The Philosophy of Mathematics

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Abstract Is there a philosophy of mathematics in classical Islam? If so, what are the conditions and the scope of its presence? To answer these questions, hitherto left unnoticed, it is not sufficient to present the philosophical views on mathematics, but one should examine the interactions between mathematics and theoretical philosophy. These interactions are numerous, and mainly foundational. Mathematics has provided to theoretical philosophy some of its central themes, methods of exposition and techniques of argumentation. The aim of this chapter is to study some of these interactions, in an effort to give some answers to the questions raised above. The themes which will be successively discussed are mathematics as a model for the philosophical activity (al-Kindī, Maimonides), mathematics in the philosophical syntheses (Ibn Sīnā, Naṣīr al-Dīn al-Ṭūsī), and finally the constitution of *ars analytica* (Thābit ibn Qurra, Ibn Sinān, al-Sijzī, Ibn al-Haytham).

The historians of Islamic philosophy take a particular interest in what some, at times, like to call *falsafa* (فلسفة). As they see it, it comprises the doctrines of the Being and the Soul developed by the authors of Islamic culture, indifferent to other kinds of knowledge and independent of all determination other than the link they have with religion. These philosophers would, then, be working in the Aristotelian tradition of Neo-Platonism, heirs of late antiquity under the colours of Islam. This historical bias ensures, superficially at least, a smooth passage from Aristotle, Plotinus and Proclus, among others, to the philosophers of Islam from the ninth century on. But the price is high: it often, but not always, results in a pale and impoverished image of philosophical activity and transforms the historian into an archaeologist, although one deprived of the latter's resources. Indeed, it is not uncommon for the historian to take on as his main task an excavation of the domain of Islamic philosophy, looking for the remnants of Greek works lost in their original but preserved in Arabic translation; or, for want of such a translation, to declare himself satisfied with the fragments of the ancient philosophers often studied with talent and competence by historians of Greek philosophy.

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It is true that recently, some historians have turned to doctrines elaborated in other fields beyond the wake of the Greek inheritance: the philosophy of law, developed in magisterial manner by the jurists; the philosophy of *Kalām* (علم الكلام), that is, of the philosophical theologian, refined and subtle; the Sufism of the great masters as al-Ḥallāj and Ibn ʿArabī and others. Such studies enrich and correct the picture and reflect more faithfully the philosophical activity of the time. They also allow for a better understanding the place of the Greek inheritance in Islamic philosophy.

But the sciences and mathematics have not yet received the same attention as law, the *Kalām*, linguistics or Sufism and, even today, the links—in our opinion essential—between sciences and philosophy, and notably between mathematics and philosophy are disregarded. The links between mathematics and philosophy in the works of the philosophers of Islam as al-Kindī, al-Fārābī, Ibn Sīnā, and others are sometimes tackled, but in what must be termed a totally superficial way. Their views on the links between the two domains are described in an attempt to find a connection between these views and the Platonic or Aristotelian doctrines, or sometimes the possible influence of the Neo-Pythagoreans is examined. This means that there is no attempt to understand the repercussions of the philosophers' mathematical knowledge on their philosophies, and not even the impact on their own philosophical doctrines of their activities as scientists, which of course most of them were. The historians of philosophy are not alone accountable for this deficiency; the responsibility is also that of the historians of sciences. It is true that, to examine the links between the sciences and philosophy, it is necessary to have a particularly wide scope of competence, a much finer linguistic knowledge than what suffices in geometry, syntactically elementary and lexically poor; and a knowledge of the history of philosophy itself. If to these demands we add a conception of the links between science and philosophy that is itself inherited from the present positivism, it is easier to understand the deep indifference of the historians of science in this domain. Yet—we must remind ourselves—the links between sciences and philosophy are an integral part of the history of sciences.

To be sure, the situation is a little paradoxical: for seven centuries, a scientific and mathematical research of the most advanced was elaborated in Arabic in the urban centres of Islam. Is it likely that philosophers who were sometimes themselves mathematicians, physicians, and so on, should have carried out their philosophical activity as recluses, indifferent to the changes that were taking place under their eyes, blind to a succession of scientific results that were following one another? How is this imaginable in the face of an unprecedented profusion of disciplines and successes: astronomy critical of Ptolemaic models, reformed and renewed optics, the creation of algebra, the invention of algebraic geometry, the transformation of Diophantine analysis, the discussion of the theory of parallels, the development of projective methods, and so forth—the philosophers should have been so insensitive as to remain within the relatively narrow frame of the Aristotelian tradition of Neo-Platonism? The apparent poverty of the philosophy of classical Islam is undoubtedly due to its historians rather than to history.

Nevertheless, to we examine the links between philosophy and science or philosophy and mathematics—to which we will limit ourselves here—, only as they appear in the philosophers' works, is to make only one third of the journey. It is also necessary to question mathematician-philosophers and mathematicians. But to consider mathematics alone demands an explanation at the outset, all the more so as this means of proceeding is in no way the norm in the study of Islamic philosophy.

No scientific discipline has contributed as much to the genesis of theoretical philosophy as mathematics; none has had such ancient and numerous links with philosophy. From antiquity, mathematics has constantly provided central themes for philosophical reflection; it has supplied methods of exposition, argument techniques, and even implements appropriate to its analyses. And finally, it offers itself to the philosopher as an object of study: he sets about clarifying mathematical knowledge itself by studying its object, its methods, by probing its apodictic characters. From start to finish in the history of philosophy, questions have kept recurring on the conditions of mathematical knowledge, its capacity to be extended, the nature of the certainty it reaches, and its place at the heart other kinds of knowledge. The philosophers of Islam are no exception to this rule: al-Kindī, al-Fārābī, Ibn Bājja, Maimonides among many others.

Other less obvious links have appeared between mathematics and theoretical philosophy. It is common for them to collaborate in order to elaborate a method, a logic even, as the encounter between Aristotle and Euclid over the axiomatic method, or al-Ṭūsī's appeal to combinatorial analysis to solve the philosophical problem of emanation from the One. But whatever form this link may take, there is one which is particularly noticeable and which, in this case, was created by a mathematician, not a philosopher: we mean the doctrines developed by the mathematicians to justify their own practice. The conditions most propitious for these theoretical constructions are present when a mathematician, ahead of contemporary research, is confronted with an insurmountable obstacle, as a result of the unsuitability of available mathematical techniques for the new objects that are beginning to emerge. Just think of the different variants of the theory of parallels, notably from the time of Thābit ibn Qurra (d. 901), of a kind of *analysis situs* conceived by Ibn al-Haytham, or of the doctrine of the indivisibles in the seventeenth century.

