{A}}(B)$ the relation $\blacktriangleleft$ coincides with the set-theoretic inclusion $\subseteq$, so that $\cdot$ coincides with $\cap$. At this point the result is straightforward: $X \bullet Y = \mathcal{A}(X \cdot Y) = \mathcal{A}(X \cap Y) = X \cap Y$, because $\mathbf{Sat}_{\mathcal{A}}(B)$ is closed under intersections. QED

It is possible to prove the converse of *Proposition* 15.14.14:

Proposition 15.14.15. *Let $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ be a semi-topological formal system. If $\langle \Omega_{\mathcal{A}}(B), \cdot \rangle$ is a meet semilattice with ordering $\subseteq$, then $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ is a left quasi-topological formal system.*

Corollary 15.14.4. *Let $\langle \wp(U), \cap, U, \blacktriangleleft, \perp \rangle$ be a pseudo-topological formal system. Then $\langle \Omega_{\mathcal{A}}(\wp(U)), \bullet \rangle$ (and $\langle \Omega_{\mathcal{A}}(\wp(U)), \cdot \rangle$, as well) is a meet semilattice with ordering $\subseteq$.*

Lemma 15.14.5. *Let $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ be a semi-topological formal system. If (left) and (right) hold, then (A-distributivity) holds for all $X, Y \subseteq B$.*

Proof. We have to prove that for any $X, Y \subseteq B$, $\mathcal{A}(X) \cap \mathcal{A}(Y)) = \mathcal{A}(X \cdot Y)$.

(a) $\mathcal{A}(X \cdot Y) \subseteq \mathcal{A}(X) \cap \mathcal{A}(Y)$: since (*left*) holds, $X \cdot Y \blacktriangleleft X$ and $X \cdot Y \blacktriangleleft Y$, so that $\mathcal{A}(X \cdot Y) \subseteq \mathcal{A}(X)$ and $\mathcal{A}(X \cdot Y) \subseteq \mathcal{A}(Y)$, thanks to *Proposition* 15.14.1.(2). (b) $\mathcal{A}(X) \cap \mathcal{A}(Y) \subseteq \mathcal{A}(X \cdot Y)$: clearly $\mathcal{A}(X) \cap \mathcal{A}(Y) \subseteq \mathcal{A}(X)$ and $\mathcal{A}(X) \cap \mathcal{A}(Y) \subseteq \mathcal{A}(Y)$. Therefore, from *Proposition* 15.14.1. (3) $\mathcal{A}(X) \cap \mathcal{A}(Y) \blacktriangleleft \mathcal{A}(X)$ and $\mathcal{A}(X) \cap \mathcal{A}(Y) \blacktriangleleft \mathcal{A}(Y)$. Since (*right*) holds, we have $\mathcal{A}(X) \cap \mathcal{A}(Y) \blacktriangleleft \mathcal{A}(X) \cdot \mathcal{A}(Y)$ which is equivalent to $\mathcal{A}(\mathcal{A}(X) \cap \mathcal{A}(Y)) \subseteq \mathcal{A}(\mathcal{A}(X) \cdot \mathcal{A}(Y))$. But $\mathcal{A}(X) \cap \mathcal{A}(Y)$ is already saturated, thus by applying (*stability*) on the right term of the inequality we obtain $\mathcal{A}(X) \cap \mathcal{A}(Y) \subseteq \mathcal{A}(X \cdot Y)$ (we recall that (*stability*) is derivable from (*left*) jointly with (*right*)).   QED

Corollary 15.14.5. *Let $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ be a semi-topological formal system. If (left) and (right) hold, then for any $X, Y \in \Omega_{\mathcal{A}}(B)$, $X \cap Y = X \bullet Y$.*

So we have seen that in topological formal systems $\langle \Omega_{\mathcal{A}}(B), \bullet \rangle$ is a meet semilattice with ordering $\subseteq$, and $\mathbf{Sat}_{\mathcal{A}}(B)$ is a complete lattice with complete distributivity. We pack all this discussion in the following propositions:

Proposition 15.14.16. *Let $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ be a semi-topological formal system. Then $\langle \wp(B), \cdot, \cup \rangle$ is a distributive lattice.*

Proposition 15.14.17. *Let $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ be a pre-topological formal system. Then $\langle \Omega_{\mathcal{A}}(B), \bullet, \vee \rangle$ is a distributive lattice.*

Proposition 15.14.18. *Let $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ be a semi-topological formal system. Then the following are equivalent:*

1. *$\langle \Omega_{\mathcal{A}}(B), \bullet \rangle$ is a meet semilattice with ordering $\subseteq$.*

2. *(Left) and idempotence of $\cdot$ on B hold.*

Proposition 15.14.19. *Let $\langle B, \cdot, 1, \blacktriangleleft, \bot \rangle$ be a semi-topological formal system. Then the following are equivalent:*

1. *$\langle \Omega_{\mathcal{A}}(B), \bullet, \vee, B, \mathcal{A}(\bot) \rangle$ is a complete lattice with complete distributivity and with ordering $\subseteq$.*

2. *(Left) and (right) hold.*

3. *(Left), idempotence of $\cdot$ on B and (stability) hold.*

	$\langle \wp(B), \cdot, \cup \rangle$	$\langle \Omega_{\mathcal{A}}(B), \cap, \vee \rangle$	$\langle \Omega_{\mathcal{A}}(B), \cdot, \vee \rangle$	$\langle \Omega_{\mathcal{A}}(B), \bullet, \vee \rangle$
semi	distributes	complete		complete
left quasi	distributes	complete	complete (and $\cdot = \cap$)	complete (and $\bullet = \cap$)
pre-topology	distributes	complete	complete	distributes
topology	distributes	distributes	distributes (and $\cdot = \cap$)	distributes (and $\bullet = \cap$)

Notice that the failure of the converse of *Proposition* 15.14.9 means that we can have finite neighborhood pairs $\langle A, B, R \rangle$ such that their induced semi-topological formal systems $\langle B, \cap, 1, \blacktriangleleft, \perp \rangle$ are topological but $\Omega_{int}(A)$ and, thereafter, $\Omega_{\mathcal{A}}(B)$, although *formal* topologies, are distributive lattices but not lattices of sets, so that $\langle A, \Omega_{int}(A) \rangle$ and $\langle B, \Omega_{\mathcal{A}}(B) \rangle$ could fail to be topological spaces. This is a consequence of the fact that (*right*) does not imply **N3**.

To sum up, when we want to grasp the concrete topological features from an abstract point of view, we must observe that **Id** and **N1** are immaterial, while (*left*) is equivalent to **N2**. Moreover, all three properties are embedded in the formal behaviour of the relation $\blacktriangleleft$, except as to the behaviour induced by (*right*). Therefore, it seems that the distinguishing principle of formal topologies is (*right*), as it appears to be in the latest development of the topic (see [Sambin, 1989] and [Sambin & Gebellato, 1998]).

15.14.3   A New Age: Getting Rid of $\cdot$ and $\perp$.

The theory of Intuitionistic formal spaces was introduced by a series of papers initiated by [Sambin, 1987], on the ground of suggestions after Martin-Löf's Intuitionistic Type Theory. We shall see below that the level of abstraction on which this business is carried on, makes it possible to translate the formal properties of neighborhood systems into Gentzen systems. Particularly it is shown that formal pre-topology provide models for a fragment of Linear Logic.

As we have seen, Sambin's notion of a pre-topology is much more specific than the notion that we have used in this Part (indeed, a pre-topology corresponds to a formal neighborhood system of type $\mathcal{N}_{2Id}$ in which (*stability*) holds). However, in recent years formal properties have been made even more formal by getting rid of the monoidal operator $\cdot$ and the subset $\perp$ (cf. [Sambin, 1989]). Let us start with $\perp$.

Consider a basic pair $\langle A, B, R \rangle$. Instead of the set $\perp$ one can use a consistency predicate $Pos(\_)$ whose meaning is given by an evaluation *eval*, such that $eval(Pos(b)) = true$ if and only if $\exists a \in A(\langle a, b \rangle \in R)$

(i.e. $\langle R \rangle(b) \neq \emptyset$). That is, b is consistent if and only if b is a neighborhood of at least a point in A (this is recorded by saying that b is "inhabited").

The advantage of the predicate Pos, from a constructivistic point of view, is that it can be described purely on the abstract side, without mentioning the set A and its elements, in the following way:

$$\frac{Pos(b) \qquad b \blacktriangleleft Y}{\exists y \in Y \ \& \ Pos(y)} \ (monotonicity) \qquad \qquad \frac{\neg Pos(b)}{b \blacktriangleleft Y}$$

$$(ex \ falso \ sequitur \ quodlibet)$$

A second note concerns the possibility to get rid of the monoidal operator "$\cdot$". This is achieved using a binary operator $\downarrow$, defined by:

$$b \downarrow b' =_{def} \{c : \langle R \rangle(c) \subseteq \langle R \rangle(b) \cap \langle R \rangle(b')\}.$$

Then a formal system is topological, in the sense that $\{\langle R \rangle(X) : X \subseteq B\}$ is a topology, if $\langle R \rangle(\emptyset) = \emptyset$ and if for all $b, b' \in B$, $\langle R \rangle(b) \cap \langle R \rangle(b') = \langle R \rangle(b \downarrow b')$. In fact, by distributivity of $\cap$ over $\cup$, this equation holds if and only if the following holds: $\langle R \rangle(Y) \cap \langle R \rangle(Y') = \langle R \rangle(\{c : \langle R \rangle(c) \subseteq \langle R \rangle(Y) \cap \langle R \rangle(Y')\})$. But this is exactly the closure of the $\langle R \rangle$-images under finite intersections (closure under unions is given by the definition of $\langle R \rangle(Y)$ as $\bigcup_{y \in Y} \langle R \rangle(y)$).

Using these notions, we can readily see that a concrete topological space (that is, a topological space defined in terms of points), is a structure $CT = \langle A, B, R \rangle$ satisfying:

(B1) $\forall a \exists b(\langle a, b \rangle \in R)$; \qquad (B2) $\dfrac{\langle a, b \rangle \in R \qquad \langle a, b' \rangle \in R}{\langle a, b \downarrow b' \rangle \in R}$.

But in view of the above considerations, and defining the relation $\blacktriangleleft$ as usual by means of the operator $\mathcal{A}$, this definition may be completely shifted to the pure abstract side in the following way:

Definition 15.14.11. *A formal topology is a structure $FT = \langle B, \blacktriangleleft, Pos \rangle$, where B is a set, $\blacktriangleleft$ and Pos relations between elements of B and subsets of B, such that:*

1. $\dfrac{b \in Y}{b \blacktriangleleft Y}$ *(reflexivity)*.

2. $\dfrac{b \blacktriangleleft Y \qquad Y \blacktriangleleft Y'}{b \blacktriangleleft Y'}$ $(\blacktriangleleft - transitivity)$.

3. $\dfrac{b \blacktriangleleft Y \quad b \blacktriangleleft Y'}{b \blacktriangleleft Y \downarrow Y'}$ $(\downarrow - right)$ $- \text{ where } Y \downarrow Y' = \bigcup\limits_{y \in Y, y' \in Y'} y \downarrow y'.$

4. $\dfrac{\langle b, Y \rangle \in Pos}{b \in Y}$ $(antireflexivity)$.

5. $\dfrac{\langle b, Y \rangle \in Pos \ \forall b(\langle b, Y \rangle \in Pos \Rightarrow b \in Y')}{\langle b, Y \rangle \in Pos}$ $(Pos - transitivity)$.

6. $\dfrac{\langle b, Y \rangle \in Pos \qquad b \blacktriangleleft Y'}{\exists b'(b' \in Y' \ \& \ \langle b', Y \rangle \in Pos)}$ $(compatibility)$.

If we start from a formal topology and define, as usual, $\mathcal{A}(Y) = \{b : b \blacktriangleleft Y\}$ and, moreover, $\mathcal{C}(Y) = \{b : \langle b, Y \rangle \in Pos\}$, then we obtain $\mathcal{A}(Y \downarrow Y') = \mathcal{A}(Y) \cap \mathcal{A}(Y')$ and $\mathcal{C}(Y) \bowtie \mathcal{A}(Y') \Rightarrow \mathcal{C}(Y) \bowtie Y'$, where $Y \bowtie Y'$ if and only if $\exists a \in A(a \in Y \ \& \ a \in Y')$.

This translation makes it possible to explain $(antireflexivity)$. In fact $\langle b, Y \rangle \in Pos$ means that in $\langle R \rangle(b)$ there are elements x such that $R(x) \subseteq Y$. That is, $\langle b, Y \rangle \in Pos$ not only if b is inhabited, but if it is inhabited at least by an element x which belongs to Y with all its neighbors. This makes $\mathcal{C}$ into an interior operator (as we already know – explicitly: $b \in \mathcal{C}(Y)$ iff $\exists a \in \langle R \rangle(b) \ \& \ R(a) \subseteq Y$. But if $a \in \langle R \rangle(b)$, then $b \in R(a)$. It follows that $b \in Y$). Therefore, antireflexivity states that $\mathcal{C}(Y) \subseteq Y$: if $b \in \mathcal{C}(Y)$ then $b \in U$.

It follows that if we start from a concrete topological space $\langle A, B, R \rangle$, then we obtain a formal topology $\langle B, \blacktriangleleft, Pos \rangle$ by defining $\mathcal{A}$ and $\mathcal{C}$ in the usual way.

Points are defined in formal systems as follows.

If we are dealing with a topological formal system $TFS = \langle B, \cdot, \blacktriangleleft, 1, \perp \rangle$, a formal point is any subset $\alpha \subseteq B$, such that:

(i) α is inhabited: $1 \in \alpha$; (ii) α is convergent: $\dfrac{b \in \alpha \qquad b' \in \alpha}{b \cdot b' \in \alpha}$;

(iii) α splits $\blacktriangleleft$: $\dfrac{b \in \alpha \quad b \blacktriangleleft Y}{\exists c(c \in Y \ \& \ c \in \alpha)}$; (iv) α is consistent: $a \in \alpha \Rightarrow a \neq \perp$.

It is possible to prove that if $TFS = \langle B, \cdot, \blacktriangleleft, 1, \perp \rangle$ is a topological formal system, then there is a bijection between formal points of TFS and completely prime filters of $\mathbf{Sat}_{\mathcal{A}}(B)$. This result is the formal

counterpart of the construction of abstract points in pointless topology (see Introduction).

If we deal with a formal topology $FT = \langle B, \blacktriangleleft, Pos \rangle$, then $\alpha \subseteq B$ is a formal point provided:

(i*) α is inhabited: $\exists b \in B \ \& \ b \in \alpha$;

(ii*) α is convergent: $\dfrac{b \in \alpha \qquad b' \in \alpha}{b \downarrow b' \in \alpha}$;

(iii*) α splits $\blacktriangleleft$: $\dfrac{b \in \alpha \qquad b \blacktriangleleft Y}{\exists c (c \in Y \ \& \ c \in \alpha)}$;

(iv*) α is consistent: $\dfrac{b \in \alpha \qquad \alpha \subseteq Y}{\langle a, Y \rangle \in Pos}$.

15.14.4   The Logical Interpretation of the Pre-Topology Formal Approach

To begin with, notice that since $\mathbf{Sat}_{\mathcal{A}}(B)$ is a complete lattice with complete distributivity, then it is a Heyting algebra. Therefore, we can introduce a relative pseudo-complementation $\Longrightarrow$ in the following manner: for any $X, Y \in \mathbf{Sat}_{\mathcal{A}}(B)$,

$$X \Longrightarrow Y =_{def} \{b : b \cdot X \blacktriangleleft Y\}. \qquad (15.14.7)$$

Even without (*stability*) $\Longrightarrow$ preserves $\cap$: $X \Longrightarrow (Y \cap Z) = \{b : b \cdot X \blacktriangleleft (Y \cap Z)\} = \{b : \mathcal{A}(b \cdot X) \subseteq \mathcal{A}(Y \cap Z)\}$. But $\mathcal{A}(Y \cap Z) \subseteq \mathcal{A}(Y) \cap \mathcal{A}(Z)$, thus $X \Longrightarrow (Y \cap Z) = \{b : \mathcal{A}(b \cdot X) \subseteq \mathcal{A}(Y)\} \cap \{b : \mathcal{A}(b \cdot X) \subseteq \mathcal{A}(Z)\} = X \Longrightarrow Y \cap X \Longrightarrow Z$. Hence $\Longrightarrow$ has a lower adjoint. Moreover, we already know that in the presence of (*stability*) $\cdot$ preserves $\cup$ and $\cup$ is the dual of $\cap$. Hence $\cdot$ has a upper adjoint. This leads to a suspicion as $\cdot$ and $\Longrightarrow$ are linked by the following adjoint relation

$$X \cdot Z \blacktriangleleft Y \ \textit{iff} \ Z \blacktriangleleft X \Longrightarrow Y \qquad (\textit{imp-adjoint})$$

Actually, this is what happens even in pre-topological formal systems, as we are going to see.

Proposition 15.14.20. *In any pre-topological system (imp-adjoint) holds.*

Proof. The proof is based on the following lemma:

Lemma 15.14.6. *In any semi-topological space* $X \cdot (X \Longrightarrow Y) \blacktriangleleft Y$, *for all* $X, Y \subseteq \wp(B)$.

Proof. Trivially, because $X \cdot (X \Longrightarrow Y) = X \cdot \{b : X \cdot b \blacktriangleleft Y\}$.
Now, by (*stability*) we obtain $(b' \blacktriangleleft X \,\&\, b \blacktriangleleft X \Longrightarrow Y) \Rrightarrow (b' \cdot b \blacktriangleleft X \cdot (X \Longrightarrow Y))$ which, by the above *Lemma*, reduces to $(b' \blacktriangleleft X \,\&\, b \blacktriangleleft X \Longrightarrow Y) \Rrightarrow (b' \cdot b \blacktriangleleft Y)$, and by generalization we obtain $(Z' \blacktriangleleft X \,\&\, Z \blacktriangleleft X \Longrightarrow Y) \Rrightarrow (Z' \cdot Z \blacktriangleleft Y)$. If we put $Z' = X$ we obtain $(X \blacktriangleleft X \,\&\, Z \blacktriangleleft X \Longrightarrow Y) \Rrightarrow (X \cdot Z \blacktriangleleft Y)$, and since $X \blacktriangleleft X$ is an identity, finally we have $(Z \blacktriangleleft X \Longrightarrow Y) \Rrightarrow (X \cdot Z \blacktriangleleft Y)$. Conversely, suppose $b \cdot X \blacktriangleleft Y$. Then, by definition, $b \in X \Longrightarrow Y$. Hence $\{b\} \subseteq X \Longrightarrow Y$, so that $b \blacktriangleleft X \Longrightarrow Y$. By generalization we obtain $(Z \cdot X \blacktriangleleft Y) \Rrightarrow (Z \blacktriangleleft X \Longrightarrow Y)$.   QED

Therefore, in pre-topological formal systems (*imp-adjoint*) holds.

The connection of pre-topological and topological formal systems with logic is further developed as follows.

An evaluation ϕ from the language of Linear Logic to a pre-topology $\mathbf{P} = \langle B, \bullet, \vee, 1, \perp \rangle$ is a function from propositional variables into $\mathbf{Sat}_{\mathcal{A}}(B)$, which is extended to all formulas by:

α	$\varphi(\alpha)$
$\top$	B
1	$\mathcal{A}(1)$
$\perp$	$\perp$
0	$\mathcal{A}(\emptyset)$
$a \otimes b$	$\varphi(a) \bullet \varphi(b)$
$a \& b$	$\varphi(a) \cap \varphi(b)$
$a \oplus b$	$\varphi(a) \vee \varphi(b)$
$a \multimap b$	$\varphi(a) \Longrightarrow \varphi(b)$
$\neg a$	$a \multimap \perp = \varphi(a) \Longrightarrow \perp$

One should remember that in general $\bullet$ does not coincide with $\cap$ in $\mathbf{Sat}_{\mathcal{A}}(B)$, in pre-topological formal systems differently from topological formal systems.

Having set this interpretation, we can give pre-topologies the structure of a Gentzen (sequent) calculus by putting:

$$\Gamma \vdash c \;\; \textit{iff} \;\; \phi(\otimes_{a \in \Gamma} \{a\}) \blacktriangleleft \phi(c)$$

A rule of inference will be said to be valid in **P** if for any evaluation ϕ on **P**, the conclusion is valid whenever all the premises are valid.

It is easy to verify that the *exchange rule* $\frac{\Gamma \vdash C}{\Gamma' \vdash C}$, where Γ' is any permutation of Γ, is valid because $\cdot$ is commutative.

Now let us note that (*right*) and (*left*) are equivalent to the following properties (R) and, respectively, (L):

$$\frac{W \cdot A \cdot A \blacktriangleleft Z}{W \cdot A \blacktriangleleft Z} \, (R) \qquad \frac{W \blacktriangleleft Z}{A \cdot W \blacktriangleleft Z} \, (L)$$

The logical interpretation is striking, because the above rules (R) and (L) turn into the following structural rules, respectively:

$$\frac{\Gamma, a, a \vdash b}{\Gamma, a \vdash b} \, (contraction) \qquad \frac{\Gamma \vdash b}{a, \Gamma \vdash b} \, (weakening)$$

Since in general pre-topologies are such that the relation $\blacktriangleleft$ is a precover and not a cover, then pre-topologies provide a suitable semantic for logics with limited structural rules.

Moreover, (R) jointly with (L), in turn, is equivalent to the idempotence of $\cdot$ with respect to the preorder $\blacktriangleleft$, which in $\mathbf{Sat}_{\mathcal{A}}(B)$ means that $\cdot$, $\bullet$ and $\cap$ coincide.

This means that in the presence of (L) and (R) we cannot any longer distinguish between the additive connectives $\cap$, $\vee$, $\top$ and 0, and the multiplicative connectives $\otimes$, $\multimap$, 1 and $\bot$. On the contrary, without (R) and (L) we have the following rules for $\otimes$ and $\&$:

$$\otimes l : \frac{\Gamma, a, b \vdash c}{\Gamma, a \otimes b \vdash c}; \quad \otimes r : \frac{\Gamma_1 \vdash a \qquad \Gamma_2 \vdash b}{\Gamma_1, \Gamma_2 \vdash a \otimes b}$$

$$\& l : \frac{\Gamma, a \vdash c}{\Gamma, a \,\&\, b \vdash c}, \quad \frac{\Gamma, b \vdash c}{\Gamma, a \,\&\, b \vdash c}; \otimes r : \frac{\Gamma \vdash a \qquad \Gamma \vdash b}{\Gamma \vdash a \,\&\, b}.$$

Notice that in view of (*A-stability*) $\phi(a_1 \otimes \ldots \otimes a_n) \blacktriangleleft\blacktriangleright \phi(a_1) \cdot \ldots \cdot \phi(a_n)$, so that both $\otimes$ and "," correspond to the operation (*point-meet*) ".". Therefore, $\otimes r$ is evaluated as $\dfrac{\phi(\otimes\Gamma_1) \blacktriangleleft \phi(a) \qquad \phi(\Gamma_2) \blacktriangleleft \phi(b)}{\phi(\Gamma_1) \cdot \phi(\Gamma_2) \blacktriangleleft \phi(a) \cdot \phi(b)}$ which is exactly (*stability*). In turn, $\otimes l$ is evaluated as $\dfrac{\phi(\Gamma, a, b) \blacktriangleleft \phi(c)}{\phi(\Gamma) \cdot \phi(a \otimes b) \blacktriangleleft \phi(c)}$. But $\phi(a \otimes b) = \phi(a) \bullet \phi(b) = \phi(a) \cdot \phi(b)$ so that $\phi(\Gamma) \cdot \phi(a \otimes b) = \phi(\Gamma) \cdot \phi(a) \cdot \phi(b)$ which is the same evaluation as $\phi(\Gamma, a, b)$. $\& r$ is evaluated as $\dfrac{\phi(\Gamma) \vdash \phi(a) \quad \phi(\Gamma) \vdash \phi(b)}{\phi(\Gamma) \vdash \phi(a \,\&\, b)}$. But $\phi(a \,\&\, b) =$

$\phi(a) \cap \phi(b)$, so that $\&r$ is evaluated as $\dfrac{\phi(\Gamma) \blacktriangleleft \phi(a) \ \phi(\Gamma) \blacktriangleleft \phi(b)}{\phi(\Gamma) \blacktriangleleft \phi(a) \cap \phi(b)}$ that, since for saturated sets $a \blacktriangleleft b$ implies $a \subseteq b$, corresponds to $\dfrac{\phi(\Gamma) \subseteq \phi(a) \ \phi(\Gamma) \subseteq \phi(b)}{\phi(\Gamma) \subseteq \phi(a) \cap \phi(b)}$, which trivially holds. Similarly for $\&l$.

15.14.5   A Sample Formal Neighborhood System

Consider the neighborhood systems of Example 12.4.4. This system is induced by the basic pair $\langle U, \wp(U), R \rangle$, where $U = \{a, b, c\}$ and $R(a) = \{\{a, b\}, \{a, c\}, U\}$, $R(b) = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, U\}$, $R(c) = \{\{c\}, \{a, c\}, U\}$. It induces the following structures, ordered by the subset relation:
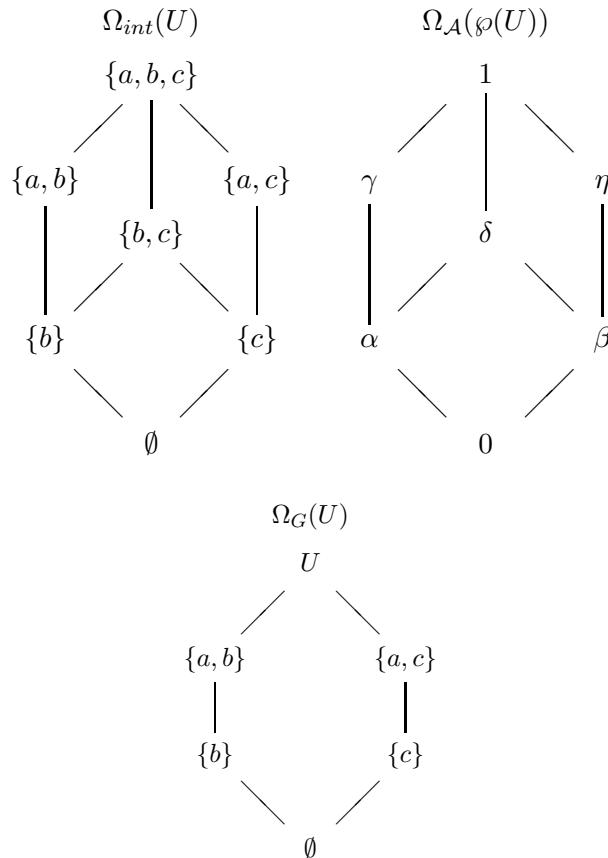
where:

(a) In $\Omega_{int}(U)$, $\emptyset = int(\emptyset) = int(\{a\})$, $\{b\} = int(\{b\})$, $\{c\} = int(\{c\})$, $\{a,b\} = int(\{a,b\})$, $\{b,c\} = int(\{b,c\})$, $\{a,c\} = int(\{a,c\})$, $U = int(U)$.

(b) [In what follows, given a set of sets $\mathcal{X}$, $\mathbf{C}_{\mathcal{X}}$ stands for any combination without repetition of elements from $\mathcal{X}$. For instance, if $\mathcal{X} = \{\{a\}, \{b\}\}$ then $\mathbf{C}_{\mathcal{X}} = \{\{a\}\}$ or $\mathbf{C}_{\mathcal{X}} = \{\{b\}\}$ or $\mathbf{C}_{\mathcal{X}} = \{\{a\}, \{b\}\}$]. In $\Omega_{\mathcal{A}}(\wp(U))$, $0 = \{\emptyset\} = \mathcal{A}(\emptyset) = \mathcal{A}(\{\emptyset\})$; $\alpha = 0 \cup \{\{a\}, \{b\}, \{b,c\}\} = \mathcal{A}(\mathbf{C}_{\{\{a\},\{b\},\{b,c\}\}})$; $\beta = 0 \cup \{\{c\}\} = \mathcal{A}(\{\{c\}\} \cup \mathbf{C}_0)$; $\gamma = \alpha \cup \{\{a,b\}\} = \mathcal{A}(\{\{a,b\}\} \cup \mathbf{C}_{\alpha})$; $\delta = \alpha \cup \beta = \mathcal{A}(\mathbf{C}_{\alpha} \cup \mathbf{C}_{\beta})$; $\eta = \beta \cup \{\{a,c\}\} = \mathcal{A}(\{\{a,c\}\} \cup \mathbf{C}_{\beta})$; $1 = \gamma \cup \eta \cup \{U\} = \mathcal{A}(\{U\} \cup \mathbf{C}_{\gamma} \cup \mathbf{C}_{\eta})$.

