Geometric Algebra Approach to Singularity of Parallel Manipulators with Limited Mobility

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Abstract. Geometric algebra is employed for the analysis of the singularity of parallel manipulators with limited mobility. The rotations and translations of vectors and screws are performed in the degenerate geometric algebra $G_{3,0,1}$. The condition for singularity is obtained using the language of geometric algebra. The approach is applied to two parallel manipulators with limited mobility.

Key words: parallel manipulator, geometric algebra, kinematics, singularity.

1 Introduction

This paper presents an application of geometric algebra for the analysis of the singularity of parallel manipulators that do not have full mobility, i.e. spatial parallel manipulators with less than six degrees of freedom (dof) or planar ones with less than three dof. The analysis of the singular configurations of the parallel manipulators is an essential part of the process of design and control. In a singular configuration the moving platform of the parallel manipulator has an uncontrollable instant mobility and the manipulator can not sustain a certain wrench applied to the moving platform.

Recently, some "non-standard" methods have been introduced to robot kinematics. For example, the Grassmann geometry was used by Merlet (1989) and the Grassmann–Cayley algebra has been applied to robotics by several researchers: White (1994) analysed the motion of serial robot using Grassmann–Cayley algebra. The Grassmann–Cayley algebra was employed in Staffetti and Thomas (2000) and Ben-Horin and Shoham (2006). The Clifford algebra was used in Collins and McCarthy (1998) and Selig (2000). In Zamora-Esquivel and Bayro-Corrochano (2006) and in Tanev (2006) the geometric algebra was applied.

The Grassmann and Clifford algebras were created in the 19th century. In the second half of the 20th century Clifford algebras have been "rediscovered" and further developed into a unified language named "geometric algebra" in Hestenes (1999), Lasenby et al. (2000), Dorst and Mann (2002), and some other authors.

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In this paper, the geometric algebra is used for obtaining the singularity conditions for parallel manipulators with fewer than six degrees of freedom. This approach is applied to two parallel manipulators – a simple planar one and a spatial 5-dof parallel manipulator.

2 Kinematics of Parallel Manipulators Using Geometric Algebra

The different types of geometric algebra distinguished by the different signatures can be denoted by $G_{p,g,r} = G(p, q, r)$. This geometric algebra has $n = q + q + r$ orthonormal basis vectors e_i $(i = 1, \ldots, n)$ which obey the following rule:

$$e_i \cdot e_j = \begin{cases} 1, \; i = j = 1, \ldots, p, \\ -1, \; i = j = p + 1, \ldots, p + q, \\ 0, \; i = j = p + q + 1, \ldots, p + q + r, \\ 0, \; i \neq j, \end{cases}$$
$$e_i \wedge e_i = 0. \tag{1}$$

In this paper, the transformations for the kinematic analysis are performed in the degenerate geometric algebra (Hestenes et al., 1999). In this case, the translation can be represented as a spinor:

$$T = e^{e_0 \mathbf{a}/2} = 1 + \frac{1}{2} e_0 \mathbf{a}, \tag{2}$$

where e_0 $(e_0 \cdot e_0 = 0)$ is a null vector orthogonal to R^3; $\mathbf{a}$ is a vector in G_3.

Here, the point x is represented as a trivector in $G_{3,0,1}$ similar to the form given in Selig (2000), i.e.,

$$x = (1 + e_0 \mathbf{x}) I_3, \tag{3}$$

where $I_3 = e_1 e_2 e_3$ is the unit pseudoscalar of G_3; $\mathbf{x} = a_1 e_1 + a_2 e_2 + a_3 e_3$ is a vector in G_3.

The points denoted as italic characters are represented by vectors in $G_{3,0,1}$, and points denoted as boldface characters are represented by vectors in G_3.

