Alternative Forms for Displacement Screws and Their Pitches

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Abstract. It is well established that finite displacement screws effective for the (incompletely specified) relocation of a *body with symmetries* form linearly combined sets if they are of a *sin*-screw form $\hat{\mathbf{S}} = \sin \frac{1}{2}\hat{\theta}\,\hat{\mathbf{s}}$, characterised by pitch $P_S = \frac{1}{2}\sigma/\tan\frac{1}{2}\theta$. This paper shows that screws of indefinitely many other functional forms may be derived, each with a correspondingly distinct definition of pitch, which in the same kinematical situations will also form sets of screws that are linearly combined with dual coefficients. As example, screws of form $\hat{\mathcal{S}} = \sin\hat{\theta}\,\hat{\mathbf{s}}$, of pitch $P_{\hat{\mathcal{S}}} = d/\tan\theta$, are evaluated that describe displacement of a *point-line*.

Key words: kinematics, screw theory, finite displacement screw, pitch.

1 Introduction

Screws of a particular *sin*-screw form, $\hat{\mathbf{S}} = \sin\frac{1}{2}\hat{\theta}\,\hat{\mathbf{s}}$, characterised by pitch $P_S = \frac{1}{2}d/\tan\frac{1}{2}\theta$, have recently found use in representing the finite displacement of a rigid body through a dual angle $\hat{\theta} = \theta + \varepsilon\, d$, $-\pi < \theta \leq \pi$, about a *screw-axis* sited in the unit line $\hat{\mathbf{s}}$, with $|\hat{\mathbf{s}}| = 1$. Using that sin-screw form it is found, when a body *with spatial symmetries of figure* is relocated – or, equivalently, when a displacement is *incompletely specified* – that the (possibly infinite) set of screws available to the body in achieving the relocation is described by linear combination of a small basis of screws [2, 6, 7].

In this paper we show that these properties are by no means unique to the 'sin half-angle' screw form and that, for any such kinematic context, we may derive an indefinitely large number of screw forms of quite different definition (and pitch), each of which occur in similarly constituted linear combinations, formed with dual coefficients in general.

It is not our purpose to advocate use of screw forms other than the sin-screw form: that form, which – as the vector sub-component – has intimate connection with the *unit biquaternion* for the displacement (see Eq. (4)), appears to represent the displacement with least sign-ambiguity and to be the simplest to manipulate in formal analysis. Rather, the purpose is to point out that when – in the course of

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exploratory geometric analysis – a screw of different definition or a different pitch-form appears, it does not necessarily betoken a distinct physical phenomenon but may simply be, in effect, an *alias* for the sin-screw under the kind of derivation just mentioned.

A number of investigations have recently been made into the formal underpinnings of finite displacements which have turned up suggestive measures of pitch for the helicoidal vector fields under study. In one of these [4], for example, the pitch measure $P = d/\sin\theta$ has emerged which, though not that of the sin-screw, is recognisable as the pitch-form associated with the screw $\hat{\mathbf{T}} = \tan\frac{1}{2}\hat{\theta}\,\hat{\mathbf{s}}$, obtained from the sin-screw by the simple step of dividing by $\cos\frac{1}{2}\hat{\theta}$ [3, 9, 11].

To show the ease of creating screw-forms, a new screw-form defined in terms of the *full-* rather than *half*-dual angle, viz. $\hat{\mathcal{S}} = \sin\hat{\theta}\,\hat{\mathbf{s}}$, characterised by pitch $P_{\hat{\mathcal{S}}} = d/\tan\theta$, is used in a typical kinematic context.

2 Notation and Basic Geometry

We write a *screw* $\hat{\mathbf{S}}$ as a 3-vector of dual numbers

$$\left.\begin{aligned} \hat{\mathbf{S}} &= |\hat{\mathbf{S}}|\,(1+\varepsilon\,p)\,\hat{\mathbf{s}}, \quad \hat{\mathbf{s}} = \mathbf{l}+\varepsilon\,\mathbf{M} \\ \hat{\mathbf{s}}^{\mathbf{2}} &= \mathbf{l}^{\mathbf{2}}+\varepsilon\,2\,\mathbf{l}\cdot\mathbf{M} = 1+\varepsilon\,0, \quad \mathbf{l}\times\mathbf{M} = \mathbf{R} \end{aligned}\right\} \tag{1}$$

