Stiffness Matrix of Compliant Parallel Mechanisms

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Abstract. Starting from the definition of a stiffness matrix, the authors present the Cartesian stiffness matrix of parallel compliant mechanisms. The proposed formulation is more general than any other stiffness matrix found in the literature since it can take into account the stiffness of the passive joints and remains valid for large displacements. Then, the conservative property, the validity,and the positive definiteness of this matrix are discussed.

Key words: stiffness matrix, compliant parallel mechanisms, kinemato-static model.

1 Introduction

The stiffness matrix of a mechanism is defined as the Hessian matrix of a potential. For example, the Cartesian stiffness matrix is the square matrix of second-order partial derivatives of potential ξ_f associated with wrench $\mathbf{f}$ with respect to the vector of Cartesian coordinates, noted $\mathbf{x}$:

$$\mathbf{K}_C = \frac{\partial^2 \xi_f}{\partial \mathbf{x}^2}. \tag{1}$$

Thus by definition, a stiffness matrix is a symmetric matrix [3]. A stiffness matrix is also conservative [1]. And since the Hessian matrix of a potential is used to determine the stability of an equilibrium [5], a stiffness matrix can be either positive-definite or negative-definite.

In this paper, a stiffness matrix that considers the external loads, the changes of the geometry of the mechanism and the stiffness of any joint – even the passive ones – is presented. The kinematic model of a parallel mechanism that takes into account the passive joints is first introduced. Then, expressions for the potential energy are derived in order to obtain a general form of the Cartesian stiffness matrix of a compliant mechanism. The correctness and the properties of this matrix are then discussed and applied in a simple parallel mechanism.

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2 Model of a Parallel Mechanism

2.1 Geometric Constraints

In a parallel mechanism, some geometrical constraints between the joint coordinates corresponding to the closure of loops formed by legs must always be satisfied. These constraints are written as $\mathcal{K}(\boldsymbol{\theta}) = \mathbf{0}$, where $\boldsymbol{\theta}$ is the joint coordinate vector of the mechanism.

Therefore, a vector of generalized coordinates $\boldsymbol{\chi}$ is defined such that $\boldsymbol{\lambda}$, the vector of the kinematically constrained coordinates and $\boldsymbol{\theta}$, the complete joint coordinate vector of the mechanism, always satisfy the geometric constraints. One has:

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{\chi}) \quad \text{and} \quad \boldsymbol{\theta} = \boldsymbol{\theta}(\boldsymbol{\chi}) = \begin{bmatrix} \boldsymbol{\chi} \\ \boldsymbol{\lambda} \end{bmatrix}, \tag{2}$$

where $\boldsymbol{\lambda} = [\lambda_1, \cdots, \lambda_c]^T$ – c being the number of constrained coordinates – and $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_m]^T$ with m the number of joints in the mechanism and θ_k the coordinate of the kth joint.

2.2 Kinematic Constraints

The variation of the kinematically constrained (dependent) joint coordinates is described by a matrix $\mathbf{G}$ and a matrix $\mathbf{R}$ defined as

$$\mathbf{G} = \frac{d\boldsymbol{\lambda}}{d\boldsymbol{\chi}} \quad \text{and} \quad \mathbf{R} = \frac{d\boldsymbol{\theta}}{d\boldsymbol{\chi}} = \begin{bmatrix} \mathbf{1}_l \\ \mathbf{G} \end{bmatrix}, \tag{3}$$

where $\mathbf{1}_l$ stands for the $l \times l$ identity matrix. The relations between the variation of the joint coordinates and the variation of the generalized coordinates are expressed as $d\boldsymbol{\lambda} = \mathbf{G}d\boldsymbol{\chi}$ and $d\boldsymbol{\theta} = \mathbf{R}d\boldsymbol{\chi}$.