We give some example of calculation of $\mathcal{A}$ (we use all the canonical parentheses):

$\mathcal{A}(\{\{a\}\}) = [R^{\smile}]\langle R\rangle(\{\{a\}\}) = [R^{\smile}](\{b\}) = \{\emptyset, \{a\}, \{b\}, \{b,c\}\}$.
$\mathcal{A}(\{\{b\}, \{c\}\}) = [R^{\smile}](\{b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b,c\}\}$.

One can notice:

A $\Omega_{int}(U)$ is not closed under intersections (for instance, $\{a,b\} \cap \{a,c\} = \{a\} \notin \Omega_{int}(U)$). In fact, we need the operation $\wedge$: $\{a,b\} \wedge \{a,c\} = int(\{a,b\} \cap \{a,c\}) = int(\{a\}) = \emptyset$.

B $\Omega_{\mathcal{A}}(\wp(U))$ is not closed under unions (for instance, $\beta \cup \gamma = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b,c\}, \{a,b\}\{a,c\}\}$, which is not an element of $\Omega_{\mathcal{A}}(\wp(U))$). Indeed, we need the operation $\vee$: $\beta \vee \gamma = \mathcal{A}(\beta \cup \gamma) = 1$.

C We can easily verify the following correspondence:

sets in U	sets in $\wp(U)$	symbol
$\emptyset$	$\{\emptyset\}$	0
$\{b\}$	$\{\emptyset, \{a\}, \{b\}, \{b,c\}\}$	α
$\{c\}$	$\{\emptyset, \{c\}\}$	β
$\{a,b\}$	$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$	γ
$\{a,c\}$	$\{\emptyset, \{c\}, \{a,c\}\}$	η
$\{b,c\}$	$\{\emptyset, \{a\}, \{b\}, \{c\}, \{b,c\}\}$	δ
U	$\{\emptyset\} \cup \wp(U)$	1

We can therefore verify, for instance, that η is the top element of an equivalence class modulo ◄►. In fact $\eta = \{\emptyset, \{c\}, \{a,c\}\}$ and it is the top element of the equivalence class $[\{\{a,c\}\}, \{\{c\}, \{a,c\}\}, \{\emptyset, \{a,c\}\}, \{\emptyset, \{c\}, \{a,c\}\}]$, i.e. $\{\{a,c\}\} \cup \mathbf{C}_{\beta}$.

D (*Left*) Fails: $\{a,c\} \blacktriangleleft \{\{a,c\}\}$, but $\{a,c\} \cdot \{a,b\} = \{a,c\} \cap \{a,b\} = \{a\}$, and $\{a\}$ is not semicovered by $\{\{a,c\}\}$. In fact $\langle R\rangle(\{a\}) = R^{\smile}(\{a\}) = \{b\}$, while $\langle R\rangle(\{\{a,c\}\}) = R^{\smile}(\{a,c\}) = \{a,c\}$ (and notice that **N2** fails in the neighborhood system $\{R(x)\}_{x\in U}$).

E (*Right*) Does not hold. Notice for instance, that $\{b,c\} \blacktriangleleft \{\{a\}\}$ and $\{b,c\} \blacktriangleleft \{\{b\}\}$, but $\{b,c\}$ is not semicovered by $\{\{a\}\} \cdot \{\{b\}\} = \{\emptyset\}$, because $\langle R\rangle(\{b,c\})$ (*viz.* $\{b\}$) is not included in $\langle R\rangle(\{\emptyset\})$ (*viz.* $\emptyset$) (remember that $\langle R\rangle(\{b,c\})$ is short for $\langle R\rangle(\{\{b,c\}\})$).

This also shows that:

F (*Stability*) Does not hold. We show this by exhibiting a counterexample of (*A-stability*).
In fact, $\mathcal{A}(\mathcal{A}(\{\{a\}\})\cdot\mathcal{A}(\{\{b\}\})) = \mathcal{A}(\{\emptyset,\{a\},\{b\},\{b,c\}\}\cdot\mathcal{A}(\{\emptyset,\{a\}, \{b\},\{b,c\}\}) = \{\emptyset,\{a\},\{b\},\{b,c\}\}$. On the contrary, $\mathcal{A}(\{\{a\}\} \cdot \{\{b\}\}) = \mathcal{A}(\emptyset) = \{\emptyset\}$.

The operations on $\Omega_{\mathcal{A}}(U)$ are:

$\bullet$	0	α	β	γ	δ	η	1
0	0	0	0	0	0	0	0
α	0	α	β	α	δ	δ	α
β	0	β	β	β	β	β	β
γ	0	α	β	γ	δ	δ	1
δ	0	δ	β	δ	δ	δ	δ
η	0	δ	β	δ	δ	η	η
1	0	α	β	1	δ	η	η

$\vee$	0	α	β	γ	δ	η	1
0	0	α	β	γ	δ	η	1
α	α	α	δ	γ	δ	1	1
β	β	δ	β	1	δ	η	1
γ	γ	γ	1	γ	1	1	1
δ	δ	δ	δ	1	δ	1	1
η	η	1	η	1	1	η	1
1	1	1	1	1	1	1	1

$\cap$	0	α	β	γ	δ	η	1
0	0	0	0	0	0	0	0
α	0	α	0	α	α	0	α
β	0	0	β	0	β	β	β
γ	0	α	0	γ	α	0	γ
δ	0	α	β	α	δ	β	δ
η	0	0	β	0	β	η	η
1	0	α	β	γ	δ	η	1

For instance, $\gamma \bullet \eta = \mathcal{A}(\{\emptyset,\{c\},\{a\}\}) = [R^{\smile}](\{b,c\}) = \{\emptyset,\{a\},\{b\},\{c\}, \{b,c\}\} = \delta$.

Clearly, for any $\lambda, \mu \in \Omega_{\mathcal{A}}(\wp(U))$, $\lambda \vee \mu = \lambda$ if and only if $\mu \subseteq \lambda$. Indeed, $\lambda = \lambda \vee \mu$ means $\mathcal{A}(\lambda) = \mathcal{A}(\lambda \cup \mu)$. But $\mathcal{A}(\lambda \cup \mu) \supseteq \mathcal{A}(\lambda) \cup \mathcal{A}(\mu)$. Therefore, $\lambda \supseteq \lambda \cup \mu$, so that $\lambda \supseteq \mu$.

From this observation and the shape of $\Omega_{\mathcal{A}}(\wp(U))$, it is immediate to infer that $\langle \Omega_{\mathcal{A}}(U), \vee, \cap \rangle$ is not a distributive lattice (indeed, (*stability*) does not hold). Moreover, not even $\bullet$ distributes over $\vee$. For instance, $\eta \bullet (\gamma \vee \delta) = \eta \bullet 1 = \eta$, whereas $(\eta \bullet \gamma) \vee (\eta \bullet \delta) = \delta \vee \delta = \delta$.

15.14.6 A Pseudo-Topological Neighborhood System

Consider the neighborhood system of Example 12.6.3. This system is induced by the basic pair $\langle U, \wp(U), R \rangle$, where $U = \{a, b, c\}$ and $R(a) = \{\{a\}, \{a, b\}, \{a, c\}, U\}, R(b) = \{\{b\}, \{a, b\}, \{b, c\}, U\}, R(c) = \{\{a, c\}, \{b, c\}, U\}$. It induces the following structures:
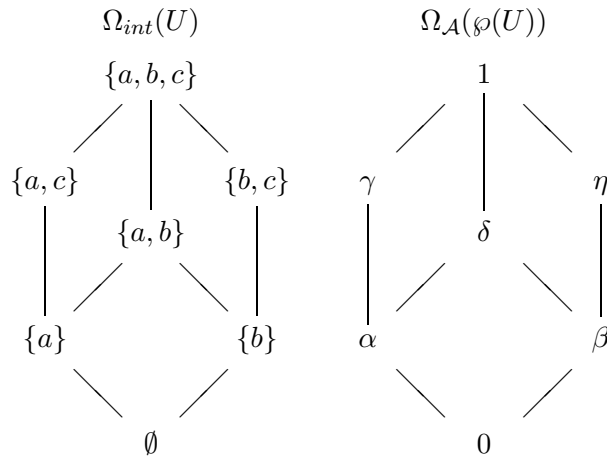
We have:

(a) $\emptyset = int(\emptyset) = int(\{c\})$, and $X = int(X)$, in the remaining cases.

(b) $0 = \{\emptyset, \{c\}\} = \mathcal{A}(\mathbf{C}_0)$; $\alpha = 0 \cup \{\{a\}\} = \mathcal{A}(\{\{a\}\} \cup \mathbf{C}_0)$; $\beta = 0 \cup \{\{b\}\} = \mathcal{A}(\{\{b\}\} \cup \mathbf{C}_0)$; $\gamma = \alpha \cup \{\{a, c\}\} = \mathcal{A}(\{\{a, c\}\} \cup \mathbf{C}_\alpha)$; $\delta = \alpha \cup \beta \cup \{\{a, b\}\} = \mathcal{A}(\{\{a, b\}\} \cup \mathbf{C}_\alpha \cup \mathbf{C}_\beta)$; $\eta = \beta \cup \{\{b, c\}\} = \mathcal{A}(\{\{b, c\}\} \cup \mathbf{C}_\beta)$; $1 = \gamma \cup \eta \cup \delta \cup U = \mathcal{A}(\{\{a, c\}\} \cup \mathbf{C}_\beta) = \mathcal{A}(\{\{a, c\}\} \cup \mathbf{C}_\beta \cup \mathbf{C}_\alpha) = \ldots = \mathcal{A}(\{U\})$.

Notice that $\mathcal{A}(\mathbf{C}_0) = \mathcal{A}(\{\emptyset\})$ and $\{c\} \in \mathcal{A}(\mathbf{C}_0)$ because in the formula for $[R^{\smile}]$ the left side of the implication is false when the argument is $\{c\}$.

Although the shape of $\Omega_{int}(U)$ is the same as that of the set of int-saturated elements of the formal neighborhood system of Example 15.14.5, nonetheless they differ in important respects. First, in the present system int and G (and g) coincide because $\{R(x)\}_{x \in U}$ is a neighborhood system of type $\mathcal{N}_{2Id}$.

Second, we can observe that:

A (*Left*) Holds, because $\{R(x)\}_{x \in U}$ is a neighborhood system in which **N2** holds.

B As a consequence, the operations $\bullet$ and $\cap$ coincide, in this case.

C (*Right*) Fails. For instance, $\{a, c\} \blacktriangleleft \{\{a, b\}, \{b, c\}\}$ and $\{a, c\} \blacktriangleleft \{\{a, c\}\}$, but $\langle R \rangle(\{a, c\}) = \{a, c\}$ which is not included in $\{a\} = \langle R \rangle(\{a\}, \{c\}) = \langle R \rangle(\{\{a, b\}, \{b, c\}\} \cdot \{\{a, c\}\})$. This corresponds to the fact that in $\Omega_{int}(U)$, $\{a, c\} \subseteq \{a, b\} \cup \{b, c\}$ and $\{a, c\} \subseteq \{a, c\}$, but $\{a, c\}$ is not included in $\{a\} = \{\{a, b\} \wedge \{a, c\}\} \cup \{\{b, c\} \wedge \{a, c\}\}$. This, in $\Omega_{\mathcal{A}}(\wp(U))$, is due to the following fact:

D $\cap$ and $\vee$, and hence $\bullet$ and $\vee$, do not distribute over each other. This is an effect of the failure of (*stability*).

E In fact (*stability*) fails. Obviously, because idempotence of $\cdot$ and (*stability*) gives (*right*) but (*right*) fails and idempotence of $\cdot$ holds. Therefore, (*stability*) cannot hold. Indeed, the counterexample for (*right*) is also a counterexample for (*stability*).
Another counterexample is the following: $\{\{a, c\}, \{b, c\}\} \blacktriangleleft \{\{a, b\}, \{b, c\}\}$ and $\{\{a, c\}, \{b, c\}\} \blacktriangleleft \{\{a, c\}, \{a, b\}\}$, but $\{\{a, c\}, \{b, c\}\} \cdot \{\{a, c\}, \{b, c\}\} = \{\{a, c\}, \{b, c\}, \{c\}\}$ which is not semicovered by $\{\{a\}, \{b\}, \{a, b\}\} = \{\{a, b\}, \{b, c\}\} \cdot \{\{a, c\}, \{a, b\}\}$. In $\Omega_{int}(U)$ this effect corresponds to the fact that $\{a, b, c\} = \{a, b\} \cup \{b, c\}$, $\{a, b, c\} = \{a, c\} \cup \{a, b\}$, but $\{a, b, c\} \neq \{\{a, b\} \wedge \{a, c\}\} \cup \{\{a, b\} \wedge \{a, b\}\} \cup \{\{b, c\} \wedge \{a, c\}\} \cup \{\{b, c\} \wedge \{a, b\}\} = \{a, b\}$ (notice that $\{b, c\} \wedge \{a, c\}) = int(\{b, c\} \cap \{a, c\}) = int(\{c\}) = \emptyset$). This, in turn, is due to the fact that $\wedge$ and $\cup$ do not distribute over each other.

15.14.7 A Topological Formal Neighborhood System in which N3 does not Hold

Consider the neighborhood system induced by the basic pair $\langle U, \wp(U), R \rangle$, where $U = \{a, b, c\}$ and $R(a) = \{\{a, b\}, \{a, c\}, U\}$,

$R(b) = \{\{a,b\}, U\}$, $R(c) = \{\{a,c\}, U\}$. It induces the following structures, where $\Omega_{int}(U) = \Omega_G(U) = \Omega_g(U)$:

$$\Omega_{int}(U) \qquad\qquad\qquad \Omega_{\mathcal{A}}(\wp(U))$$
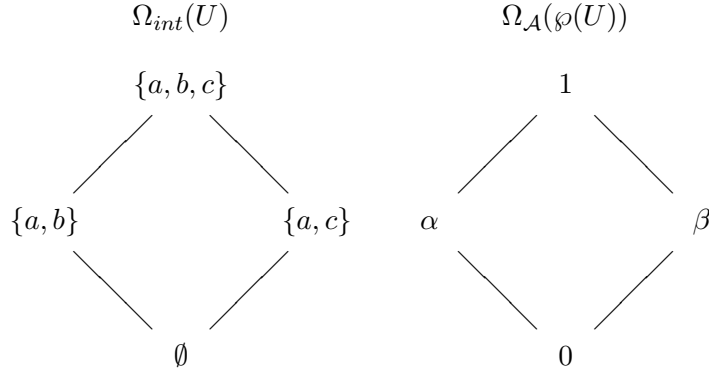
In $\Omega_{\mathcal{A}}(\wp(U))$, $0 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b,c\}\}$, $\alpha = 0 \cup \{\{a,b\}\}$, $\beta = 0 \cup \{\{a,c\}\}$ and $1 = \alpha \cup \beta \cup \{U\}$.

We can observe what follows:

A In the neighborhood system $\{R(x)\}_{x \in U}$, **N1**, **N2** and **Id** hold but **N3** does not hold (because $\{a\} \notin R(a)$). Nevertheless (*right*) holds. In fact, for any $X \subseteq U$ and $\mathcal{Y} \subseteq \wp(U)$, $X \blacktriangleleft \mathcal{Y}$ if and only if $X \in \mathcal{Y}$ or $U \in \mathcal{Y}$ (this is a counterexample of the converse of *Proposition* 15.14.9).

B It follows that $\langle U, \wp(U), R\rangle$ induces a topological formal neighborhood system (because **N2** implies (*left*)).

C Although $\langle U, \wp(U), R\rangle$ gives rise to a topological formal system, $\langle U, \Omega_{int}(U)\rangle$ is not a topological space. Indeed, it is a distributive lattice, but it is not a lattice of sets ($\{a,b\} \cap \{a,c\} \notin \Omega_{int}(U)$).

Therefore, this is an example that a formal neighborhood pair $\langle U, \wp(U), R\rangle$ may induce a topological formal system even if **N3** does not hold in $\{R(x)\}_{x \in U}$.

15.14.8 A Topological Formal Neighborhood System in which N1 does not Hold

Do not deduce that $\Omega_{int}(U) = \Omega_G(U) = \Omega_g(U)$ from the fact that a basic pair induces a topological formal system. This deduction is incorrect because the coincidence of *int* and G relies on **N2**, **N1** and **Id** and the latter two are principles with no formal counterparts. Consider

the neighborhood system $R(a) = \{\{b\}, \{a,b\}\}$, $R(b) = \{\{a,b\}\}$, on $U = \{a,b\}$. Here **N1** and **Id** does not hold (indeed, $a \notin \{b\} \in R(a)$ and $a \in \{a\} = G(\{b\}) \notin R(a)$), while **N2** and **N3** hold. Hence the basic neighborhood pair $\langle U, \wp(U), R \rangle$ induces a topological formal system, anyway. However, in this system $int(\{a\}) = \{a\}$, because $G(\{b\}) \subseteq \{a\}$ and $\{b\} \in R(a)$. On the contrary, $g(\{a\}) = G(\{a\}) = \emptyset$. Moreover, $int(\{b\}) = \emptyset$, while $g(\{b\}) = G(\{b\}) = \{a\}$.

With the following example we show that **N2** and **N3** do not imply extensionality. Consider the neighborhood system $\mathcal{N}_a = \{\{a\}, \{a,b\}, \{a,c\}, U\}$, $\mathcal{N}_b = \{U\}$, $\mathcal{N}_c = \{\{a,c\}, U\}$. This system is of type $\mathcal{N}_{3Id}$. Nevertheless it is not extensional, because $G(\{a,b\}) = \{a\}$.

15.14.9 A Non-Topological Formal Neighborhood System such that $\Omega_{int}(U)$ is a Heyting Algebra of Sets

Let $U = \{a,b,c\}$, $R(a) = \{\{a\}, U\}$, $R(b) = \{\{b\}, U\}$, $R(c) = \{U\}$. We obtain the following lattices:
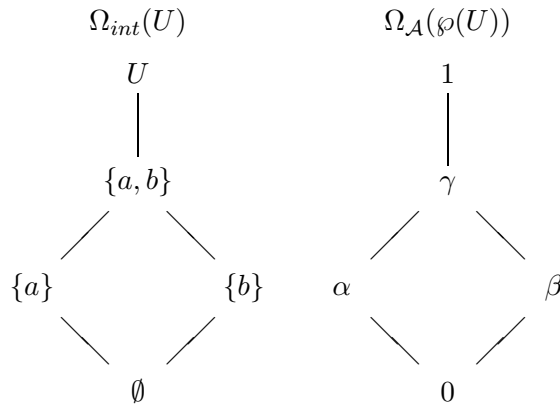
$$\Omega_{int}(U) \qquad\qquad \Omega_{\mathcal{A}}(\wp(U))$$

where: $0 = \{\emptyset, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$, $\alpha = \{\{a\}\} \cup 0$, $\beta = \{\{b\}\} \cup 0$, $\gamma = \{\{a\}, \{b\}\} \cup 0$ and $1 = \{\{a\}, \{b\}, U\} \cup 0$.

Therefore, $\langle U, \Omega_{int}(U) \rangle$, as well as $\langle \Omega_{\mathcal{A}}(\wp(U)) \rangle$, is a topological space. Nonetheless, $G \neq int$ (for instance, $G(\{a,b\}) = \emptyset \neq \{a,b\} = int(\{a,b\})$. Therefore, either **Id**, **N1** or **N2** fail in $\mathcal{N}(U) = \{R(a)\}_{a \in U}$. Indeed both **Id** and **N2** do not hold. It follows that (*left*) cannot hold. For example, $\{a,b\} \blacktriangleleft \{a,b\}$, but $\{a,b\} \cdot \{a\} = \{a,b\} \cap \{a\} = \{a\}$ and $\langle R \rangle(\{\{a\}\}) = \{a\}$, while $\langle R \rangle(\{\{a,b\}\}) = \emptyset$, so that $\{a,b\} \cdot \{a\}$ is not semicovered by $\{a,b\}$. It follows that $\langle U, \wp(U), R \rangle$ does not induce a

topological formal system. Incidentally, thanks to this example we can appreciate the connection between **N2** and (*left*). In fact, if (*left*) held, $\langle a, \{a\}\rangle \in R$ would imply $\langle a, \{a, b\}\rangle \in R$, too. Hence we would have $\langle R\rangle(\{\{a\}\}) \subseteq \langle R\rangle(\{\{a, b\}\})$.

Moreover, neither a pre-topological formal system is induced by $\langle U, \wp(U), R\rangle$, because (*stability*) fails. For instance, since both $\{a, b\}$ and $\{b, c\}$ belong to 0 so that $\langle R\rangle(\{\{a, b\}\}) = \langle R\rangle(\{\{b, c\}\}) = \emptyset$, one has $\{a, b\} \blacktriangleleft \{a, b\}$ and $\{b, c\} \blacktriangleleft \{a, b\}$. But $\langle R\rangle(\{a, b\} \cdot \{b, c\}) = \langle R\rangle(\{b\}) = \{b\} \nsubseteq \langle R\rangle(\{\{a, b\}\} \cdot \{\{a, b\}\}) = \langle R\rangle(\{\{a, b\}\}) = \emptyset$.

The same example shows that if $\langle U, \Omega(U)\rangle$ is a topological space and we put $\langle x, X\rangle \in R$ if and only if $X \in \Omega(U)$ and $x \in X$, for $x \in U$ and $X \subseteq U$, then $\langle U, \wp(U), R\rangle$ generally does not induce a topological formal system (trivially, put in the above example $\Omega(U) = \Omega_{int}(U)$).

Notice that $0 \bullet 0 = \gamma$. Therefore, since 0 is a subset of any $\mathcal{A}$-saturated element, for all $\lambda, \mu \in \Omega_{\mathcal{A}}(\wp(U))$, $\lambda \bullet \mu = \gamma$ except for the case $1 \bullet 1 = 1$. Therefore it is not difficult to prove that $\bullet$ and $\vee$ distribute over each other. For $\chi \bullet (\lambda \vee \mu)$, suppose that $\lambda = 1 = \chi$. Then $\chi \bullet (\lambda \vee \mu) = 1 \bullet 1 = 1$. But $\chi \bullet \lambda = 1$, too, so that $(\chi \bullet \lambda) \vee (\chi \bullet \mu) = 1$. If $\chi \neq 1$, then $\chi \bullet (\lambda \vee \mu) = \gamma = \gamma \vee \gamma = (\chi \bullet \lambda) \vee (\chi \bullet \mu)$. For $\chi \vee (\lambda \bullet \mu)$, suppose $\chi = 1$ then $\chi \vee (\lambda \bullet \mu) = 1 = 1 \bullet 1 = (\chi \vee \lambda) \bullet (\chi \vee \mu)$. Next, suppose $\chi \neq 1$. If $\lambda = \mu = 1$, then $\chi \vee (\lambda \bullet \mu) = \chi \vee 1 = 1 = 1 \bullet 1 = (\chi \vee \lambda) \bullet (\chi \vee \mu)$. If $\lambda \neq 1$ or $\mu \neq 1$, then $\chi \vee \lambda \neq 1$ or $\chi \vee \mu \neq 1$. In both cases $(\chi \vee \lambda) \bullet (\chi \vee \mu) = \gamma = \chi \vee \gamma = \chi \vee (\lambda \bullet \mu)$.

Therefore, $\langle \Omega_{\mathcal{A}}(\wp(U)), \bullet, \vee\rangle$ is a distributive lattice, although (*stability*) fails and (*left*) either (indeed $\bullet \neq \cap$). On the contrary, (*right*) holds because **N3** holds in $\mathcal{A}(U)$.

Form this example and Example 15.14.7 one is not allowed to infer that right quasi-topological systems give rise to distributive lattices of saturated sets.

In fact, $\vee$ distributes on $\cap$ if for all $\alpha, \beta, \gamma \in \Omega_{\mathcal{A}}(U)$, $\alpha \vee (\beta \cap \gamma) = (\alpha \vee \beta) \cap (\alpha \vee \gamma)$, *viz.* if (i) $\mathcal{A}((\alpha \cup \beta) \cap (\alpha \cup \gamma)) = \mathcal{A}(\alpha \cup \beta) \cap \mathcal{A}(\alpha \cup \gamma)$. Since $\mathcal{A}$ is a closure operator, the first term is included in the second. Put $(\alpha \cup \beta) = X$ and $(\alpha \cup \gamma) = Y$. From (*right*) we can deduce (ii) $\mathcal{A}(X) \cap \mathcal{A}(Y) \subseteq \mathcal{A}(X \cdot Y)$. In fact, $\mathcal{A}(X) \cap \mathcal{A}(Y) \subseteq \mathcal{A}(X)$ and $\mathcal{A}(X) \cap \mathcal{A}(Y) \subseteq \mathcal{A}(Y)$ give $\mathcal{A}(X) \cap \mathcal{A}(Y) \blacktriangleleft X$ and $\mathcal{A}(X) \cap \mathcal{A}(Y) \blacktriangleleft Y$. Thus from (*right*) and *Proposition* 15.14.1.(2) we obtain (ii). But $X \cap Y \subseteq X \cdot Y$. Hence $\mathcal{A}(X \cap Y) \subseteq \mathcal{A}(X \cdot Y)$. About the reverse inclusion, hence about (iii) $\mathcal{A}(X \cap Y) = \mathcal{A}(X \cdot Y)$ which would give

the inclusion of the second term of (i) into the first term, (*right*) and idempotence of · are silent.

However, incidentally (iii) holds in the present example and in Example 15.14.7, (easy inspection).

Nonetheless, the above discussion actually gives at least one statement linking distributivity of the saturated structures and (*right*), but in the opposite direction:

Proposition 15.14.21. *If* $\Omega_{\mathcal{A}}(B)$ *is distributive, then in* $\langle B, \cdot, 1, \blacktriangleleft, \perp \rangle$ *(right) holds.*

Proof. If (*right*) does not hold, then we cannot prove $\mathcal{A}(X \cap Y) \subseteq \mathcal{A}(X \cdot Y)$ and, therefore, $\mathcal{A}(X \cap Y) = \mathcal{A}(X \cdot Y)$ either, which on the contrary must hold to obtain distributivity.   QED

15.14.10   A Logical Interpretation of Concrete Properties

Pre-topological formal systems provide models for limited resource logics. Referring to [Sambin, 1989], we recall that the multiplicative conjunction $\otimes$ is interpreted by ·, the additive conjunction & is interpreted by $\cap$, while $\blacktriangleleft$ gives the interpretation of the sequent relation $\vdash$ and formulas are evaluated over saturated subsets. One can easily verify that if (*left*) and (*right*) hold, then the behaviour of $l\otimes$ and l& coincide. Indeed, since in this case $\bullet$ and · coincide, from $\dfrac{A \blacktriangleleft C}{A \cdot B \blacktriangleleft C}$ we immediately obtain the validity of $\dfrac{\Gamma, a \vdash c}{\Gamma, a \otimes b \vdash c}$ which coincides with l&. As for (*right*), it makes the behaviour of $r\otimes$ and r& coincide: from $\dfrac{A \blacktriangleleft B \quad A \blacktriangleleft C}{A \blacktriangleleft B \cdot C}$ we immediately obtain the validity of $\dfrac{\Gamma \vdash a \quad \Gamma \vdash b}{\Gamma \vdash a \otimes b}$, which is the same as r&. It follows that in logics modeled by topological formal systems, additive and multiplicative conjunctions coincide.

But it is worth underlining that **N2** and **N3** already embed the logical power of (*left*) and, respectively, (*right*). Indeed, **N3** may be read in the following form $\dfrac{N \in \mathcal{N}_x \quad N' \in \mathcal{N}_x}{N \cap N' \in \mathcal{N}_x}$, which corresponds to the introduction of &, hence to r&. In turn, **N2** may have the following

reading: $\dfrac{N \cap N' \in \mathcal{N}_x}{N \in \mathcal{N}_x}$ and $\dfrac{N \cap N' \in \mathcal{N}_x}{N' \in \mathcal{N}_x}$, which correspond to the elimination of &, hence to $l\&$.

Finally, in view of the fact that in topological formal systems $\mathcal{A}(X) \cap \mathcal{A}(Y) = \mathcal{A}(X \cdot Y)$, we can notice that $\mathcal{A}$ is a meet-preserving closure operator from $\langle \wp(B), \cdot, \blacktriangleleft \rangle$ to $\langle \Omega_{\mathcal{A}}(B), \cap, \subseteq \rangle$. Then $\mathcal{A}$ is a Grothendieck closure operator (see Part II).

15.15   Frame – Modal Structures and Pre-Topological Spaces

15.15.1   Modal Algebras and Neighborhood Systems

In ([Došen, 1987]) Kosta Došen developed a duality theory between neighborhood frames and modal algebras, in the line of [Goldblatt, 1976] (in what follows we use the symbolism and the concepts introduced in this Part plus the additional symbols and concepts used in Došen's work).