The rigid displacement can be written in spinor representation, i.e.,

$$Q = TR, \tag{4}$$

where the spinor

$$R = e^{-(1/2) I_3 \mathbf{a}} = \cos\left(\frac{1}{2}\mathbf{a}\right) - I_3 \sin\left(\frac{1}{2}\mathbf{a}\right).$$

Thus, the linear transformation is written as:

$$Q(x) = QxQ^{\dagger}, \tag{5}$$

where $Q^{\dagger} = R^{\dagger}T^{\dagger}$ is the reverse of Q.

This representation has the great advantage of reducing the group composition to the geometric product.

The screw axes (lines) of the joints can be expressed in the geometric algebra as follows: any oriented line $\mathbf{L}$ is uniquely determined by given its direction $\mathbf{u}$ and its moment $\mathbf{m}$ and in the geometric algebra G_3 of 3-D vector space V^3 with the basis $\{e_1, e_2, e_3\}$ it can be written as (Hestenes, 1999):

$$\begin{aligned} \mathbf{L} &= \mathbf{u} + \mathbf{m} \\ &\equiv \mathbf{u} + \mathbf{r} \wedge \mathbf{u} = u_1 e_1 + u_2 e_2 + u_3 e_3 + m_1 e_2 \wedge e_3 + m_2 e_3 \wedge e_1 + m_3 e_1 \wedge e_2, \end{aligned} \tag{6}$$

where $\mathbf{r}$ is the position vector of a point on the line; u_i $(i = 1, 2, 3)$ and m_i $(i = 1, 2, 3)$ are scalar coefficients.

The transformation of a line can be performed in the same way as a vector (see Selig, 2000) and for that reason the line can be written as a bivector in $G_{3,0,1}$, i.e.,

$$\begin{aligned} L^{(4)} &= (\mathbf{u} + \mathbf{m} e_0) I_3 \\ &= u_1 e_2 \wedge e_3 + u_2 e_3 \wedge e_1 + u_3 e_1 \wedge e_2 + m_1 e_1 \wedge e_0 + m_2 e_2 \wedge e_0 + m_3 e_3 \wedge e_0, \end{aligned} \tag{7}$$

where the superscript (4) indicates that the screw is written in $G_{3,0,1}$.

A general screw can be expressed in $G_{3,0,1}$ in a similar way, i.e.,

$$S^{(4)} = v_1 e_2 \wedge e_3 + v_2 e_3 \wedge e_1 + v_3 e_1 \wedge e_2 + b_1 e_1 \wedge e_0 + b_2 e_2 \wedge e_0 + b_3 e_3 \wedge e_0, \tag{8}$$

where v_i $(i = 1, 2, 3)$ and b_i $(i = 1, 2, 3)$ are scalar coefficients.

Then, the transformation of a screw (or line) can be written as

$$S^{(4)} = QS^{(4)}Q^{\dagger}. \tag{9}$$

2.1 Velocity

In this section, the screws are expressed as vectors in G_6. In other words, in the geometric algebra G_6 of 6-D vector space V_6 with the basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$, a screw can be written as a vector (grade 1), i.e.,

$$S = v_1 e_1 + v_2 e_2 + v_3 e_3 + b_1 e_4 + b_2 e_5 + b_3 e_6, \tag{10}$$

where the coefficients are the same as in Eq. (8).

The following notation of a screw is adopted: an upper case letter without superscript (S, D) denotes a screw written as a vector in G_6 of 6-D space, otherwise a superscript indicates the type of the geometric algebra in which the screw is described; letters with a tilde mark ($\tilde{S}$) denote the elliptic polars of the screws S (Lipkin and Duffy, 1985).

The moving platform and the base of a parallel manipulator are connected with n-legs, which can be considered as serial chains. The velocity of the moving platform can be expressed as a linear combination of the joint instantaneous twists (for example, see Rico and Duffy, 1996):

$$V_p = \sum_{i=1}^{f} {}^j\omega_i \, {}^jS_i, \quad j = 1, 2, \ldots, n, \tag{11}$$

where ${}^j\omega_i$ denotes the joint rate and ${}^jS_i$ represents the normalized screw associated with the ith joint axis of the jth leg; f is the dof of the jth leg. The left leading superscript denotes the leg number.