in which ε is a quasi-scalar such that $(a+\varepsilon\,b = c+\varepsilon\,d) \Leftrightarrow (a = c) \wedge (b = d)$ for all real a, b, c, and d, and satisfying $\varepsilon^2 = 0$. $|\hat{\mathbf{S}}|$ is the real *magnitude* and p is the real *pitch* of the screw $\hat{\mathbf{S}}$, and $\hat{\mathbf{s}}$ (written in lower case) is its *normalised line* which, regarded as a screw in its own right, has unit magnitude and zero pitch. The line $\hat{\mathbf{s}}$ of the screw is spatially located by the *direction 3-vector* of direction cosines $\mathbf{l} = (l,\ m,\ n)$, and by the *moment 3-vector* $\mathbf{M} = (P,\ Q,\ R)$ which determines its *origin-radius* 3-vector $\mathbf{R}$.

Two screws $\hat{\mathbf{S}}_\mathbf{1}$, $\hat{\mathbf{S}}_\mathbf{2}$ are *perpendicular* if $\mathbf{l_1}\cdot\mathbf{l_2} = 0$, and *orthogonal* if $\hat{\mathbf{S}}_\mathbf{1}\cdot\hat{\mathbf{S}}_\mathbf{2} = 0$, which implies that each intersects the other at right angles. The *cross product* $\hat{\mathbf{S}}_\mathbf{1}\times\hat{\mathbf{S}}_\mathbf{2}$ is sited in their *common perpendicular*.

We represent the typical right-handed reference frame by orthogonal normalised axial lines $\hat{\mathbf{x}}_\mathbf{i}$, $\hat{\mathbf{y}}_\mathbf{i}$, and $\hat{\mathbf{z}}_\mathbf{i}$ for which

$$\left.\begin{aligned} &\hat{\mathbf{x}}_\mathbf{i}\cdot\hat{\mathbf{y}}_\mathbf{i} = \hat{\mathbf{y}}_\mathbf{i}\cdot\hat{\mathbf{z}}_\mathbf{i} = \hat{\mathbf{z}}_\mathbf{i}\cdot\hat{\mathbf{x}}_\mathbf{i} = 0, \quad \hat{\mathbf{x}}_\mathbf{i}^\mathbf{2} = \hat{\mathbf{y}}_\mathbf{i}^\mathbf{2} = \hat{\mathbf{z}}_\mathbf{i}^\mathbf{2} = 1, \\ &\hat{\mathbf{x}}_\mathbf{i}\times\hat{\mathbf{y}}_\mathbf{i} = \hat{\mathbf{z}}_\mathbf{i}, \quad \hat{\mathbf{y}}_\mathbf{i}\times\hat{\mathbf{z}}_\mathbf{i} = \hat{\mathbf{x}}_\mathbf{i}, \quad \hat{\mathbf{z}}_\mathbf{i}\times\hat{\mathbf{x}}_\mathbf{i} = \hat{\mathbf{y}}_\mathbf{i}. \end{aligned}\right\} \tag{2}$$

If, in some common frame, we know a general screw $\hat{\mathbf{S}}$ and the i-frame axes $\hat{\mathbf{x}}_\mathbf{i}$, $\hat{\mathbf{y}}_\mathbf{i}$, $\hat{\mathbf{z}}_\mathbf{i}$, we *transform* that screw into i-frame coordinates by

$$\hat{\mathbf{S}}_\mathbf{i} = \begin{bmatrix} \hat{\mathbf{x}}_\mathbf{i}^\mathbf{T} \\ \hat{\mathbf{y}}_\mathbf{i}^\mathbf{T} \\ \hat{\mathbf{z}}_\mathbf{i}^\mathbf{T} \end{bmatrix} \hat{\mathbf{S}} = \begin{bmatrix} \hat{\mathbf{x}}_\mathbf{i}\cdot\hat{\mathbf{S}} \\ \hat{\mathbf{y}}_\mathbf{i}\cdot\hat{\mathbf{S}} \\ \hat{\mathbf{z}}_\mathbf{i}\cdot\hat{\mathbf{S}} \end{bmatrix}$$

in which the first-written matrix is 3×3 dual orthogonal.