We have seen that a modal algebra $\mathbf{A} = \langle A, \wedge, \vee, \sim, 0, 1, \square \rangle$ is a non-degenerate Boolean algebra equipped with a unary operator $\square$, without any specific property. A *neighborhood frame* is a structure $\mathbf{F} = \langle U, \mathcal{N}(U), D^F \rangle$, where D^F is a family of subsets of U closed under finite intersections, complementation, and under the core map G defined on the basis of a neighborhood system $\mathcal{N}(U)$. In turn, $\mathcal{N}(U)$ is such that for all $x \in U$, $\mathcal{N}_x \subseteq D^F$. We notice that there is a maximal and a minimal way for building a neighborhood frame, given a neighborhood system $\mathcal{N}(U)$. In the maximal we take $D^F = \wp(U)$, while in the minimal we take the inductive closure of $\bigcup \mathcal{N}(U) \cup \{G(X)\}_{X \subseteq U}$ under finite intersections and complementation.

Došen describes how to "spread" a neighborhood frame $F(\mathbf{A}) = \langle U^{\mathbf{A}}, \mathcal{N}^{\mathbf{A}}(U^{\mathbf{A}}), D^{F(\mathbf{A})} \rangle$ over a modal algebra $\mathbf{A}$:

$U^{\mathbf{A}} = \{X \subseteq A : X \text{ is a maximal filter in } \mathbf{A}\}$ (if $\mathbf{A}$ is finite, then $U^{\mathbf{A}} = \mathcal{J}(\mathbf{A}) = atom(\mathbf{A})$);

$D^{F(\mathbf{A})} = \{q(a) : a \in A\}$, where $q : A \longmapsto \wp(\wp(U^{\mathbf{A}})); q(a) = \{X \in U^{\mathbf{A}} : a \in X\}$ (if $\mathbf{A}$ is finite, then $q(a) = \{x \in atom(\mathbf{A}) : x \leq a\}$);

$\mathcal{N}^{\mathbf{A}} : U^{\mathbf{A}} \longmapsto \wp(\wp(U^{\mathbf{A}})); \mathcal{N}^{\mathbf{A}}_X = \{q(a) : \square(a) \in X\}$ (if $\mathbf{A}$ is finite, then $\mathcal{N}^{\mathbf{A}}_x = \{q(a) : x \leq \square(a)\}$).

Conversely, given a neighborhood frame $\mathbf{F} = \langle U, \mathcal{N}(U), D^F \rangle$ we can define a modal algebra $A(\mathbf{F}) = \langle A^{\mathbf{F}}, \cap, \cup, -, \emptyset, U, \square^{\mathbf{F}} \rangle$ such that $A^{\mathbf{F}} = D^F$ and $\square^{\mathbf{F}} = G$.

EXAMPLE

Consider a modal algebra $\mathbf{A} = \langle \mathbf{B}, \square \rangle$, where $\mathbf{B}$ is the Boolean algebra depicted below together with the table for $\square$:

x	0	a	b	c	d	e	f	1
$\square(x)$	b	0	0	c	b	0	0	d
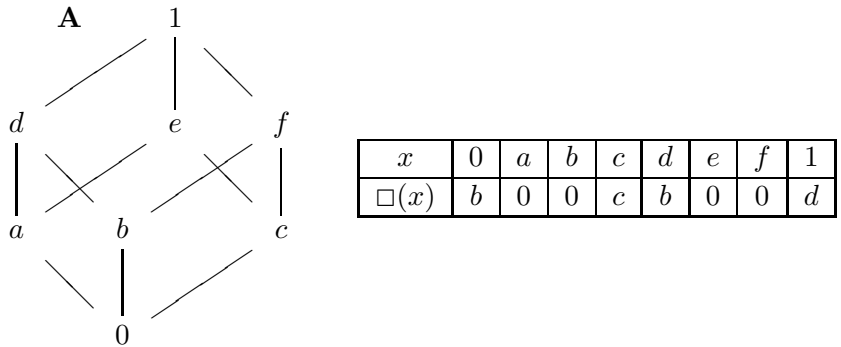
Then, the neighborhood frame $F(\mathbf{A})$ is built in the following manner:

$$U^{\mathbf{A}} = \{\uparrow a, \uparrow b, \uparrow c\};$$

Recalling that $\Uparrow \{X\} = \{\uparrow x\}_x \in X$, $q(x)$ is given by

x	0	a	b	c	d	e	f	1
$q(x)$	$\emptyset$	$\Uparrow \{a\}$	$\Uparrow \{b\}$	$\Uparrow \{c\}$	$\Uparrow \{a,b\}$	$\Uparrow \{a,c\}$	$\Uparrow \{b,c\}$	$\Uparrow U$

$D^{F(\mathbf{A})} = \wp(U^{\mathbf{A}})$ (notice that this holds in the present case, while it is not uniformly true in every case).

$\mathcal{N}^{\mathbf{A}}_{\uparrow a} = \{q(x) : \square(x) \in \uparrow a\} = \{q(x) : x \in \{1\}\} = \{q(1)\} = \{\Uparrow U\} = \{\{\uparrow a, \uparrow b, \uparrow c\}\}$; $\mathcal{N}^{\mathbf{A}}_{\uparrow b} = \{q(x) : \square(x) \in \uparrow b\} = \{q(x) : x \in \{0, d, 1\}\} = \{q(0), q(d), q(1)\}) = \{\emptyset, \Uparrow \{a, b\}, \Uparrow U\}$; $\mathcal{N}^{\mathbf{A}}_{\uparrow c} = \{\Uparrow \{c\}\}$.

Clearly, since in this case $\mathbf{A}$ is finite we can substitute $\Uparrow U$ for $\{a, b, c\}$ and obtain $\mathcal{N}^{\mathbf{A}}_a = \{U\}$, $\mathcal{N}^{\mathbf{A}}_b = \{\emptyset, \{a, b\}, U\}$, $\mathcal{N}^{\mathbf{A}}_c = \{\{c\}\}$.

Conversely, consider a neighborhood frame $\mathbf{F} = \langle U, \mathcal{N}(U), D^F \rangle$, where $U = \{a, b, c\}$ and $\mathcal{N}(U)$ is given by:

x	a	b	c
$\mathcal{N}_x$	$\{\{a, c\}, U\}$	$\{\{b\}, U\}$	$\{\emptyset, \{c\}, \{a, c\}\}$

and D^F is the inductive closure of $\bigcup \mathcal{N}(U)$ under G, $\cap$ and $-$. The map G is given by the following table:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U
$G(x)$	$\{c\}$	$\emptyset$	$\{b\}$	$\{c\}$	$\emptyset$	$\{a,c\}$	$\emptyset$	$\{a,b\}$

Therefore, D^F is given by the following steps: (i) $\bigcup \mathcal{N}(U) = \{\emptyset, \{b\}, \{c\}, \{a,c\}, U\}$; (ii) $\bigcup \mathcal{N}(U) \cup \{G(X)\}_{X \subseteq U} = \{\emptyset, \{b\}, \{c\}, \{a,b\}, \{a,c\}, U\}$; (iii) the closure of this set under $\cap$ and $-$ is $\wp(U)$ (notice that this is not a necessary consequence of the construction, as some counterexamples will show). Since $A^\mathbf{F} = D^F$, in $A(\mathbf{F})$ we have $A^\mathbf{F} = \wp(U)$. The unary operator $\square^\mathbf{F}$ equals G.

It is possible to prove that q is an isomorphism between $\mathbf{A}$ and $A(F(\mathbf{A}))$. Hence $\mathbf{A} \cong A(F(\mathbf{A}))$.

Can we prove $\mathbf{F} \cong F(A(\mathbf{F}))$?

First of all, let us define what an isomorphism between neighborhood frames must be. So, let $\mathbf{F}_1 = \langle U_1, \mathcal{N}^1(U_1), D^{F_1} \rangle$ and $\mathbf{F}_2 = \langle U_2, \mathcal{N}^2(U_2), D^{F_2} \rangle$ be two neighborhood frames and let $f : U_1 \longmapsto U_2$. We recall that given any $X \subseteq U_2$ we denote $\{x : f(x) \in X\}$ by $f^{\leftarrow}(X)$. Then, f is a *frame morphism* between $\mathbf{F}_1$ and $\mathbf{F}_2$ if the following conditions ate satisfied:

F1 $f^{\leftarrow}(X_2) \in D^{F_1}$, for all $X_2 \in D^{F_2}$.

F2 $f^{\leftarrow}(X_2) \in \mathcal{N}^1_{x_1}$ if and only if $X_2 \in \mathcal{N}^2_{f(x_1)}$.

A *frame isomorphism* is a frame morphism f such that f is 1-1, onto (i.e. surjective) and $f^{\leftarrow}$ is a frame morphism. Since $D^{F_1} = A^{\mathbf{F}_1}$ and $D^{F_2} = A^{\mathbf{F}_2}$, an other way to understand the matter is that $f : U_1 \longmapsto U_2$ is a frame morphism between $\mathbf{F}_1$ and $\mathbf{F}_2$ if and only if $f^{\leftarrow}$ is a homomorphism between $A(\mathbf{F}_2)$ and $A(\mathbf{F}_1)$.

A good candidate as a frame isomorphism between a frame $\mathbf{F} = \langle U, \mathcal{N}(U), D^F \rangle$ and $F(A(\mathbf{F}))$ is the following map p:

$$p : U \longmapsto \wp(\wp(U)); p(x) = \{X \in D^F : x \in X\}$$

Indeed, it easy to verify that for all $x \in U$, $p(x)$ is a proper maximal filter in $\langle D^F, \subseteq \rangle$. Since the carrier of $A(\mathbf{F})$, $A^\mathbf{F}$, is D^F, then $p(x)$ is a proper maximal filter in $A(\mathbf{F})$. But in the neighborhood frame $F(A(\mathbf{F}))$, the universe $U^{A(\mathbf{F})}$ is exactly the set of proper maximal filters of $A(\mathbf{F})$. Therefore $p : U \longmapsto U^{A(\mathbf{F})}$.

For instance, in the previous Example we have:

$$p(a) = \{X \in D^F : a \in X\} = \{\{a\}, \{a, b\}, \{a, c\}, U\} = \uparrow \{a\}.^{[10]}$$

Is p a frame isomorphism, always? Actually, it is possible to prove that p is a frame isomorphism if the following two conditions hold:

P1 $p(x) = p(y) \Rightarrow x = y$ (i.e. p is 1-1).

P2 $\forall X \in U^{A(\mathbf{F})}, \exists x \in U(X = p(x))$ (thus p is onto).

If this happens, then $\mathbf{F}$ is called a *descriptive neighborhood frame* (cf. Frame 15.15.2).

EXAMPLE
We give an example of a non-descriptive frame.

Consider $U = \{a, b, c\}$ and the following neighborhood system $\mathcal{N}(U)$:

x	a	b	c
$\mathcal{N}_x$	$\{\emptyset, \{a, b\}, U\}$	$\{\emptyset, \{a, b\}, U\}$	$\{\{c\}\}$

Then, the core map G is given by:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U
$G(x)$	$\{a, b\}$	$\emptyset$	$\emptyset$	$\{c\}$	$\{a, b\}$	$\emptyset$	$\emptyset$	$\{a, b\}$

Therefore, if we define D^F as the inductive closure of $\bigcup \mathcal{N}(U)$ under G, $\cap$ and $-$, we have $D^F = \{\emptyset, \{c\}, \{a, b\}, U\}$.

It is straightforward to check that $p(a) = p(b) = \{\{a, b\}, U\} = \uparrow \{a, b\}$, although $a \neq b$.

One could guess that this happens because we have two points with the same neighborhood system (viz. $\mathcal{N}_a = \mathcal{N}_b$). Thus, we give immediately a counterexample of this conjecture. Consider the following neighborhood system $\mathcal{N}(U)$:

x	a	b	c
$\mathcal{N}_x$	$\{\{c\}\}$	$\{\emptyset, \{a, b\}, U\}$	$\{\{c\}\}$

In this case $\mathcal{N}_a = \mathcal{N}_c$. However, G is given by:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U
$G(x)$	$\{b\}$	$\emptyset$	$\emptyset$	$\{a, c\}$	$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$

---

[10]Mind: do not confuse $\uparrow \{a\}$ and $\Uparrow \{a\} = \{\uparrow a\}$.

It follows that the inductive closure of $\bigcup \mathcal{N}(U)$ under G, $\cap$ and $-$ is $\wp(U)$, so that $A^{\mathbf{F}} = D^F = \wp(U)$ and for any pair of elements $x, y \in U$, $x \neq y \Rightarrow p(x) = \uparrow \{x\} \neq \uparrow \{y\} = p(y)$. Obviously, the map p is onto, too.

Moreover, Došen proves that for any modal algebra $\mathbf{A}$, $F(\mathbf{A})$ is a descriptive frame, so that we always have $F(\mathbf{A}) \cong F(A(F(\mathbf{A})))$. And we can also notice that if $\mathbf{F}$ is finite and non-descriptive, then we can make $\mathbf{F}$ into a descriptive frame with the same neighborhood system (hence without altering the core map G), by extending D^F to $\wp(U)$ (this move is not uniformly valid if D^F is infinite – think of a Boolean algebra without isolated points.[11])

We recall that a modal algebra $\mathbf{A}$ is normal if and only if for all $a, b \in A$, $\Box(a) \wedge \Box(b) = \Box(a \wedge b)$ and $\Box(1) = 1$. In view of the analysis developed in this Part it is not a surprise at all, that K. Došen proves that $\mathbf{A}(A(\mathbf{F}))$ is normal if and only if in $F(\mathbf{A}(A(\mathbf{F})))$ any $\mathcal{N}_x$ is a filter (not necessarily proper; indeed normal modal algebras are not required to feature $\Box(0) = 0$).

In such a case, $F(\mathbf{A}(A(\mathbf{F})))$ is called a *filter frame*.

A neighborhood frame $\mathbf{F} = \langle U, \mathcal{N}(U), D^F \rangle$ is called a *hyperfilter frame*, if

$$\forall x \in U, \forall X \in D^F, \quad \bigcap \mathcal{N}_x \subseteq X \Rightarrow X \in \mathcal{N}_x \tag{15.15.8}$$

Notice that if $\bigcap \mathcal{N}_x \notin D^F$, then it does not belong to $\mathcal{N}_x$ (actually, $\bigcap \mathcal{N}_x \in D^F$ if D^F is closed under arbitrary intersections); thus, hyperfilter frames do not necessarily correspond to our neighborhood systems of type $\mathcal{N}_4$ (which, on the contrary, are variants of the so called "augmented frames" cited in Frame 15.13.3). Nonetheless, suppose $\mathbf{F}$ is a *hyperfilter frame*. Then, as with $\mathcal{N}_4$-type neighborhood systems, we can T-associate with $\mathbf{F}$ a relation R such that

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[11]We have seen in the Introduction, Section 5.1, that an infinite distributive lattice $\mathbf{D}$ may be not isomorphic to its soberification $S(\mathbf{D})$ because we can have less abstract points than open subsets. Moreover given a topological space τ its soberification may be not homeomorphic to τ, because we can have less abstract point than concrete points.

To some extent, descriptive frames are those neighborhood frames in which both situations do not happen. Otherwise stated, if we think of D^F as a set of properties that can be enjoyed by the elements of U (and that are really enjoyed by a point x when they belong to $\mathcal{N}_x$), and of an ultrafilter in D^F as the approximation of a point by means of its properties, then a neighborhood frame is descriptive if for each approximation there is one and only one concrete point from U.

H1 For any $X \in D^F$ and $x \in U$, $x \in G(X)$ if and only if $R(x) \subseteq X$.

H2 For any $X \in D^F$ and $x, y \in U$, $(x \in G(X) \Rightarrow y \in X) \Rightarrow \langle x, y \rangle \in R$.

H3 For any $x \in U$, $\mathcal{N}_x = \{X \in D^F : R(x) \subseteq X\}$.

H2 makes it possible to recover R from G. This is not always possible in frames featuring just H1 (called *general relational frames* in Došen's work). General relational frames in which H2 holds, are called *reducible frames*.

Clearly, if in a reducible frame **F** we define $\mathcal{N}_x$ by means of H3, then (15.15.8) holds and R is the relation T-associated with the neighborhood system $\mathcal{N}(U)$ of **F**.

We have also to mention that Došen proves that every descriptive filter frame is a hyperfilter frame.

15.15.2 Modal Systems, Pre-Monadic Boolean Algebras and Kripke Frames

The construction of Section 12.1 may be summed up as follows. Given a pre-monadic Boolean algebra of sets $\mathbf{A} = \langle \mathbf{B}(U), L_R \rangle$ we found a Kripke frame $K(\mathbf{A}) = \langle U, R \rangle$ such that for any formula α and $x \in U$, $x \Vdash \alpha$ iff $x \in X$ for any $X \in \mathbf{B}(U)$ such that $X \vDash \alpha$. Or, rather redundantly, the pre-monadic Boolean algebra $A(K(\mathbf{A}))$, defined over $K(\mathbf{A})$, coincides, of course, with **A** (coherently with Došen's results that we have reported above). In *Definition* 11.5.1 we answered to a more abstract question: given an abstract pre-monadic Boolean algebra $\mathbf{A} = \langle \mathbf{A}, L \rangle$ such that $\mathbf{L}(\mathbf{A})$ is distributive (finite) lattices, find a Kripke frame $\mathbf{F} = \langle U, R \rangle$ such that $A(\mathbf{F}) = \langle \mathbf{B}(U), L_R \rangle$ is isomorphic to **A**. The required Kripke frame is implicit in the representation procedure. Namely it is $\langle \mathcal{J}(\mathbf{A}), \preceq \rangle$, where $\mathcal{J}(\mathbf{A})$ is the set of co-prime elements (the atoms, in this case) of **A** and $\preceq$ is the specialization preorder induced by $\mathbf{L}_{\mathbf{L}(\mathbf{A})}$ on $\mathcal{J}(\mathbf{A})$ (where $\mathbf{L}_{\mathbf{L}(\mathbf{A})}$ is the isomorphic image of $\mathbf{L}(\mathbf{A})$ in $\mathbf{F}(\mathbf{J}(\mathbf{A}))$, the dual of **A**). Without effort, it is possible to generalise this result to arbitrary pre-monadic Boolean algebras: U must be the collection of ultrafilters of **A** (instead of the collection of the co-prime elements, as in the finite case) and one must set $\langle S, S' \rangle \in R$ if and only if for all $a \in \mathbf{A}$, $L(a) \in S$ implies $a \in S'$ (in the finite case this means $\langle x, y \rangle \in R$ if and only if for all $a \in \mathbf{A}$, $x \leq L(a)$ implies $y \leq a$) (which is the algebraic companion of the usual clause in Kripke

models). The resulting Kripke frame $F(\mathbf{A}) = \langle U, R \rangle$ is called the dual frame of $\mathbf{A}$. The pre-monadic Boolean algebra $A(F(\mathbf{A}))$ defined over $F(\mathbf{A})$, is isomorphic to $\mathbf{A}$.

On the contrary, the converse problem is not that easy, as we have seen in the above Frame. It is formulated as follows: given a Kripke frame $\mathbf{F} = \langle U, R \rangle$ find a pre-monadic Boolean algebra $A(\mathbf{F})$ such that its dual Kripke frame $F(A(\mathbf{F}))$ is isomorphic to $\mathbf{F}$. It has been shown that a frame $\mathbf{F}$ is isomorphic to its bi-dual $F(A(\mathbf{F}))$ if and only if $\mathbf{F}$ is *descriptive*, that is, if and only if $\mathbf{F}$ is isomorphic to $F(A(\mathbf{F}'))$ for some frame $\mathbf{F}'$. However, this restriction is not satisfactory, because in the infinite case the move from a non descriptive frame to its equivalent descriptive frame does not preserve equivalence with respect to consequence. This was noted in [Sambin & Vaccaro, 1988], were the notion of a "*refined frame*" is shown to be the best compromise to find a type of frames solving the isomorphism $\mathbf{F} \cong F(A(\mathbf{F}))$. Refined frames are defined over Hausdorff spaces and require a particular topological closure property to hold for the accessibility relation. The interested reader is encouraged to go through the quoted paper (having in mind that pre-monadic Boolean algebras are named "modal algebras", there).

15.16   Frame – Duality of Operations and Algebraic Structures

Let us consider a Rough Set System $\mathbf{RS}(U) = \langle RS(U), \wedge, \vee, \neg, \sim, \neg, \longrightarrow, \supset, 0, 1 \rangle$ induced by an Approximation Space $\mathbf{AS}(U)$.

We know that $\mathbf{N}(\mathbf{AS}(U)) = \langle RS(U), \wedge, \vee, \sim, \neg, \longrightarrow, 0, 1 \rangle$ is a semi-simple Nelson algebra and that the operations $\supset$ and $\neg$ are definable in this fragment of $\mathbf{RS}(U)$. We can easily verify that given $\mathbf{N}(\mathbf{AS}(U))$, if we set, for any $a, b \in RS(U)$, (i) $L(a) = \neg\neg a = \sim \neg a$ and (ii) $a \Rightarrow b = L(a \supset b)$ (where "$\Rightarrow$" is the extensional implication defined in Section 14.3), we can make $\mathbf{RS}(U)$ into a pre-Rough algebra. Vice-versa, given a pre-Rough algebra $\langle A, \leq, \wedge, \vee, \sim, L, \Rightarrow, 0, 1 \rangle$, if we set, for any $a, b \in A$, (iii) $\neg a = \sim L(a)$ and (iv) $a \longrightarrow b = \neg a \vee b = \sim L(a) \vee b$, we obtain a semi-simple Nelson algebra (see Part II).

These transformations are just particular cases of the mathematical mechanism explained from a purely logical point of view in Frame 10.21 of Part II.

Indeed, the extensional implication $\Rightarrow$ defined in pre-Rough algebras is an operation definable by means of the following duality:

$$\lnot_d(\lnot_d(a \to b)) \land \lnot_d(\lnot_d(a \to b)) =_L \longrightarrow_L \land_M \longrightarrow_M.$$

At this point a note on the extensional implications in Rough algebras is in order. We say that a dyadic function $f(x,y)$ in a Rough algebra (Nelson algebra, and the like) is extensional when the following conditions are fulfilled: $f(x,y) = 1$ & $f(y,x) = 1 \Rightarrow x = y$. Therefore, $f(x,y) = 1$ if and only if $x \leq y$. We immediately observe that there can be more than one such function in a Rough algebra. Indeed in the least non-trivial Rough algebra with carrier $\{1, \delta, 0\}$ (see Frame 14.2.1), f is extensional whenever in the corresponding truth-table the lower left triangle is set to 1, while the remaining elements are 0 or δ. Following this intuition, there are four possible extensional implication operations that are consistent with the material implication (the consistency property reduces here to the requirement that $f(1,0) = 0$ – more details about this topic are in Frame 10.20 of Part II).

The first is the implication $\Rightarrow$. The second is the pseudo-complementation $\supset$, while the third is the contrapositional implication $\rightarrowtail$ that we have defined as $a \rightarrowtail b = (a \longrightarrow b) \land (\sim b \longrightarrow \sim a)$. The last one will be temporarily denoted by $\rightsquigarrow$ (and we leave its interpretation as a simple open problem):

$\Rightarrow$	1	δ	0
1	1	0	0
δ	1	1	0
0	1	1	1

$\supset$	1	δ	0
1	1	δ	0
δ	1	1	0
0	1	1	1

$\rightarrowtail$	1	δ	0
1	1	δ	0
δ	1	1	δ
0	1	1	1

$\rightsquigarrow$	1	δ	0
1	1	0	0
δ	1	1	δ
0	1	1	1

15.17 Frame – Computing Dependency Relations in a Fragment of Intuitionistic Logic

In Information Systems without information gaps, functional dependencies may be characterised by means of the intuitionistic implication.

It is worthwhile noticing that given an A-system $\mathbf{A} = \langle U, At, Val \rangle$, in order to algebraically grasp a dependency relation between sets of attributes we have to equip the Boolean algebra $\mathbf{B}(At) = \langle \wp(At), \cap, \cup, -, At \rangle$ with a binary relation R, meaning that if $\langle A, B \rangle \in R$ holds for $A, B \subseteq At$, then the indiscernibility relation E_A is finer than the indiscernibility relation E_B. Therefore the resulting structure $\langle \mathbf{B}(At), R(\mathbf{B}(At)) \rangle$, where $R(\mathbf{B}(At)) = \{R(X) : X \subseteq At\}$, is a modal system because R is a preorder. It follows that a logic for dependency relations must embed both a Boolean part, for $\mathbf{B}(At)$, and a non-classical part, for $R(\mathbf{B}(At))$.

This intuition was used to develop algebraic systems for dependence relations in [Pagliani, 1993a], while it was used to formalise a syntactic system in [Rauszer, 1984]. In her work, Cecylia Rauszer uses two systems, the first with Gentzen rules for Classical propositional logic, with operations denoted as $\cup, \cap$ and $-$, call it $G(\mathbf{B})$, and the second with the Gentzen rules for the $\{\wedge, \Longrightarrow\}$ fragment of Intuitionistic logic, call it $G(R)$.

The language $\mathcal{L}_{FD}$ consists of two levels: $Terms = \mathbf{B}(At)$ and $Atomic$ $Formulas = \{[X] : X \subseteq \wp(At)\}$, where, intuitively, $[X]$ stands for E_X. The set of all formulas of $\mathcal{L}_{FD}$ is the inductive closure of $Atomic$ $Formulas$ with respect to the operations $\wedge$ and $\Longrightarrow$. In what follows we denote by $\vdash$ the sequent relation for $G(\mathbf{B})$ and by $\Vdash$ the sequent relation for $G(R)$. $G(\mathbf{B})$ accounts for sequents of terms, while $G(R)$ accounts for sequents of formulas. The empty set of attributes is denoted by $\bot$. The only two axiom schemata are:

$$A \vdash A \ and \ \bot \vdash A$$

Notably, the two systems are linked by means of the following specific inference rule:

$$\frac{A \vdash B, \Delta}{[\Delta], [B] \Vdash [A]} \tag{15.17.9}$$

Rule (15.17.9) is justified by the fact that if $A \subseteq B$ then $[B] \subseteq [A]$. The resulting system is called FD-logic.

Using this apparatus it is possible two prove, for instance that $[A] \wedge [B] \Longrightarrow [A \cup B]$:

$$(r-weakening) \; \dfrac{A \vdash A \qquad B \vdash B}{\dfrac{\dfrac{\dfrac{A \vdash A, B \quad B \vdash A, B}{A \cup B \vdash A, B}}{\dfrac{[A], [B] \Vdash [A \cup B]}{\dfrac{[A] \wedge [B] \Vdash [A \cup B]}{\Vdash [A] \wedge [B] \Longrightarrow [A \cup B]}}}}} \; \begin{matrix} (r-weakening+exchange) \\ \\ (\cup \vdash) \\ (15.17.9) \\ (\wedge \Vdash) \\ (\Vdash \Longrightarrow) \end{matrix}$$

The semantic of FD-logic is obtained by considering the equivalence classes of the subsets of At modulo the identity of their indiscernibility relation, namely $A \approx B$ iff $[A] = [B]$. Let us denote such equivalence classes by $A_\approx$ and denote the collection of all these equivalence classes by $\mathfrak{A}$. Unfortunately $\mathfrak{U}$ is not closed under the relative psuedo-complementation $\Longrightarrow$.

This fact was amended in [Pagliani, 1993a] by considering $\mathbf{P} = \langle \mathfrak{A}, \precsim \rangle$ as a Kripke frame, where $\precsim$ is given by $[A]_\approx \precsim [B]_\approx$ iff $[A] \supseteq [B]$. In this algebraic framework it is possible to prove that in an Attribute System a set A of attributes is absolutely superfluous if $\neg\neg([[A]]) = 1$, where $[[A]]$ is the transformation of A into an element of the Heyting algebra dual to $\mathbf{P}$.[12]

In Rauszer's approach, we have to take the subset $\mathfrak{A}^\Rightarrow$ closed under the operation $\Longrightarrow$. Then $\langle \mathfrak{A}^\Rightarrow, \Longrightarrow, \wedge, 1 \rangle$, where $1 = [\emptyset]_\approx$ and $\wedge$ is the set-theoretical meet, is called an FD-*algebra of information systems.*

It is straightforward to prove that for any $[A]_\approx, [B]_\approx \in \mathfrak{A}^\Rightarrow$, $[A]_\approx \Longrightarrow [B]_\approx = 1$ iff $[A] \subseteq [B]$.

But this means that the subset B depends on A.

Thanks to a completeness theorem between FD-algebras and FD-logic, we have that in FD-logic we can derive $\Vdash \alpha \Longrightarrow \beta$, for α and β formulas of $\mathcal{L}_{FD}$ if and only if the subset of attributes represented by β depends on that represented by α. For instance the above deduction (together with the provable reverse implication) tells us that the indiscernibility relation induced by the union of two subsets of attributes equals the indiscernibility relation given by the intersection of the indiscernibility relation induced separately by the two sets.

In [Rauszer, 1984] it is also proved that classically valid formulas such as Peirce's law $((([A] \Longrightarrow [B]) \Longrightarrow [A]) \Longrightarrow [A]$ are not derivable, so that FD-logic is not a formulation of Classical propositional logic.