A leg with full mobility and a leg with less than six dof could be treated in a similar way. For that reason the necessary extra dummy joints are added to the leg with less than six dof so that it becomes a leg with full mobility. The dummy joints are considered as driven but locked ones. Then, taking the outer product of five screws of the jth leg gives the following 5-blade:

$${}^jA_k = {}^jS_1 \wedge {}^jS_2 \wedge \cdots \wedge {}^jS_{k-1} \wedge {}^jS_{k+1} \wedge \cdots \wedge {}^jS_6, \tag{12}$$

where ${}^jS_i$ are the normalized joint axes of the jth leg.

The 5-blade ${}^jA_k$ from Eq. (12) involves five screws (out of six with the exception of the ${}^jS_k$ screw), where the kth joint is active. In a non-degenerate space, the dual of a blade represents the orthogonal complement of the subspace represented by the blade. The dual of the above 5-blade ${}^jA_k$ is given by the following geometric product:

$${}^jD_k = {}^jA_k I_6^{-1} = (-1)^{n(6-n)} I_6^{-1} \, {}^jA_k, \tag{13}$$

where $I_6 = e_1e_2e_3e_4e_5e_6$ is a unit pseudoscalar of the G_6 and I_6^{-1} is its inverse; $n = 5$ (in case of 6-dof limb).

Pre-multiplying (inner product) both sides of Eq. (11) by ${}^jD_k$ one obtains:

$${}^j\omega_k = \frac{1}{{}^jD_k \cdot {}^jS_k} \, {}^jD_k \cdot V_p \quad \text{or} \quad {}^j\omega_k = \frac{1}{{}^j\tilde{R}_k \cdot {}^jS_k} \, {}^jR_k \cdot \tilde{V}_p, \tag{14}$$

where ${}^jR_k \equiv {}^j\tilde{D}_k$ is a screw reciprocal to the joint screws ${}^jS_1, {}^jS_2, \ldots, {}^jS_{k-1}, \ldots, {}^jS_6$, and $\tilde{V}_p$ is the velocity of the moving platform with interchanged primary and secondary parts.

The result in Eq. (14) is obtained having in mind that ${}^jS_i \cdot {}^jD_k = 0$ ($i \neq k$) and ${}^jS_k \cdot {}^jD_k = {}^jc_k$ (providing the joint screws of the jth leg are linearly independent); ${}^jc_k$ is a scalar.

3 Wrenches of Constraints and Singularity

Let n be the number of the manipulator legs and m ($m = 6 - q$) be the degrees of freedom of the parallel manipulator. We suppose that the remaining q degrees of freedom are represented by dummy joints (or driven but locked joints) and associated with them dummy screws. So, in non-singular configuration the driven joints and the geometry (or the dummy joints) of the manipulator sustain a general wrench applied to the moving platform. Therefore, the singular configuration can occur when all dual (or reciprocal, respectively) screws ${}^jD_k$ (${}^jR_k$) from Eq. (14), representing active and dummy joints, are linearly dependent. Using the language of the geometric algebra, the condition of singularity for the parallel manipulator with less than six degrees of freedom (but with dummy joints) can be expressed in the following way

$$D_{a_1} \wedge \cdots \wedge D_{a_k} \wedge D_{d_1} \wedge \cdots \wedge D_{d_q} = 0, \quad k + q = 6, \tag{15}$$

where D_{a_i} is a dual vector (grade 1-blade) associated to the ith active joint and D_{d_i} is a dual vector (grade 1-blade) associated to the ith dummy joint. In this case each leg has a full mobility. Here the dummy joints are considered as active but locked.

In the following sections, the approach is applied to two particular parallel manipulators.