3 Specification of a Finite Displacement Screw

We represent the general finite displacement of a body – comprising *translation* through distance d and *rotation* through angle θ, $-\pi < \theta \leq \pi$, about the unit *screw axis* $\hat{\mathbf{s}}$ ($\hat{\mathbf{s}}^2 = 1$) – by constructing the dual angle

$$\tfrac{1}{2}\hat{\theta} = \tfrac{1}{2}\theta + \varepsilon\,\tfrac{1}{2}d \quad \text{so that} \quad \sin\tfrac{1}{2}\hat{\theta} = \sin\tfrac{1}{2}\theta + \varepsilon\,\tfrac{1}{2}d\cos\tfrac{1}{2}\theta,$$

and by then writing the *sin-screw*

$$\hat{\mathbf{S}} = \sin\tfrac{1}{2}\hat{\theta}\,\hat{\mathbf{s}} = \sin\tfrac{1}{2}\theta\,(1+\varepsilon\,P_S)\,\hat{\mathbf{s}} \quad \text{where} \quad P_S = \tfrac{1}{2}d/\tan\tfrac{1}{2}\theta. \tag{3}$$

The sin-screw resultant, $\hat{\mathbf{S}}$, of successively applying two such screws, first $\hat{\mathbf{S}}_1 = \sin\frac{1}{2}\hat{\theta}_1\,\hat{\mathbf{s}}_1$ and then $\hat{\mathbf{S}}_2 = \sin\frac{1}{2}\hat{\theta}_2\,\hat{\mathbf{s}}_2$, is conveniently written

$$\begin{bmatrix} \cos\frac{1}{2}\hat{\theta} \\ \hat{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \cos\frac{1}{2}\hat{\theta}_1\cos\frac{1}{2}\hat{\theta}_2 \;-\; \hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2 \\ \cos\frac{1}{2}\hat{\theta}_2\,\hat{\mathbf{S}}_1 \;+\; \cos\frac{1}{2}\hat{\theta}_1\,\hat{\mathbf{S}}_2 \;-\; \hat{\mathbf{S}}_1\times\hat{\mathbf{S}}_2 \end{bmatrix}, \tag{4}$$

which comprises the *biquaternion product* rule [1].

An alternative form of finite displacement screw, which derives from the sin-screw on division by $\cos\frac{1}{2}\hat{\theta}$, is the *tan-screw*, written [11]

$$\hat{\mathbf{T}} = \tan\tfrac{1}{2}\hat{\theta}\,\hat{\mathbf{s}} = \tan\tfrac{1}{2}\theta\,(1+\varepsilon\,P_T)\,\hat{\mathbf{s}} \quad \text{where} \quad P_T = d/\sin\theta. \tag{5}$$

Such manipulation of Eq. (4) yields the corresponding tan-screw resultant [11] of applying two tan-screws, first $\hat{\mathbf{T}}_1$, then $\hat{\mathbf{T}}_2$, viz.

$$\hat{\mathbf{T}} = \frac{\hat{\mathbf{T}}_1 \;+\; \hat{\mathbf{T}}_2 \;-\; \hat{\mathbf{T}}_1\times\hat{\mathbf{T}}_2}{1 \;-\; \hat{\mathbf{T}}_1\cdot\hat{\mathbf{T}}_2}. \tag{6}$$

Using the sin-screw form of Eq. (3) it is found, when a body *with spatial symmetries of figure* is relocated – or, equivalently, when a displacement is *incompletely specified* in some coordinate(s) – that the (possibly infinite) set of screws available to the body in achieving the relocation is described by linear combination of a small basis of screws [2, 6, 7]. These findings, each dealing with particular kinematic instances, have been generalised to treatment of the *symmetry screws* of *any* body shape, and have been shown to extend equally to unit biquaternions in linear combinations [8, 10]. This generalisation has been re-expressed in terms of the tan-screw form of Eq. (5) [9].