---

[12]Let $\mathbf{I} = \langle U, At, Val, i \rangle$ be an Attribute System. Let $Ind(At)$ be the family of equivalence classes induced by At. If $S \subsetneq At$ is such that $Ind(S) = Ind(At)$ then S is called a *reduct* of $\mathbf{I}$. Let $RD(\mathbf{I})$ be the union of all reducts of $\mathbf{I}$. If $A \subseteq At \cap -RD(\mathbf{I})$, then A is said to be *absolutely superfluous*. The same characterization of absolutely superfluous subsets of attributes as dense elements, was later proved in [Düntsch & Gediga, 1997].

It is clear that the algebraic approach has an almost pure theoretical meaning, since in practice, in order to compute the model all dependencies must be calculated in advance. On the contrary, the syntactic approach may account for computational purposes.

15.18 Frame – Approximation, Formal Concepts, Modalities and Relation Algebras

Approximation Systems and modal systems in a broader sense, have been studied also from the point of view of *Relation Algebras.* In this setting, any element of the language is interpreted as a relation or an operation on relations.

This approach makes it possible both to reach new advances and to implement functions to compute the operations described by the theory.

The calculus of relations was created by Augustus De Morgan, Charles Sanders Peirce and Ernst Schröder. The field of Relation Algebras was pioneered by Alfred Tarski, Louise H. Chin, Leon Henkin, J. Donald Monk and Bjarni Jónsson.

A *relation algebra* is a structure $\langle A, +, 0, \cdot, 1, -, ; , 1', ^\smile \rangle$ such that $\langle A, +, 0, \cdot, - \rangle$ is a Boolean algebra and $\langle A, ; , 1', ^\smile \rangle$ is an involuted monoid, (that is, (i) the operation; is associative, (ii) $(x; y)^\smile = y^\smile; x^\smile$, (iii) $x^{\smile\smile} = x$, (iv) $x; 1' = x = 1; x$, any $x, y \in A$), (v) the operations ; and $^\smile$ distribute over $+$ (that is, $(x + y); z = x; z + y; z$, (v) $(x + y)^\smile = x^\smile + y^\smile$, any $x, y, z \in A$).

Moreover, the following so-called Schröder rule holds for any x, $y \in A$:

$$x^\smile; -(x; y) \leq -y \qquad (15.18.10)$$

where $x \leq y$ iff $x + y = y$.

More recently, relation algebras were studied intensively by Hajnal Andréka, Chris Brink, Roger D. Maddux, Maarten Marx, István Németi, Maarten de Rijke, Ildikó Sain, Renate A. Schmidt, and Yde Venema, (and, of course, other researchers we have no room to mention).

As to connections with logics, relation algebras were explored particularly by Ewa Orłowska (see [Orłowska, 1988b, 1990b] – also, cf.

[Orłowska & Szalas, 2001]). E. Orłowska was able to extend relation algebraic techniques to a number of logical systems, such as any kind of regular modal logics ([Orłowska, 1990b, 1991a]), Non-classical Logics ([Orłowska, 1994]), Relevant Logics, Dynamic Logics and other sorts of logics of programs (see later on for references).

Robert E. Kent ([Kent, 1993, 1996]), Ivo Düntsch ([Düntsch, 1997]) and Piero Pagliani ([Pagliani, 1996, 1998b,c]), applied the relation algebra approach to Rough Set Theory and Formal Concept Analysis.

In this Frame we want to introduce the relational algebraic interpretation of modal algebras and to exploit it in order (a) to formally compare Formal Concept Analysis and Rough Set Theory and make them interact, and (b) to find easily implementable functions to compute rough functional dependencies in Information Systems.

15.18.1 Relation Algebras

As usual, given a set U, we shall represent a binary relation $R \subseteq U \times U$ as a square matrix $\mathbf{R}$ of type $U \times U$ such that the element at row i and column j is 1 if $\langle i, j \rangle \in R$; it is 0 otherwise. Here i and j stay for the $i - th$ and, respectively $j - th$ element of U given an arbitrary enumeration of its elements. It is worth noticing that the enumeration for the rows and that for the columns may be different. We assume for convenience that the enumeration coincides:

If we apply the operations defined in Mathematical toolkit 16.5, on the family of all binary relations over the same set, then we can define the following instance of a relation algebra.

Definition 15.18.1. *By a* full algebra of binary relations *over a set W, we mean an algebra*

$$fullREL(W) = (\wp(W \times W), \cup, \cap, -, \mathbf{1}, \otimes, {}^{\smile}, \mathbf{1}')$$

where $(\wp(W \times W), \cup, \cap, -, \mathbf{1})$ is a Boolean algebra of sets, $\otimes$ is the relational composition, ${}^{\smile}$ is the inverse and $\mathbf{1}'$ is the identity relation.

An explanation of these operations is in order.

Let R and S be two binary relations over W. Hence $R, S \subseteq W \times W$.

1. $R \cup S = \{\langle x, y \rangle \in W \times W : \langle x, y \rangle \in R \text{ or } \langle x, y \rangle \in S\}$.
   It follows that the element at row i, column j in $\mathbf{R} \cup \mathbf{S}$ is 1 if the same element is 1 in $\mathbf{R}$ or it is 1 in $\mathbf{S}$.

2. $R \cap S = \{\langle x, y \rangle \in W \times W : \langle x, y \rangle \in R \text{ and } \langle x, y \rangle \in S\}$.
   It follows that the element at row i, column j in $\mathbf{R} \cap \mathbf{S}$ is 1 if the same element is 1 in both $\mathbf{R}$ and $\mathbf{S}$.

3. $-R = \{\langle x, y \rangle \in W \times W : \langle x, y \rangle \notin R\}$.
   Hence the element at row i, column j in $-\mathbf{R}$ is 1 if and only if the same element is 0 in $\mathbf{R}$.

4. Since $\mathbf{1}$ is the top element in the Boolean algebra of the set $\wp(W \times W)$, then $\mathbf{1}$ is obviously $W \times W$.

5. $R^{\smile} = \{\langle y, x \rangle \in W \times W : \langle x, y \rangle \in R\}$.
   Thus in order to obtain $\mathbf{R}^{\smile}$ from $\mathbf{R}$, one has to transpose the matrix.

6. $R \otimes S = \{\langle x, y \rangle \in W \times W : \exists z \in W(\langle x, z \rangle \in R \text{ and } \langle z, y \rangle \in S)\}$.
   Composition is simply the Boolean pointwise multiplication of matrices. Thus to obtain $\mathbf{R} \otimes \mathbf{S}$ we compare pointwise row i with column j; if the pointwise Boolean multiplication gives 1 for at least one point, then element at row i and column j of $\mathbf{R} \otimes \mathbf{S}$ is 1. It is 0 otherwise.

7. Since $\mathbf{1}'$ is the identity relation, its representation is the identity matrix, that is the matrix such that the element at row i and column j is 1 if and only if $i = j$.

EXAMPLE 3
Let $W = \{a, b, c, d\}$, and

$\mathbf{R}$	a	b	c	d
a	1	1	1	1
b	0	1	1	0
c	0	0	1	0
d	0	0	0	1

$\mathbf{S}$	a	b	c	d
a	1	1	1	0
b	0	0	0	1
c	0	1	0	0
d	1	0	0	1

$\mathbf{P}$	a	b	c	d
a	1	1	1	1
b	0	0	0	0
c	0	0	0	0
d	1	1	1	1

We have:

1	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
d	1	1	1	1

1'	a	b	c	d
a	1	0	0	0
b	0	1	0	0
c	0	0	1	0
d	0	0	0	1

$\mathbf{R}^{\smile}$	a	b	c	d
a	1	0	0	0
b	1	1	0	0
c	1	1	1	0
d	1	0	0	1

$\mathbf{R} \otimes \mathbf{S}$	a	b	c	d
a	1	1	1	1
b	0	1	0	1
c	0	1	0	0
d	1	0	0	1

To compute, for instance, the element at row 3 column 2 of $\mathbf{R} \otimes \mathbf{S}$, first we take row 3 of $\mathbf{R}$, 0010, and column 2 of $\mathbf{S}$, 1010. Then we apply component-wise the logical $\wedge$ to these two Boolean vectors obtaining 0010. Finally we apply to these elements the logical summation. And we obtain 1.

Following these steps, the reader can easily verify that $\mathbf{P} \otimes \mathbf{1} = \mathbf{P}$. A relation enjoying this property is called a *right ideal element.*

15.18.2 Modalizing Relations by Means of Relations

TERMINOLOGY AND NOTATION. From now on if $R \subseteq W \times W'$ for two sets W and W', then we say that $W \times W'$ is the *type* of the relation R and we shall denote this fact with $R : W \times W'$. Moreover, we shall say that W is the "internal dimension" of R and W' the external dimension.

Now, we have to find a uniform relational representation of the notions we have to work with. Therefore, we need:

1. A way for representing extensions of formulae in a universe of possible worlds.

2. A way for representing accessibility relations.

3. A way for representing worlds.

4. A way for representing the forcing relation between possible worlds and sets.

The Relational Approach aims at representing all of these components in a uniform way implementable on computers.

In what follows we shall refer to some general facts of relation algebras. Because of the introductory character of this Frame, we omit the proofs. Instead we shall focus on the intuitive and conceptual core of the approach.

1. REPRESENTING RELATIONS

Obviously, the relational representation of a relation R is the relation R itself.

EXAMPLE 1

Let us consider the *A-system* depicted in the table *Hypothermic Post Anesthesia Patients*, of Example 5.2.1 of Introduction.

As we know any subset of attributes induces an indiscernibility relation $E_A \subseteq U \times U$, where $U = \{a, b, c, d, e, f, g, h, i\}$. For instance, the set $\{Temperature, Hemoglobin\}$ induces the equivalence relation TH: $\{a, b\}, \{c, d\}, \{e, f\}, \{g\}, \{h\}, \{i\}$.

TH	a	b	c	d	e	f	g	h	i
a	1	1	0	0	0	0	0	0	0
b	1	1	0	0	0	0	0	0	0
c	0	0	1	1	0	0	0	0	0
d	0	0	1	1	0	0	0	0	0
e	0	0	0	0	1	1	0	0	0
f	0	0	0	0	1	1	0	0	0
g	0	0	0	0	0	0	1	0	0
h	0	0	0	0	0	0	0	1	0
i	0	0	0	0	0	0	0	0	1

2. REPRESENTING SETS

We have to represent objects as binary relations. So we have to find a way for transforming sets into binary relations. A suitable notion to this end is that of a *right ideal element* obtained from sets *via right cylindrification*:

Definition 15.18.2. *Let W be a set and let $X \subseteq W$; the* right cylindrification *of X (with respect to the unity $\mathbf{1} = W \times W$), denoted by X^c, is $X \times W$. The relation X^c is called a* right cylinder.

Therefore, $X^c = \{\langle x, y \rangle : x \in X \ \& \ y \in W\}$. It follows that X^c is a relation of type $W \times W$.

EXAMPLE 2

Given the *A-system* of Example 1, and the property $\alpha =$ "Comfort $=$ medium", we have $[\![\alpha]\!] = \{d, e, f, g\}$. The subset $[\![\alpha]\!]$ is represented as the right cylinder $[\![\alpha]\!]^c$:

$[\![\alpha]\!]^c$	a	b	c	d	e	f	g	h	i
a	0	0	0	0	0	0	0	0	0
b	0	0	0	0	0	0	0	0	0
c	0	0	0	0	0	0	0	0	0
d	1	1	1	1	1	1	1	1	1
e	1	1	1	1	1	1	1	1	1
f	1	1	1	1	1	1	1	1	1
g	1	1	1	1	1	1	1	1	1
h	0	0	0	0	0	0	0	0	0
i	0	0	0	0	0	0	0	0	0

Right cylinders represent sets: from a conceptual point of view a set is a property, hence a unary relation, hence a right cylinder in $W \times W$ is a unary relation embedded into the two-dimensional universe $\mathbf{1} = W \times W$. More technically, any right cylinder X^c is (right) *persistent* (like any good object) in this universe because $X^c \otimes \mathbf{1} = X^c$. Otherwise stated, any right cylinder is a right ideal element.

By $RI(W)$ we shall denote the set of right ideal elements of type $W \times W$. Moreover any right cylinder $X^c : W \times W$ is biunivocally linked to a subset of W by means of the right Peirce product $X^{c\smile}(W)$. In fact, for any right cylinder X^c we have:

$$(X^{C\smile}(W))^c = X^c \text{ and } X^{c\smile}(W) = X. \qquad (15.18.11)$$

3. REPRESENTING ELEMENTS

Any element of a universe W will be denoted by a singleton set, hence as a right ideal element with a single row whose elements are set to 1.

4. REPRESENTING FORCING

And now the core of our representation.

As to Boolean formulas, the interpretation is immediate: since $[\![\alpha]\!]^c$ and $[\![\beta]\!]^c$ are the right cylinders representing the set of possible worlds forcing α, respectively, β, we have $[\![\alpha]\!]^c \cap [\![\beta]\!]^c = [\![\alpha \wedge \beta]\!]^c$, $[\![\alpha]\!]^c \cup [\![\beta]\!]^c = [\![\alpha \vee \beta]\!]^c$ and $-[\![\alpha]\!]^c = [\![\sim\alpha]\!]^c$ (it is trivial to verify that these operations turn right cylinders into right cylinders).

As to modalised formulae, let us look again at the forcing clauses listed in Frame 4.13.1 of Part I. Given a Kripke frame $\langle W, R \rangle$, we have:

$$w \models [R](\alpha) \ \textit{iff} \ \forall w' \in W((\langle w, w' \rangle \in R) \Rightarrow (w' \models \alpha)). \qquad (15.18.12)$$

$$w \models \langle R \rangle(\alpha) \ \textit{iff} \ \exists w' \in W((\langle w, w' \rangle \in R) \ \& \ (w' \models \alpha)). \qquad (15.18.13)$$

Hence the validity domains of the modalised formulae are:

$$\begin{aligned}
[\]\!] &= \{w \in W : w \models [R](\alpha)\} \\
&= \{w \in W : \forall w' \in W((\langle w, w' \rangle \in R) \Rightarrow (w' \in [\![\alpha]\!]))\}. \\
& \hspace{8cm} (15.18.14)
\end{aligned}$$

$$\begin{aligned}
[\![\langle R \rangle(\alpha)]\!] &= \{w \in W : w \models \langle R \rangle(\alpha)\} \\
&= \{w \in U : \exists w' \in U((\langle w, w' \rangle \in R) \ \& \ (w' \in [\![\alpha]\!]))\}. \\
& \hspace{8cm} (15.18.15)
\end{aligned}$$

What we have now to do is to translate 15.18.12 and 15.18.13 into pure relational terms. The steps are straightforward.

1. Instead of the element w we will use the right ideal element $\{w\}^c$. Thus since any element of W is now in relation $\{w\}^c$ with w, we can select a generic representative $u \in W$ and set:

$$\langle w, u \rangle \models [R](\alpha) \ \textit{iff} \ \forall w' \in W((\langle w, w' \rangle \in R) \Rightarrow (\langle w', u \rangle \models \alpha)).$$
$$(15.18.16)$$

$$\langle w, u \rangle \models \langle R \rangle(\alpha) \ \textit{iff} \ \exists w' \in W((\langle w, w' \rangle \in R) \ \& \ (\langle w', u \rangle \models \alpha)).$$
$$(15.18.17)$$

Thus:

$$\begin{aligned}
[\]\!]^c \ &= \ \{\langle w, u \rangle \in W \times W : \forall w' \in W((\langle w, w' \rangle \in R) \\
&\Rightarrow \ (\langle w', u \rangle \in [\![\alpha]\!]^c))\}. \hspace{3cm} (15.18.18)
\end{aligned}$$

$$\begin{aligned}
[\![\langle R \rangle(\alpha)]\!]^c \ &= \ \{\langle w, u \rangle \in W \times W : \exists w' \in W((\langle w, w' \rangle \in R) \\
&\& \ (\langle w', u \rangle \in [\![\alpha]\!]^c))\}. \hspace{3cm} (15.18.19)
\end{aligned}$$

So far, we have been given a good description of logical objects in relational terms. However, we want a set of operations which are able to compute modalities for any given binary relation $R \subseteq W \times W$ and any given right ideal element $C \in RI(W)$, for any finite set W.

So the question is now: is there any operation between relations which is able to compute 15.18.18 and 15.18.19?

The answer is "Yes". And these operations are not *ad hoc*, but they are specialization of mathematical concepts we have introduced in Part I.

Let us define a couple of operations on arbitrary binary relations. In what follows we assume $R : W \times W'$ and $S : U \times U'$. Then we have:

$$R \longrightarrow S = -(R^{\smile} \otimes -S), \text{ right residuation of } S \text{ with respect to } R.$$
$$(15.18.20)$$

This operation is defined only if $W = U$.

$$Y \longleftarrow R = -(-S \otimes R^{\smile}), \text{ left residuation of } S \text{ with respect to } R.$$
$$(15.18.21)$$

This operation is defined only if $W' = U'$.

These operations may be depicted in the following manner:
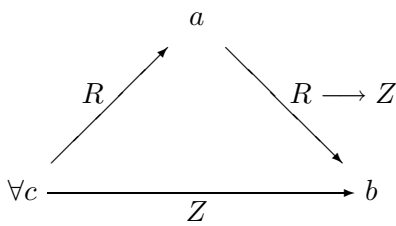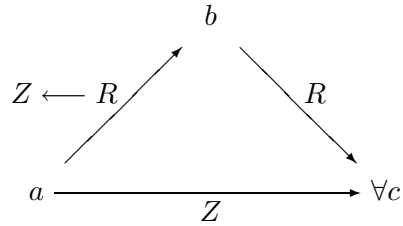
Diagram 1   *Diagram* 2

We have that if R and S are binary relations on a set W, then:

Proposition 15.18.1.

1. $R \longrightarrow S = \{\langle a, b \rangle \in W \times W : \forall c \in W (\langle c, a \rangle \in R) \Rrightarrow (\langle c, b \rangle \in S)\}$.

2. $S \longleftarrow R = \{\langle a, b \rangle \in W \times W : \forall c \in W (\langle b, c \rangle \in R) \Rrightarrow (\langle a, c \rangle \in S)\}$.

Indeed in view of (15.18.20), (15.18.21) and Schröder inequality (15.18.10), $R \longrightarrow S$, is the largest relation Z on W such that $R \otimes Z \leq S$. On the other hand $S \longleftarrow R$, is the largest relation Z such that $Z \otimes R \leq S$.

The reader is invited to notice that both residuations are sorts of divisors (see Subsection 1.4.2 of Chapter 1). In usual mathematics if we are given the equation $n \times x = m$, for given numbers m and n, then we solve the equation by a division, $x = m/n$.
Residuations solve the same problem for relations. But since $\otimes$ is a non-commutative operation, we have to distinguish left and right divisors.

It is easy to verify that they are an extension to relations of the definitions of right and left residuations given in Mathematical toolkit 16.5.

If we now compare 15.18.18 with *Proposition* 15.18.1.(1) we can notice that right residuation is *almost* the operation we are looking for. We say 'almost', since the direction of R in 15.18.18 is opposite to the direction of R in *Proposition* 15.18.1.(1).

On the contrary, the operation $\otimes$ provide us with the required interpretation in relation algebras of the possibility modalisation.

It follows that:

Proposition 15.18.2. *Given any Kripke frame* $\langle W, R \rangle$, *For any formula* α *of a propositional language,*

1. $[\]\!]^c = R^{\smile} \longrightarrow [\![\alpha]\!]^c = -(R \otimes -[\![\alpha]\!]^c)$.

2. $[\![\langle R \rangle(\alpha)]\!]^c = R \otimes [\![\alpha]\!]^c$.

In view of (15.18.11) we have $[\]\!] = (R^{\smile} \longrightarrow [\![\alpha]\!]^c)^{\smile}(W)$ and $[\![\langle R \rangle(\alpha)]\!] = (R \otimes [\![\alpha]\!]^c)^{\smile}(W)$.

In the case of an Indiscernibility Space, things are simpler, obviously:

Corollary 15.18.1. *If* E *is a symmetric relation on* W, *then* $[E](\alpha)]\!]^c = E \longrightarrow [\![\alpha]\!]^c$.

Notice that for any right cylinder C and any relation Z of type $W \times W'$, $R \otimes C$ is a right cylinder (whenever the compositions is applicable).

Clearly, if we confine to relations on the same set, then we do not have problems of dimension. Therefore we can define a logico-algebraic structure made up of only the right ideal elements of type $W \times W$ and a privileged relation $R \subseteq W \times W$:

Definition 15.18.3. *Given a modal frame* $\mathbf{K} = \langle W, R \rangle$, *by a* full modal algebra of relations determined by $\mathbf{K}$, *denoted by* $fullMREL(\mathbf{K})$, *we intend the structure:*

$$\langle \{X \times W : X \subseteq W\}, \cup, \cap, -, \mathbf{1}, [\mathbf{R}], \langle \mathbf{R} \rangle \rangle,$$

where $\mathbf{1} = W \times W$ *and the operators* $[\mathbf{R}]$ *and* $\langle \mathbf{R} \rangle$ *are defined as follows for each right cylinder* $C \in \{X \times W : X \subseteq W\}$:
(1) $[\mathbf{R}](C) = R^{\smile} \longrightarrow C;$ (2) $\langle \mathbf{R} \rangle(C) = R \otimes C$.

In view of *Definition* 15.18.1.(1) and the definition of a right Peirce product, it is not difficult to see that $[\mathbf{R}]$ and $\langle\mathbf{R}\rangle$ are the "implementation" of the modal clauses of Kripke models in the realm of relation algebras.

In fact from (15.18.11) we immediately obtain
$\langle\mathbf{R}\rangle(X^c)^{\smile}(W) = (R \otimes X^c)^{\smile}W = R^{\smile}(X) = \langle R\rangle(X)$ and
$[\mathbf{R}](X^c)^{\smile}(W) = (R^{\smile} \longrightarrow X^c)^{\smile}W = R^{\smile} \Longrightarrow X = [R](X).$

REMARKS. Do not confuse $[R]$, $\langle R\rangle$, $\Longrightarrow$ and $\Longleftarrow$, which apply to sets, with $[\mathbf{R}]$, $\langle\mathbf{R}\rangle$, $\longrightarrow$ and $\longleftarrow$, which apply to relations. Moreover, do not confuse R and $R^{\smile}$, which are relations, with $R(X)$ and $R^{\smile}(X)$, which are sets.

We have almost all the ingredients to achieve our goals. Indeed, from the above equations one immediately can show:

Proposition 15.18.3. *Let $\langle G, E\rangle$ be an Indiscernibility Space. Then for any $X \subseteq G$,*

(1) $(lE)(X) = ([\mathbf{E}](X^c))^{\smile}G$; (2) $(uE)(X) = (\langle\mathbf{E}\rangle(X^c))^{\smile}G.$

15.18.3 Comparing Formal Concepts and Rough Sets

So far we have dealt with operations between relations of the same type (i.e. with *homogeneous* relation algebras). But at this point we need to define operations between relations of different types. We have seen that in order to be defined, binary operations will require the congruence of dimensions: if $R : W \times W'$ and $Z : U \times U'$, then $R \otimes Z$ is defined only if $W' = U$ (that is, the external dimension of R equals the internal dimension of Z. If this happens, then $R \otimes Z : W \times U'$.

Moreover, we need *right modalisations*, too.

Definition 15.18.4. *Let W, W', U, U' be sets and $R : W \times W'$, $Z : U \times U'$:*

1. $[\mathbf{R}](Z) = R^{\smile} \longrightarrow Z$ *is called the* left necessitation *of Z by means of R.*

2. $(Z)[\mathbf{R}] = Z \longleftarrow R^{\smile}$ *is called the* right necessitation *of Z by means of R.*

3. $\langle\mathbf{R}\rangle(Z) = R \otimes Z$ *is called the* left possibility *of Z by means of R.*

4. $(Z)\langle \mathbf{R} \rangle = Z \otimes R$ is called the right possibility *of Z by means of R.*

whenever the operations are defined (clearly $\langle \mathbf{R} \rangle(Z) = (R)\langle \mathbf{Z} \rangle$).

We have observed in Subsection 2.2.1 of Chapter 2 that the basic constructors $[[e]]$ and $[[i]]$ are orthogonal to $[e]$ and, respectively, $[i]$. That is, we can notice an inversion of the predicates with respect to the implication.

This leads to an inversion of the universal quantifiers in the definition of *est* with respect to that of *(lR)*. Indeed, for any *P-system* $\langle G, M, R \rangle$, for all $A \subseteq G$:

15.18.3.1. $est(A) = \{g \in G : \forall m \in M, \forall g' \in G((g' \in A \Rightarrow \langle g', m \rangle \in R) \Rightarrow \langle g, m \rangle \in R)\}$;

15.18.3.2. $(lR)(A) = \{g \in G : \forall g' \in G, \forall m \in M((\langle g', m \rangle \in R \Longleftarrow\Rightarrow \langle g, m \rangle \in R) \Rightarrow g' \in A)\}$.

This is an informal observation, supported to some extent by the fact that *est* and *(lE)* have also opposite behaviours: the former is a closure operator, while the latter is an interior operator. In this Subframe we want to formalise these clear appearances.

We start with by noticing that on the one hand any Indiscernibility Space, (U, E) is a modal frame and on the other hand in any *P-system*, $\langle G, M, R \rangle$, if $\langle g, m \rangle \in R$ then we can say that the object g *forces* the property m, $g \models m$. That is R is a *forcing relation*.

So, on the one side we have a *modal frame (U, E) without a forcing relation*, while on the other we have *a forcing relation R without a modal frame* (here the language $\mathcal{L}$ is defined from M, as set of atomic formulas, and conjunctions of atomic formulas are represented by subsets of M). Now we have some tesserae of our puzzle. But we must find the others: the language and the forcing relation in the case of Indiscernibility Spaces; the accessibility relation, in the case of Formal Contexts. Moreover, in both cases we have to determine at least a modal operator making all these pieces work.

Let us first deal with $[[R]]$ and $[[R^\smile]]$.

Lemma 15.18.1. *Let $\mathbf{K} = \langle G, M, R \rangle$ be a P-system, then for any $X \subseteq G$, for any $Y \subseteq M$,*

1. *$([[R^\smile]](X))^c = (R^\smile)[\mathbf{X^c}]$.*

2. *$([[R]](Y))^c = (R)[\mathbf{Y^c}]$.*

Proof. $(R^{\smile})[\mathbf{X^c}] = R^{\smile} \longleftarrow X^{c\smile} = \{\langle m, x \rangle \in M \times G : \forall g \in G(\langle x, g \rangle \in X^{c\smile} \Rightarrow \langle m, g \rangle \in R^{\smile})\} = \{\langle m, x \rangle \in M \times G : \forall g \in G(\langle g, x \rangle \in X^c \Rightarrow \langle g, m \rangle \in R)\}$, where x is any object. Hence, from this and the definition of $[[R^{\smile}]]$ we obtain immediately the proof.

Similarly, $(R)[\mathbf{Y^c}] = R \longleftarrow Y^{c\smile} = \{\langle g, x \rangle \in G \times M : \forall m \in M(\langle x, m \rangle \in Y^{c\smile} \Rightarrow \langle g, m \rangle \in R)\} = \{\langle g, x \rangle \in G \times M : \forall m \in M(\langle m, x \rangle \in Y^c \Rightarrow \langle g, m \rangle \in R)\}$, where x is any attribute. At this point, we obtain the result from the definition of $[[R]]$.   QED

Using Diagrams 1 and 2, we can depict $(R)[\mathbf{Y^c}]$ (i.e. $[[\mathbf{R}]](Y^c)$) and $(R^{\smile})[\mathbf{X^c}]$ (i.e. $[[\mathbf{R^{\smile}}]](X^c)$), as follows:
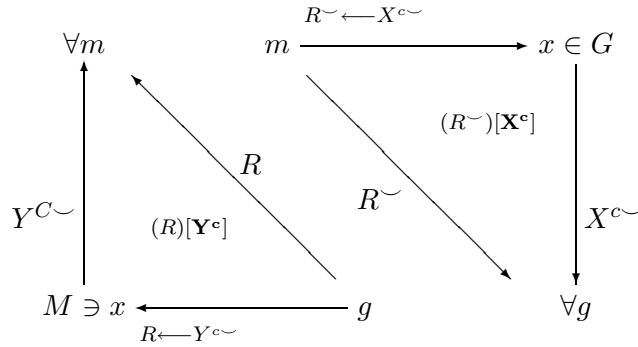
Diagram 3

From the above results we can infer that there is a neat symmetry between (lE) and $[[R^{\smile}]]$: in fact (lE) corresponds to a modalisation *on the left of a set of objects by means of a relation*, whereas $[[R^{\smile}]]$ corresponds to a modalisation *on the right of a relation by means of a set of objects.*

Let us try to give an explanation. In the case of an Indiscernibility Space $\mathbf{IS} = \langle G, E \rangle$ induced by an *A-system*, two objects $g, g' \in G$ are $E - accessible$ if and only if they cannot be discerned by means of the properties definable on the ground of the information provided by $\mathbf{IS}$. In a sense, we have no evolution of our quanta of information, but we have different individuals either carrying exactly the same amount of information or not. But g, g' carry the same amount of information if $g \models \alpha$ iff $g' \models \alpha$, α any set of attribute-value pairs or any set of properties. Therefore, a forcing relation $\models$ between objects and properties (or nominalised attributes), is in fact synthesized by the accessibility relation E. And this forcing relation is *locally and virtually* recovered

whenever we consider any $X \subseteq G$. Indeed one can think of X as the extension $[\![\alpha]\!]$ of a property α (that is, as the set of objects forcing α). It follows that $g \in [\mathbf{E}](X^c)$ means that for all g' such that $\langle g, g' \rangle \in E$, $g' \models \alpha$.