3.1 Example of Four-Bar Mechanism as a One-dof Parallel Manipulator

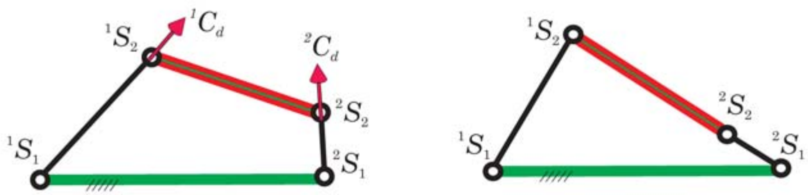
Firstly, in order to illustrate the approach, it is applied to a very simple example, i.e. to the four-bar mechanism (Figure 1a), whose singular configurations are known (e.g., Zlatanov et al., 2002). It is considered as a planar parallel manipulator with two RR-legs and the coupler as a moving platform. The mechanism has one driven joint with the joint axis ${}^1S_1$. In order to have full mobility, we suppose that one dummy joint is added to each leg. Then, the duals corresponding to the active and dummy joints for the first and the second leg, respectively, are as follows:

$${}^1D_1 = ({}^1S_2 \wedge {}^1S_d \wedge e_{126})I_6^{-1}; \quad {}^1D_d = ({}^1S_1 \wedge {}^1S_2 \wedge e_{126})I_6^{-1}, \tag{16}$$

$${}^2D_d = ({}^2S_1 \wedge {}^2S_2 \wedge e_{126})I_6^{-1}, \tag{17}$$

where $e_{126} = e_1 \wedge e_2 \wedge e_6$ is a 3-blade representing the restricting subspace, i.e., it restricts the mechanisms to move only in the horizontal (X–Y) plane; ${}^jD_d$ denotes the dual corresponding to the dummy joint and ${}^jS_d$ is a screw associated with the dummy joint.

After applying some identities of the geometric algebra (see Hestenes et al., 1999) one obtains



(a) The manipulator and the constraint forces

(b) A singular configuration

Fig. 1 Four-bar mechanism (one-dof parallel manipulator).

$$^1D_1 \wedge {}^1D_d = c({}^1S_2 \wedge e_{126})I_6^{-1}, \tag{18}$$

where $c = {}^1D_1 \cdot {}^1S_1$ is a scalar.

The blade from Eq. (18) is a blade of non-freedom for the first leg. In this case, the wrenches of constraint associated with the dummy joints for the first and the second leg, respectively, are ${}^1C_d = {}^1\tilde{D}_d$, ${}^2C_d = {}^2\tilde{D}_d$ (derived from Eqs. (16) and (17)) and the third one can be obtained by factoring the 2-blade from Eq. (18).

In this case the wrenches of constraints are pure forces, Notice, that the two constraint forces ${}^1C_d$ and ${}^2C_d$, associated with the dummy joints, are unique (along the legs, Figure 1a).

The condition for singularity of the manipulator can be written as

$${}^1D_1 \wedge {}^1D_d \wedge {}^2D_d = 0 \quad \text{or} \quad {}^1\tilde{D}_1 \wedge {}^1\tilde{D}_d \wedge {}^2\tilde{D}_d = 0. \tag{19}$$

Again, applying the identities of the geometric algebra and keeping in mind Eq. (18), the left-hand blade (the singularity condition) from Eq. (19) becomes

$$\begin{aligned} &[c({}^1S_2 \wedge e_{126})I_6^{-1}] \wedge [({}^2S_1 \wedge {}^2S_2 \wedge e_{126})I_6^{-1}] \\ &= -c({}^2S_1 \wedge {}^2S_2 \wedge e_{126} \wedge {}^1S_2)I_6^{-1}e_{126}I_6^{-1} \\ &= c({}^2S_1 \wedge {}^2S_2 \wedge {}^1S_2 \wedge e_{126})e_{126} = 0. \end{aligned} \tag{20}$$

Therefore, bearing in mind that $e_1 \wedge e_2 \wedge e_6 \neq 0$, it is clear from Eq. (20) that the condition for singularity can be written as

$${}^2S_1 \wedge {}^2S_2 \wedge {}^1S_2 = 0. \tag{21}$$

Eq. (21) implies that the mechanism is in singular configuration if the three lines (joint axes, which are parallel) are linearly dependent, i.e., lie in a single plane, defined by any two of the lines (Figure 1b). Eq. (21) involves only the screw axis of the passive joints. Therefore, in case of changing the driven joint (for example from ${}^1S_1$ to ${}^2S_1$), the configuration shown in Figure 1b will be no longer singular.