The links between theoretical philosophy and mathematics are to be found mainly in four types of works: the works of philosophers; those of the mathematician-philosophers as al-Kindī, Muḥammad ibn al-Haytham (not to be mistaken for al-Ḥasan ibn al-Haytham [see Rashed, 1993b, II, pp. 8–19; 2000, III, pp. 937–941]); those of the philosopher-mathematicians as Naṣīr al-Dīn al-Ṭūsī, and others; and those of mathematicians as Thābit ibn Qurra, his grandson Ibrāhīm ibn Sinān, al-Qūhī, Ibn al-Haytham, and others. Therefore to limit oneself to one group or another when examining the links between philosophy and mathematics is to condemn oneself to the loss of an essential dimension of the field of study.

We have tried on several occasions now to provide an exposition of some of the themes of the philosophy of mathematics; these are but a few soundings intended to reveal the riches of a domain rather more soundings, in fact, than a systematic examination of the domain. Such a project deserves a substantial volume, a volume which has yet to be written. The fact remains that the way that seems best suited to the task differs from merely setting out the views the philosophers may have expressed on mathematics and its importance; rather, it considers which themes were tackled, the intimate links between mathematics and philosophy and their role in the elaboration of doctrines and systems—that is to say the organisational role of mathematics. Notably, we will show how mathematician-philosophers set about solving philosophical problems mathematically, a fruitful approach generating new doctrines, new disciplines even. We will bring out the attempts of mathematicians to resolve mathematical problems philosophically and we shall see it constitutes an investigation which is profound and necessary. I will deal with the following topics:

1. Mathematics as the condition and source of models for philosophical activity. From the numerous philosophers who may illustrate this theme, we have selected just two: a mathematician philosopher and a philosopher who without being a mathematician was yet knowledgeable in mathematics: al-Kindī and Maimonides.
2. Mathematics in philosophical synthesis. It is with the first known synthesis, that of Ibn Sīnā, that mathematics as such intervenes in philosophical works. One of the results—and by no means the least—is the "formal" turn in ontology; which permitted the mathematical treatment of a philosophical problem. Naturally, we will consider here the contribution of Ibn Sīnā, a philosopher well-read in mathematics, which was continued by the mathematician Naṣīr al-Dīn al-Ṭūsī.
3. The third topic, mainly cultivated by mathematicians dealing with the problem of mathematical invention, is *ars inveniendi* and *ars analytica* with Thābit ibn Qurra, Ibrāhīm ibn Sinān, al-Sijzī and Ibn al-Haytham.

1 Mathematics as Conditions and Models of Philosophical Activity: al-Kindī, Maimonides

The links between philosophy and mathematics are essential to the reconstitution of al-Kindī's system (the ninth century); it is indeed such a dependence that the philosopher advertises when he writes a book entitled *Philosophy can only be acquired through mathematical discipline* (al-Nadīm, ed. 1971, p. 316), and when in his epistle on *The quantity of Aristotle's books* (*الرسائل* (i.e. *Rasā'il*), Al-Kindī, 1950, pp. 363–384), he presents mathematics as a propaedeutic to philosophical teaching. He even goes as far as calling out to the student in philosophy, warning him that he is facing the following alternative: to begin with the study of mathematics before tackling Aristotle's books, according to the order given by al-Kindī—and then he can hope to become a true philosopher; or to do without mathematics and come merely to parrot philosophy, if he is capable of memorising by heart. Having mentioned Aristotle's different groups of books, al-Kindī writes:

> This is the number of his books, that we have already mentioned, and which a perfect philosopher needs to know, after mathematics, that is to say, the mathematics I have defined by name. For if somebody is lacking in mathematical knowledge, that is, arithmetic, geometry, astronomy and music, and thereafter uses these books throughout his life, he will not be able to complete his knowledge of them, and all his efforts will allow him only to master the 'ability' to repeat if he can remember by heart. As for their deep knowledge

and the way to acquire it, these are absolutely non existent if he has no knowledge of mathematics (ibid., I, pp. 369–370).

For al-Kindī, then, mathematics is at the base of the philosophical programme. By going deeper into its role in al-Kindī's philosophy—which is not our purpose here—one will be able to understand more rigorously the specificity of his work, which indeed historians often approach in two different ways. According to the first interpretation, al-Kindī presents himself as a Muslim representative of the Aristotelian tradition of Neo-Platonism, a philosopher of a doubly late antiquity. The second interpretation sees in him a follower of philosophical theology (*Kalām*), a theologian who would have liked to change its language for that of Greek philosophy. But if we give back to mathematics the role which has been devolved on it in the elaboration of his philosophy, al-Kindī's fundamental options will open up before our eyes. One of them comes from his Islamic convictions, as they were explained and set out in the tradition of philosophical theology, notably that of *al-Tawḥīd* (the doctrine of God's unicity), that Revelation delivers us the truth, which is unique and rational. The second one refers us back to Euclid's elements as method and model: what is rational can be reached in a concise, very condensed and almost instantaneous way by Revelation, and can equally be derived through collective and cumulative work—that of philosophers—from truths of reason, independent of Revelation, which should satisfy the criteria of geometric proof. These truths of reason, which are used as primitive notions and postulates, were provided at the time of al-Kindī by the Aristotelian tradition of Neo-Platonism. They were chosen to replace the truths that Revelation offers in philosophical theology since they could fulfil the requirements of geometric thought and make possible an axiomatic style of exposition. The "mathematical examination (الفحص الرياضي)" became then the instrument of metaphysics.

That is in fact the case for the epistles in theoretical philosophy, such as for example *First Philosophy*, and the *Epistle for Explaining the Finitude of the Body of the World* (Rashed and Jolivet, 1988). To take the latter text as an example, al-Kindī proceeds methodically to prove the inconsistency of the concept of an infinite body. He begins by defining primitive terms: *magnitude* and *homogenous magnitudes.* He then introduces what he calls "a certain proposition (قضية حق)" (ibid., p. 161, *l.* 16), or, as he explains elsewhere, "the first true and immediately intelligible premises (المقدمات الأولى الحقية المعقولة بلا توسط)" (*First Philosophy*, ibid., p. 29, *l.* 8), or else "the first obvious true and immediately intelligible premises" (*On the Unicity of God and the Finitude of the Body of the World*, ibid., p. 139, *l.* 1), i.e. tautological propositions. These are expressed in terms of primitive notions, of order relations on them, of union and separation operations on them, of predications: finite and infinite. The following statements illustrate such propositions: homogeneous magnitudes which are no bigger than each other are equal; or, if one of equal homogeneous magnitudes is added to a magnitude which is homogeneous to it, then they become unequal (ibid., p. 160). Finally, al-Kindī uses a process of proof, *reductio ad absurdum*, by adopting a hypothesis: the part of an infinite magnitude is necessarily finite.