2.3 Kinematic Model

2.3.1 Pose of the Platform

Represented by a vector $\mathbf{x}$, it is defined as the average pose of the end-effector of all legs of the mechanism, namely

$$\mathbf{x} = \frac{1}{n}\sum_{i=a}^{n} \mathbf{x}_i, \quad i \in \{a, \cdots, n\}, \tag{4}$$

where $\mathbf{x}_i = \left[\mathbf{c}_i^T, \mathbf{q}_i^T\right]^T$ is the pose vector of the ith leg and where $\mathbf{c}_i$ is the position vector of a chosen point on the platform while $\mathbf{q}_i$ is a quaternion vector describing the orientation of the platform. All legs are indexed from a to n.

2.3.2 Jacobian Matrix $\mathbf{J}_{\boldsymbol{\theta}}$

The Jacobian matrix $\mathbf{J}_\theta$ of a parallel mechanism in which all joints – even the passive ones – are considered is written as

$$\mathbf{J}_\theta = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} = \frac{1}{n}\sum_{i=a}^{n} \frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}. \tag{5}$$

2.3.3 Jacobian Matrix J

In this Jacobian matrix, only the generalized coordinates are considered. It is defined as

$$\mathbf{J} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\chi}} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}}\frac{\partial \boldsymbol{\theta}}{\partial \boldsymbol{\chi}} = \mathbf{J}_\theta \mathbf{R}. \tag{6}$$

2.3.4 Kinematic Model

Thus, the kinematic model of the complete mechanism can be written in different equivalent forms, namely

$$d\mathbf{x} = \mathbf{J}_\theta d\boldsymbol{\theta} = \mathbf{J}_\theta \mathbf{R} d\boldsymbol{\chi} = \mathbf{J} d\boldsymbol{\chi}. \tag{7}$$

2.3.5 Inverse Kinematic Model

From Eq. (7), the inverse kinematic model of the mechanism is expressed as

$$d\boldsymbol{\chi} = \mathbf{J}^{-1} d\mathbf{x}, \quad d\boldsymbol{\lambda} = \mathbf{G}\mathbf{J}^{-1} d\mathbf{x} \quad \text{and} \quad d\boldsymbol{\theta} = \mathbf{R}\mathbf{J}^{-1} d\mathbf{x}. \tag{8}$$

If the number of components in $\mathbf{x}$ is larger than six, then $\mathbf{J}^{-1}$ should be replaced by the Moore–Penrose generalized inverse.

3 Cartesian Stiffness Matrix of a Compliant Mechanism

3.1 Potential Energy of a Mechanism

The potential energy stored in the compliant joints of a compliant mechanism, noted ξ_θ, is calculated as

$$\xi_\theta = \int_{\boldsymbol{\theta}_0}^{\boldsymbol{\theta}} \boldsymbol{\tau}_\theta^T d\boldsymbol{\theta} = \int_{\boldsymbol{\chi}_0}^{\boldsymbol{\chi}} \boldsymbol{\tau}_\chi^T d\boldsymbol{\chi} + \int_{\boldsymbol{\lambda}_0}^{\boldsymbol{\lambda}} \boldsymbol{\tau}_\lambda^T d\boldsymbol{\lambda}, \tag{9}$$

where $\boldsymbol{\tau}_\theta$, $\boldsymbol{\tau}_\chi$ and $\boldsymbol{\tau}_\lambda$ are the vectors of joint torques/forces respectively associated with the joints corresponding to vectors $\boldsymbol{\theta}$, $\boldsymbol{\chi}$ or $\boldsymbol{\lambda}$ and where $\boldsymbol{\theta}_0$, $\boldsymbol{\chi}_0$ and $\boldsymbol{\lambda}_0$ correspond to the unloaded configuration of the mechanism. In the particular – but frequent – case of compliant joints with constant stiffness, ξ_θ is written as

$$\xi_\theta = \frac{1}{2}(\boldsymbol{\chi} - \boldsymbol{\chi}_0)^T \mathbf{K}_\chi (\boldsymbol{\chi} - \boldsymbol{\chi}_0) + \frac{1}{2}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)^T \mathbf{K}_\lambda (\boldsymbol{\lambda} - \boldsymbol{\lambda}_0), \tag{10}$$

where $\mathbf{K}_\chi$ and $\mathbf{K}_\lambda$ are the (diagonal) joint stiffness matrices.