On the contrary, we can explain the behaviour of the operator $[R^\smile]$ in the following way: $\langle g, m \rangle \in R$ means $g \models m$. So R is not an accessibility relation between objects, but the forcing relation itself. Therefore, we need an accessibility relation. But we can notice that in Formal Concept Analysis an *accessibility relation* E_X is *locally and virtually* defined whenever we take into consideration a set of objects $X \subseteq G$: in fact let us set $\langle w, w' \rangle \in E_X$ if and only if both $w, w' \in X$ or both $w, w' \in -X$. Thus the perspective is here perfectly reversed with respect to Indiscernibility Spaces: by applying $[[R^\smile]](X)$ we collect the set of properties that are necessarily linked, *via* the forcing relation, with respect to the accessibility relation E_X. We support this claim by proving that when we modalise on the right *via* $[\mathbf{X^c}]$ or on the left (as for any usual accessibility relation) *via* $[\mathbf{E_X}]$, we obtain the same results:

Proposition 15.18.4. *For any P-system* $\mathbf{K} = (G, M, R)$, $\forall X \subseteq G$,

$$((R^\smile)[\mathbf{X^c}])^\smile(X) = ([\mathbf{E_X}](R))(X) = [[R^\smile]](X).$$

Proof. E_X is an equivalence relation. Hence $[\mathbf{E_X}](R) = E_X \longrightarrow R = \{\langle g, m \rangle : \forall g'(\langle g', g \rangle \in E_X \Rightarrow \langle g', m \rangle \in R\}$. In turn, $(R^\smile)[\mathbf{X^c}] = \{\langle m, g \rangle : \forall g'(\langle g', g \rangle \in X^c \Rightarrow \langle g', m \rangle \in R\}$. Hence $[\mathbf{E_X}](R) = (R^\smile)[\mathbf{X^c}]$. Since we are dealing with a right cylinder, here g is any element of G. Clearly $\forall g' \in X$, $\langle g', g \rangle \in E_X \Longleftrightarrow \langle g', g \rangle \in X^c$ (for the whole G the right to left implication is not true, but it is immaterial because of the dummy nature of g in the right cylinder X^c). Hence we have the result (incidentally, $([\mathbf{E_X}](R))(-X) = [R^\smile](-X)$).   QED

Therefore the above noticed symmetry can be explained as follows: $(lE)(X)$ *computes the set of objects* that necessarily force (the extension of) a *given property* α. On the other side, $[[R^\smile]](X)$ starts with a *given set of objects* that necessarily force some (possibly non elementary) property β and *computes this property*.

Now let us go on and analyse *est*.

Proposition 15.18.5. *For any P-system* $\mathbf{K} = (G, M, R)$, *for any* $X \subseteq G$,

$$(est(X))^c = R \longleftarrow (X^c \longrightarrow R);$$

Proof. $R \longleftarrow (X^c \longrightarrow R) = \{\langle g, g' \rangle : \forall m \in M(\langle g', m \rangle \in X^c \longrightarrow R \Rightarrow \langle g, m \rangle \in R)\} = \{\langle g, g' \rangle : \forall m \in M, \forall g'' \in G((\langle g'', g' \rangle \in X^c \Rightarrow \langle g'', m \rangle \in R) \Rightarrow \langle g, m \rangle \in R)\}$. Thus we have the proof (notice that since $X^c \longrightarrow R : G \times M$ and $R : G \times M$, then $R \longleftarrow (X^c \longrightarrow R) : G \times G$). If in Diagram 3 we substitute $R^\smile \longleftarrow X^{c\smile}$ for Y^c by rotating and translating the upper triangle until $Y^{c\smile}$ and $(R^\smile \longleftarrow X^{c\smile})^\smile$ coincide (renaming the variable x and reversing accordingly all the arrows of the upper triangle), then we obtain:
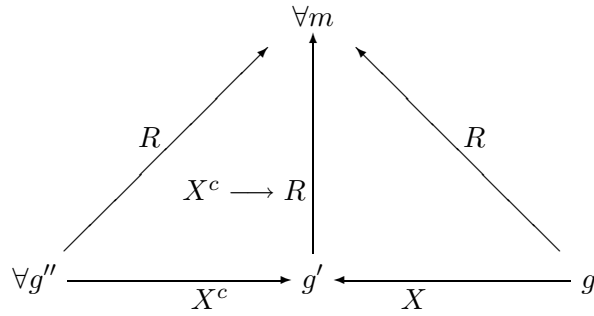
Diagram 4

Diagram 4 commutes for $X = R \longleftarrow (X^c \longrightarrow R)$, giving the diagrammatic version of the proof.   QED

Dually, with respect to the relation R, $(\mathcal{ITS}(Y))^c = R^\smile \longleftarrow (Y^c \longrightarrow R^\smile)$, $\forall Y \subseteq M$.

If $R \subseteq W \times W'$ is thought of as a set of transitions in a computer from a sets of states in W to a set of states in W' and $Z \subseteq U \times W'$ is thought of as a set of transitions from U to W', then $R \longleftarrow Z$ is the largest set of transitions from W to U that are required before Z to approximate R.

Roughly speaking, this is the idea behind the semantics of *weakest pre-specification* and *weakest post-specification*, proposed in [Hoare & He, 1986]. Therefore, *Proposition* 15.18.1 tells us that a weakest pre-specification is a form of sufficiency operator, as was noted by Ewa Orłowska (see also [Demri et al., 1994]).

15.18.4   Approximation of Relations

Modalisation of relations by means of relations makes it possible to move from approximation of sets to approximations of relations. First of all we need to generalise the notion of an Indiscernibility Space.

Definition 15.18.5. *A n-ary Relational Approximation Triple is a tuple $\mathbf{RA}(U) = \langle U, \mathfrak{R}, Z \rangle$, where $U = \{U_i\}_{1 \leq i \leq n}$ is a family of sets, $\mathfrak{R} = \{R_i\}_{1 \leq i \leq n}$ is a family of binary relations such that for any $1 \leq i \leq n$, $R_i \subseteq U_i \times U_i$ and a modalizing relation Z is point-wise defined on $\prod_{i=1}^{n} U_i$ by:*

$$\langle \langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle \rangle \in Z \ \text{iff} \ \langle x_i, y_i \rangle \in R_i, \ \text{all} \ i.$$

Thus for any $R \subseteq \prod_{i=1}^{n} U_i$: we set:

$$(lZ)(R) = \{\langle x_1, \ldots, x_n \rangle : \forall \langle y_1, \ldots, y_n \rangle (\langle \langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle \rangle$$
$$\in Z \Rrightarrow \langle y_1, \ldots, y_n \rangle \in R \}.$$

In particular, when any R_i is an equivalence relation, under some constraint we obtain the notion $\mathcal{L}(\mathcal{R}, R)$ introduced in Frame 4.7.3 of Part I.

   If in a Relational Approximation Triple $n = 2$, then a modal relational characterization can be given:

Proposition 15.18.6. *For any binary Relational Approximation Triple $\langle U, \mathfrak{R}, Z \rangle$, for any $R \subseteq U_1 \times U_2$, $(lZ)(R) = [\mathbf{R_1}](R)[\mathbf{R_2^{\smile}}]$.*

Proof. In what follows, $a, a' \in U_1$ and $b, b' \in U_2$. $([\mathbf{R_1}](R))[\mathbf{R_2^{\smile}}] = (R_1^{\smile} \longrightarrow R) \longleftarrow R_2 = \{\langle a, b \rangle : \forall b'(\langle b, b' \rangle \in R_2 \Rrightarrow \langle a, b' \rangle \in R_1^{\smile} \longrightarrow R))\} = \{\langle a, b \rangle : \forall b'(\langle b, b' \rangle \in R_2 \Rrightarrow \forall a'(\langle a', a \rangle \in R_1^{\smile} \Rrightarrow \langle a', b' \rangle \in R))\} = \{\langle a, b \rangle : \forall b', \forall a'((\langle b, b' \rangle \in Z_2 \wedge \langle a, a' \rangle \in R_1) \Rrightarrow (\langle a', b' \rangle \in R)\} = (lZ)(R)$. Moreover by easy computation one can prove $(R_1^{\smile} \longrightarrow R) \longleftarrow R_2 = R_1^{\smile} \longrightarrow (R \longleftarrow R_2)$.

   (Following [Van Benthem, 1991], we can comment on this fact by saying that in this formula we can change the temporal but not the spatial application order of the two residuals).
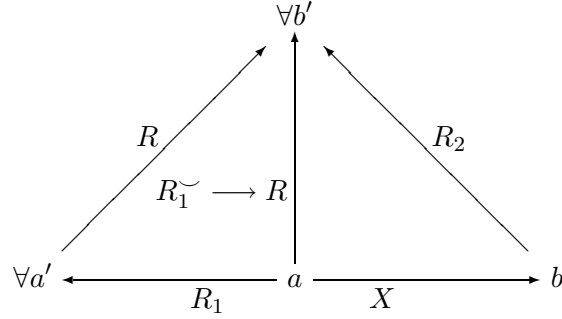
Diagram 5

The diagram commutes for $X = R_1^\smile \longrightarrow R \longleftarrow R_2$ QED

In particular when $U_1 = U_2$ and $R_1 = R_2$ is an equivalence relation the following holds:

Corollary 15.18.2. *For any binary Relational Approximation Triple $\langle U, \mathfrak{R}, Z \rangle$ such that $U_1 = U_2$, $R_1 = R_2$ and R_1 is an equivalence relation, for any $R \subseteq U_1 \times U_1$, $(lZ)(R) = [R_1](R)[R_1]$.*

REMARKS. It can be shown (see [Pagliani, 1996]) that Proposition 15.18.3 is a specialization of this Corollary to the case in which R is a right cylinder.

Comparing now this result with Proposition 15.18.5 we are able to see again that between lower approximations of relations and formal concept extents there is a clear symmetry:

$$(est(X))^c = R \longleftarrow (X^c \longrightarrow R), ((lZ)(R))^c = Z_1 \longrightarrow R \longleftarrow Z_1.$$

15.18.5 Making Formal Concepts and Rough Sets Interact

Now, we address the reader to the definitions of an upper approximation $\langle G, M, \Vdash^{cl^{\mathbf{E}}} \rangle$ and a lower approximation $\langle G, M, \Vdash^{int^{\mathbf{E}}} \rangle$ of a formal context $\langle G, M, \Vdash \rangle$ given in *Definition* 4.9.1 and *Definition* 4.9.2 of Frame 4.9.3 of Part I.

It is not difficult to prove that given an Indiscernibility Space $\langle G, E \rangle$ and a *P-system* $\langle G, M, \Vdash \rangle$ the following equations hold:

$$\Vdash^{cl^{\mathbf{E}}} = \langle \mathbf{E} \rangle (\Vdash), \quad \Vdash^{int^{\mathbf{E}}} = [\mathbf{E}](\Vdash) \qquad (15.18.22)$$

15.18.6   What this Approach may Suggest

1. Though we think that the above introduced notion of $n-$ary Relational Approximation Triple is a natural extension of that contained in [Skowron & Stepaniuk, 1993] (in view for instance of the approach described in [Yao & Lin, 1997]), nevertheless we could think of them as interacting Kripke frames, so going towards some connection with *fibred semantics* (cf. [Gabbay, 1996]).

2. We did not quote [Van Benthem, 1991] by chance: indeed we have seen in *Proposition* 15.18.6 that in any binary Relational Triple, (lZ) $(R) = R_1^{\smile} \longrightarrow R \longleftarrow R_2$. We can notice that this expression resembles the semantic type of "*and*" in Categorial Grammar (cf. for instance [Van Benthem, 1991; Desclés, 1990] or [Abrusci, 1990]), that is $(a\backslash a)/a$ (indeed our expression is a sort of "and" – see the proof of Proposition 15.18.6). But this expression is closer related to the semantic type of transitive verbs: $(e\backslash t)/e$, where the two e are nouns (the subject on the left and the direct object on the right) and t is a verb. Hence $(lZ)(R)$ should live in the semantic type of *two-place relations between individuals.*

Under this interpretation the above expression could make an interesting fact explicit also: the role of the subject, $R_1^{\smile}$, is opposite to the role of the complement, R_2, but this is no longer true if the verb is reflexive, that is, if with respect to this verb the role of the subject is reflexive and equals that of the object: this is the case described in Proposition 15.18.2.

On its own rights, the relational interpretation of $[R^{\smile}](Y)$ look like the semantic type of determiners, t/e, where t is a noun phrase and e a noun. In our case this determiner should be "*every*" so that the relational interpretation of $est(X)$ is a sort of "*every*(*every*)" (and indeed it is), where the internal "*every*" must be switched to the type $e\backslash t$ (one-place predicate on individuals) in order to be applied by the external one (see the proof of *Proposition* 15.18.5). Eventually we have $t/(e\backslash t)$. Hence $est(X)$ should live in the semantic type of *second-order properties of predicates.*

So, formal language theory could help us to understand the syntactic form of our operators. One can try to provide a very initial explanation of these formal analogies, by considering some connections with another linguistic framework: *terminological languages* (see for instance [Brachman & Levesque, 1987] and [Schmidt, 1993]).

15.18.7   Computing Dependencies in Relation Algebras

We have seen that a good framework to compute upper and lower approximations are modal algebras of relations. Let us start with an example.

EXAMPLE 4

The steps for computing the lower approximation $(lTH)(\llbracket \alpha \rrbracket)$ of Examples 1 and 2 are the following:

1. Switch the Boolean values of $\llbracket \alpha \rrbracket^c$:

$-\llbracket \alpha \rrbracket^{\mathbf{c}}$	a	b	c	d	e	f	g	h	i
a	1	1	1	1	1	1	1	1	1
b	1	1	1	1	1	1	1	1	1
c	1	1	1	1	1	1	1	1	1
d	0	0	0	0	0	0	0	0	0
e	0	0	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	0	0
g	0	0	0	0	0	0	0	0	0
h	1	1	1	1	1	1	1	1	1
i	1	1	1	1	1	1	1	1	1

2. Compute $\mathbf{TH} \otimes -\llbracket \alpha \rrbracket^{\mathbf{c}}$:

$\mathbf{TH} \otimes -\llbracket \alpha \rrbracket^{\mathbf{c}}$	a	b	c	d	e	f	g	h	i
a	1	1	1	1	1	1	1	1	1
b	1	1	1	1	1	1	1	1	1
c	1	1	1	1	1	1	1	1	1
d	1	1	1	1	1	1	1	1	1
e	0	0	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	0	0
g	0	0	0	0	0	0	0	0	0
h	1	1	1	1	1	1	1	1	1
i	1	1	1	1	1	1	1	1	1

3. Now (last step) switch the values of $\mathbf{TH} \otimes -\llbracket \alpha \rrbracket^{\mathbf{c}}$: we obtain $[\mathbf{TH}](\llbracket \alpha \rrbracket^c)$ (remember that $TH = TH^{\smile}$ because TH is an equivalence relation):

$-(\mathbf{TH} \otimes -[\![\alpha]\!]^{\mathbf{c}})$	a	b	c	d	e	f	g	h	i
a	0	0	0	0	0	0	0	0	0
b	0	0	0	0	0	0	0	0	0
c	0	0	0	0	0	0	0	0	0
d	0	0	0	0	0	0	0	0	0
e	1	1	1	1	1	1	1	1	1
f	1	1	1	1	1	1	1	1	1
g	1	1	1	1	1	1	1	1	1
h	0	0	0	0	0	0	0	0	0
i	0	0	0	0	0	0	0	0	0

This matrix is the right ideal element representing the set $\{e, f, g\}$: indeed $(lTH)([\![\alpha]\!]) = \{e, f, g\}$.

So, we have seen how to exploit relational algebra in order to compute upper and lower approximations.

Now we are about to show that the same machinery makes it possible to compute functional dependencies among subsets of attributes.

From *Definition* 3.2.1 and *Definition* 3.2.2, of Chapter 3 we have that given an Information System $\mathbf{I} = \langle U, At, Val \rangle$, given $A, B \subseteq At$ and $u \in U$, B is dependent on C at point u, if and only if $\forall u' \in U(\langle u, u' \rangle \in E_A \Rightarrow \langle u, u' \rangle \in E_B)$.

Then the reader can immediately see that this definition may be interpreted by means of the relational right residuation.

In fact, consider $POS_A(B)$ as defined in the Remarks after *Proposition* 10.20.1 in Frame 10.20. We know that $POS_A(B)$ is the set of all the elements of U which support the functional dependency $A \twoheadrightarrow B$. From this observation $I(B, A) = \{(lE_A)(X)\}_{X \in IND(B)}$ is the set of the equivalence classes of $IND(A)$ which are included in some equivalence class of $IND(B)$. If we are denote by $E(A, B)$ the equivalence relation E such that $IND(E) = I(B, A)$, then it is not difficult to understand that $E(A, B)$ is nothing else but the lower approximation of the relation E_B with respect to the relation E_A, *viz.* $(lE_A)(E_B)$.

Hence the step will be performed by exploiting the above technique for approximating relations by means of relations and, specifically, *Corollary* 15.18.2.

Proposition 15.18.7. *For any Information System $\langle U, At, Val \rangle$, for any $A, B \subseteq At$,*

$$E(A, B) = (lE_A)(E_B) = E_A \longrightarrow E_B \longleftarrow E_A = -(E_A \otimes -E_B \otimes E_A).$$

Clearly, if $\langle u, u' \rangle \in E(A, B)$, then $A \twoheadrightarrow_u B$ and $A \twoheadrightarrow_{u'} B$. Therefore we have just to transform the equivalence relation $(lE_A)(E_B)$ into a right ideal element, obtaining the relational representation of the set $POS_A(B)$:

Proposition 15.18.8. *For any Information System* $\langle U, At, Val \rangle$, $A, B \subseteq At$,

$$((lE_A)(E_B)) \otimes \mathbf{1} = (POS_A(B))^c.$$

EXAMPLE

Continuing *Example 1*, we see that attribute *Comfort* is considered a decision. Hence the problem of computing, for instance $POS_{\{Temperature,Hemoglobin\}}(\{Comfort\})$ naturally arises. Remember that we set $TH = E_{\{Temperature,Hemoglobin\}}$ and we put $C = E_{\{Comfort\}}$. Since $IND(E_{\{Comfort\}}) = \{\{a, b, c\}, \{d, e, f, g\}, \{h, i\}\}$, we have:

$-\mathbf{C}$	a	b	c	d	e	f	g	h	i
a	0	0	0	1	1	1	1	1	1
b	0	0	0	1	1	1	1	1	1
c	0	0	0	1	1	1	1	1	1
d	1	1	1	0	0	0	0	1	1
e	1	1	1	0	0	0	0	1	1
f	1	1	1	0	0	0	0	1	1
g	1	1	1	0	0	0	0	1	1
h	1	1	1	1	1	1	1	0	0
i	1	1	1	1	1	1	1	0	0

1. Let us first compute $TH \otimes -C$:

$\mathbf{TH} \otimes -\mathbf{C}$	a	b	c	d	e	f	g	h	i
a	0	0	0	1	1	1	1	1	1
b	0	0	0	1	1	1	1	1	1
c	1	1	1	1	1	1	1	1	1
d	1	1	1	1	1	1	1	1	1
e	1	1	1	0	0	0	0	1	1
f	1	1	1	0	0	0	0	1	1
g	1	1	1	0	0	0	0	1	1
h	1	1	1	1	1	1	1	0	0
i	1	1	1	1	1	1	1	0	0

2. Now we have to compute $TH \otimes -C \otimes TH$:

$\mathbf{TH} \otimes -\mathbf{C} \otimes \mathbf{TH}$	a	b	c	d	e	f	g	h	i
a	0	0	1	1	1	1	1	1	1
b	0	0	1	1	1	1	1	1	1
c	1	1	1	1	1	1	1	1	1
d	1	1	1	1	1	1	1	1	1
e	1	1	1	1	0	0	0	1	1
f	1	1	1	1	0	0	0	1	1
g	1	1	1	1	0	0	0	1	1
h	1	1	1	1	1	1	1	0	0
i	1	1	1	1	1	1	1	0	0

3. Finally we obtain $(l\,TH)(lC)$ by applying the Boolean negation:

$-(\mathbf{TH} \otimes -\mathbf{C} \otimes \mathbf{TH})$	a	b	c	d	e	f	g	h	i
a	1	1	0	0	0	0	0	0	0
b	1	1	0	0	0	0	0	0	0
c	0	0	0	0	0	0	0	0	0
d	0	0	0	0	0	0	0	0	0
e	0	0	0	0	1	1	1	0	0
f	0	0	0	0	1	1	1	0	0
g	0	0	0	0	1	1	1	0	0
h	0	0	0	0	0	0	0	1	1
i	0	0	0	0	0	0	0	1	1

4. The last step gives the right cylinder $(POS_{\{Temperature,Hemoglobin\}}(\{Comfort\}))^c$:

$-(\mathbf{TH} \otimes -\mathbf{C} \times \mathbf{TH}) \otimes \mathbf{1}$	a	b	c	d	e	f	g	h	i
a	1	1	1	1	1	1	1	1	1
b	1	1	1	1	1	1	1	1	1
c	0	0	0	0	0	0	0	0	0
d	0	0	0	0	0	0	0	0	0
e	1	1	1	1	1	1	1	1	1
f	1	1	1	1	1	1	1	1	1
g	1	1	1	1	1	1	1	1	1
h	1	1	1	1	1	1	1	1	1
i	1	1	1	1	1	1	1	1	1

Hence $POS_{\{Temperature,Hemoglobin\}}(\{Comfort\}) = \{a, b, e, f, g, h, i\}$.

Notice that in order to strictly obtain the right ideal element, steps 2, 3 and 4 are not necessary, because $(POS_{\{Temperature,Hemoglobin\}}(\{Comfort\}))^c = -(TH\otimes-C)\otimes\mathbf{1}$ (cf. Remarks after *Corollary* 15.18.2).

However, step 3 provides us with some useful information. Namely, we are given the values of the functional dependence. In fact, step 3 tells us that this dependence is based on three classes: $\{a,b\}, \{e,f,g\}, \{h,i\}$ and since these classes refer respectively to *Low Comfort*, *Medium Comfort* and *Very Low Comfort*, by looking at the attribute-values for their elements we find the following laws:

1. *Low Temperature and Fair Hemoglobin* imply *Low Comfort.*

2. *Low Temperature and Good Hemoglobin* or *Normal Temperature and Fair Hemoglobin* imply *Medium Comfort.*

3. *Normal Temperature and Poor Hemoglobin* or *High Temperature and Good Hemoglobin* imply *Very Low Comfort.*

15.19   Frame – Relational Proof Theory

The relation algebraic approach suggested developing relational deductive systems. Indeed, if relation algebras are models for modal logics (and other kinds of logics), it is straightforward to think of deductive systems transforming sequences of relations into sequences of relations.

Ewa Orłowska devoted several works to this topic. In what follows we give an initial taste of this techniques, based essentially on [Orłowska, 1988b].

First, we have to define a language $\mathcal{LR}$ in which well formed formulas are relations. The ingredients of $\mathcal{LR}$ are:

- A set $varOb = \{x, y, z, \ldots\}$ of variables for objects

- A set of operations $opRel = \{\cap, \cup, -, \otimes, \smile, \longrightarrow\}$

- A set $conRel = \{r_1, r_2, \ldots\}$ of constants for relations

- The closure $eRel$ of $conRel$ with respect to $opRel$

Then $\mathcal{LR} = \{xPy : x, y \in varOb, P \in eRel\}$. For instance, $x(r_1\otimes(-r_2\cap r_3^{\smile}))y$ is a wff of $\mathcal{LR}$. A model $\mathcal{M}$ of $\mathcal{LR}$ is a triple $\langle w, \{R_{r_i}\}_{r_i\in conRel}, m\rangle$ such that (a) $W \neq \emptyset$, (b) $r_i \subseteq W \times W$, any $r_i \in conRel$, (c) m is a

meaning function $m : \mathcal{M} \longmapsto \mathcal{LR}$ such that:

$(i)\ m(r_J) = R_{r_j},\ (ii)\ m(P{\bullet}Q) = m(P){\bullet}M(Q),\ (iii)\ m({\circ}P) = {\circ}m(P),$

$$(15.19.23)$$

for all $r_j \in conRel$, $P, Q \in eRel$, binary operation $\bullet$ and unary operation $\circ$ in $opRel$ (for example, $m(P^{\smile}) = (m(P))^{\smile}$ and $m(P \longrightarrow Q) = m(P) \longrightarrow m(Q)$).

Therefore a model of $\mathcal{LR}$ is a relation algebra. We shall denote the unity of this algebra by $1_{\mathcal{M}}$.

Now we have to interpret object variables. We set an evaluation map $v : varOb \longmapsto W$.

Definition 15.19.1. *Given a relational language $\mathcal{LR}$, for any relational model $\mathcal{M}$, atomic formula xPy, well formed formula α of $\mathcal{LR}$ and set of well formed formulae Γ,*

1. *$\mathcal{M} \models_{v} xPy$ if $\langle v(x), v(y) \rangle \in P$ (v satisfies xPy in model $\mathcal{M}$).*

2. *$\models_{\mathcal{M}} \alpha$ iff $\mathcal{M} \models_{v} xPy$, for all evaluations v (α is true in $\mathcal{M}$).*

3. *$\Gamma \models_{\mathcal{M}} \alpha$ if $\models_{\mathcal{M}} \Gamma \Rightarrow \models_{\mathcal{M}} \alpha$ (Γ implies α; notice that $\models_{\mathcal{M}} \Gamma$ means $\models_{\mathcal{M}} \gamma$ for all $\gamma \in \Gamma$).*

4. *$\models_{\mathcal{LR}} \alpha$ if $\models_{\mathcal{M}} \alpha$ for all models $\mathcal{M}$ of $\mathcal{LR}$ (α is valid).*

For example, one can prove (a) $\models_{\mathcal{M}} xPy$ iff $m(P) = 1_{\mathcal{M}}$ or (b) $\models_{\mathcal{M}} x(-P \cup Q)y$ iff $m(P) \subseteq m(Q)$. Notice that (b) is a clue to verify if P is transitive. In fact P is transitive if $P \otimes P \subseteq P$ (cf. property 4 of the table of Section 12.2). Therefore, P is transitive if $\models_{\mathcal{M}} x(-(P{\otimes}P)\cup P)y$.

So far we have a language and a class of models for this language. Now we have to define a deductive system SR such that for any formula α, $\models_{\mathcal{LR}} \alpha$ iff $\vdash_{S\mathcal{R}} \alpha$.

Following a method developed by Helena Rasiowa and Roman Sikorski, E. Orłowska was able to set rules which decompose sequences of relational formulae into sequences of less complex relational formulae. As a result we obtain either a 1-term sequence or a pair of sequences.

We assume that Γ and Δ are, possibly empty, sequences of formulae.

$$(\cup\, dec): \ \frac{\Gamma, x(P \cup Q)y, \Delta}{\Gamma, xPy, xQy, \Delta}, \quad (-\cup\, dec): \ \frac{\Gamma, x(-(P \cup Q))y, \Delta}{\Gamma, x(-P)y, \Delta \quad \Gamma, x(-Q)y, \Delta},$$

$$(\cap\, dec): \ \frac{\Gamma, x(P \cap Q)y, \Delta}{\Gamma, xPy, \Delta \quad \Gamma, xQy, \Delta}, \quad (-\cap\, dec): \ \frac{\Gamma, x(-(P \cap Q))y, \Delta}{\Gamma, x(-P)y, x(-Q)y, \Delta},$$

$$(- - \ dec): \ \frac{\Gamma, x(- - P)y, \Delta}{\Gamma, xPy, \Delta}, \quad (\smile \ dec): \ \frac{\Gamma, xP^{\smile}y, \Delta}{\Gamma, yPx, \Delta} \ ,$$

$$(-\smile \ dec): \ \frac{\Gamma, x(-(P^{\smile}))y, \Delta}{\Gamma, y(-P)x, \Delta}$$

$$(\otimes \ dec): \ \frac{\Gamma, x(P \otimes Q)y, \Delta}{\Gamma, xPz, \Delta, x(P \otimes Q)y \quad \Gamma, zQy, \Delta, x(P \otimes Q)y},$$

where z is a variable,

$$(- \otimes \ dec): \ \frac{\Gamma, x(-(P \otimes Q))y, \Delta}{\Gamma, x(-P)z, z(-Q)y, \Delta}, \quad (\longrightarrow \ dec): \ \frac{\Gamma, x(P \longrightarrow Q)y, \Delta}{\Gamma, z(-P)x, zQy, \Delta},$$

where z is a variable which does not appear in any formula above the line,

$$(- \longrightarrow dec): \ \frac{\Gamma, x(-(P \longrightarrow Q))y, \Delta}{\Gamma, zPx, \Delta, x(-(P \longrightarrow Q))y \quad \Gamma, z(-Q)y, \Delta, x(-(P \longrightarrow Q))y},$$

where z is a variable.