This is the path al-Kindī follows in his other writings. As in his *First Philosophy*, he proceeds *more geometrico* in his epistle *On the Quiddity of What Cannot*

Be Infinite and of What is called Infinite, this is how al-Kindī wants to prove the impossibility that the world and time are infinite. Al-Kindī begins here once again by stating four premises: (1) “Of anything from which some thing is taken away, what remains is smaller than what was before the subtraction was carried out”; (2) “Anything from which some thing is taken away, if what is taken away is given back to the former, it goes back to the original quantity”; (3) “For all finite things, if they are put together, a finite thing is obtained”; (4) “If there are two things such that one is smaller than the other, then the smaller measures the bigger or measures a part of it, and if it entirely measures it, then it measures a part of it” (Rashed and Jolivet, 1998, p. 150). From these premises, inspired directly by Euclid’s *Elements*, al-Kindī intends to establish his philosophical assertion. He then assumes an infinite body from which some finite thing is taken away, and the question is whether what remains is finite or infinite. He then shows that both hypotheses lead to contradictions, and concludes that no infinite body can exist. He goes on, showing that it is the same for the body’s accidents, notably time. And time, movement and the body are reciprocally involved. He then shows that there is no infinite time *a parte ante* and that neither the body, movement, nor time are eternal. There is therefore no eternal thing, and the infinite is only potential, as in the case of numbers. These examples, briefly mentioned, show how al-Kindī articulated simultaneously mathematical principles and methods, and philosophy according to the Aristotelian tradition of Neo-Platonism. It should be noted that al-Kindī the philosopher was also a mathematician as his works in optics (Rashed, 1996) and mathematics (Rashed, 1993a) testify. In philosophy, he was also familiar not only with Aristotle’s accounts and those of the Aristotelian and Neo-Platonist tradition, but also with Aristotelian commentators such as Alexander.

Maimonides (1135–1204), while not productive in mathematics like al-Kindī, was informed about the subject. He obviously has enough knowledge of mathematics to try to read, pen in hand, perhaps even to teach and to comment on, mathematical works as Apollonius’ *Conics*, which is to say, works of the highest level at the time. But his commentary never bears on the fundamental ideas, on the properties really studied in the work; he is interested only in the elementary proof techniques taught, for the most part, in the first six books of Euclid’s *Elements*. Put bluntly, his commentary is nowhere near the level of the works commented upon. But why did Maimonides spend so much time and energy for so meagre an outcome? We can certainly invoke—in Maimonides’ own words—the role of mathematics in training the mind (ترويض الذهن) to reach human perfection (Maimonides, 1972, p. 80). But there is more: it has to do with the other connections between mathematics and philosophy. We will confine ourselves to the most important of these.

One must to bear in mind that the starting point of Maimonides is dogma and not philosophy: “to elucidate (as he says) the difficulties of dogma (مُشكلات الشَّريعة), and to make plain its hidden truths, which are far above the comprehension of the multitude.” (ibid., p. 282). This has been one of the major tasks of philosophy since al-Kindī (see his epistle *On the Quantity of Aristotle’s Books*), and consists in reaching the truth passed on by the Scriptures through reason, that is to say, philosophical speculation. To accomplish this task, even simply to initiate it, a perfect concordance had to be assumed between the two kinds of truth, that of the Scriptures and that of reason and philosophy. This "concordance" lies on a principle formulated by Ibn Rushd as follows (1126–1198): "a truth does not contradict a truth but accords with it and testifies for it" (Ibn Rushd, 1983, pp. 31–32). In this respect, the means for which Maimonides opted is the same as that with which his predecessors were equipped: "the method based on indubitable proof (الطريق الذي لا ريب فيه)" (Maimonides, 1972, p. 187), i.e. to establish by the "true proof (البرهان الحقيقي)" the truth of dogma: the existence of God, His unity and His incorporeality. For these philosophers, this proof can only be conceived of as a mathematical model. And to do so, a language other than that of the Revelation had to be used, a language whose concepts, defined by reason alone, are endowed with a certain ontological neutrality.

The "true proof", that is, according to the mathematical model, is the way necessary for the truths of Revelation to obtain further the status of truths of reason, which is in no way peculiar to a particular religion, revealed or not. Such is the first connection between mathematics and philosophy. But these connections, as we shall see, occur at different levels. First of all, Maimonides' general approach consists in borrowing notions from the Aristotelian philosophy of his predecessors, and proof and exposition techniques from mathematics; it is this approach which has been effectively used, for example, in the major part of the second book of the *Guide*. The method follows that of geometers, to whom he owes certain proof techniques—mainly *reductio ad absurdum*—to establish each element of his exposition. In the *Guide*, there are twenty-five such elements, twenty-five lemmas most of which are quoted, but all of which are taken by Maimonides to have been rigorously proved by his predecessors. To these lemmas, he adds one postulate, and from these twenty-six propositions he infers his "principal theorem": GOD EXISTS, HE IS UNIQUE, AND HE IS NEITHER A BODY NOR IN A BODY. The importance of this passage is due not so much to the strength of the proof as to the deliberate metaphysical arrangement of a *more geometrico* exposition. The first lemmas were the potential subject of a logical and mathematical treatment since Aristotle, revived by al-Kindī, then picked up by several metaphysicians like Ibn Zakariyā al-Rāzī, Abū al-Barakāt al-Baghdādī (11th–12th), Fakhr al-Dīn al-Rāzī (1150–1210), Naṣīr al-Dīn al-Ṭūsī (1201–1274), among others; finally, they are put together in the commentary of the *Guide* by al-Tabrīzī and later, in that of Hasdai Crescas (1340-ca 1414). They concern the impossibility of the existence of an infinite magnitude, and the impossibility of the coexistence of an infinite number of finite magnitudes. The third lemma states the impossibility of the existence of an infinite chain of causes and effects, material or not—thus condemning in advance the infinite regression of causes. Three propositions follow the three lemmas. The first deals with change; four categories are subject to change: substance, quantity, quality, and place. The second concerns motion: motion implies change and transition from potentiality to actuality. The third proposition enumerates the different kinds of motion. The seventh lemma is stated as follows: "Things which are changeable are, at same the time, divisible. That is why everything that moves is divisible, and necessarily corporeal; but that which is indivisible cannot move, and cannot therefore be corporeal" (Maimonides, 1972, p. 249).