The potential energy ξ_f associated to the external wrench $\mathbf{f}$ is equal to the work provided by $\mathbf{f}$ and is defined as

$$\xi_f = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{f}^T d\mathbf{x}, \tag{11}$$

where $\mathbf{x}_0$ corresponds to the unloaded configuration.

The potential energy due to the external wrench ξ_f is equal – apart from a constant ξ_0 – to the energy stored in the mechanism. ($\xi_f = \xi_\theta + \xi_0$). From Eq. (8), this can be written as

$$\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{f}^T d\mathbf{x} = \int_{\mathbf{x}_0}^{\mathbf{x}} \boldsymbol{\tau}_\chi^T \mathbf{J}^{-1} d\mathbf{x} + \int_{\mathbf{x}_0}^{\mathbf{x}} \boldsymbol{\tau}_\lambda^T \mathbf{G}\mathbf{J}^{-1} d\mathbf{x} + \xi_0. \tag{12}$$

3.2 Cartesian Static Equilibrium

Differentiating Eq. (12) with respect to the pose $\mathbf{x}$ leads to the Cartesian static equilibrium of a compliant mechanism. It is written as

$$\frac{d\xi_f}{d\mathbf{x}} = \frac{d\xi_\theta}{d\mathbf{x}} + \frac{d\xi_0}{d\mathbf{x}} \Leftrightarrow \mathbf{f} = \mathbf{J}^{-T}\boldsymbol{\tau}_\chi + \mathbf{J}^{-T}\mathbf{G}^T\boldsymbol{\tau}_\lambda. \tag{13}$$

In the most general case, the stiffness of these joints is not constant and the corresponding forces/torques are defined as

$$\begin{cases} \boldsymbol{\tau}_\chi = \int_{\boldsymbol{\chi}_0}^{\boldsymbol{\chi}} \mathbf{K}_\chi d\boldsymbol{\chi} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{K}_\chi \mathbf{J}^{-1} d\mathbf{x}, \\ \boldsymbol{\tau}_\lambda = \int_{\boldsymbol{\lambda}_0}^{\boldsymbol{\lambda}} \mathbf{K}_\lambda d\boldsymbol{\lambda} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{K}_\lambda \mathbf{G}\mathbf{J}^{-1} d\mathbf{x}. \end{cases} \tag{14}$$

3.3 Cartesian Stiffness Matrix

The definition of the Cartesian stiffness matrix of a mechanism is given in Eq. (1) and is equivalent to $\mathbf{K}_C = d\mathbf{f}/d\mathbf{x}$. Therefore, using Eqs. (13) and (14), it is obvious that $\mathbf{K}_C$ is not constant and depends on the stiffness of the joints and the geometric configuration of the mechanism. To obtain this function, the right-hand side of Eq. (13) is differentiated with respect to $\mathbf{x}$

$$\frac{d^2\xi_\theta}{d\mathbf{x}^2} = \frac{d}{d\mathbf{x}}\left(\mathbf{J}^{-T}\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{K}_\chi\, \mathbf{J}^{-1} d\mathbf{x} + \mathbf{J}^{-T}\mathbf{G}^T \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{K}_\lambda \mathbf{G}\mathbf{J}^{-1} d\mathbf{x}\right), \tag{15}$$

which leads to

$$\frac{d^2\xi_\theta}{d\mathbf{x}^2} = \mathbf{A} + \mathbf{B} + \mathbf{J}^{-T}\mathbf{K}_\chi \mathbf{J}^{-1}, \tag{16}$$