4 Generalisation to Other Screw Forms

We now show that this property – the capacity to form kinematically significant linearly combined sets – is by no means retricted to sin-screws and tan-screws, or to screws of their pitch-forms. For suppose that some kinematic situation (such as those cited above) is described by such a set of sin-screws so that the typical screw of the set is characterised by the dual-linear form

$$\hat{\mathbf{S}} = \sin\tfrac{1}{2}\hat{\theta}\,\hat{\mathbf{s}} = \hat{\mathcal{X}}(\ldots)\,\hat{\mathbf{x}} \;+\; \hat{\mathcal{Y}}(\ldots)\,\hat{\mathbf{y}} \;+\; \hat{\mathcal{Z}}(\ldots)\,\hat{\mathbf{z}}, \tag{7}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are mutually intersecting orthogonal unit lines for which

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0,\;\; \hat{\mathbf{x}}^{\mathbf{2}} = \hat{\mathbf{y}}^{\mathbf{2}} = \hat{\mathbf{z}}^{\mathbf{2}} = 1,$$

and the dual-valued coefficients $\hat{\mathcal{X}}(\ldots)$, $\hat{\mathcal{Y}}(\ldots)$, $\hat{\mathcal{Z}}(\ldots)$ are functions of one or more spatial variables applying in the particular kinematic context.

Since $\hat{\mathbf{s}}^{\mathbf{2}} = 1$, from Eq. (7) we readily derive

$$\sin^2\tfrac{1}{2}\hat{\theta} = \hat{\mathcal{X}}^2(\ldots)+\hat{\mathcal{Y}}^2(\ldots)+\hat{\mathcal{Z}}^2(\ldots). \tag{8}$$

If we assume that the sign of $\sin\frac{1}{2}\theta$ has been incorporated into the direction specified for the line $\hat{\mathbf{s}}$, so that we may safely adopt the positive square root of $\sin^2\frac{1}{2}\theta$ wherever it occurs, we may multiply both sides of Eq. (7) by arbitrary powers g, h of $\sin\frac{1}{2}\hat{\theta}$, $\cos\frac{1}{2}\hat{\theta}$ to obtain

$$\begin{aligned}\sin^{g+1}\tfrac{1}{2}\hat{\theta}\cos^{h}\tfrac{1}{2}\hat{\theta}\,\hat{\mathbf{s}} &= \left\{\hat{\mathcal{X}}^2()+\hat{\mathcal{Y}}^2()+\hat{\mathcal{Z}}^2()\right\}^{g/2}\left\{1-\hat{\mathcal{X}}^2()-\hat{\mathcal{Y}}^2()-\hat{\mathcal{Z}}^2()\right\}^{h/2}\\ &\quad\times\left\{\hat{\mathcal{X}}()\,\hat{\mathbf{x}} \;+\; \hat{\mathcal{Y}}()\,\hat{\mathbf{y}} \;+\; \hat{\mathcal{Z}}()\,\hat{\mathbf{z}}\right\},\end{aligned}$$

which, on the right-hand side, is again a linear combination of the same orthogonal basis screws with coefficient functions parameterised in the same spatial variables. The item on the left, when dual angles are expanded, takes the form

$$\begin{aligned}&\sin^{g+1}\tfrac{1}{2}\theta\left\{1+\varepsilon\,\tfrac{1}{2}d\cot\tfrac{1}{2}\theta\right\}^{g+1}\cos^{h}\tfrac{1}{2}\theta\left\{1-\varepsilon\,\tfrac{1}{2}d\tan\tfrac{1}{2}\theta\right\}^{h}\hat{\mathbf{s}}\\ &\quad= \sin^{g+1}\tfrac{1}{2}\theta\cos^{h}\tfrac{1}{2}\theta\left\{1+\varepsilon\,\tfrac{1}{2}d\left\{(g+1)\cot\tfrac{1}{2}\theta-h\tan\tfrac{1}{2}\theta\right\}\right\}\hat{\mathbf{s}},\end{aligned}$$

which is a screw of pitch $p = \frac{1}{2}d\,\{(g+1)\cot\frac{1}{2}\theta-h\tan\frac{1}{2}\theta\}$.

We may, further, observe that sums of such terms, with arbitrarily chosen exponents g_i and h_i, and combined with arbitrarily chosen dual-valued functions $C_i()$, of the same spatial parameters, such as

$$\begin{aligned}&\textstyle\sum_i C_i()\sin^{g_i+1}\tfrac{1}{2}\hat{\theta}\cos^{h_i}\tfrac{1}{2}\hat{\theta}\,\hat{\mathbf{s}}\\ &\quad= \textstyle\sum_i C_i()\left\{\hat{\mathcal{X}}^2()+\hat{\mathcal{Y}}^2()+\hat{\mathcal{Z}}^2()\right\}^{g_i/2}\left\{1-\hat{\mathcal{X}}^2()-\hat{\mathcal{Y}}^2()-\hat{\mathcal{Z}}^2()\right\}^{h_i/2}\\ &\quad\times\left\{\hat{\mathcal{X}}()\,\hat{\mathbf{x}} \;+\; \hat{\mathcal{Y}}()\,\hat{\mathbf{y}} \;+\; \hat{\mathcal{Z}}()\,\hat{\mathbf{z}}\right\},\end{aligned} \tag{9}$$

yield similar linear combinations.