The eighth lemma asserts that: "anything that moves accidentally will necessarily come to rest" (ibid., p. 251). The ninth, that "a body that sets another corporeal thing in motion can only effect this by setting itself in motion at the time" (ibid., p. 252). The exposition of the preliminary propositions goes on in like manner; the fourteenth postulates that locomotion precedes all motions, and the twenty-fifth that each compound substance consists of matter and form.

These twenty-five lemmas, some of which have just been mentioned, all belong to the Aristotelian philosophy. But they are not homogeneous: their origin separates them as much as their logical complexity. Maimonides acknowledges this heterogeneity, since he generally gives us his sources: "*Physics* and its commentaries", and "*Metaphysics* and its commentary". The books of *Physics* and *Metaphysics* are easy to identify: the third and the eighth book of *Physics* and the tenth and the eleventh of *Metaphysics*. But to identify exactly which commentaries on *Physics*, and which commentary on *Metaphysics*, is another matter, though not our concern here. The logical complexity of the lemmas is described by Maimonides as follows: "some lemmas are obvious by the least reflection and by demonstrative premises and by primary intelligible notions or by those close to them", while "others require more proofs, many premises, all of which, however, have been established by indubitable proofs" (Maimonides, 1972, p. 272). In other words, there are lemmas which are so close to axioms that they become self-evident by applying only the "merest reflection (التأمل الأيسر)"; others which are so remote that their proof requires many intermediary propositions, a task which has been accomplished by Aristotle, his commentators and his successors. The twenty-five lemmas of the system belong to one type or the other.

Maimonides is aware that, to be worth the name, a proof has to be both universal and compelling. But that is not the case for the question examined here regarding the irreducible opposition between the two truths, revealed and philosophical, concerning the eternity of the world. For the proof to have the form of a mathematical proof, that is, be truly apodictic, it should always be valid, whether one believes in the eternity of the world or not. Maimonides thus introduces into the system, as a mathematician so to speak, and also against his own conviction, the eternity of the world as a postulate, bringing the number of the preliminary propositions up to twenty-six. Regarding this, he says without the slightest ambiguity:

> To the above lemmas one lemma must be added which enunciates that the universe is eternal, which is held by Aristotle to be true, and which has to be believed first and foremost. We therefore admit it by convention (على جهة التقرير) only for the purpose of demonstrating our theorem (ibid., p. 272).

Maimonides thus introduces the eternity of the world as a necessary postulate for the completion of the system and, subsequently, for the deduction of his "theorem". The conventional—but non-arbitrary—aspect of the proposition is in sharp contrast with his rejection of the doctrine of the eternity of the world. Here, for example, is what he has to say on this matter:

> The true method, which is based on a logical and indubitable proof, consists, in my opinion, in demonstrating the existence of God, His unity, and His incorporeality by philosophical methods, but founded on the theory of the eternity of the universe; I do not propose

> this method as though I believed in the eternity of the universe, for I do not follow the philosophers on this point, but because by the aid of this method the proof can be valid; and certainty can be reached concerning these three principles, viz., the existence of God, His unity and His incorporeality, irrespectively of the question as to whether the universe is eternal or created (ibid., p. 187).

In fact, Maimonides knew that the problem of the eternity of the universe cannot have a positive solution. Some were to say later that dialectical reason comes up against an antinomy, since the properties of things which do not yet exist have be determined.

The architectonic of this part of the *Guide* is surely conceived of as a mathematical exposition, following the order of geometry. In fact, this order appears to be a condition for the certainty of metaphysical knowledge, namely that of God, of His existence and of His incorporeality. This seminal idea, already present in al-Kindī, will be found later in Spinoza. But, as noted by Crescas, the big problem still remains as to whether these twenty-five propositions have effectively been proved; and, whether, even then, the "theorem" can really be deduced. These two questions will keep on haunting Maimonides' successors. Al-Tabrīzī's commentary is designed to prove these propositions, and Crescas' attempt is motivated by the same intention. Maimonides himself attempts this deduction, which we will expound in broad terms, while emphasising the spirit in which it is carried out.

According to the twenty-fifth lemma, each composite individual substance needs for its existence a motor which properly prepares matter and enables it to receive form. But, according to the fourth lemma, there exists necessarily another motor which can be of a different class and which precedes the first motor. Following the third lemma, this chain of motors/mobiles is necessarily finite: motion finishes in the celestial sphere and then comes to rest. The celestial sphere establishes the act of locomotion, since this motion precedes all the other kinds of motion for the four categories of change, according to the fourteenth lemma. But the celestial sphere must have a motor since each moving object has necessarily a motor according to the seventeenth lemma. And this motor either resides within or without the moving object. This is a necessary division. If the motor is outside, then either it is an object outside the celestial sphere, or it is not in an object; in the latter case, the motor is said to be "separate" from the sphere. If the motor is within, it must be either a force distributed throughout, or an indivisible force, like soul in man. Four cases have then to be examined; three of them have been rejected by Maimonides since he shows their impossibility with the help of different lemmas. He is then left with only one possibility, of an incorporeal object outside and separate which is the cause of locomotion of the celestial sphere in space. Maimonides concludes his long proof in these words:

> It is therefore proved (فقد تبرهن) that the motor of the first Orb, if its motion be eternal and continuous, is necessarily neither itself corporeal nor does it reside as a *potentia* in a corporeal object for this motor to move, either of its own accord or accidentally; that is why it must be indivisible and unchangeable, as it has been mentioned in the fifth and the seventh lemmas. This prime Motor of the sphere is God, praised be His name. It is impossible that He could be two or more […]. That is what had to be proved (ibid., p. 276).

We have just shown that according to Maimonides, mathematics can be considered as a condition for metaphysical knowledge in three senses. The most obvious one is that mathematics is an exercise for the mind. In the second place, it offers a construction model—an architectonic—which can lead to certainty. And finally, it provides theoretical-proof techniques, mainly, the apagogic method. But these are not the only connections between mathematics and metaphysics that we can find in the *Guide*. We have quite recently drawn attention to another connection which is by no means less important: mathematics can play the role of an argumentation method in metaphysics. The most famous example, and the most relevant, is precisely taken from Apollonius' *Conics*: the problem of the relation between imagination and conception can best be dealt with by taking the example of an asymptote to an equilateral hyperbola. In his criticism of *Kalām*, Maimonides intends to refute the following thesis: "everything conceived by imagination is admitted by the intellect as possible". His strategy is to establish the negation of the thesis: there are unimaginable things, that is, things that can in no way be imagined though their existence can be proved. This shows that, for Maimonides, there is no principle which licenses a move from imagination to the metaphysical reality. He expresses his thesis as follows:

> Know that there are certain things, which would appear impossible, if tested by man's imagination, being as inconceivable as the co-existence of two opposite properties in one object; yet the existence of those same things, which cannot be represented by imagination, can nevertheless be established by proof, and their reality brought about (ibid., p. 214).