where

$$\mathbf{A} = \frac{d\mathbf{J}^{-T}}{d\mathbf{x}}(\boldsymbol{\tau}_\chi + \mathbf{G}^T\boldsymbol{\tau}_\lambda) \quad \text{and} \quad \mathbf{B} = \mathbf{J}^{-T}\frac{d\mathbf{G}^T}{d\mathbf{x}}\boldsymbol{\tau}_\lambda + \mathbf{J}^{-T}\mathbf{G}^T\mathbf{K}_\lambda\mathbf{G}\mathbf{J}^{-1} \tag{17}$$

3.3.1 Matrix A

First, the derivative of the inverse of a matrix can be written as

$$\frac{d\mathbf{J}^{-T}}{d\mathbf{x}} = -\mathbf{J}^{-T}\frac{d\mathbf{J}^T}{d\mathbf{x}}\mathbf{J}^{-T}. \tag{18}$$

Thus, matrix $\mathbf{A}$ can be expressed, using Eq. (13) as

$$\mathbf{A} = -\mathbf{J}^{-T}\frac{d\mathbf{J}^T}{d\mathbf{x}}(\mathbf{J}^{-T}\boldsymbol{\tau}_\chi + \mathbf{J}^{-T}\mathbf{G}^T\boldsymbol{\tau}_\lambda) = -\mathbf{J}^{-T}\frac{d\mathbf{J}^T}{d\mathbf{x}}\mathbf{f}. \tag{19}$$

Using the chain rule, the derivative is written as

$$\frac{d\mathbf{J}^T}{d\mathbf{x}}\mathbf{f} = \left(\frac{d\mathbf{J}^T}{d\boldsymbol{\chi}}\mathbf{f}\right)\frac{d\boldsymbol{\chi}}{d\mathbf{x}} = \left(\frac{d\mathbf{J}^T}{d\boldsymbol{\chi}}\mathbf{f}\right)\mathbf{J}^{-1}. \tag{20}$$

Hence, a matrix that captures the effect of the external wrench can be defined as

$$\mathbf{K}_E = -\frac{d\mathbf{J}^T}{d\boldsymbol{\chi}}\mathbf{f} = -\left[\left(\frac{d\mathbf{J}^T}{d\chi_1}\mathbf{f}\right) \cdots \left(\frac{d\mathbf{J}^T}{d\chi_m}\mathbf{f}\right)\right], \tag{21}$$

where χ_i is the ith joint coordinate of $\boldsymbol{\chi}$ and $(d\mathbf{J}^T/d\chi_i)\mathbf{f}$ is a vector forming the ith column of $l \times l$ matrix $\mathbf{K}_E$. Indeed matrix $\mathbf{K}_E$ is equal to the opposite of the matrix noted $\mathbf{K}_G$ in [1].

Therefore from Eqs. (19) and (20), the matrix $\mathbf{A}$ introduced in Eq. (16) is equal to

$$\mathbf{A} = -\mathbf{J}^{-T}\left(\frac{d\mathbf{J}^T}{d\boldsymbol{\chi}}\mathbf{f}\right)\mathbf{J}^{-1} = \mathbf{J}^{-T}\mathbf{K}_E\mathbf{J}^{-1}. \tag{22}$$

3.3.2 Matrix B

Using the chain rule, the right-hand element of matrix $\mathbf{B}$ introduced in Eq. (16), can be differentiated as

$$\frac{d\mathbf{G}^T}{d\mathbf{x}}\boldsymbol{\tau}_\lambda = \left(\frac{d\mathbf{G}^T}{d\boldsymbol{\chi}}\boldsymbol{\tau}_\lambda\right)\frac{d\boldsymbol{\chi}}{d\mathbf{x}} = \left(\frac{d\mathbf{G}^T}{d\boldsymbol{\chi}}\boldsymbol{\tau}_\lambda\right)\mathbf{J}^{-1}. \tag{23}$$