We may, therefore, synthesise an arbitrarily large set of functions $\hat{\mathcal{F}}(\hat{\theta})$ of the dual angle $\hat{\theta}$ to serve as multipliers of the screw-axis line $\hat{\mathbf{s}}$ in screw forms $\hat{\mathcal{F}}(\hat{\theta})\,\hat{\mathbf{s}}$ which can be expressed as sets of screws deriving as linear combinations of the chosen basis.

5 The Full-Angle Sin Screw

To exemplify the results of the preceding section we now introduce a new screw form – an "$\hat{\mathcal{S}}$-screw" – defined in terms of the *full* dual angle of displacement; thus, in terms of quantities defined in Section 3,

$$\hat{\mathcal{S}} = \sin\hat{\theta}\,\hat{\mathbf{s}} = \sin\theta\,(1+\varepsilon\,P_{\hat{\mathcal{S}}})\,\hat{\mathbf{s}} \quad \text{where} \quad P_{\hat{\mathcal{S}}} = d/\sin\theta. \tag{10}$$

(whereas the definition of $\hat{\mathbf{S}}$ at Eq. (3) involved the dual *half*-angle). Since $\hat{\mathcal{S}} = 2\cos\frac{1}{2}\hat{\theta}\,\hat{\mathbf{S}}$, we may write the resultant, $\hat{\mathcal{S}}$, of successively applying two such screws, first $\hat{\mathcal{S}}_1 = \sin\hat{\theta}_1\,\hat{\mathbf{s}}_\mathbf{1}$ and then $\hat{\mathcal{S}}_2 = \sin\hat{\theta}_2\,\hat{\mathbf{s}}_\mathbf{2}$, as the doubled product of the two entries on the right in Eq. (4), viz.

$$\begin{aligned}\hat{\mathcal{S}} = \cos\hat{\theta}_2\,\hat{\mathcal{S}}_1 + \cos\hat{\theta}_1\,\hat{\mathcal{S}}_2 - \tfrac{1}{2}\left[1 - \frac{\hat{\mathcal{S}}_1\cdot\hat{\mathcal{S}}_2}{(\cos\hat{\theta}_1+1)\,(\cos\hat{\theta}_2+1)}\right]\hat{\mathcal{S}}_1\times\hat{\mathcal{S}}_2 \\ -\tfrac{1}{2}\left[\frac{\hat{\mathcal{S}}_1}{\cos\hat{\theta}_1+1} - \frac{\hat{\mathcal{S}}_2}{\cos\hat{\theta}_2+1}\right]\times(\hat{\mathcal{S}}_1\times\hat{\mathcal{S}}_2).\end{aligned} \tag{11}$$

To provide an exemplary set of $\hat{\mathcal{S}}$-screws in linear combination, we could adapt a general method used elsewhere [8–10] in order to generate them directly from the *symmetry $\hat{\mathcal{S}}$-screws* of a body undergoing displacement. But the visible growth in complexity of Eq. (11) – a necessary component in the development – when it is contrasted with its analogue in Eq. (4), makes this a tortuous course to follow.

Instead, we shall broadly follow the prescription of the preceding section. Having outlined the known solution to a particular kinematic context as it is expressed in sin-screws $\hat{\mathbf{S}}$, we shall convert those sin-screws to $\hat{\mathcal{S}}$-screws by multiplying by an expression for $2\cos\frac{1}{2}\hat{\theta}$.

6 Finite Displacement of a Body with Symmetries

We will restrict attention to properties of the *point-line* object, for which the set of *symmetry screws* – the totality of finite displacements (screws) which leave the the object invariant – consists of all possible pure rotations of the object about the line component i.e. all sin-screws of the form

$$\hat{\mathbf{S}} = \sin\hat{\psi}\,\hat{\mathbf{l}}, \quad -\pi < \psi \leq \pi, \tag{12}$$

where $\hat{\mathbf{l}}$ is the unit line component of the point-line object.