We have had the opportunity of showing (Rashed, 1987) that in these terms Maimonides takes up the problem of proving what cannot be conceived, a problem posed in the tenth century by the mathematician al-Sijzī. The example invoked by Maimonides to make his point is the same as the one discussed by his predecessor—proposition II. 14 of Apollonius' *Conics* concerning asymptotes to an equilateral hyperbola: the curve and its asymptotes will always come closer to each other if they are prolonged indefinitely, but they never meet.

> This is a fact, writes Maimonides, which cannot easily be conceived, and which does not come within the scope of imagination. Of these two lines the one is straight, the other curved, as stated in the aforementioned book. One has consequently proved the existence of what cannot be perceived or imagined, and would be found impossible if tested solely by imagination (ibid., p. 215).

The imagination invoked here by Maimonides is the mathematical imagination: nothing ensures even the way to metaphysical reality. But it can be stated with certainty that what is true for the mathematical imagination is *a fortiori* also true for all other forms of this faculty. Invoking the *Conics* proposition seems, in Maimonides' mind, to have more force than just that of mere example: it is an argumentation technique that the metaphysician borrows from mathematics.

To conclude: as did his predecessors from the time of al-Kindī, Maimonides finds in mathematics an architectonic model, proof techniques and model argumentation methods. The role of mathematics is in no way reduced to that of a propaedeutic to philosophical teaching: if Maimonides devoted time and energy to acquiring a mathematical knowledge—however modest one—it is because he conceived of it, as did his predecessors, as a deeply philosophical task: that of resolving metaphysical problems mathematically.

2 Mathematics in the Philosophical Synthesis and the "formal" Modification of Ontology: Ibn Sīnā and Naṣīr al-Dīn al-Ṭūsī

In his monumental *al-Shifā'*, as in his book *al-Najāt*, and in his *Danish-Nameh*, Ibn Sīnā gives mathematics a particular prominence. To take the *Shifā'* alone, Ibn Sīnā (980–1037) devotes no fewer than four books to mathematical sciences. To this must be added some independent chapters in astronomy and music. In all these writings, it has not been sufficiently understood that the presence of mathematics is significant in two respects. We have seen that al-Kindī was interested in mathematics on two accounts, in his capacity as a philosopher, and as a mathematician. So when he treats of burning mirrors, optics, sundials, astronomy, and when he comments on Archimedes, he does so as a mathematician. Mathematics is also a source of inspiration and an argumentation model for the philosopher. While al-Kindī's tradition survived him in the writings of Muḥammad ibn al-Haytham, Ibn Sīnā belongs only in part to this tradition. His mathematical knowledge, as one can see, is fairly wide-ranging though traditional. He probably knew the works of Euclid, of Nicomachus of Gerasa, and of Thābit ibn Qurra on the amicable numbers. He was also familiar with elementary algebra, with the theory of numbers and with certain works in Diophantine analysis. He seems not to have been well informed about contemporary research, as is shown by his claims about the regular heptagon. We can say, then, without fear of contradiction that Ibn Sīnā had a solid mathematical knowledge which allowed him to deal with certain applications, though not to undertake true mathematical research. This means that it is just as inaccurate to reduce his mathematical knowledge to Euclid's *Elements* and to Nicomachus of Gerasa's *Introduction to Mathematics*, as it is to represent him as a major mathematician of the tenth century. For this great logician, metaphysician and physician, mathematics plays a different role from that in al-Kindī since it is not only a source of inspiration for philosophical research but an integral part in a philosophical system. This explains the presence of four books in *al-Shifā'* devoted successively to the disciplines of the *quadrivium.* The question therefore is to assess the philosophical implications of this state of affairs.

If we consider Ibn Sīnā's theoretical views on the status of mathematics, the nature of its objects and the number of disciplines of which it is composed, we can conclude that he is the direct heir to a tradition: the status of mathematics is defined accordance with the Aristotelian theory of the classification of sciences, itself founded on the famous doctrine of Being; its objects are defined thanks to abstraction theory; as for the number of its disciplines, it is the well-known number passed on by the ancient Greek tradition. This concerns the three disciplines of the intermediary science (العلم الأوسط), which make up theoretical philosophy the objects of which are distributed among physics, mathematics and metaphysics—an order that the composition of *al-Shifā'* follows as a function of the materiality and mobility of the objects studied. Therefore mathematics considers objects abstracted from experience, separated from mobile, material and physical objects. The four disciplines which form mathematics are called the Quadrivium: Arithmetic, Geometry, Astronomy and Music. Ibn Sīnā always comes back to this doctrine, in the *Isagoge* as well as in the *Metaphysics* of *al-Shifā's*, and also in an opuscule devoted to the classification of sciences, among other writings.

> The types of sciences set out to consider beings either as moving objects, according to their conception and constitution, and as having to do with particular species and matters; either as separated from matters, according to the conception but not the constitution; or as separated according to the constitution and the conception. The first part of these sciences is physics; the second part is pure mathematics which includes the famous theory of numbers. As for the nature of numbers as numbers, they do not belong to this science. The third part is metaphysics. Since beings are by nature according to the three parts, theoretical philosophical sciences are those ones. Practical philosophy has to do either with the teaching of opinions whose use makes it possible to order the participation in common human things, and <this part> is known as the city's organisation; it is called politics; or with what makes it possible to order the participation in private human things, and <this part> is known as the home's organisation, <economics>; or finally what makes it possible to order the state of one person in order to build his soul: that is called ethics, p. 14).

There is nothing new in this conception. If we stop at this Aristotelian bias of Ibn Sīnā, the real role that mathematics plays in *al-Shifā'* cannot be captured. Perhaps we should wonder, first and foremost, whether such a position of principle corresponds to the philosopher's mathematical knowledge and whether the theoretical classification reflects a possible *de facto* classification. But to assess and to understand the distance, if it exists, between these two classifications, it is necessary to refer first to Ibn Sīnā's mathematical studies. Only arithmetic will be considered, even if geometry provides the philosopher with further opportunities for reflection (the fifth postulate for example, as in *Danish-Nameh*).