REMARKS. Notice that the constraint on the variable z in $(- \otimes \ dec)$ and $(\longrightarrow \ dec)$ corresponds to the universal quantification in $\longrightarrow$ (see *Proposition* 15.18.1). Moreover, $(\longrightarrow \ dec)$ is justified by relation algebra. Indeed, $P \longrightarrow Q = -(P^{\smile} \otimes -Q)$, therefore if we substitute $P^{\smile}$ for P and $-Q$ for Q in $(- \otimes \ dec)$, and apply $(- \otimes \ dec)$, $(\smile \ dec)$ and $(- - \ dec)$), we obtain $(\longrightarrow \ dec)$.

A formula is said to be *indecomposable* if it is of the form xPy or $x(-P)y$ for $P \in conRel$, $x, y \in varOb$.

A sequence of formulae is indecomposable if every element is an indecomposable formula.

A sequence of formulae is said to be *fundamental* if it contains only indecomposable formulae or $x1y$.

A sequence Γ of formulae is said to be *valid* if there is a formula $\alpha \in \Gamma$ such that $\models_{\overline{\mathcal{LR}}} \alpha$.

One can prove:

Proposition 15.19.1. *(a) Every fundamental sequence is valid. (b) The empty sequence is not valid.*

A rule $\frac{\Gamma}{\Delta, \Delta'}$ is said to be *admissible* whenever Γ is valid if and only if both Δ and Δ' are valid. We have that all the above decomposition rules are admissible.

The application of these rules produces a tree such that each node of the tree generates at most two branches. A branch is called *fundamental* if it contains a fundamental sequence. A decomposition tree is fundamental if all its branches are fundamental.

One can prove:

Proposition 15.19.2. *A formula α of $\mathcal{LR}$ is valid if and only if there is a fundamental decomposition tree for α.*

Now we define a language $\mathcal{LMR}$ to deal with modal logic in a relational fashion. So we need:

(a) A set $varProp = \{p, p_1, p_2, q, \ldots\}$ of propositional variables.

(b) A set $conAcc = \{k_1, k_2, \ldots\}$ of *accessibility relation constants*.

(c) The set $eAcc = \{c_1, c_2, \ldots\}$ which is obtained by closing $conAcc$ with respect to the operations in *opRel*.

(d) The logical constants $\neg, \vee, \wedge, \Longrightarrow, [c_i]$ and $\langle c_j \rangle$, where $c_i, c_j \in eAcc$.

(e) The set *forMod* of modal formulas, obtained by closing *varProp* with respect to the logical constants.

For example, if $k_1, k_2, k_3 \in conAcc$ then $\langle (k_1 \cap k_2) \otimes -k_3 \rangle (p \wedge \neg q)$ is a well formed formula of $\mathcal{LMR}$.

A model $\mathcal{M}$ of $\mathcal{LMR}$ is obtained by extending a model for $\mathcal{LR}$ to form a triple $\mathcal{M} = \langle W, \{R_{k_i}\}_{k_i \in conAcc}, \vDash \rangle$, where $W \neq \emptyset$, $R_{k_i} \subseteq W \times W$, any $k_i \in conAcc$, and $\vDash$ is a forcing relation between elements of W and elements of $eAcc$ of formulae of $\mathcal{LRM}$.

Therefore, if p is a propositional variable, then the set-up of $\vDash$ reads: $\vDash (p) \subseteq W$. For any accessibility relation k_i, $\vDash (k_i) = R_{k_i}$. Regarding the inductive step, as to elements of $eAcc$, $\vDash$ works as like as the meaning function m in (15.19.23), while as to formulae $\vDash$ works along the usual forcing clauses for modal logics (see Frame 4.13.1 of Part I), starting with the basic step $s \vDash p$ iff $s \in \vDash (p)$, for $s \in W$ and p propositional variable.

Particularly, the forcing clauses for modalised formulae read:

$$s \vDash [c_i](\alpha) \ \textit{iff} \ \forall s' \in W (\langle s, s' \rangle \in \vDash (c_i) \Rightarrow s' \vDash \alpha),$$

$$s \vDash \langle c_i \rangle(\alpha) \ \textit{iff} \ \exists s' \in W (\langle s, s' \rangle \in \vDash (c_i) \ \& \ s' \vDash \alpha).$$

As usual, we say that a formula α is true in a model $\mathcal{M}$, $\vDash_{\overline{\mathcal{M}}} \alpha$ if and only if $[\![\alpha]\!] = W$. Otherwise stated, $\vDash_{\overline{\mathcal{M}}} \alpha$ if and only if all $s \vDash \alpha$ for all $s \in W$.

A formula α is valid, $\models_{\overline{\mathcal{LMR}}} \alpha$ if it is true in all models.

It is possible to prove that given a model $\mathcal{M}$ and a formula α of $\mathcal{LMR}$,

$$[\]\!] = \{s \in W : \exists t \in W \ \& \ \langle s, t \rangle \in ((\models (c_i))^{\smile} \longrightarrow ([\![\alpha]\!] \otimes W)\}.$$

This amounts to saying that $[\]\!]$ is the right Peirce product with respect to W of $R^{\smile} \longrightarrow [\![\alpha]\!]^c = [R]([\![\alpha]\!]^c)$, provided $\models (c_i) = R$. Which is not a surprise in view of *Lemma* 15.18.1.

To use the deductive machine described above we have to interpret formulae of $\mathcal{LMR}$ into formulae or the relation language $\mathcal{LR}$.

First, we need an interpretation of the atomic elements of the language. So we set a translation t (preserving 1): $t : conAcc \longmapsto conRel$ and $t : varProp \longmapsto conRel$.

Second, we extend t to a translation $\ulcorner \urcorner$ of both accessibility expressions and propositional formulas: $\ulcorner \urcorner : eAcc \longmapsto eRel$ and $\ulcorner \urcorner : forMod \longmapsto eRel$.

This translation works as usual. That is,

1. $\ulcorner k \urcorner = t(k)$ for $k \in conAcc$; $\ulcorner c_1 \bullet c_2 \urcorner = \ulcorner c_1 \urcorner \bullet \ulcorner c_2 \urcorner$, $\ulcorner \circ c \urcorner = \circ \ulcorner c \urcorner$, for $c_1, c_2, c \in eAcc$, $\bullet \in \{\otimes, \cap, \cup\}$ and $\circ \in \{-, \smile\}$.

2. $\ulcorner p \urcorner = t(p) \otimes 1$, for $p \in varProp$; $\ulcorner \alpha \vee \beta \urcorner = \ulcorner \alpha \urcorner \cup \ulcorner \beta \urcorner$, $\ulcorner \alpha \wedge \beta \urcorner = \ulcorner \alpha \urcorner \cap \ulcorner \beta \urcorner$, $\ulcorner \alpha \implies \beta \urcorner = \ulcorner \neg \alpha \urcorner \cup \ulcorner \beta \urcorner$, $\ulcorner \neg \alpha \urcorner = - \ulcorner \alpha \urcorner$, for $\alpha, \beta \in forMod$.

3. Finally, $\ulcorner \langle c_i \rangle(\alpha) \urcorner = \ulcorner c_i \urcorner \otimes \ulcorner \alpha \urcorner$ and $\ulcorner [c_i](\alpha) \urcorner = \ulcorner c_i \urcorner^{\smile} \longrightarrow \ulcorner \alpha \urcorner$.

Since in (2) a propositional variable is interpreted as a right cylinder, and right cylindrification is preserved by the Boolean operators, it follows that any formula is interpreted as a right cylinder.

Given this translation, one can prove:

Proposition 15.19.3. $\models_{\overline{\mathcal{LMR}}} \alpha$ *if and only if* $\models_{\overline{\mathcal{LR}}} x \ulcorner \alpha \urcorner y$.

On the basis of this machinery, one can add rules in order to define a deductive system for specific logics.

For instance, if we assume that accessibility relations are transitive, then the following rule shall be added:

$$(trans \ R) : \quad \frac{\Gamma, xRy, \Delta}{\Gamma, xRz, \Delta, xRy \quad \Gamma, zRy, \Delta, xRy},$$

where z is a variable.

Along this line one can define the deductive system of an impressive number of logical systems with modalities, such as normal Modal Logics ([Orłowska, 1996b]), Dynamic Logics ([Orłowska, 1993c]), some program semantics, and non classical logics such as Post n-valued logic ([Orłowska, 1991c]).

Moreover, using ternary relations instead of binary relations makes it possible to encompass Relevant Logics ([Orłowska, 1992]) and, in general, substructural logics ([MacCaull, 1997]).

15.20   Frame – Some History of the Algebraic Concepts used in this Part

We cannot provide the reader with an exhaustive list of authors and works which are related with the algebraic concepts used in this Part.

Therefore, we just propose a few suggestions.

In [Monteiro, 1967] a method is given to construct a quotient algebra by factorisation through a relation that is the counterpart of rough equality, by starting with a monadic Boolean algebra. The algebra $\langle \wp(U)/\approx, \overline{\sqsubseteq}, \sqcap, \sqcup, \neg, L, [\emptyset], [U] \rangle$ of Chapter 14 can also be obtained from the monadic Boolean algebra $\langle \wp(U), \subseteq, \cap, \cup, -, M, \emptyset, U \rangle$, following the same construction (M stands for upper approximation). It should be mentioned that the definitions of operators in $\wp(U)$ leading to the operators $\sqcap$ and $\sqcup$ are, however, different.

The construction of $\mathbf{TQ(B)}$ from $\mathbf{B}$ is a special case of a general methods given by Moisil (cf. [Boicescu et al., 1991]).

The notion of a Rough algebra and a pre-Rough algebra and Rough Set logic were introduced and developed by Mihir Chakraborty and Mohua Banerjee (see [Banerjee & Chakraborty, 1993, 1994] and [Banerjee & Chakraborty, 1996]).

The notion of topological quasi-Boolean algebra first appeared in [Banerjee & Chakraborty, 1993] during the Lindenbaum like construction in rough logic and was later given a formal definition by Banerjee and Wasilewska in [Wasilewska & Banerjee, 1995] where a natural representation theorem was established. In a subsequent paper ([Vigneron & Wasilewska, 1996]) an automatic prover was proposed.

15.21   Solutions

- Exercise 11.1

  (a) (i) Since $1 \notin \mathbf{L}''$, $\mathcal{J}(\mathbf{L}) \notin \mathbf{L_{L''}}$. Indeed, $T^* = \bigcup \{\emptyset, \{a\}, \{e\},$ $\{a, b, e\}\} = \{a, b, e\}$. (ii) The specialization preorder $\preceq$ on T^* is:

$\preceq$	a	b	e
a	1	0	0
b	0	1	1
e	0	1	1

  Hence we have: $k^*(\{a\}) = \{a\}, k^*(\{b\}) = \{b, e\}$ and so on, obtaining $k(0) = 0, k(a) = a, k(b) = e, k(c) = a, k(d) = g, k(e) = e, k(g) = g, k(1) = g$, so that in this case the relation $x \leq k(x)$ does not hold for any x.

  (b) (i) Yes, g is a knowledge map. (ii) $T^* = \{a, 1\} = \mathcal{J}(\mathbf{A})$ while $\mathbf{L}_{A'}$ and its specialization preorder $\preceq$ are:

  $$\{a, 1\}$$
  $$| \quad \mathbf{L'_A}$$
  $$\emptyset$$

$\preceq$	a	1
a	1	1
1	1	1

  Hence, $k^*(\emptyset) = \emptyset, k^*(\{a\}) = k^*(\{a, 1\}) = \{a, 1\}$. It follows that $k(0) = 0, k(a) = k(1) = 1$. Therefore, g and k do not coincide.

  (c) (i) Applying the representation procedure we obtain the following representation of $\mathbf{L}'''$, $\mathbf{L_{L'''}}$:
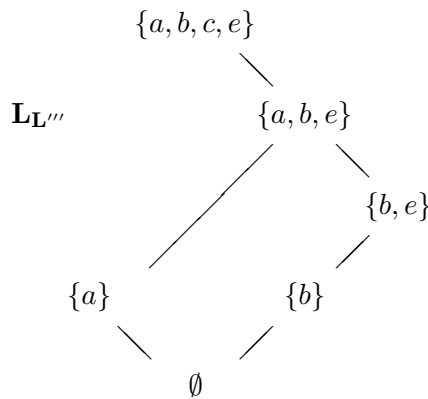
$\mathbf{L_{L'''}}$ induces the following specialization preorder on T^*:

$\preceq$	a	b	c	e
a	1	0	0	0
b	0	1	0	0
c	1	1	1	1
e	0	1	0	1

It follows that $k^*(\{a, b\}) = \{a, b\}$, but $\{a, b\}$ is not an element of $\mathbf{L_{L'''}}$.

(ii) The dual lattice of the preordered space $\mathbf{P} = \langle T^*, \preceq \rangle$ is:

```
                    {a, b, c, e}
                         \
                          {a, b, e}
                          /      \
              {a, b}              {b, e}
              /      \           /
       {a}            {b}
           \         /
              ∅
```
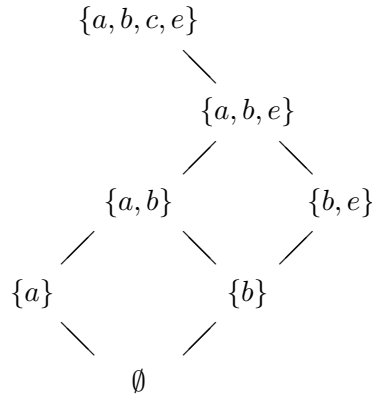
In this lattice the element $\{a, b\}$ appears as union of $\{a\}$ and $\{b\}$. In fact, the dual construction from the pre-order $\preceq$ provides $\{a, b\}$, too (actually, the fact that $\mathbf{F}(\mathbf{P}) \neq \mathbf{L_{L'''}}$, is another proof that $\mathbf{L_{L'''}}$ – hence $\mathbf{L}$ – is not distributive. Of course, the most immediate proof is the fact that $\mathbf{L'''}$ contains the 5-element sublattice $\{0, a, b, g, e\}$).

(iii) No. The problem concerns the elements a and b. Since a and b belong to $\mathbf{L'''}$, there must be $x, y \in \mathbf{L}$ such that $k(x) = a$ and $k(y) = b$. But $k(x) \vee k(y) = k(x \vee y)$. It follows that $d = k(x \vee y)$. But $d \notin \mathbf{L'''}$. In fact, $\phi(d) = \{a, b\}$.

(iv) The answer is "No", because k^* is not onto: $k^*(X) = \{b, e\}$ for no X and $\{b, e\}$ is the image of e in $\mathbf{L_{L''''}}$. This is due to the fact that $\mathbf{L'''}$, although a distributive lattice is not a sublattice of $\mathbf{L}$. Hence, $\phi'(\mathbf{L_{L''''}})$ is not a lattice of sets. To verify this statement, use the preorder induced by $\phi'(\mathbf{L''''})$:

$\preceq$	a	b	c	e
a	1	1	0	0
b	1	1	0	1
c	1	1	1	1
e	0	1	0	1

(v) The following is a knowledge map: $k(0) = 0, k(a) = f, k(b) = k(e) = e, k(c) = k(f) = k(1) = 1, k(d) = k(g) = g$.

- Exercise 12.1

  (A) Continuity: in the definition of R-neighborhood applied to $R(X)$ and $R(Y)$, consider the initial parts "$\exists x(x \in X \ldots$" and "$\exists x(x \in Y \ldots$". Now, consider that in first order logic $\exists x(x \in X) \vee \exists x(x \in Y) \equiv \exists x(x \in X \vee x \in Y)$. (B) Co-discontinuity: start the proof as before, but now consider that for all first order formulas φ and ψ, $\exists x(\varphi(x) \wedge \psi(x))$ implies $\exists x(\varphi(x)) \wedge \exists x(\psi(x))$, while the converse is not valid. (C) Normality: it is a corollary of continuity, because in any complete lattice $0 = \bigvee \emptyset$. (D) Isotonicity: it is a corollary of continuity and the fact that if $X \subseteq Y$ then $X \cup Y = Y$.

  Nonetheless, notice that in a sense we used some form of adjunction. In fact, $\exists$ is a lower adjoint, hence it is additive (cf. Frame 15.9).

- Exercise 12.2

  (a) Table for M_0:

x	0	a	b	c	d	e	f	1
$M_0(x)$	0	b	a	c	d	1	e	1

  (b) Example of $M-$discontinuity: $M_0(a) \vee M_0(c) = b \vee c = f \neq 1 = M_0(e) = M_0(a \vee c)$.

  (c) The answer is "No". For any binary relation R, neighboring is additive. Thus M_R must be continuous.

- Exercise 12.3

  (a) Table for M_1:

x	0	a	b	c	d	e	f	1
$M_1(x)$	0	f	d	c	1	f	1	1

  (c) Example of M-co-discontinuity: $M_1(d \wedge c) = M_1(0) = 0 \neq c = c \wedge 1 = M_1(c) \wedge M_1(d)$.

  (d) Non validity of the monadic M-co-continuity property: $M_1(a \wedge M_1(b)) = M_1(a \wedge d) = M_1(a) = f \neq b = f \wedge d = M_1(a) \wedge M_1(b)$.

(e) In order to find a representation for $\langle \mathbf{A}, \mathbf{L}_1 \rangle$ we cannot apply the Representation Procedure, because $\mathbf{L}_1$ is not distributive. In the first step we have to start with a set of eligible points of $\mathbf{A}'$; this is, nonetheless, the set of co-prime elements of $\mathbf{A}$, i.e. the set of atoms $\mathcal{J}(\mathbf{A}) = \{a, b, c\}$. Consider $\mathbf{J}(\mathbf{A}) = \langle \mathcal{J}(\mathbf{A}), \leq \rangle$ (where, as we know, $\leq$ is contravariant with respect to the order of $\mathbf{A}$), the dual lattice $\mathbf{F}(\mathbf{J}(\mathbf{A}))$ and the isomorphism $\phi : \mathbf{A} \longmapsto \mathbf{F}(\mathbf{J}(\mathbf{A}))$. The second step is lead by two facts about the operator M: (i) the operator M makes it possible to directly compute the requested relation R between the atoms, such that $\phi(M(x)) = M_R(\phi(x))$, for any atom x. Indeed, from its definition by means of the existential quantifier, if $x \in M_R(\phi(y))$, then $\langle x, y \rangle \in R$; therefore assumed that $\phi(M(x)) = M_R(\phi(x))$, we obtain $\langle x, y \rangle \in R$ if and only if $x \in \phi(M(y))$, for any $x, y \in \mathcal{J}(\mathbf{A})$. That is, if and only if $x \in M_R(\{y\})$; (ii) M−continuity and R−neighboring continuity then guarantees that running this procedure on the element of $\mathcal{J}(\mathbf{A})$ is sufficient.

So, consider $\mathbf{F}(\mathbf{J}(\mathbf{A}))$ and the isomorphism $\phi : \mathbf{A} \longmapsto \mathbf{F}(\mathbf{J}(\mathbf{A}))$. We will have: for any $x, y \in \mathcal{J}(\mathbf{A}), \langle x, y \rangle \in R$ if and only if $x \in \phi(M(\phi^{-1}(\{y\})))$.

For instance, let us consider the co-prime element a: $\phi^{-1}(\{a\}) = a$, $M_1(\phi^{-1}(\{a\})) = f$, $\phi(M_1(\phi^{-1}(\{a\}))) = \{b, c\}$. It follows that since $b, c \in \phi(M_1(\phi^{-1}(\{a\})))$, $\langle b, a \rangle, \langle c, a \rangle \in R$. In the same manner we obtain the entire relation

R	a	b	c
a	0	1	0
b	1	1	0
c	1	0	1

Now, let us compute, for instance, $L_R(\{b, c\})$: $L_R(\{b, c\}) = -M(-\{b, c\}) = -M(\{a\}) = -\{b, c\} = \{a\}$. Indeed, $\phi^{-1}(\{a\}) = a$ and $\phi^{-1}(\{b, c\}) = f$. But, actually, $L(f) = a$.

(f) The above relation R is not reflexive: for instance $\langle a, a \rangle \notin R$; it is not transitive: for instance $\langle c, a \rangle \in R, \langle a, b \rangle \in R$, but $\langle c, b \rangle \notin R$; it is not symmetric: for instance $\langle c, a \rangle \in R$, but $\langle a, c \rangle \notin R$.

- Exercise 12.4
  (a) By easy inspection.

(b) By applying the Representation Procedure we find what follows:

$\mathbf{L_{L_2}}$

$$\{a,b,c\}$$
$$\{b,c\}$$
$$\{b\} \qquad \{c\}$$
$$\emptyset$$

$\preceq$	a	b	c
a	1	1	1
b	0	1	0
c	0	0	1

x	$k^*(x)$
$\emptyset$	$\emptyset$
$\{a\}$	$\{a,b,c\}$
$\{b\}$	$\{b\}$
$\{c\}$	$\{c\}$
$\{a,b\}$	$\{a,b,c\}$
$\{a,c\}$	$\{a,b,c\}$
$\{b,c\}$	$\{b,c\}$
$\{a,b,c\}$	$\{a,b,c\}$
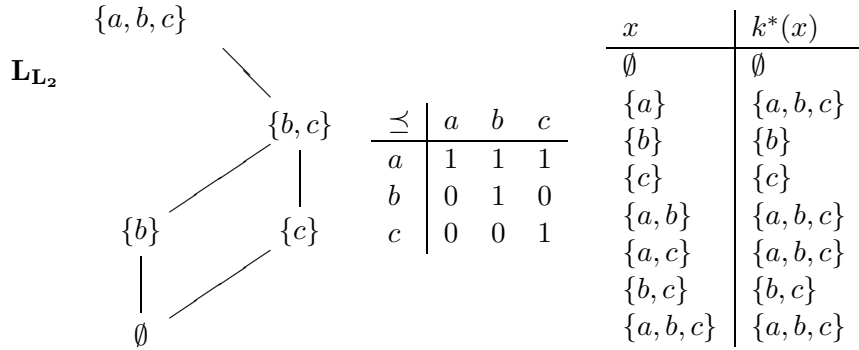
(c) The specialization preorder $\preceq$ is reflexive and transitive.

(d) The system $\langle \mathbf{A}, L_2 \rangle$ is not a Monadic Boolean algebra: the L-deflationary property is validated but the monadic L-continuity property is invalidated by $L_2(d \vee L_2(f)) = L_2(d \vee f) = L_2(1) = 1$, while, $L_2(d) \vee L_2(f) = b \vee f = f$. Also, this proves that the two properties are independent.

- Exercise 12.5

(a) By applying the Representation Procedure we find:

$$\{a,b,c\}$$
$$\{b\} \qquad \{a,c\}$$
$$\emptyset$$

$\preceq$	a	b	c
a	1	0	1
b	0	1	0
c	1	0	1

x	$k^*(x)$
$\emptyset$	$\emptyset$
$\{a\}$	$\{a,c\}$
$\{b\}$	$\{b\}$
$\{c\}$	$\{a,c\}$
$\{a,b\}$	$\{a,b,c\}$
$\{a,c\}$	$\{a,c\}$
$\{b,c\}$	$\{a,b,c\}$
$\{a,b,c\}$	$\{a,b,c\}$
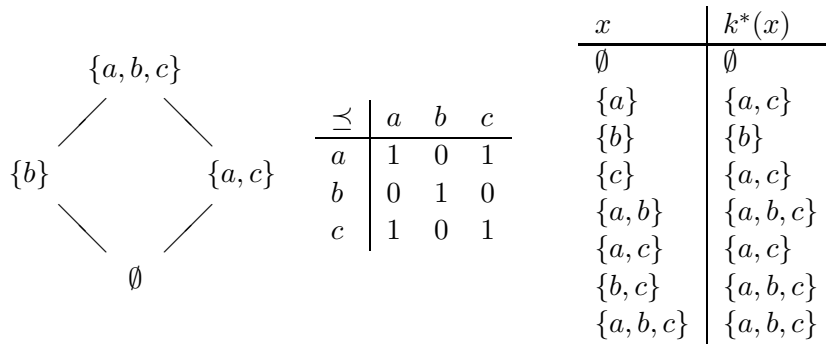
(b) The specialization preorder $\preceq$ is reflexive, symmetric and transitive. In other terms $\preceq$ is an equivalence relation.

- Exercise 12.6

(a) (Reflexivity) Since for any x, $\langle x, x \rangle \in R$, from the definition $x \in [\![L(\alpha)]\!]$ *if and only if* $\forall x', \langle x, x' \rangle \in R \Rightarrow x' \in [\![\alpha]\!]$, we immediately obtain that $x \in [\![L(\alpha)]\!]$ implies $x \in [\![\alpha]\!]$. Hence $[\![L(\alpha)]\!] \subseteq [\![\alpha]\!]$ so that $L(\alpha) \to \alpha$.

(b) (Transitivity) Assume $x \in [\![L(\alpha)]\!]$. Then for any x', $\langle x, x' \rangle \in R$ implies $x' \in [\![\alpha]\!]$. But for any x'', iff $\langle x', x'' \rangle \in R$ then $\langle x, x'' \rangle \in R$, by transitivity. It follows that $x'' \in [\![\alpha]\!]$, because our assumption. Therefore, $x' \in [\![L(\alpha)]\!]$, too. So, we obtain $x \in [\![L(L(\alpha))]\!]$.

- Exercise 12.7

  (a) The answer is "No" in both cases. In fact, consider the family $\varkappa_a = \{\{a, b\}, \{a, c\}, U\}$, from the contraction map $\varkappa$ of Example 12.4.8. The intersection $\{a, b\} \cap \{a, c\}$ does not belong to $\varkappa_a$. Consider, now, a contraction map $\varkappa'$, defined on subsets of the universe $\{a, b, c, d\}$ and such that $\varkappa'(\{a\}) = \emptyset, \varkappa'(\{a, b\}) = \{a\}$, $\varkappa'(\{a, c\}) = \{a\}, \varkappa'(\{a, d\}) = \{d\}, \varkappa'(\{a, b, c\}) = \{b, c\}$. We have the family $\varkappa'_a = \{\{a, b\}, \{a, c\}, \{a, b, c, d\}\}$ and $\{a, b\} \cup \{a, c\} = \{a, b, c\}$ does not belong to $\varkappa'_a$.

  (b) The answer is "No". For instance, by easy inspection of $\varkappa_c$: since $\{b, c\}$ does not belong to $\varkappa_c$, $\varkappa_c$ is not stable under superset formation.

  (c) Table of ε:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	U
$\varepsilon(x)$	$\emptyset$	$\{a, c\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	U	U

  (i) The above map ε is not continuous. Indeed, $\varepsilon(\{b\}) \cup \varepsilon(\{c\}) = \{b, c\} \neq U = \varepsilon(\{b, c\}) = \varepsilon(\{b\} \cup \{c\})$. Intuitively, this happens because taken singularly, b is connected only with b, and c only with c, while taken together we get suddenly a new connection with a.

  (ii) The set $\{a\}$ does not have a closure. Indeed, the closed sets containing $\{a\}$ are $\{a, b\}, \{a, c\}$ and U; but their intersection is $\{a\}$ itself which is not closed.

- Exercise 12.8

  (a) Consider the pre-topological space of Example 12.4.5. It is easy to see that $\{x : \exists X(X \in \varkappa_x \ \& \ X \subseteq \{b, c\}\} = \{b, c\} \neq \{b\} = \{x : \{b, c\} \in \varkappa_x\} = \varkappa(\{b, c\})$.

  Another example is given by the neighborhood system $\mathcal{N}(U)$ of Example 12.4.2, where $\{x : \exists X(X \in \mathcal{N}_x \ \& \ X \subseteq \{a, b\}\} = U \neq \{b, c\} = \{x : \{a, b\} \in \mathcal{N}_x\} = \varkappa(\{a, b\})$.

(b) Let us compute $\varepsilon(\{a,b\})$ by exploiting the formula $\varepsilon(\{a,b\}) = \{x : \forall X(X \in \varkappa_x \Rightarrow X \cap \{a,b\} \neq \emptyset)\}$: a and b are obviously in $\varepsilon(\{a,b\})$. So test $\varkappa_c = \{\{a,c\},\{b,c\},\{a,b,c\}\}$: (i) $\{a,b\} \cap \{a,c\} = \{a\}$; (ii) $\{a,b\} \cap \{b,c\} = \{b\}$; (iii) $\{a,b\} \cap \{a,b,c\} = \{a,b\}$. Henceforth, $\varepsilon(\{a,b\}) = \{a,b,c\}$.