However, it will better serve to provide geometric context and to illustrate the provenance of many linearly combined sets of screws if we consider the symmetry screws of a body to be written more generally as

$$\hat{\mathbf{S}} = \hat{L}\,\hat{\mathbf{i}} + \hat{M}\,\hat{\mathbf{j}} + \hat{N}\,\hat{\mathbf{k}},$$

where $\hat{L}$, $\hat{M}$, $\hat{N}$ are dual-valued coefficient functions and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are orthonormal axial lines fixed in the body.

Consider that we observe such a body to undergo a relocation which – because its symmetries restrict our ability to distinguish apparently equivalent locations – is, in effect, an *incompletely specified displacement*. Our goal is to identify all finite displacement screws which are capable of producing the observed relocation. We may generate the typical screw

- by, firstly, applying some particular displacement (screw) that carries the body from its initial location into one of its symmetrically equivalent final locations; this, so called, *pilot screw* is singled out for this role only and is not otherwise distinguished among the screws effective for the displacement;
- by then applying to the body – in that final location – a typical member of the set of *symmetry screws* which leaves the body apparently invariant by carrying it into a symmetrically equivalent final location.

Thus, every screw effective for the observed relocation may be obtained as the *resultant* of applying the *pilot screw* and one such *symmetry screw*.

Let the pilot displacement comprise translation d_Z and rotation θ_Z, $-\pi < \theta_Z \leq \pi$, about a unit line $\hat{\mathbf{z}}$ so, by Eq. (10), the pilot screw is

$$\hat{\mathbf{S}}_{\mathbf{Z}} = \sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}} \quad \text{where} \quad \hat{\theta}_Z = \theta_Z + \varepsilon\, d_Z.$$

We adopt $\hat{\mathbf{z}}$ and two further lines $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ – which together satisfy orthonormality conditions of Eqs. (2) – as the reference frame for all results.

Now the pilot displacement carries this xyz-frame, embedded in the body, from an *initial location* $\hat{\mathbf{x}}_{\mathbf{i}}$, $\hat{\mathbf{y}}_{\mathbf{i}}$, $\hat{\mathbf{z}}_{\mathbf{i}}$ to a *final location* $\hat{\mathbf{x}}_{\mathbf{f}}$, $\hat{\mathbf{y}}_{\mathbf{f}}$, $\hat{\mathbf{z}}_{\mathbf{f}}$ with the z-axis $\hat{\mathbf{z}} = \hat{\mathbf{z}}_{\mathbf{i}} = \hat{\mathbf{z}}_{\mathbf{f}}$ in common. It greatly simplifies later working to define the orthonormal axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ to be the *mean lines* of those extremum axes, which are then expressed in that reference location by

$$\begin{aligned}
\hat{\mathbf{x}}_{\mathbf{i}} &= \cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{x}} - \sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{y}}, & \hat{\mathbf{x}}_{\mathbf{f}} &= \cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{x}} + \sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{y}}, \\
\hat{\mathbf{y}}_{\mathbf{i}} &= \sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{x}} + \cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{y}}, & \hat{\mathbf{y}}_{\mathbf{f}} &= -\sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{x}} + \cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{y}}, \\
\hat{\mathbf{z}}_{\mathbf{i}} &= \hat{\mathbf{z}}, & \hat{\mathbf{z}}_{\mathbf{f}} &= \hat{\mathbf{z}}.
\end{aligned}$$

We may now evaluate the resultant of applying the *pilot screw*,

$$\hat{\mathbf{S}}_{\mathbf{1}} = \sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}},$$

and the *symmetry screw* $(\hat{L},\ \hat{M},\ \hat{N})$ as expressed at the *final location*,

$$\hat{\mathbf{S}}_{\mathbf{2}} = \hat{L}(\cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{x}}+\sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{y}})-\hat{M}(\sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{x}}-\cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{y}})+\hat{N}\hat{\mathbf{z}},$$

for which $\sin^2\tfrac{1}{2}\hat{\theta}_2 = \hat{L}^2+\hat{M}^2+\hat{N}^2$, so that $\cos\tfrac{1}{2}\hat{\theta}_2 = \sqrt{1-\hat{L}^2-\hat{M}^2-\hat{N}^2}$. For the resultant sin-screw, by use of Eq. (4) we find