A matrix $\mathbf{K}_{IG}$ that captures the effect of the changes of geometry of the kinematic constraints, is defined as

$$\mathbf{K}_{IG} = \frac{d\mathbf{G}^T}{d\boldsymbol{\chi}}\boldsymbol{\tau}_\lambda = \left[\left(\frac{d\mathbf{G}^T}{d\chi_1}\boldsymbol{\tau}_\lambda\right) \cdots \left(\frac{d\mathbf{G}^T}{d\chi_m}\boldsymbol{\tau}_\lambda\right)\right], \tag{24}$$

where $(d\mathbf{G}^T/d\chi_i)\boldsymbol{\tau}_\lambda$ is a vector forming the ith column of $l \times l$ matrix $\mathbf{K}_{IG}$. Moreover, another matrix noted $\mathbf{K}_{IK}$ that captures the effect of the stiffness of the kinematically constrained joints, is defined as

$$\mathbf{K}_{IK} = \mathbf{G}^T\mathbf{K}_\lambda\mathbf{G}. \tag{25}$$

Matrices $\mathbf{K}_{IG}$ and $\mathbf{K}_{IK}$ are functions of the generalized coordinates and they represent the contribution of the kinematically constrained joints to the stiffness of the mechanism. This contribution is assembled in a matrix $\mathbf{K}_I$, defined as

$$\mathbf{K}_I = \mathbf{K}_{IG} + \mathbf{K}_{IK} = \frac{d\mathbf{G}^T}{d\boldsymbol{\chi}}\boldsymbol{\tau}_\lambda + \mathbf{G}^T\mathbf{K}_\lambda\mathbf{G}. \tag{26}$$

Thus, according to Eqs. (17), (23), (25) and (26), $\mathbf{B}$ is equal to

$$\mathbf{B} = \mathbf{J}^{-T}\mathbf{K}_I\mathbf{J}^{-1}. \tag{27}$$

3.3.3 Cartesian Stiffness Matrix

Finally, combining eqs. (16), (22) and (27), the Cartesian stiffness matrix of a compliant mechanism is written as

$$\mathbf{K}_C = \mathbf{J}^{-T}\left(\mathbf{K}_\chi + \mathbf{K}_I + \mathbf{K}_E\right)\mathbf{J}^{-1}. \tag{28}$$

This matrix includes the three contributions that determine the stiffness of a mechanism according to our initial assumption (no gravity and no dynamical effects), namely: the stiffness of the kinematically unconstrained joints ($\mathbf{K}_\chi$), the stiffness due to the passive joints and the internal torques/forces ($\mathbf{K}_I$) and the stiffness due to the external loads ($\mathbf{K}_E$).

3.4 Stiffness Matrix Expressed in Generalized Coordinates

In the domain of generalized coordinates, the stiffness of the mechanism is described by a matrix $\mathbf{K}_M$ defined as

$$\mathbf{K}_M = \mathbf{K}_\chi + \mathbf{K}_I + \mathbf{K}_E. \tag{29}$$

Therefore, the relation between the stiffness in the generalized domain and in the Cartesian domain is written under a familiar form, namely

$$\mathbf{K}_C = \mathbf{J}^{-T}\mathbf{K}_M\mathbf{J}^{-1} \quad \text{or} \quad \mathbf{K}_M = \mathbf{J}^T\mathbf{K}_C\mathbf{J}. \tag{30}$$

4 Properties of the Stiffness Matrix

4.1 Conservativity of the Matrix

Since the Cartesian stiffness matrix has been calculated by differentiating three torques/forces, namely $\mathbf{f}$, $\boldsymbol{\tau}_\chi$ and $\boldsymbol{\tau}_\lambda$, which are in turn expressed as the derivative of a potential function, $\mathbf{K}_C$ is by definition a conservative matrix. Thus, $\mathbf{K}_C$ is proved symmetric and satisfying the exact differential condition [1].