Notice, however, that $\varkappa_c$ is not a filter. It follows that some difference from topologies must be somewhere. Indeed, ε is not continuous: $\varepsilon(\{a\}) \cup \varepsilon(\{b\}) = \{a\} \cup \{b\} = \{a,b\} \neq \{a,b,c\} = \varepsilon(\{a,b\})$.

(c) A minimal basis for the pre-topology $\mathbf{P}_1$ of Example 12.6.2: $\varkappa_a = \uparrow \{a\}$, $\varkappa_b = \uparrow \{b\}$, $\varkappa_c = \uparrow \{a,c\}$. Therefore the required basis is given by $\mathcal{B}_a = \{\{a\}\}, \mathcal{B}_b = \{\{b\}\}, \mathcal{B}_c = \{\{a,c\}\}$.

(d) As X we can use the set $\bigcup\{\varkappa_x : x \in U\}$ in a P−system $\langle U, X, \in \rangle$:

$\in$	$\{a\}$	U	$\{b,c\}$	$\{a,c\}$	$\{b\}$
a	1	1	0	1	0
b	0	1	1	0	1
c	0	1	1	1	0

As usual, given a subset A of $\{a,b,c\}$, we denote the Galois closure of A by $ext(A)$. It is easy to verify that the operator ext corresponds to the operator ε of the pre-topology $\mathbf{P}_1$ of Example 12.6.2. For instance: (i) $[[i]](\{a,b\}) = \{B \in X : \forall g(g \in \{a,b\} \Rightarrow g \in B)\} = \{\{a,b,c\}\}$; (ii) $[[e]][[i]](U) = \{g \in \{a,b,c\} : \forall B(B \in \{\{a,b,c\}\} \Rightarrow g \in B)\} = ext(\{a\}) = \{a,b,c\}$.

- Exercise 12.9
  (a) The family $\mathscr{F} = \{\varkappa_a, \varkappa_b, \varkappa_c\}$ from the pre-topology $\mathbf{P}_2$ of Example 12.6.7: $\varkappa_a = \{\{a,b\}, U\}, \varkappa_b = \{\{a,b\}, U\}, \varkappa_c = \{\{b,c\}, U\}$.

  (b) By trivial inspection.

  (c) Compute a minimal basis for the pre-topology $\mathbf{P}_2$: $\mathcal{B}_a = \{\{a,b\}\}, \mathcal{B}_b = \{\{a,b\}\}, \mathcal{B}_c = \{\{b,c\}\}$.

  (d) Consider $\varkappa_c = \{\{b,c\}, U\}$. Take $\{b,c\}$: $b \in \{b,c\}$, but $\{b,c\} \notin \varkappa_b$. A fortiori, $\{b,c\}$ it is not even a $\varkappa$-neighborhood of all the elements of U.

- Exercise 12.10

  (a) Let us compute ε^1, ε^2, $\varkappa^1$ and $\varkappa^2$ by starting with the two families of bases $\mathfrak{B}^1 = \{\mathcal{B}^1_x\}_{x \in U_4}$ and $\mathfrak{B}^2 = \{\mathcal{B}^2_x\}_{x \in U_4}$:

  (a.1) (i) Let us compute $\{\mathcal{B}^1_x\}_{x \in U_4}$ and $\{\mathcal{B}^2_x\}_{x \in U_4}$: for any $x \in U_4$, $\mathcal{B}^1_x = \{E_1(x) \cup E_2(x)\}$ while $\mathcal{B}^2_x = \{E_1(x), E_2(x)\}$. Therefore we obtain:

$\mathcal{B}^m_x$	$\mathcal{B}^m_a$	$\mathcal{B}^m_b$	$\mathcal{B}^m_c$	$\mathcal{B}^m_d$
$\mathcal{B}^1_x$	$\{\{a,b,c\}\}$	$\{\{a,b,c\}\}$	$\{\{a,b,c,d\}\}$	$\{\{c,d\}\}$
$\mathcal{B}^2_x$	$\{\{a,b\},\{a,b,c\}\}$	$\{\{a,b\},\{a,b,c\}\}$	$\{\{c,d\},\{a,b,c\}\}$	$\{\{d\},\{c,d\}\}$

  In view of *Proposition* 12.6.7 the operators ε^1 and ε^2 are computed from $\mathfrak{B}^1$ and, respectively, $\mathfrak{B}^2$ using the definition $\varepsilon^m(A) = \{x : \forall X(X \in \mathcal{B}^m_x \Rightarrow X \cap A \neq \emptyset)\}$. For instance, let us compute $\varepsilon^2(\{a\})$:

  (i) Every element of $\mathcal{B}^2_a$ and of $\mathcal{B}^2_b$ contains a. Hence $a, b \in \varepsilon^2(\{a\})$.

  (ii) No element of $\mathcal{B}^2_c$ or $\mathcal{B}^2_d$ contains a. Hence $c, d \notin \varepsilon^2(\{a\})$.

  (iii) It follows: $\varepsilon^2(\{a\}) = \{a,b\}$.

  The operators $\varkappa^1$ and $\varkappa^2$ are computed from $\mathfrak{B}^1$ and, respectively, $\mathfrak{B}^2$ using the definition $\varkappa^m(A) = \{x : \exists X(X \in \mathcal{B}^m_x \ \& \ X \subseteq A)\}$. For instance, let us compute $\varkappa^2(\{a,d\})$:

  (i) No subset of $\mathcal{B}^2_a$, $\mathcal{B}^2_b$ or $\mathcal{B}^2_c$ is included in $\{a,d\}$. Hence $a,b,c \notin \varkappa^2(\{a,d\})$.

  (ii) The subset $\{d\}$ belongs to $\mathcal{B}^2_d$ and is included in $\{a,d\}$. Hence $d \in \varkappa^2(\{a,d\})$.

  (iii) It follows $\varkappa^2(\{a,d\}) = \{d\}$.

  Continuing this procedure we obtain:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	...	$\{c,d\}$	...	U_4
$\varepsilon^1(x)$	$\emptyset$	$\{a,b,c\}$	$\{a,b,c\}$	U_4	$\{c,d\}$	$\{a,b,c\}$	U_4	U_4	...	U_4	...	U_4
$\varkappa^1(x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	...	$\{d\}$	...	U_4
$\varepsilon^2(x)$	$\emptyset$	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{a,b,c\}$	U_4	...	$\{c,d\}$	...	U_4
$\varkappa^2(x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{d\}$	$\{a,b\}$	$\emptyset$	$\{d\}$	...	$\{c,d\}$	...	U_4

(a.2) The relation $E_* = E_1 \cap E_2$ runs as follows:

E_*	a	b	c	d
a	1	1	0	0
b	1	1	0	0
c	0	0	1	0
d	0	0	0	1

If we consider the rough set generated by the subset $\{c\}$, we trivially obtain $\langle \{c\}, \{c\} \rangle$. On the contrary, $\langle \varepsilon^1(\{c\}), \varkappa^1(\{c\}) \rangle = \langle U_4, \emptyset \rangle$ and $\langle \varepsilon^2(\{c\}), \varkappa^2(\{c\}) \rangle = \langle \{c\}, \emptyset \rangle$. It follows that neither $\langle U_4, \varepsilon^1, \varkappa^1 \rangle$ nor $\langle U_4, \varepsilon^2, \varkappa^2 \rangle$ coincides with the Approximation Space induced by $\langle U_4, E_* \rangle$.

Moreover, $\langle U_4, \varepsilon^2, \varkappa^2 \rangle$ cannot be an Approximation Space. Indeed an Approximation Space is a topological space, while $\Uparrow \mathcal{B}_c^2$ is not a filter, so that $\langle U_4, \varepsilon^2, \varkappa^2 \rangle$ is not even of type $\mathfrak{V}_D$.

(b) Let us follow the usual procedure and compute, first, the bases $\mathcal{B}_x^1$ and $\mathcal{B}_x^2$:

$\mathcal{B}_x^m$	$\mathcal{B}_a^m$	$\mathcal{B}_b^m$	$\mathcal{B}_c^m$
$\mathcal{B}_x^1$	$\{\{a,b\}\}$	$\{\{a,b\}\}$	$\{\{b,c\}\}$
$\mathcal{B}_x^2$	$\{\{a,b\}\}$	$\{\{b\}, \{a,b\}\}$	$\{\{b\}, \{c\}\}$

(b.1) Let us then compute ε^1, ε^2, $\varkappa^1$ and $\varkappa^2$ using $\mathscr{F}_x^n$:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	U_3
$\varepsilon^1(x)$	$\emptyset$	$\{a,b\}$	U_3	$\{c\}$	U_3	U_3	U_3	U_3
$\varkappa^1(x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\emptyset$	$\{c\}$	U_3
$\varepsilon^2(x)$	$\emptyset$	$\{a\}$	$\{a,b\}$	$\emptyset$	$\{a,b\}$	$\{a\}$	U_3	U_3
$\varkappa^2(x)$	$\emptyset$	$\emptyset$	$\{b,c\}$	$\{c\}$	U_3	$\{c\}$	$\{b,c\}$	U_3

It is easy to verify that $\langle U_3, \varepsilon^1, \varkappa^1 \rangle$ is indeed a pre-topological space. On the contrary $\langle U_3, \varepsilon^2, \varkappa^2 \rangle$ is not a pre-topological space. In fact, $\varepsilon^2(\{a,c\}) = \{a\}$. Hence ε^2 is not an inflationary operator. Dually $\varkappa^2(\{b\}) = \{b,c\}$, so that $\varkappa^2$ is not a deflationary operator. Why? It happens that R_1 is not a reflexive relation: $\langle c, c \rangle \notin R_1$. This fact prevents the construction of any pre-topological space upon R_1 or any family $\{R_i\}_{i \in I}$ of relations containing R_1, at least

when we deal with the finest operators ε^m and $\varkappa^m$, where m is the cardinality of I.

We conclude that on the one hand not every pre-topology is connected with a binary relation, and, on the other hand, not every binary relation induces a pre-topology.

- Exercise 12.11
  Directed graphs representing $R_1, R_2, R^B(\mathbf{P}^1)$ and $R^B(\mathbf{P}^2)$ of Example 12.7.4:
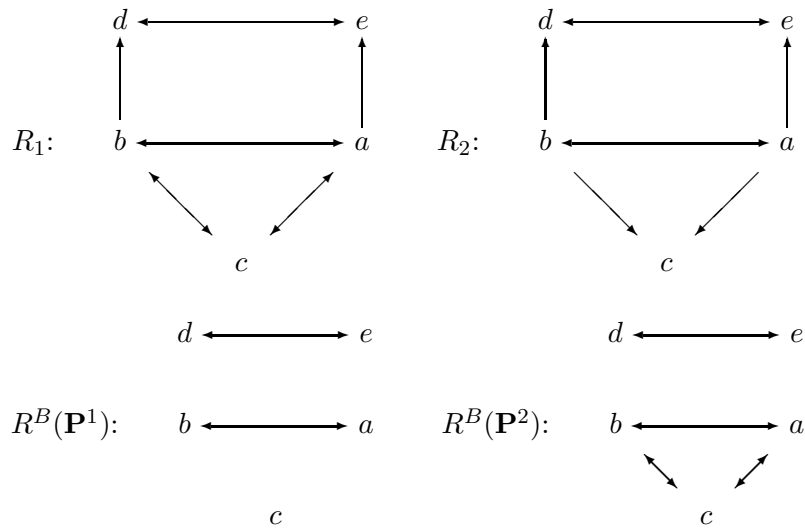
- Exercise 12.12

  Directed graphs representing R and $R^B(\mathbf{P}(R))$ of Example 12.7.4:
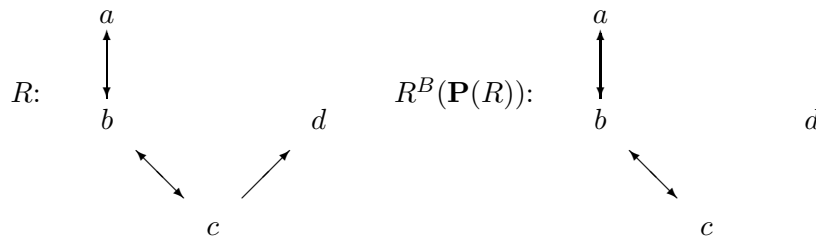
- Exercise 12.13

  (A)$\Rightarrow$: Let $\mathbf{P}$ be a topological space. Let $\mathfrak{B} = \{\mathcal{B}_x\}_{x \in U}$ be a base for $\mathbf{P}$. In view of the definition of $R^T(\mathbf{P})$, i.e. $\langle x, y \rangle \in R^T(\mathbf{P})$ if and

only if $y \in \bigcap \varkappa_x$, and in view of the equality $\varkappa_x = \Uparrow \mathcal{B}_x$, we can set $\langle x, y \rangle \in R^T(\mathbf{P})$ if and only if $y \in \bigcap \mathcal{B}_x$. Thus we can derive the properties of $R^T(\mathbf{P})$ from those of $\mathcal{B}_x$. First, $x \in \bigcap \mathcal{B}_x$, because $\mathbf{P}$ is a topological space, so that $x \in B$ for any $B \in \mathcal{B}_x$. Thus, $\langle x, x \rangle \in R^T(\mathbf{P})$ so that $R^T(\mathbf{P})$ is reflexive. Moreover, we have (i) $\bigcap \mathcal{B}_x \in \mathcal{B}_x$, because $\Uparrow \mathcal{B}_x$ is a filter, since $\mathbf{P}$ is a topological space. Suppose now $y \in \bigcap \mathcal{B}_x$, so that $\langle x, y \rangle \in R^T(\mathbf{P})$, and $z \in \bigcap \mathcal{B}_y$, so that $\langle y, z \rangle \in R^T(\mathbf{P})$. From *Proposition* 12.8.3 $\bigcap \mathcal{B}_x$ is open. Thus $\bigcap \mathcal{B}_x \in \mathcal{B}_w$ for any $w \in \bigcap \mathcal{B}_x$. Therefore $\bigcap \mathcal{B}_x \in \mathcal{B}_y$, which implies $\bigcap \mathcal{B}_x \supseteq \bigcap \mathcal{B}_y$. It follows that $z \in \bigcap \mathcal{B}_x$ and we obtain $\langle x, z \rangle \in R^T(\mathbf{P})$. Thus $R^T(\mathbf{P})$ is transitive.

(B) $\Leftarrow$: Conversely, suppose $R^T(\mathbf{P})$ is a preorder and $\mathbf{P}$ is of type $\mathfrak{V}_S$. From *Proposition* 12.7.8 $\mathbf{P} = \mathbf{P}(R^T(\mathbf{P}))$, because $\mathbf{P}$ is of type $\mathfrak{V}_S$. Therefore, from *Proposition* 12.8.2 we obtain immediately that $\mathbf{P}$ is a topological space.

A direct proof of this part runs as follow: Let $y \in \bigcap \varkappa_x$ and suppose $z \in \bigcap \varkappa_y$. Then $\langle x, y \rangle \langle y, z \rangle \in R^T(\mathbf{P})$. But $R^T(\mathbf{P})$ is a preorder, so that $\langle x, z \rangle \in R^T(\mathbf{P})$, by transitivity. Hence $z \in \bigcap \varkappa_x$. Thus, $\bigcap \varkappa_y \subseteq \bigcap \varkappa_x$. Moreover, $\bigcap \varkappa_y \in \varkappa_y$ because $\mathbf{P}$ is of type $\mathfrak{V}_S$. Therefore, $\bigcap \varkappa_x \in \varkappa_y$, because $\varkappa_y$ is a filter. It follows that $\bigcap \varkappa_x \in \varkappa_w$ for any $w \in \bigcap \varkappa_x$. Let A be any element of $\varkappa_x$. Since $A \supseteq \bigcap \varkappa_x$, $A \in \varkappa_w$ for any $w \in \bigcap \varkappa_x$, too. This means that for any $X \in \varkappa_x$, there is a $Y \in \varkappa_x$ such that $X \in \varkappa_y$ for any $y \in Y$. Which is exactly property (τ).   QED

- Exercise 12.14

  (a) The required graph G for $\Omega_\varkappa(U)$, where $\mathbf{P}$ is the pre-topological space of Example 12.4.8, is
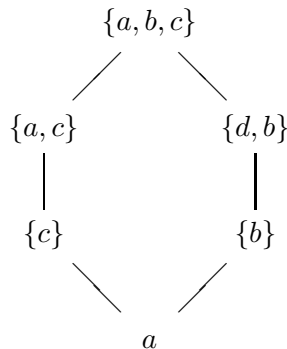
(b) $\Omega_{\not\preceq}(U)$ is not a lattice of sets. (c) $\Omega_{\not\preceq}(U)$ is not distributive.
(d) $\Omega_{\not\preceq}(U)$ is an ortholattice.

- Exercise 14.1 Trivial.

- Exercise 14.2 Immediate.

- Exercise 14.3.

  (a) According to *Exercise* 12.5, $\mathbf{L}'(\mathbf{A}) = \mathbf{AS}(G/R)$ (in this equation $\mathbf{AS}(G/R)$ is the lattice of the Example of Frame 10.4 of Part II). These two lattices are Boolean algebras with atoms given by the equivalence classes of an equivalence relation R on $\{a,b,c\}$ (i.e. the specialization preorder $\preceq$ of *Exercise* 12.5) such that $R = \{\{a,b\},\{c\}\}$. It follows that $L_R(X) = (lR)(X)$ and $M_R(X) = (uR)(X)$, any $X \subseteq \{a,b,c\}$. We know from Frame 10.4 that $\mathbf{L}(\mathbf{AS}(G/R)) \cong \mathbf{L_3}$, where $\mathbf{L}$ is the operator of *Proposition* 8.3.1. Hence, the carrier of $\mathbf{L}(\mathbf{AS}(G/R))$ is $\mathcal{RS}_{\{b\}}(\mathbf{AS}(G/R)) = \{\langle (uR)(X), (lR)(X)\rangle : X \subset \{a,b,c\}\}$. Therefore, $\mathbf{ML}(\mathbf{L}'(\mathbf{A})) \cong \mathbf{L}(\mathbf{AS}(G/R)) \cong \mathbf{L_3}$.

  (b) The diagram of $\mathbf{TQ}(\mathbf{A})$ is the following:

$$
\begin{array}{ccccc}
 & & \langle 1,1\rangle & & \\
 & \langle 1,e\rangle & & \langle 1,b\rangle & \\
\langle e,e\rangle & & \langle 1,0\rangle & & \langle b,b\rangle \\
 & \langle e,0\rangle & & \langle b,0\rangle & \\
 & & \langle 0,0\rangle & &
\end{array}
$$
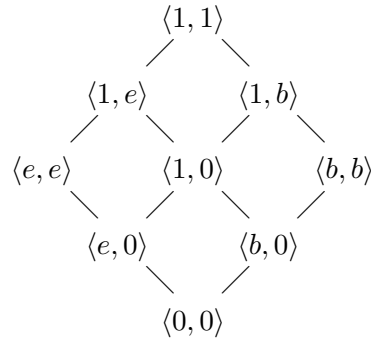
  In this diagram we have the elements $\langle 1,e\rangle, \langle 1,0\rangle$ and $\langle b,0\rangle$ which are isomorphic to the elements $\langle\{a,b,c\},\{a,c\}\rangle, \langle\{a,b,c\},\emptyset\rangle$ and $\langle\{b\},\emptyset\rangle$. But these ordered pairs are not of the form $\langle (lR)(X), (lR)(X)\rangle$ since both the components do not coincide on the set $\{b\}$ (i.e. the union of the – only one in this case- singletons).

  (c) A necessary and sufficient condition to have $\mathbf{TQ}(\mathbf{A}) = \mathbf{ML}(\mathbf{L})$ is that for any atom p of $\mathbf{A}$, $L(p) = p$. Indeed, in $\mathbf{L_L}$ we have that for any singleton (atom) X, $(lR)(X) = X$ (where R is the equivalence relation – i.e. specialization preorder – induced by

L$_L$). In this case $B = \bigcup\{X \in \mathbf{L_L} : card(X) = 1\} = \mathcal{J}(\mathbf{A})$, that is, the union of exact pieces of information coincide with the entire universe of discourse, and the results of Part II provide us with the result.

- Exercise 15.1

Suppose R is Serial, Symmetric and Euclidean and that we are given two points w and z (the case with one point is trivial). If neither wRz nor zRw, then since R is serial we must have both wRw and zRz, and we are done (R is also reflexive). Suppose now wRz (or zRw). Then:

$$\frac{1:\ wRz\ (hypothesis)\ \ 2:\ wRz\ (hypothesis)}{zRz\ \ (1,2:\ from\ the\ Euclidean\ property)}$$

Since R is also symmetric, we obtain, analogously, wRw, that is, R is also reflexive. Notice that seriality is essential.

Chapter 16

Mathematical Toolkits

16.1 A Mathematical Toolkit: Orders

1. A relation $\preccurlyeq$ on a set A is said to be a *preorder* if and only if it is (i) *transitive* ($a \preccurlyeq b$ and $b \preccurlyeq c$ imply $a \preccurlyeq c$) and (ii) *reflexive* ($a \preccurlyeq a$).

2. A *partial order* is a preorder fulfilling, in addition, (iii) *antisymmetry* ($a \preccurlyeq b$ and $b \preccurlyeq a$ imply $a = b$).

3. An *equivalence relation* is a preorder fulfilling, in addition (iv) *symmetry* ($a \preccurlyeq b$ implies $b \preccurlyeq a$).

4. If $\langle A \leq \rangle$ is a partial order, for any $X \subseteq A$ we set $min(X) = a$ if and only if for all $x \in X, a \leq x$ and, dually, $max(X) = a$ if and only if for all $x \in X, a \geq x$. Clearly, $min(X)$ and $max(X)$ may not exist for a given X.

5. For any preordered set $\mathbf{A} = \langle A, \preccurlyeq \rangle$, for any $X \subseteq A$ and $x \in A$, we define:

   (a) $\uparrow X = \{y : \exists x (x \in X \ \& \ x \preccurlyeq y)\}$ – *order filter* generated by X. In particular if $p \in A$ then $\uparrow p = \uparrow \{p\}$ is called the *principal order filter generated by p* and if $\mathbf{A}$ is a partial order then $p = min(\uparrow p)$.

   (b) $\downarrow X = \{y : \exists x (x \in X \ \& \ y \preccurlyeq x)\}$ – *order ideal* generated by X. In particular if $\forall p \in A$ then $\downarrow p = \downarrow \{p\}$ is called the *principal order ideal generated by p* and if $\mathbf{A}$ is a partial order then $p = max(\downarrow p)$.

Let $\mathbf{A}$, $\mathbf{A}'$ and $\mathbf{A}''$ be three partially ordered sets and $\phi : A \longmapsto A'$ a map. Then,

6. ϕ is called an *order-homomorphism* if and only if $x \leq y$ implies $\phi(x) \leq' \phi(y)$ (isotonicity);

7. ϕ is called an *order-anti-homomorphism* if and only if $x \leq y$ implies $\phi(x) \geq' \phi(y)$ (antitonicity);

8. if $\phi : A \longmapsto A'$, $\psi : A' \longmapsto A''$ and both ϕ and ψ are isotone, then $\phi \circ \psi$ is isotone.

_ _

16.2   A Mathematical Toolkit: Functions

1. If $f : A \longmapsto B$ is a function, then A is called the *domain* of f and B the *codomain* of f. The term *map* is sometimes used as a synonymous of "function".

2. If $f : A \longmapsto B$ and both A and B are sets, then in some contexts we shall call f a *morphism*. If A and B are equipped with a structure, then the term "morphism" will denote a structure-preserving mapping.

3. If $f : A \longmapsto A$, then f is called an *endomorphism* (of A).

4. If $f : A \longmapsto B$ and both A and B are ordered sets, then f is said to be *monotonically increasing* (or *monotone*, *isotone*, *monotonic*, *order preserving*), if for all $x, y \in A$, $x \leq y$ implies $f(x) \leq f(y)$.
   If $x \leq y$ implies $f(x) \geq f(y)$ then f is said to be *monotonically decreasing* or *order reversing*.
   If $x \leq y$ if and only if $f(x) \leq f(y)$, then f is called *order embedding*. Finally, if f is order embedding and surjective, then it is called an *order isomorphism*.

5. If $g : B \longmapsto C$ is another function, then with $(f \circ g)(x)$ or, equivalently, $g(f(x))$ we denote the *composition of g after f*, with $x \in A$. The composition $f \circ g$ is allowed only if the domain of g coincides with the codomain of f.
   In what follows, we shall denote a string of applications $f(g(h(...)))$ with $fgh(...)$ as well. We shall write $h \circ g \circ f(...)$ or use parenthesis just to put in evidence some particular string.

6. If $f : A \longmapsto B$ is a function, then we can extend f to subsets of A in two canonical ways:

   (a) $f^{\rightarrow} : \wp(A) \longmapsto \wp(B); f^{\rightarrow}(X) = \{f(a) : a \in X\}$ – the *direct image* of X via f,

   (b) $f^{\leftarrow} : \wp(B) \longmapsto \wp(A); f^{\leftarrow}(Y) = \{a : f(a) \in Y\}$ – the *inverse image of Y via f*.

   The set $f^{\rightarrow}(A)$ is denoted by Im_f and $f^{\leftarrow}(B)$ is also called the *pre-image of f*. If f is a total function, $(f^{\rightarrow} \circ f^{\leftarrow})(A) = A$. If f' is another function with domain A and for all $a \in A$, $f(a) = f'(a)$, then we say that $f = f'$ (clearly, this happens only if $Im_{f'} = Im_f$). Sometimes instead of $f^{\leftarrow}$ we can use f^{-1} (specially if f is injective – see below).

7. The map $1_A : A \longmapsto A; 1_A(x) = x$, is called the *identity function on A*.

8. If $f : A \longmapsto A$ and $f \circ f = f$, then f is said to be *idempotent* on A.

9. A function $f : A \longmapsto B$ is said to be *onto* or *surjective* or *epic* if $Im_f = B$, hence if $f^{\leftarrow} \circ f^{\rightarrow} = 1_B^{\rightarrow}$. It is said to be *into*, or *injective or monic*, if $f^{\rightarrow} \circ f^{\leftarrow} = 1_A^{\rightarrow}$. In particular, if $A \subseteq B$ and $f = 1_A$, then f is called an *inclusion* and denoted by the symbol *"in"*.

   f is said to be *bijective* or *isomorphisms* or *iso* if f is both epic and monic.

   The map $f^o : A \longmapsto Im_f; f^o(a) = f(a)$ is called the *corestriction* of f to Im_f.

   It is immediate that f^o is surjective.

   Dually, the map $f_o : Im_f \longmapsto B; f_o(b) = b$ is called the *inclusion* of Im_f into B. Clearly, f_o is injective, $f_o = f \circ in$ and $f = f^o \circ f_o$. Moreover, if f is idempotent on A, then $f_o \circ f^o = 1_{Im_f}$.

10. Given a function $f : A \longmapsto B$, the relation $k_f = \{\langle a, a' \rangle : f(a) = f(a')\}$ is called the *kernel* of f or the *fibred product* $A \times_B A$ obtained by pulling back f along itself. From the very definition, the kernel of a function is an equivalence relation. If E is an equivalence relation, let us denote with $[a]_E$ the equivalence class of a modulo E and with nat_E the *natural map* $A \longmapsto$

$\wp(A); nat_E(a) = [a]_E$. One can prove that any function f can be decomposed starting from its kernel k_f and the induced natural map nat_{k_f}. Three steps are enough: first, send all the elements of the domain into the appropriate equivalence class modulo the kernel k_f, via the natural map nat_{k_f}. Then send back the kernel to the image of the function f by means of a bijection g from the quotient space modulo the kernel A_{k_f}. Finally, embed the image of the function into its codomain, via an inclusion function in. In fact, let $f : A \longmapsto B$ be a function and g the map $g : A/_{k_f} \longmapsto B$; $g([a]_{k_f}) = f(a)$. Then, g is *iso* and for all $a \in A$, $nat_{k_f}(a) \circ g = f(a)$ (see below Relations 16.5.1). This is the content of the so-called *First Homomorphism Theorem* (see later on at the beginning of Section 1.4.3). The informational content of this result is quite evident: if two elements a and a' are B-evaluated in the same manner, then we group them under the same collective name, or class. Therefore the family of collective names and that of possible evaluation are clearly linked by a 1-1 map.

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16.3   A Mathematical Toolkit: Lattices

1. Let $\mathbf{O} = \langle A, \leq \rangle$ be a partial ordered set, $a, a_1, ...b...$ arbitrary elements of A and X a subset of A. Let us set an operation *sup* (or $\vee$ or *join*, or *least upper bound*, or *lub*) and an operation *inf* (or $\wedge$, or *meet*, or *greatest lower bound*, or *glb*), as follows:

   (a) $X^u = \{a : \forall x(x \in X \Rightarrow x \leq a)\}$, upper bounds of X.