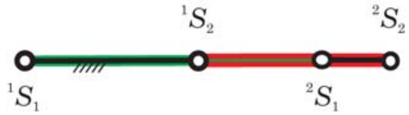


Fig. 2 Constraint singular configuration of the four-bar mechanism.

If we consider the blade formed only by the duals associated with the dummy joints, the condition for the so-called constraint singularity (the term introduced by Zlatanov et al., 2002) can be obtained, i.e.,

$$^1D_d \wedge {}^2D_d = 0 \quad \text{or} \quad {}^1\tilde{D}_d \wedge {}^2\tilde{D}_d = 0. \tag{22}$$

Therefore, the constraint singularity occurs when these two lines $^1C_d = {}^1\tilde{D}_d$ and $^2C_d = {}^2\tilde{D}_2$ coincide (Figure 2).

From Eqs. (19) and (22) it can be seen that the constraint singularity is a subset of general singularity. It is clear from Eqs. (16), (17) and (22) that the condition for constraint singularity involves all joint axes, which fact implies that the mechanism remains in constraint singular configuration even when the driven joint is changed.

3.2 Singularity of a 5-dof Parallel Manipulator

In this section a type of parallel manipulator with five degrees of freedom is introduced and its singular configurations are analyzed using the geometric algebra. The considered parallel manipulator has four legs; the first leg has RRPRR structure and the other three legs have identical SPS structure (Figure 3). The driven (active) joints are the four prismatic joints of the legs and a revolute joint R attached to the base of the RRPRR leg. In this case, the RRPRR (R ⊥ R ⊥ P ⊥ R ⊥ R) leg has two driven (active) joints: the first one (R – revolute joint attached to the base) and the prismatic joint (P).

The SPS legs have full mobility and each one has one driven joint (the P joint). In this case the SPS (or UPS) leg has only one possible dual screw, or reciprocal screw to the joint axis associated with the U and S joints. This reciprocal screw is a line along the SPS leg. The RRPRR leg has five degrees of freedom and in order to have full mobility one extra dummy joint (denoted by a superscript d in the equations) is added, which can be considered as active but locked. The dual screws associated with the active and dummy joints of the first (RRPRR) leg are as follows:

$$^1D_1 = (^1S_2 \wedge {}^1S_3 \wedge {}^1S_4 \wedge {}^1S_5 \wedge {}^1S_d)I_6^{-1}, \tag{23}$$

$$^1D_3 = (^1S_1 \wedge {}^1S_2 \wedge {}^1S_4 \wedge {}^1S_5 \wedge {}^1S_d)I_6^{-1}, \tag{24}$$

$$^1D_d = (^1S_1 \wedge {}^1S_2 \wedge {}^1S_3 \wedge {}^1S_4 \wedge {}^1S_5)I_6^{-1}, \tag{25}$$

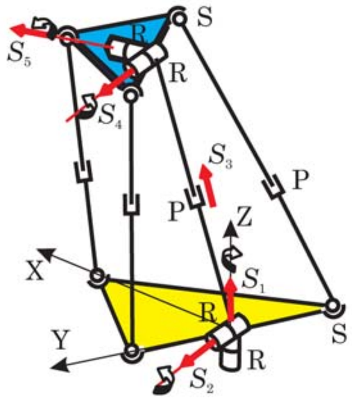


Fig. 3 The 5-dof parallel manipulator.

where ${}^1S_1$ is the axis of the joint attached to the base and ${}^1S_5$ is the axis of the joint connected to the moving platform of the RRPRR leg (Figure 3).