$$\cos\tfrac{1}{2}\hat{\theta} = \sqrt{1-\hat{L}^2-\hat{M}^2-\hat{N}^2}\,\cos\tfrac{1}{2}\hat{\theta}_Z-\hat{N}\sin\tfrac{1}{2}\hat{\theta}_Z, \tag{13}$$

$$\hat{\mathbf{S}} = \hat{L}\hat{\mathbf{x}}+\hat{M}\hat{\mathbf{y}}+\left[\sqrt{1-\hat{L}^2-\hat{M}^2-\hat{N}^2}\,\sin\tfrac{1}{2}\hat{\theta}_Z+\hat{N}\cos\tfrac{1}{2}\hat{\theta}_Z\right]\hat{\mathbf{z}}, \tag{14}$$

the second of which shows, in its generalised terms, the provenance of many linearly combined sets of screws of the kind considered in this paper.

7 Half-Angle Sin Screws for Displacement of the Point-Line

We can now make these results specific to the *point-line* symmetry object. Within the displacing xyz-frame, we consider the *line*-component of the element to lie parallel with the $\hat{\mathbf{y}}$-axis. Generality is lost if the *line* is constrained to lie *on* the $\hat{\mathbf{y}}$-axis since some *point* of the line is then required – atypically – to traverse the screw axis $\hat{\mathbf{z}}$ itself during the course of the displacement. Instead, we specify that the *line*-component intersects the axis $\hat{\mathbf{x}}$ in a point at distance τ from $\hat{\mathbf{y}}$, and we adopt that point as the *point*-component.

So located, the symmetry screws of the point-line in Eq. (12) are expressed in the functional forms

$$\begin{bmatrix}\hat{L}\\ \hat{M}\\ \hat{N}\end{bmatrix} = \sin\psi\begin{bmatrix}0\\ 1\\ \varepsilon\,\tau\end{bmatrix},$$

in which the real parameter ψ, $-\pi < \psi \le \pi$, may be arbitrarily chosen.

Since $\hat{L}^2+\hat{M}^2+\hat{N}^2 = \sin^2\psi$, so that $\sqrt{1-\hat{L}^2-\hat{M}^2-\hat{N}^2} = \cos\psi$, with these values Eqs. (13, 14) simplify to:

$$\cos\tfrac{1}{2}\hat{\theta} = \cos\psi\cos\tfrac{1}{2}\hat{\theta}_Z \;-\; \varepsilon\,\tau\sin\psi\sin\tfrac{1}{2}\hat{\theta}_Z, \tag{15}$$

$$\hat{\mathbf{S}} = \sin\psi\big[\hat{\mathbf{y}} \;+\; \varepsilon\,\tau\cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}}\big] \;+\; \cos\psi\,\sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}}, \tag{16}$$

the second of which shows the *two-system* [5] of screws expected for the displacement of a point-line [7], as generated by linear combination of the mutually perpendicular, but not intersecting, sin-screws

$$\hat{\mathbf{y}} \;+\; \varepsilon\,\tau\cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}},$$
$$\sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}},$$

with the real coefficient functions $\sin\psi$, $\cos\psi$ parameterised by the variable ψ, $-\pi < \psi \leq \pi$.

This basis pair, found earlier as both sin- and tan-screws [8,9], has been made the subject of a theorem in a recent treatment of the point-line situation which adopts their directions, and the mid-point between them, as defining a *canonical system* for this kinematic context [12]. This adoption is not based on any fundamental characteristic of the screws which distinguishes them from any others in the two-system, but by the convenience for the human observer of indentifying the locations of those particular screws within the physical reality of a practical situation.

At the values $\sin\psi = \pm\sin\frac{1}{2}\hat{\theta}_Z/\sqrt{1+\sin^2\frac{1}{2}\hat{\theta}_Z}$, $\cos\psi = 1/\sqrt{1+\sin^2\frac{1}{2}\hat{\theta}_Z}$, the two central *principal screws* [5] of the two-system are selected, viz.