If we first consider purely biographical details, we know that while receiving his philosophical teaching, Ibn Sīnā was learning Indian arithmetic and algebra. It is only later that he was to learn logic, Euclid's *Elements* and the *Almagest*; an account given by many biobibliographers such as al-Bayhaqī, Ibn al-ʿImād, Ibn Khallikān, al-Qifṭī and Ibn Abī Uṣaybiʿa. Al-Bayhaqī reports for example:

> When he was ten years old, he knew certain fundamental texts of literature by heart. His father was studying and reflecting upon an opuscule of the Brothers of the Purity. He also reflected over it. His father took him to a greengrocer named Maḥmūd al-Massāḥ who knew Indian calculation and algebra and al-muqābala (Al-Bayhaqī, 1946, p. 53).

Ibn al-ʿImād gives this biographical anecdote in the same words and, quoting Ibn Khallikān, he writes: "When he was ten years old, he improved his knowledge in the science of the Glorious Qur'ān, literature, and he knew certain religious foundations by heart, Indian calculation and algebra and al-muqābala" (Ibn al-ʿImād, n.d., III, p. 234; see also Ibn Khallikān, 1969, II, pp. 157–158). As for Ibn Sīnā himself, he writes: "My father took me to a greengrocer who practised Indian arithmetic so that he could teach me." (al-Qifṭī, *Ta'rīkh al-Ḥukamā'*, p. 413 and Ibn Abī Uṣaybiʿa, 1965, p. 437).

But these new disciplines—Indian arithmetic and algebra—unknown to the Alexandrians, cannot find their place in the traditional framework of the classification of sciences without at least changing its general outline, if not changing drastically its underlying conceptions. But in Ibn Sīnā's classification, they appear under the sole title of "secondary parts of arithmetic (الأقسام الفرعية)". Ibn Sīnā gives no explanation whatsoever of this notion; contents himself simply with their enumeration. Here is what he writes:

> The secondary parts of mathematics — branches of the [science of] numbers: the science of addition and of separation of the Indian arithmetic; the science of algebra and al-muqābala. And the branches of the science of geometry: the science of measurement, the science of ingenious devices the science of the traction of heavy bodies; the science of weights and scales; the science of instruments specific to arts; the science of perspectives and mirrors; the hydraulic science. And the branches of astronomy: the science of astronomical tables and of calendars. And the branches of music: the use of wonderful and curious instruments as the organ and the like (*Parts of rational sciences*, p. 112).

Thus we learn only that arithmetic has as secondary parts Indian arithmetic and algebra. But the number of arithmetic disciplines invoked by Ibn Sīnā is not limited to the last two given in his classification of sciences. We have in fact already mentioned the volume that he devotes, in *al Shifā'*, to the science of calculation called *al-Arithmāṭīqī*. To this two further disciplines have yet to be added: one, though named, has never had its status fixed by Ibn Sīnā—it is al-Ḥisāb; the other is only present through its objects: integral Diophantine analysis.

The theory of numbers, *al-Arithmāṭīqī*, Indian arithmetic, algebra, al-Ḥisāb and integral Diophantine analysis: six disciplines which overlap and which are sometimes superimposed to cover the study of numbers. The reality is thus obviously much more complex than it looks in the classificatory schema of sciences. But to disentangle these disciplines and to elucidate their connections, we must briefly recall the works of the mathematicians at the time. The latter in fact distinguished, by denoting them under two different names, the Hellenistic tradition of arithmetic and its Arabic development: the number theory (علم الأعداد) on the one hand, and the discipline denoted by the phonetic transcription of ἡ ἀριθμητική on the other. If their connotation was not altogether unrelated, each of these terms did however refer to a distinct tradition. The expression "number theory (علم الأعداد)" referred to the arithmetic books of Euclid's *Elements*, and also to later works such as those of Thābit ibn Qurra, for example. Meanwhile, the phonetic transcription of ἡ ἀριθμητική (*al-arithmāṭīqī*) denoted the arithmetic tradition of the Neo-Pythagoreans, that is, the tradition as Nicomachus of Gerasa understands it in his *Introduction*; a term translated nevertheless by Ibn Qurra under the title *Introduction to the Number theory* (المدخل إلى علم العدد) (see Nicomachus, 1958). Without being systematic, the terminological difference between the ninth and tenth centuries seems to measure the gap which separated the two disciplines at the time. To understand how this gap was perceived later, let us read what Ibn al-Haytham writes.

> There are two ways in which the properties of numbers appear: the first is induction, since if we follow numbers one by one, and if we distinguish them, we find all their properties by distinguishing and by considering them, and to find the number in this way is called *al-arithmāṭīqī*. This is shown by [Nicomachus'] *al-arithmāṭīqī*. The other way in which the properties of numbers appear is by proofs and deductions. All the properties of numbers seized by proofs are contained in these three books [of Euclid] or in what is related to them (Rashed, 1980, p. 236).

This eminent mathematician deems both approaches to be scientific; a remark all the more important since Ibn al-Haytham demanded, everywhere and without restriction, rigorous proofs. And in fact, from the tenth century at least, these two traditions offered mathematicians the same conception of the object of arithmetic: an integer arithmetic represented by line segments. But while in number theory the norm of proof is restrictive, in *al-arithmāṭīqī* a simple induction can be used. For scientists of the tenth century, the difference between the two traditions was reduced to a distinction between methods and norms of rationality.

It is precisely this conception of the connection between the two disciplines which is expressed by Ibn Sīnā. In *al-Shifā'*, arithmetic appears twice: the first time in the geometry of *al-Shifā'* in which he merely summarises Euclid's books on arithmetic. On the second occasion, he writes his own book of *al-arithmāṭīqī*—which will be read and taught for many centuries—and whose real foundations, according to the author himself, can be mainly found in the *Elements*. Perhaps it is also this vision of the relationship between the two disciplines which explains why, in his *al-arithmāṭīqī*, Ibn Sīnā is not content with a simple summary of Nicomachus, as he had been for the theory of numbers, with Euclid's *Elements*. It would thus become clear how far he departs in this regard from the Neo-Pythagorean tradition. From now on, all the ontological and cosmological considerations which burdened the notion of number are *de facto* banned from *al-arithmāṭīqī*, considered thus as a science. What is left is the philosophical intention common to all branches of philosophy, whether theoretical or practical, that is, the perfection of the soul. Ibn Sīnā thus directs his attacks against the Neo-Pythagoreans:

> It is customary, for those who deal with this art of arithmetic, to appeal, here and elsewhere, to developments foreign to this art, and even more foreign to the custom of those who proceed by proof, [developments which are] closer to the exposition of rhetoricians and poets. It should be abandoned (*al-Shifā'*, *al- Arithmāṭīqī*, ed. Maẓhar, p. 60. It should be noted that few lines earlier, Ibn Sīnā clearly mentions them by their name, i.e. the Pythagoreans).