   (b) $X^l = \{a : \forall x(x \in X \Rightarrow x \geq a)\}$, lower bounds of X.

   (c) $\bigvee X = min(X^u)$; if $X = \{a_1, a_2, \ldots, a_n\}$ then we write $a_1 \vee a_2 \ldots \vee a_n$ as $\bigvee X$.

   (d) $\bigwedge X = max(X^l)$ if $X = \{a_1, a_2, \ldots, a_n\}$ then we write $a_1 \wedge a_2 \ldots \wedge a_n$ as $\bigwedge X$.

   (e) We have that $a \leq b$ if and only if $a \vee b = b$ and $a \leq b$ if and only if $a \wedge b = a$. Thus we can regard any lattice also as a partially ordered set.

(f) If $\vee$ is defined for any pair of elements of A, then $\mathbf{L} = \langle A, \vee \rangle$ is called a *sup-semilattice*.

(g) If $\wedge$ is defined for any pair of elements of A, then $\mathbf{L} = \langle A, \wedge \rangle$ is called an *inf-semilattice*.

(h) If both operations are defined for any pair of elements of A, then $\mathbf{L} = \langle A, \vee, \wedge \rangle$ is called a *lattice*.

(i) If for all $X, \emptyset \neq X \subseteq A$, $\bigwedge X$ and $\bigvee X$ exist, then the lattice is called *complete*.

(j) $\bigvee \emptyset = 0$ is called a *bottom element* and $\bigwedge \emptyset = 1$ is called a *top element*.
   We have, by applying the implication in the definition of $\vee$ and $\wedge$ to an empty set, $1 \vee x = 1, 1 \wedge x = x, 0 \wedge x = 0, 0 \vee x = x$.

(k) If $0 \in A$ then $\mathbf{L}$ is called *lower bounded*. If $1 \in A$ then $\mathbf{L}$ is called *upper bounded*. If $1, 0 \in A$ then $\mathbf{L}$ is called *bounded*.

(l) if for all $a, b, c \in L, a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, then we say that $\mathbf{L}$ is *inf-distributive* or that $\wedge$ distributes over $\vee$. If $a, b, c \in L, a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, then we say that $\mathbf{L}$ is *sup-distributive* or that $\vee$ distributes over $\wedge$.

From now on by $\mathbf{L}$ and $\mathbf{L}'$ we shall denote two arbitrary complete and bounded lattices $\mathbf{L} = \langle L, \vee, \wedge, 0, 1 \rangle$ and, respectively, $\mathbf{L}' = \langle L', \vee', \wedge', 0', 1' \rangle$. If necessary, we shall use also notations such as $\wedge_{\mathbf{L}}, 1_{\mathbf{L}'}$ and the like.

2. Let $\mathbf{L}$ be a lattice and $A \subseteq L$. If for all $a, b \in A, a \vee b \in A$ and $a \wedge b \in A$, then $\mathbf{A} = \langle A, \wedge, \vee, 1, 0 \rangle$ is said to be a *sublattice* of $\mathbf{L}$. If $\mathbf{L}'$ is a lattice included in $\mathbf{L}$ but such that the operations in $\mathbf{L}'$ and in $\mathbf{L}$ do not coincide, then $\mathbf{L}'$ is said to be a *substructure* of $\mathbf{L}$.

3. (a) If $x \in \mathbf{L}$ is such that for all $X \subseteq L, x \leq \bigvee X \implies x \in X$, then x is called a *co-prime* element. The family of co-prime elements of $\mathbf{L}$ is denoted as $\mathcal{J}(\mathbf{L})$.

   (b) Dually, if for all $X \subseteq L, \bigwedge X \leq x \implies x \in X$, then x is called a *prime* element. The family of prime elements of $\mathbf{L}$ is denoted as $\mathcal{M}(\mathbf{L})$.

4. Let $\vartheta : \mathbf{L} \longmapsto \mathbf{L}'$ be a map between two lattices and let x and y be arbitrary elements of L. Then,

(a) ϑ is a *sup-homomorphism* (or "additive") if $\vartheta(x \vee y) = \vartheta(x) \vee' \vartheta(y)$.

(b) ϑ is an *inf-homomorphism* (or "multiplicative") if $\vartheta(x \wedge y) = \vartheta(x) \wedge' \vartheta(y)$.

(c) ϑ is a *sup-anti-homomorphism* (or "anti-additive") if $\vartheta(x \vee y) = \vartheta(x) \wedge' \vartheta(y)$.

(d) ϑ is a *inf-anti-homomorphism* (or "anti-multiplicative") if $\vartheta(x \wedge y) = \vartheta(x) \vee' \vartheta(y)$.

(e) ϑ is a *0-morphism* (or "normal") if $\vartheta(0) = 0'$.

(f) ϑ is a *1-morphism* (or "co-normal") if $\vartheta(1) = 1'$.

(g) ϑ is a *0-anti-morphism* (or "anti-normal") if $\vartheta(0) = 1'$.

(h) ϑ is a *1-anti-morphism* (or "anti-co-normal") if $\vartheta(1) = 0'$.
    Clearly, if ϑ is a *sup*-homomorphism (or an *inf*-homomorphism), then ϑ is isotonic. In fact, since $x \leq y$ iff $y = x \vee y$, from *sup*-homomorphism we obtain $\vartheta(y) = \vartheta(x) \vee' \vartheta(y) \geq' \vartheta(x)$ (dually for *inf*-homomorphisms).
    Symmetrically, if ϑ is a *sup*-anti-homomorphism (or an *inf*-anti-homomorphism), then ϑ is antitonic. In fact, since $x \leq y$ iff $y = x \vee y$ iff $x = x \wedge y$, from *sup*-anti-homomorphism we obtain $\vartheta(y) = \vartheta(x) \wedge' \vartheta(y) \leq' \vartheta(x)$ (dually for *inf*-anti-homomorphisms).

(i) If $\mathbf{L} = \mathbf{L}'$ we turn the terms "*homomorphism*" and "*morphism*" into the term "*endomorphism*". Moreover, if ϑ is an endomorphism on $\mathbf{L}$,

   (i) If $x \leq \vartheta(x)$, then ϑ is said to be *increasing* (or "inflationary").

   (ii) If $x \geq \vartheta(x)$, then ϑ is said to be *decreasing* (or "deflationary").

5. Let $f : \mathbf{L} \longmapsto X$ be a function from a lattice $\mathbf{L}$ to a set X. Then the kernel k_f is a *congruence* on $\mathbf{L}$, that is, for any operation $\odot$ in $\mathbf{L}$, if $a \equiv_{k_f} b$ and $a' \equiv_{k_f} b'$ then $a \odot a' \equiv_{k_f} b \odot b'$.

6. Finally we report the so-called *First Homomorphism Theorem* (for lattices) which states:
   Let $\mathbf{L}$ and $\mathbf{L}'$ be two lattices and let $\phi : \mathbf{L} \longmapsto \mathbf{L}'$ be a homomorphism. Then the map $\psi : \mathbf{L}/_{k_\phi} \longmapsto \mathbf{L}'; \psi([a]_{k_\phi}) = \phi(a)$ is an isomorphism. Furthermore, for all $a \in \mathbf{L}$, $nat_{k_\phi}(a) \circ \psi = \phi(a)$.

16.4 A Mathematical Toolkit: Topology

1. A topological space $\langle X, \Omega(X) \rangle$ consists of a set X and a family $\Omega(X)$ of subsets of X, called a *topology* or *frame of open subsets* X, such that

   - $\emptyset \in \Omega(X)$ and $X \in \Omega(X)$.
   - A finite intersection of members of $\Omega(X)$ is in $\Omega(X)$.
   - An arbitrary union of members of $\Omega(X)$ is in $\Omega(X)$.

   The members of $\Omega(X)$ are called *open sets*.

2. Given a topological space we define a subset of X to be *closed* if it belongs to $\Gamma(X) = \{X \cap -U : U \in \Omega(X)\}$. Therefore, any closed set is the complement of an open set. The family $\Gamma(X)$ is closed under arbitrary intersections and fine unions.

3. A set which is both open and closed is called *clopen*. If $\Omega(X)$ consists of only clopen sets, then the topology is called $0 - dimensional$.

4. A topological space is said to be *connected* if its only clopen subsets are the whole space and the empty set.

5. Let $\langle X, \Omega(X) \rangle$ be a topological space. Let $\mathbb{C} : \wp(X) \longmapsto \wp(X)$ be a map such that for any $A \subseteq X$, $\mathbb{C}(A)$ is the least closed subset of the topology which includes A. Then $\mathbb{C}$ is said to be a *closure operator* induced by the topology.

6. Let $\langle X, \Omega(X) \rangle$ be a topological space. Let $\mathbb{I} : \wp(X) \longmapsto \wp(X)$ be a map such that for any $A \subseteq X$ $\mathbb{I}(A)$ is the largest open subset of the topology included in A. Then $\mathbb{I}$ is said to be an *interior operator* induced by the topology.

7. For every interior operator $\mathbb{I}$, closure operator $\mathbb{C}$ and $A \subseteq X$, the following holds:

   (a) $\mathbb{C}\mathbb{C}(A) = \mathbb{C}(A)$.

   (b) $A \subseteq \mathbb{C}(A)$.

   (c) $\mathbb{I}\mathbb{I}(A) = \mathbb{I}(A)$.

   (d) $\mathbb{I}(A) \subseteq A$.

(e) $-\mathbb{I}(A) = \mathbb{C}(-A)$.

(f) $-\mathbb{C}(A) = \mathbb{I}(-A)$.

8. Suppose $S \subset \wp(X)$ and $\emptyset \in S$. If S is closed under finite inter-
   sections, then we can obtain a topology $\Omega(X)$ by closing S under
   unions. In this case S is said to be a *basis* for $\Omega(X)$.
   If the closure of S under finite intersections is a basis for $\Omega(X)$,
   then S is said to be a *subbasis* for $\Omega(X)$.

9. Let x and y be two points in a topological space $\langle X, \Omega(X) \rangle$. We
   say that y *specializes* x, and write $x \preceq y$, if and only if for every
   open A, if $x \in A$ then $y \in A$. $\preceq$ is said to be a *specialization
   preorder*.

10. A topological space $\langle X, \Omega(X) \rangle$ is said to be a $T_0 - space$ if for any
    two distinct points $x, y \in X$ there exists an open set containing
    exactly one of them.
    A topological space is T_0 if and only if its specialization preorder
    is antisymmetric.

11. A topological space $\langle X, \Omega(X) \rangle$ is said to be T_1 if for all $x, y \in X$,
    $x \preceq y$ if and only if $x = y$ (the specialization ordering is *discrete*).

12. A topological space $\langle X, \Omega(X) \rangle$ is said to be T_2 or a *Hausdorff
    space* if for any two distinct points $x, y \in X$ there exist two open
    disjoint sets A, B such that $x \in A$ and $y \in B$. Any Hausdorff
    space is T_1.

13. A topological space $\langle X, \Omega(X) \rangle$ is said to be *totally disconnected*
    if it is Hausdorff and $A \cup B = X$.

14. Given a preordered set $\mathbf{X} = (X, \leq)$, the family of order-filters
    $\Omega_A(\mathbf{X}) =$
    $\{ \uparrow_{\leq} X' : X' \subseteq X \}$ is called the *Alexandrov topology* over $\mathbf{X}$.

15. Let $\langle X, \Omega(X) \rangle$ and $\langle Y, \Omega(Y) \rangle$ be two topological spaces, $f : X \longmapsto$
    Y a map. If $f^{\leftarrow}(A)$ is open in X whenever A is open in Y, then
    f is said to be *continuous*.

    If f is bijective and both f and f^{-1} are continuous, then f is said
    to be an *homeomorphism*.

16.5   A Mathematical Toolkit: Relations

If A and B are sets, $A \times B$ will denote the Cartesian product $\{\langle a_i, b_j \rangle : a_i \in A \ \& \ b_J \in B\}$.

Let $R \subseteq A \times B$ and $Q \subseteq C \times D$ be binary relations, $X \subseteq A$, $Y \subseteq B$, $x \in A$, $y \in B$. Then we define:

1. $R^{\smile} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$ – the *inverse relation* of R. $R^{\smile} \subseteq B \times A$ and $R^{\smile\smile} = R$.

2. $R(X) = \{y \in B : \exists x(x \in X \wedge \langle x, y \rangle \in R)\}$, – the *left Peirce product of R and X*. We shall also call $R(X)$ the *R– neighborhood* of X. In particular, if X is a singleton $\{x\}$, then we shall usually write $R(x)$ instead of $R(\{x\})$ and, clearly, $R(x) = \{y : \langle x, y \rangle \in R\}$. It is immediate to verify that $R(X) = \bigcup_{x \in X} \{R(x)\}$, for any $X \subseteq A$. The reader is invited to distinguish between the relation R and the operation $R(\ldots)$. Because of the existential quantification in its definition, the operation $R(\ldots)$ is isotonic with respect to subset relation.[1]

3. $R^{\smile}(Y) = \{x \in A : \exists y(y \in Y \wedge \langle x, y \rangle \in R)\}$ – the *right Peirce product of R and Y*, or the left Pierce product of $R^{\smile}$ and Y. Clearly, $R^{\smile}(Y)$ is an R-neighborhood, too.[2] Clearly, $a \in R^{\smile}(Y)$ if and only if $R(a) \cap Y \neq \emptyset$. In fact $a \in R^{\smile}(Y)$ if and only if $\exists y(y \in Y \ \& \ \langle a, y \rangle \in R)$. It follows that $y \in R(a)$, too.

4. $R \implies X = \{y \in B : \forall x(\langle x, y \rangle \in R \Rightarrow x \in X)\}$ – the *right residual* of R and X.

5. $R^{\smile} \impliedby X = \{y \in B : \forall x(x \in X \Rightarrow \langle x, y \rangle \in R)\}$ – the *left residual* of X and $R^{\smile}$.[3]

6. $R \otimes Q = \{\langle a, d \rangle : \exists z \in B \cap C(\langle a, z \rangle \in R \ \& \ \langle z, d \rangle \in Q)\}$ – the *right composition of R with Q* or the *left composition of Q with R*.

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[1] The left Peirce product of R and X is sometimes denoted by $X : R$. Some authors call $R(a)$ the "extension of a along R".

[2] The right Peirce product of R and X is sometimes denoted by $R : Y$.

[3] These operations, as they stand, seem new in the literature. Indeed they are compositions of residuation operations between relations of a certain kind and Peirce products, as detailed in Frame 15.18.2 of Part III.

If defined, $R \otimes Q \subseteq A \times D$. Moreover, notice that by $(R \otimes Q)(X)$, we shall intend the application $Q(R(X))$.[4]

7. The set $\Delta_A = \{\langle x, x \rangle : x \in A\}$ is called the *diagonal relation* of A.

8. If $R^\smile \otimes R \supseteq \Delta_B$ then R is called *surjective* or *onto*.

9. One can prove that R is *functional* if (i) $R \otimes R^\smile \supseteq \Delta_A$ and (ii) $R^\smile \otimes R \subseteq \Delta_B$ hold.
   Condition (ii) states, trivially, that $R(a)$ is a singleton. In fact, if $\langle a, b \rangle \in R$ and $\langle a, b' \rangle \in R$, for $b \neq b'$, then both $\langle b, a \rangle$ and $\langle b', a \rangle$ belong to $R^\smile$, so that $\langle b, b' \rangle \in R^\smile \otimes R$. Jointly, conditions (i) and (ii) state that each element of A is related with exactly one element of B. Therefore, functional relations are functions in relational guise. If R is a functional relation, we shall denote with $\hat{R}$ the corresponding function. If f is a function, $\hat{f}$ will denote the corresponding functional relation. However, if there is not risk of confusion by abuse of language we generally shall not use distinct symbols for a functional relation and its corresponding function. Let $R \subseteq A \times B$ be a functional relation between two sets A and B. Then from the fact that the corresponding functions of Δ_A and Δ_B are 1_A and, respectively, 1_B we have:

   (a) $R(A) = B$ (i.e. $\hat{R}$ is *onto* or *surjective or epic*) if $R^\smile \otimes R = \Delta_B$;

   (b) $R(x) \neq R(x')$, if $x \neq x'$ (i.e. $\hat{R}$ is *into*, or *injective*, or $1 - 1$, or *monic*) if $R \otimes R^\smile = \Delta_A$.

   If $\hat{R}$ is both monic and epic, hence an isomorphism, then R is called an *isomorphism relation*.

10. R is said to be *symmetric* if for every $x, y \in A$, $\langle x, y \rangle \in R$ implies $\langle y, x \rangle \in R$.

11. R A is said to be *antisymmetric* if for every $x, y \in A$, $\langle x, y \rangle \in R$ and $\langle y, x \rangle \in R$ implies $x = y$.

12. R is said to be *reflexive* if for every $x \in A$, $\langle x, x \rangle \in R$.

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[4]The left composition of R with Q is sometimes denoted by $R;Q$, in mathematical literature. For reasons that will be clear in Part III, we use, instead, a symbol from Non-commutative Linear Logic.

13. R is said to be *transitive* if for every $x, y, z \in A$, $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ implies $\langle x, z \rangle \in R$.

14. Let $R \subseteq A \times A$.

   (a) If R is symmetric and reflexive, then it is said to be a *tolerance relation*.

   (b) A tolerance relation which, in addition, is transitive, is called an *equivalence relation*.

   (c) If R is reflexive and transitive, then it is called a *preorder*.

   (d) A preorder which, in addition, is antisymmetric, is called a *partial order*.

   Facts:

   (S) If $R^{\smile} = R$ – equivalently $X \subseteq (R \Longrightarrow (R(X)))$, any $X \subseteq A$ – then R is symmetric.

   (R) If $R \supseteq \Delta_A$ – equivalently $X \subseteq R(X)$, any $X \subseteq A$ – then R is reflexive.

   (T) If $R \otimes R \subseteq R$ – equivalently $R(R(X)) \subseteq R(X)$, any $X \subseteq A$ – then R is transitive.

15. (a) For any relation $R \subseteq A \times B$, $R \otimes R^{\smile}$ is a tolerance relation on A. (b) If R is a functional relation then $R \otimes R^{\smile}$ is an equivalence relation on A.

   Proof. (a) Symmetry and reflexivity are proved exactly as in point 9 above. (b) Moreover, suppose R is a map and that transitivity does not hold in $R \otimes R^{\smile}$. Then for some $a, a', a'' \in A$, $\langle a, a' \rangle, \langle a', a'' \rangle \in R \otimes R^{\smile}$, but $\langle a, a'' \rangle \notin R \otimes R^{\smile}$, Thus we have at least $\langle a, b \rangle, \langle a', b \rangle, \langle a', b' \rangle, \langle a'', b' \rangle \in R$. But this is impossible, because otherwise R would not be functional on a'. Hence $\langle a'', b \rangle \in R$ so that $\langle a, a'' \rangle \in R \otimes R^{\smile}$.

16.5.1   Pull-Backs and Kernels

16.5.1.1   Categorization and Kernels

Proposition 16.5.1. *Let f be a function, then $\kappa_f = \hat{f} \otimes \hat{f}^{\smile}$, where $\hat{f}$ is the corresponding functional relation.*

The above statement is very useful since composition of binary relations is an easy operation to compute. Clearly, $\kappa_f(a)$ is a fibre, for any $a \in A$. Let us prove this statement through a pair of lemmata:

Lemma 16.5.1. *Let $\hat{f} \subseteq A \times B$ be a functional relation. Then $\hat{f} \otimes \hat{f}^{\smile}$ is an equivalence relation.*

Proof. Obviously, by definition of a function, $\hat{f} \otimes \hat{f}^{\smile} \supseteq \Delta_A$. Thus reflexivity holds. Moreover, from *Proposition* 2.1.1. $\hat{f} \otimes \hat{f}^{\smile} = (\hat{f}^{\smile})^{\smile} \otimes \hat{f}^{\smile} = (\hat{f} \otimes \hat{f})^{\smile}$. Hence symmetry holds.[5] Finally, since $\hat{f}^{\smile} \otimes \hat{f} \subseteq \Delta_B$, we have $\hat{f} \otimes (\hat{f}^{\smile} \otimes \hat{f}) \otimes \hat{f}^{\smile} \subseteq \hat{f} \otimes \Delta_B \otimes \hat{f}^{\smile} = \hat{f} \otimes \hat{f}^{\smile}$, so that transitivity holds, too.   QED

Lemma 16.5.2. *Let $\hat{f} \subseteq A \times B$ be a functional relation. Then for any $a, a' \in A$, $\langle a, a' \rangle \in \hat{f} \otimes \hat{f}^{\smile}$ if and only if $f(a) = f(a')$, where f is the corresponding function.*

Proof. $\langle a, a' \rangle \in \hat{f} \otimes \hat{f}^{\smile}$ if and only if, for some $b \in B$, $\langle a, b \rangle \in \hat{f}$ and $\langle b, a' \rangle \in \hat{f}^{\smile}$; hence $\langle a', b \rangle \in \hat{f}$. It follows that $\langle a, a' \rangle \in \hat{f} \otimes \hat{f}^{\smile}$ if and only if $f(a) = f(a')$.   QED

Finally we can prove that any function f can be decomposed starting from its kernel κ_f. Three steps are enough: first, send all the elements of the domain into the appropriate equivalence class modulo the kernel κ_f, via the natural map nat_{κ_f}. Then send back the kernel to the image of the function by means of a bijection f' from the quotient space modulo the kernel κ_f. Finally, embed the image of the function into its codomain, via an inclusion function in.

This is the meaning of the following decomposition theorem:

Theorem 16.5.1. *Let $f : A \longmapsto B$ be a function. Then $f = nat_{\kappa_f} \circ f' \circ in$, where $f' : A/\kappa_f \longmapsto Im_f$ is a bijection and $in : Im_f \longmapsto B$ is the inclusion function.*

Proof. Obviously, $Im_f \subseteq B$ and $\langle a, a' \rangle \in \kappa_f$ if and only if $f(a) = f(a')$. Therefore, if we define f' as $f'([x]_{\kappa_f}) = f(x)$ we obtain a bijection between A/κ_f and Im_f. It follows that given $a \in A$, $nat_{\kappa_f}(a) = [a]_{\kappa_f}$, $f'([a]_{\kappa_f}) = f(a)$ and, finally, $in(f(a)) = f(a)$.   QED

---

[5]The previous equations means that if $\langle a, a' \rangle \in \hat{f}$, then there is a b such that $\langle a, b \rangle \in \hat{f}$ and $\langle a', b \rangle \in \hat{f}$, so that $\langle a', a \rangle \in \hat{f}$, too.
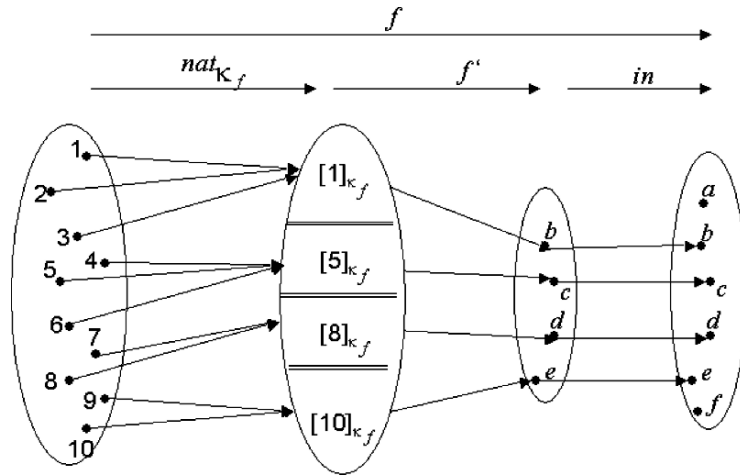
Figure 16.1: Decomposing a function through its kernel

Notice that here $f' \circ in$ is a right divisor of f by nat_{κ_f} which, in turn, is a left divisor of f by $f' \circ in$.

16.5.1.2 Categorization and Pull-Backs

The move which makes it possible to look at the inverse effect of a function and obtain its kernel is a special case of a more general operation called a *pull-back*.

Definition 16.5.1 (Pull-Back). *Let A, B and C be three sets and let $f : A \longmapsto C$ and $g : B \longmapsto C$ be two functions. Suppose that there are two functions $g' : P \longmapsto A$ and $f' : P \longmapsto B$ from a set P such that $f' \circ g = g' \circ f$, which is equivalent to the sentence stating that the following diagram commutes:*

$$
\begin{array}{ccc}
P & \xrightarrow{\ g'\ } & A \\
\Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\
B & \xrightarrow[\ g\ ]{} & C
\end{array}
$$
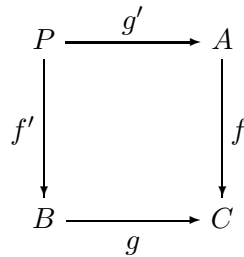
Then this diagram is said to be a pull-back (square) *or a* fibred prod-
uct*, if it fulfills the following universal property: given any set Q and
pair of functions $r : Q \longmapsto A$ and $l : Q \longmapsto B$ such that $r \circ f = l \circ g$,
there is a unique $h : Q \longmapsto P$ such that $h \circ f' = l$ and $h \circ g' = r$, that
is, such that the following diagram commutes*
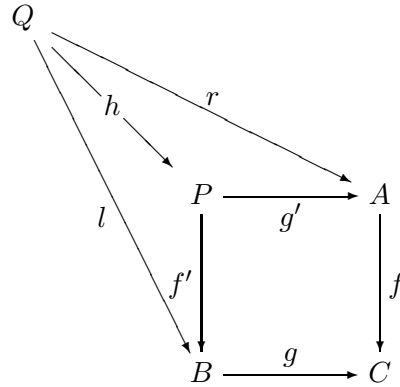
*(we shall also say that the pair of functions l and r uniquely factorises
through h).*[6]

This unique h *is usually denoted by $\langle l, r \rangle$ and P is denoted also by
$B \times_C A$:*

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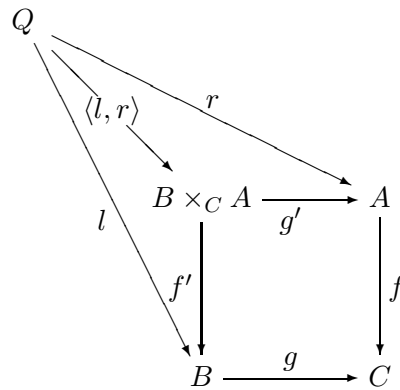
[6]The universal property of "unique factorisation" says that the operation provides
us with the most general solution, and not with a special case. "Least upper bound"
and "greatest lower bounds" are familiar examples of operations with universal
properties (indeed, we do not want an upper bound by chance, but the least upper
bound ...). The general schema of operations with universal properties are called
"limits" (and "co-limits").

Given f and g, the pull-back *or* fibred product $B \times_C A$ *of f with g is uniquely determined up to isomorphisms and one says that f' is the* pull-back *of f along g and, symmetrically, that g' is the* pull-back *of g along f.*

The fibred product is constructed as follows: $B \times_C A = \{\langle b, a \rangle : g(b) = f(a)\}$ and for all $\langle b, a \rangle \in B \times_C A$, $f'(\langle b, a \rangle) = b$ and $g'(\langle b, a \rangle) = a$. Therefore, f' and g' are called the *first* and, respectively, *second projection* of $B \times_C A$.

From the definition of a fibred product it follows immediately that if $A = B$ then $A \times_C A = \{\langle a, a' \rangle : g(a) = f(a')\}$. Hence, if moreover $f = g$, then $A \times_C A = \{\langle a, a' \rangle : f(a) = f(a')\} = \kappa_f$. Thus a pull-back of a function f along itself gives the kernel of f.

Other interesting cases of a fibred product are listed below:

- If C is a singleton $\{x\}$ then $B \times_C A = B \times A$, because for any $a \in A, b \in B, f(a) = g(b) = x$. In this case, in the pull-back diagram we can omit the f and g arrows and it is called a *product* diagram.

- If $A \subseteq C$ and f is an inclusion function, say $f(a) = a$ for all $a \in A$, then $B \times_C A = \{\langle b, a \rangle : a \in A \ \& \ g(b) = a\}$. Let us set for any $a \in A$, $G_a = \{g^{-1}(a)\}$. Therefore, $B \times_C A = \bigcup_{a \in A} \{G_a \times \{a\}\}$. Otherwise stated, $B \times_C A$ is isomorphic to $\{g^{-1}(a)\}_{a \in A}$ via the map $h(\langle b, a \rangle) = \{x : \langle x, a \rangle \in B \times_C A\}$.

- If $A, B \subseteq C$ and both f and g are inclusion functions, then $B \times_C A = \{\langle b, a \rangle : b \in B \ \& \ a \in A \ \& \ b = a\} = \{\langle x, x \rangle : x \in A \cap B\}$. Therefore, $f'(B \times_C A) = g'(B \times_C A) = A \cap B$.

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Commutative diagrams in this book were drawn with the help of Paul Taylor's Commutative Diagrams package

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Index