Applying the identities of the geometric algebra for the outer product of the duals from Eqs. (23), (24) and (25) one obtains:

$$D_{13d} = {}^1D_1 \wedge {}^1D_3 \wedge {}^1D_d = \lambda({}^1S_2 \wedge {}^1S_4 \wedge {}^1S_5)I_6^{-1}, \tag{26}$$

where $\lambda = ({}^1S_1 \cdot {}^1D_1)({}^1S_d \cdot {}^1D_d)$ is a scalar; the above result is obtained bearing in mind that ${}^1D_i \cdot {}^1S_k = 0$ $(i \neq k)$ and ${}^1D_i \cdot {}^1S_k \neq 0$ $(i = k)$.

In fact, the 3-blade from Eq. (26) is a blade of non-freedom for the RRPRR leg. One of the wrenches of constraints (${}^1C_d = {}^1\tilde{D}_d$) is uniquely defined by Eq. (25).

The algebraic condition for singularity can be written as follows:

$$\left\{[({}^1S_2 \wedge {}^1S_4 \wedge {}^1S_5)I_6^{-1}] \wedge {}^2D \wedge {}^3D \wedge {}^4D\right\}I_6^{-1} = 0, \tag{27}$$

where ${}^2D = {}^2\tilde{C}$, ${}^3D = {}^3\tilde{C}$ and ${}^4D = {}^4\tilde{C}$ are duals associated with the three SPS legs: ${}^jD = {}^jS_1 \wedge {}^jS_2 \wedge {}^jS_4 \wedge {}^jS_5 \wedge {}^jS_6$ $(j = 2, 3, 4)$. The missing joint screw axis ${}^jS_3$ is associated with the active P joint of the SPS (UPS) legs. The wrenches of constraint ${}^2C$, ${}^3C$ and ${}^4C$ for the three SPS legs can be easily obtained and in fact they are lines along the legs.

The singular configurations of the parallel manipulator can be algebraically derived from Eq. (27). Expanding Eq. (27) leads to an algebraic equation in terms of the five joint variables of the RRPRR leg (all joint screws of the parallel manipulator are expressed as functions of these five variables). The solutions of this equation give the singular configuration of the manipulator. The expanded equation is not listed here because of the limited space. Several singular configurations have been identified. Two types of singular configurations are shown in Figure 4.

The uncontrollable motion of the moving platform for the first singular configuration (Figure 4a) is a pure rotation, which axis intersects all four legs, is parallel to

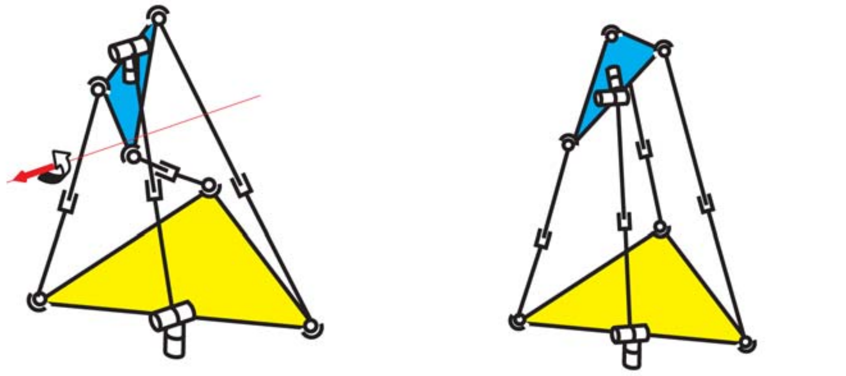


Fig. 4 Singular configurations of the 5-dof parallel manipulator.

two R joint axes of the RRPRR leg and perpendicular to P joint axis of the RRPRR leg. In the second singular configuration (Figure 4b) the uncontrollable motion is a general screw motion.

4 Conclusions

The presented approach proves to be effective in determining the singularity condition for parallel manipulators with limited mobility. This approach is applied to two parallel manipulators, which singular configurations are obtained. It has been shown that the equation for the singularity (the condition for singularity) involves the screws which represent all and only passive joints of the manipulators. This geometric algebra approach provides a good geometrical insight and efficiency in dealing with robot kinematics and singularity of parallel manipulators with fewer than six degrees of freedom.

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