He can even partly abandon traditional language, and adopt that of the algebraists, to express the successive powers of an integer. The terms "square (مال)", "cube (كعب)", "square-square (مال مال)", which used to denote the successive powers of the unknown, were thus employed by the philosopher to name the powers of an integer (ibid., p. 19).

In these conditions, nothing prevented Ibn Sīnā from including in his *al-arithmāṭīqī* theorems and results obtained elsewhere, without repeating the proof (if there was one). That is what he did when he adopted (without proof) Thābit ibn Qurra's theorem on amicable numbers, in the Thābit's pure Euclidean style. Ibn Sīnā mentions as well several problems of congruence.

If you add even-even four numbers and a unit, if you get a prime number, provided that, if the last of them is added, and if the preceding one is taken away, and if the sum and the remainder are prime, then the product of the sum by the remainder, and the total by the last added numbers, yields a number which has a friend; its friend is the number obtained by adding the sum and the remainder, multiplied by the last of the added numbers, and by adding the product to the first number which had a friend. These two numbers are amicable (after correction of some errors in the Cairo edition, p. 28).

To these two traditions, a third also mentioned by Ibn Sīnā should be added which concerns the integral Diophantine analysis. In the logical part of *al-Shifā'* devoted to the proof, Ibn Sīnā considers the example of the first case of Fermat's conjecture, already dealt with by at least two mathematicians of the tenth century, al-Khujandī and al-Khāzin. Ibn Sīnā writes:

When we wonder […] whether the sum of two cubic numbers is a cube, in the same way as the sum of two square numbers was a square, we pose then an arithmetic problem (حساب or *ḥisāb*) (Ibn Sīnā, 1956, pp. 194–195).

We realise specifically that the term *ḥisāb* seems to designate here a discipline which includes disciplines other than the Euclidean theory of numbers and *al-arithmāṭīqī*. By *ḥisāb*, Ibn Sīnā seems to mean a science which includes all those which deal with numbers, rationals or algebraic irrationals; the last paragraph of his *al-Arithmāṭīqī* is unambiguous in this respect.

That is what we meant in the science of *al-arithmāṭīqī*. Certain cases have been left aside since we consider that mentioning them here would be extrinsic to the rule of this art. There remains in the science of *al-Ḥisāb* what suits us in the use and determination of numbers. What ultimately remains in practice is like algebra and *al-muqābala*, the Indian science of addition and separation. But for the latter, it would be best to mention them among the derivative parts (Ibn Sīnā, 1975, p. 69).

Everything thus indicates that, in *al-Arithmāṭīqī* as in the summarised Euclidean arithmetic books, Ibn Sīnā, like his predecessors and contemporaries, restricts his study to natural numbers. As soon as he meets some problems which would urge him to examine the conditions of rationality, whether it comes to searching for a positive rational solution or, more generally, to considering a class of irrational numbers, he finds himself outside these two sciences. The term of *al-ḥisāb* (الحساب) thus encompasses all arithmetic researches which are carried out by such disciplines as algebra, Indian arithmetic and the like. These disciplines have consequently an instrumental and, so to speak, applied aspect which puts them in opposition to the ancient number theory. And it is precisely this instrumental and applied character which enables Ibn Sīnā, as can be verified, to distinguish in his classification the set of "derivative parts", which are then defined as such. The "derivative parts (الأقسام الفرعية)" of physics are therefore medicine, astrology, physiognomy, oneiromancy, the divinatory art, talisman, theurgy and alchemy.

To understand the distance put by Ibn Sīnā between himself and traditional, Hellenistic and Greek classifications as well as between himself and his own theoretical classification, it is worth introducing here one of his predecessors, al-Fārābī (872–950). Whether Ibn Sīnā's opuscule *The parts of rational sciences* is related to al-Fārābī's classification expounded in his *Enumeration of Sciences* is a question first posed by Steinschneider, who denied that there was any such relation.

Wiedemann (1970, p. 327) confirms this opinion, and claims that Ibn Sīnā lists only separated sciences, whereas al-Fārābī designates and characterises them by their mutual dependence; or, as he puts it "Ibn Sīnā zählt im wesentlichen die einzelnen Wissenschaften auf, während al-Fārābī sie in zusammenhängender Darstellung charakterisiert."

In fact the comparison forces itself upon us anew, since the examination of "derivative parts" of Ibn Sīnā's arithmetic shows that they are nothing but those disciplines brought together by al-Fārābī under the title "the science of ingenious techniques", which he defines as follows:

> The science of the way to proceed when we apply all whose existence is proved, by predication and proof, in the previously mentioned mathematical sciences, to physical bodies; and when we achieve and put it effectively in the physical objects (Al-Fārābī, 1968, p. 108).

According to al-Fārābī, the object of mathematics is lines, surfaces, solids and numbers that he considers as intelligible by themselves, and separate (منتزعة), that is, abstracted from physical objects. Intentionally to discover and show mathematical notions in the latter with the help of the art would require the conception of ingenious devices, the invention of techniques and methods capable of overcoming the obstacles posed by the materiality of empirical objects. In arithmetic, the ingenious devices involve, among other things, "the science known by our contemporaries under the name of algebra and *al-muqābala*, and what is similar to it" (ibid., p. 109). He also takes notice however that "this science is common both to arithmetic and geometry" and further on adds that:

> It includes the ingenious devices to determine the numbers that we try to determine and use, those which are rational and irrational the principles of which are given in Euclid's *al-Usṭuqusāt* 10th book, and those which are not mentioned by Euclid. Since the relation of rational to irrational numbers — to one another — is like the relation of numbers to numbers, each number is thus homologous with a certain rational or irrational magnitude. If we determine the numbers which are homologous with magnitude *ratios*, we then determine these magnitudes in a certain manner. That is why we postulate certain rational numbers to be homologous with rational magnitudes, and certain irrational numbers to be homologous with irrational magnitudes (ibid., p. 109).