4.2 A Matrix of a More General Application

The Cartesian stiffness matrices found in the literature can be easily obtained from the matrix presented here. The matrices for serial mechanisms [1, 4] in which there are no passive joints and no internal wrenches such that $\mathbf{K}_I = \mathbf{0}$. As well as the matrices in which the external wrench is not taken into account [2, 4] such that $\mathbf{K}_E = \mathbf{0}$.

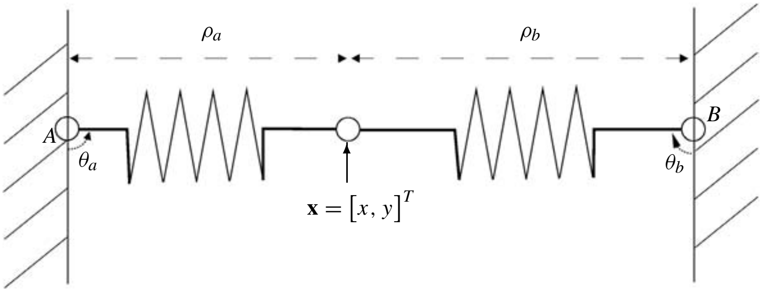

Fig. 1 2-DoF parallel mechanism in unstable static equilibrium.

4.3 Positive Definite Property

The stiffness matrix of a compliant mechanism can be positive definite, semi-positive definite or non-positive definite.

The 2-DoF mechanism presented in the configuration shown in Figure 1 is used to illustrate this property.

4.3.1 Parameters of the Mechanism

The articular coordinates of this mechanism are θ_a, θ_b, ρ_a and ρ_b, while the pose of the platform is $\mathbf{x} = [x, y]^T$. The two base points of the legs, noted A and B, are defined by vectors $\mathbf{a} = (0, 0)$ and $\mathbf{b} = (L, 0)$. The revolute joints are not compliant while both prismatic compliant joints are identical, the free length of their equivalent linear spring is noted ρ_0 and their stiffness coefficient is noted k_ρ. In the configuration presented in Figure 1, the parameters are $\theta_a = 0$, $\theta_b = \pi$, $\rho_a = \rho_b = L/2$ and the external wrench $\mathbf{f} = \mathbf{0}$.

4.3.2 Pose of the End-Effector

The coordinates of leg a are arbitrarily chosen as the generalized coordinates of the mechanism, noted $\boldsymbol{\chi}$. Then, the pose and the Jacobian matrix can be expressed as

$$\mathbf{x} = \begin{cases} \rho_a \cos\theta_a \\ \rho_a \sin\theta_a \end{cases} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} -\rho_a \sin\theta_a & \cos\theta_a \\ \rho_a \cos\theta_a & \sin\theta_a \end{bmatrix}. \tag{31}$$

4.3.3 Geometric Constraints

The two kinematically constrained joints, noted $\boldsymbol{\lambda}$, are θ_b and ρ_b. The geometric constraints that represent the condition of rigidity of the platform, are then written as

$$\mathbf{x}_a - \mathbf{x}_b = \mathbf{0} \Leftrightarrow \begin{cases} \rho_a \cos\theta_a - (\rho_b \cos\theta_b + L) = 0, \\ \rho_a \sin\theta_a - \rho_b \sin\theta_b \qquad = 0. \end{cases} \tag{32}$$

The retained solution of the latter equation is written as

$$\rho_b = \sqrt{\rho_a^2 - 2\rho_a L \cos\theta_a + L^2}, \quad \theta_b = \arctan\left(\frac{\rho 1}{\rho_b}\sin\theta_a, \frac{\rho 1}{\rho_b}\cos\theta_a\right). \tag{33}$$