$$\hat{\mathbf{S}}_{\pm} = \frac{\sin\frac{1}{2}\hat{\theta}_Z}{\sqrt{1+\sin^2\frac{1}{2}\hat{\theta}_Z}}\Big\{\pm\big[\hat{\mathbf{y}} + \varepsilon\,\tau\cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}}\big] + \hat{\mathbf{z}}\Big\}, \tag{17}$$

which are orthogonal, so that $\hat{\mathbf{S}}_+ \cdot \hat{\mathbf{S}}_- = 0$, and therefore intersect one another at right angles. In terms of these, the general member screw of Eq. (16) may be restated, thus

$$\hat{\mathbf{S}} = \frac{\sqrt{1+\sin^2\frac{1}{2}\hat{\theta}_Z}}{2}\left\{\left[\cos\psi + \frac{\sin\psi}{\sin^2\frac{1}{2}\hat{\theta}_Z}\right]\hat{\mathbf{S}}_+ + \left[\cos\psi - \frac{\sin\psi}{\sin^2\frac{1}{2}\hat{\theta}_Z}\right]\hat{\mathbf{S}}_-\right\}.$$

Normalistion aside, this equation typifies the linear combination of orthogonal basis screws with (more usually) dual-valued coefficient functions which is represented in general form at Eq. (7).

8 Full-Angle Screws for Displacement of the Point-Line

As proposed earlier, we form the full-angle $\hat{\mathcal{S}}$-screws for displacement of the point-line object by multiplying the $\cos\frac{1}{2}\hat{\theta}$ expression of Eq. (15) into the half-angle $\hat{\mathbf{S}}$-screw expression of Eq. (16), and doubling: in some respects, the least-rearranged outcome, viz.

$$\begin{aligned}\hat{\mathcal{S}} &= (\sin 2\psi\cos\tfrac{1}{2}\hat{\theta}_Z - \varepsilon\,2\tau\sin^2\psi\sin\tfrac{1}{2}\hat{\theta}_Z)\big[\hat{\mathbf{y}} + \varepsilon\,\tau\cos\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}}\big]\\ &\quad + (\cos^2\psi\sin\hat{\theta}_Z - \varepsilon\,\tau\sin 2\psi\sin^2\tfrac{1}{2}\hat{\theta}_Z)\,\hat{\mathbf{z}},\end{aligned} \tag{18}$$

is the most informative in that it preserves the identity of the lines of the basis screws – which, of course, have not changed – while revealing that the coefficients are no longer purely real.

The attempt, at those same basis screws, to allow a number of inherent references to the full-angle $\hat{\theta}_Z$ to express themselves, as represented in the rearrangement:

$$\hat{\mathcal{S}} = \sin 2\psi\left[\cos \tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{y}} \;+\; \varepsilon\,\tau\cos\hat{\theta}_Z\,\hat{\mathbf{z}}\right] \;+\; \cos^2\psi\,\sin\hat{\theta}_Z\,\hat{\mathbf{z}}$$
$$-\,2\varepsilon\,\tau\sin^2\psi\sin\tfrac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{y}}, \qquad (19)$$

not only leaves the half-angle $\frac{1}{2}\hat{\theta}_Z$ still much in evidence but also splits the identity of the basis screw $\hat{\mathbf{y}}+\varepsilon\,\tau\cos\frac{1}{2}\hat{\theta}_Z\,\hat{\mathbf{z}}$ into two less comprehensible portions. This unattractive situation is not appreciably changed if the $\hat{\mathcal{S}}$-screws are re-expressed in terms of the central orthogonal *principal screw* basis of Eq. (17).

So, while we have demonstrated the feasibility, and ease, of expressing a chosen kinematic situation in terms of full-angle $\hat{\mathcal{S}}$-screws, we do not find the representation which they offer to be as directly informative as that found earlier for half-angle sin-screws at Eq. (16).

9 Conclusion

It has been shown that in any kinematic situation which is described by a set of sin-screws in linear combination, whether with real or dual coefficients, screws of indefinitely many other functional forms may be derived, each with a correspondingly distinct definition of pitch, which in the same kinematic situation will also form sets of screws that are linearly combined, with – generally – dual coefficients.

It is clear that the scope of the demonstration could have been broadened: sin-screws were adopted as the base type from which other screw-forms might be derived because they have been central to most discussions of finite displacement screws in the recent literature; a focus was maintained on derivation of trigonometrically-related functional forms, rather than arbitrarily general forms, because these appear to be the most relevant to current and future work and might, in any case, lead to quite general forms by way of Fourier synthesis.

As stated at the outset, the purpose was not to proffer new screw forms for use but to indicate where connections may lie between screw- and pitch-forms newly arising in analysis and those which have been used before.

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