4.3.4 Kinematic Constraints

Thus, matrix **G** defined in Eq. (3) can be derived from the geometric constraints. **G** is written as

$$\mathbf{G} = \frac{d\boldsymbol{\lambda}}{d\boldsymbol{\chi}} = \mathbf{J}_b^{-1}\mathbf{J}_a. = \begin{bmatrix} \frac{-(\cos\theta_a L - \rho_a)\rho_a}{\rho_a^2 - 2\rho_a L\cos\theta_a + L^2} & \frac{-\sin\theta_a L}{\rho_a^2 - 2\rho_a L\cos\theta_a + L^2} \\ \frac{\rho_a \sin\theta_a L}{\sqrt{\rho_a^2 - 2\rho_a L\cos\theta_a + L^2}} & \frac{-(\cos\theta_a L - \rho_a)}{\sqrt{\rho_a^2 - 2\rho_a L\cos\theta_a + L^2}} \end{bmatrix}. \tag{34}$$

4.3.5 Torque/Force Vectors

The force associated to the passive compliant joint ρ_b is written as $\tau_\rho = k_\rho(\rho_b - \rho_0)$.

4.3.6 Stiffness Matrices Due to Passive Joints

The four components of $\mathbf{K}_{IK}$, defined in Eq. (25), can be analytically calculated as

$$K_{IK}(1,1) = \frac{\rho_a^2 \sin^2\theta_a L^2 k_\rho}{\rho_b^2}, \quad K_{IK}(2,2) = \frac{(\cos\theta_a L - \rho_a)^2 k_\rho}{\rho_b^2}, \tag{35}$$

$$K_{IK}(1,2) = K_{IK}(2,1) = \frac{\rho_a \sin\theta_a L k_\rho (\cos\theta_a L - \rho_a)}{\rho_b^2}. \tag{36}$$

4.3.7 Stiffness Matrices Due to Internal Wrenches

The four components of matrix $\mathbf{K}_{IG}$, defined in Eq. (24), can be analytically calculated as

$$K_{IG}(1,1) = -\frac{{\rho_a}^2 \sin^2\theta_a L^2 \tau_\rho}{\rho_b^3} + \frac{\rho_a \cos\theta_a L \tau_\rho}{\rho_b}, \tag{37}$$

$$K_{IG}(1,2) = K_{IG}(2,1) = \frac{\sin\theta_a L\tau_\rho}{\rho_b} + \frac{(\cos\theta_a L - \rho_a)\,\tau_\rho\rho_a \sin\theta_a L}{\rho_b^3}, \tag{38}$$

$$K_{IG}(2,2) = \frac{\tau_\rho}{\rho_b} + \frac{(\cos\theta_a L - \rho_a)\,\tau_\rho}{2\rho_b^2}. \tag{39}$$

4.3.8 Negative Definite Matrices

In the presented configuration, matrices $\mathbf{K}_M$ and $\mathbf{J}$ are functions of three parameters only, namely k_ρ, L and ρ_0. They are written as

$$\mathbf{K}_M = \begin{bmatrix} k_\rho L(\frac{1}{2}L - \rho_0) & 0 \\ 0 & 2k_\rho \end{bmatrix} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} 0 & 1 \\ L/2 & 0 \end{bmatrix}, \tag{40}$$

and the Cartesian stiffness matrix (Eq. (30)) is calculated as

$$\mathbf{K}_C = \begin{bmatrix} 2k_\rho & 0 \\ 0 & 2k_\rho(L - 2\rho_0)/L \end{bmatrix}. \tag{41}$$

Therefore, this formulation of the Cartesian stiffness matrix demonstrates that a stiffness matrix can be negative definite: the presented configuration is stable with respect to axis $(O\mathbf{y})$, only if $\rho_0 < L/2$, i.e., if the linear springs are in tension.

5 Conclusion

The presented formulation of the stiffness matrix is a generalization of the already existing stiffness matrices, since it can take into account non-zero external loads, non-constant Jacobian matrices and stiff passive joints, this later point being its main novelty.

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