In this text of capital importance, algebra is distinguished from science on two accounts: although—like every science—apodictic, it nevertheless represents the domain of application not only of one science but of two at the same time, arithmetic and geometry. As for its object, it includes geometric magnitudes as well as numbers, which can be both rational or algebraic irrational. In the presence of this new discipline which has to be taken into account, the new classification of the sciences which aimed at both universality and exhaustiveness has to justify in one way or another the abandonment of certain Aristotelian theses. Names such as "science of ingenious devices", "derivative parts" are coined so that a non-Aristotelian zone can be arranged within a received Aristotelian style of classification.

The philosophical impact caused by such a revision is on a larger scale and—especially—more profound than mere taxonomic modification. If algebra is in fact common to arithmetic and geometry, without in any way giving up its status as science, it is because its very object, the "algebraic unknown", that is, the "thing (الشيء, *res*)" can refer indifferently to a number or to a geometric magnitude. More than that: since a number can also be irrational, "the thing" designates then a quantity which can be known only by approximation. Accordingly the algebraists' subject matter must be general enough to receive a wide range of contents; but it must moreover exist independently of its own determinations, so that it can always be possible to improve the approximation. The Aristotelian theory is obviously unable to account for the ontological status of such an object. So a new ontology has to be made to intervene that allows us to speak of an object devoid of the character which would none the less enable us to discern what it is the abstraction of; an ontology which must also enable us to know an object without being able to represent it exactly.

This is precisely what has been developing in Islamic philosophy since al-Fārābī: an ontology which is "formal" enough, in a way, to meet the requirements mentioned above, among other things. In this new ontology, "the thing (الشيء)" has a more general connotation than the existent. This is a distinction made more precise by al-Fārābī when he writes: "the thing can be said of every thing that has a quiddity, whether it is external to the soul or [merely] conceived of in any way" whereas the "existent is always said of every thing that has a quiddity, external to the soul, and cannot be said of a quiddity merely conceived of." Therefore, according to him, the "impossible (المستحيل)" can be named a "thing" but cannot be "existent" (Al-Fārābī, 1970, p. 128).

As regards the history of mathematics, this trend has been again confirmed between al-Fārābī and Ibn Sīnā: al-Karajī particularly gives a more general status to algebra, and emphasises the extension of the concept of number. A contemporary to Ibn Sīnā, al-Bīrūnī goes even further and writes without hesitation:

> The circumference of a circle is in a given proportion to its diameter. The number of the one to the number of the other is also a proportion, even if it is irrational (Al-Bīrūnī, 1954, I, p. 303).

As regards philosophy, Ibn Sīnā—a consistent metaphysician—includes al-Fārābī's conception into a doctrine that he wants to be more systematic and which is expounded in his *al-Shifā'*. According to this doctrine, "the thing" is given in an immediate evidence or, in Ibn Sīnā's own terms, is imprinted immediately in the soul, just as "the existent" and "the necessary"; along with these two other ideas, it is the principle behind all things. While the existent signifies the same meaning as "asserted (المُثبت)" and "achieved (المُحصَّل)", the thing is, writes Ibn Sīnā, what the predication concerns (the proposition). Hence every existent is a thing but the converse is not correct, though it is impossible that a thing should exist neither as a concrete subject nor in the mind (Ibn Sīnā, 1960a, I, p. 29 sq. and p. 195 sq.). A full description of Ibn Sīnā's doctrine is outside the scope of the present chapter, but it is sufficient to recall that, being neither Platonic nor Aristotelian, this new ontology arose to, in part at least, due to the new results in mathematical sciences. If mathematics leads Ibn Sīnā to shift his ontology in a "formal" direction, so to speak, it acts in the same way on his conception of the ontology of emanation, as we shall see later with al-Ṭūsī's commentary.

The emanation from the One of Intelligences and celestial orbs and the other worlds—that of nature and corporeal things—, is one of the central doctrines of Ibn Sīnā's metaphysics. This doctrine raises both ontological and noetic questions: how can a multiplicity emanate from one unique and simple being, a multiplicity which is also a complex, ultimately containing the matter of things as well as the form of bodies and human souls? This ontological and noetic duality sets up the question as an obstacle, as both a logical and metaphysical tangle that must be unravelled. From that point we understand, in part at least, why Ibn Sīnā returns tirelessly to this doctrine and implicitly to this question in his different writings.

The study of the historical evolution of Ibn Sīnā's thought on this problem through his different writings would show how he was able to amend his initial formulation as a function of this difficulty. To limit ourselves to *al-Shifā'* and *al-Ishārāt*, Ibn Sīnā expounds the principles of the doctrine and the rules of the emanation of multiples from one simple unity. His explanation looks like an articulated and ordered exposition but does not constitute a rigorous proof: Ibn Sīnā does not in fact give the syntactic rules capable of matching the semantics of emanation. This is precisely where the difficulty of the derivation of the multiplicity from the One lies. This derivation has long been seen as a problem and examined as such. The mathematician, philosopher and commentator of Ibn Sīnā, Naṣīr al-Dīn al-Ṭūsī, not only grasped the difficulty, but wanted to offer the syntactic rules that were lacking.

To understand this contribution, we have at the outset to go back to Ibn Sīnā to recall the elements of his doctrine and also to grasp, however weakly, the formal principle in his synthetic and systematic exposition whose presence has made possible the introduction of the rules of combinatorial analysis. In fact, this principle allows Ibn Sīnā to develop his exposition in a deductive style. He has to ascertain on the one hand the unity of Being, which is said of everything in the same sense and, on the other, the irreducible difference between the First Principle and His creation. He then develops a somewhat "formal" general conception of the Being: considered as a being, he is not the subject of any determination, not even that of modalities; it is just a being. It is not a genus, but a "state" of whatever there is, and can only be grasped in its opposition to non-being, without nevertheless being preceded in time by the latter—this opposition is only according to the order of reason. On the other hand, only the First Principle receives His existence from Himself (Ibn Sīnā distinguishes between existence and essence for all other beings; on this point, see Goichon, 1957; D. Saliba, 1926; Verbeke, 1977). So this existence is what is necessary, and it is in this case that existence coincides with essence. All other beings receive their existence from The First Principle by emanation. This ontology and the cosmogony that goes with it provide the three points of view under which a being is envisaged: as a being, as an emanation (see Gardet 1951, Heer 1992, Hasnawi 1990, Druart 1992, Morewedge 1992, Marmura 1992, Owens 1992) of the First Principle, and as being its quiddity (viewed from the first two angles, the necessity of the being imposes itself while its contingency reveals the third). These are, briefly mentioned, the three notions on which Ibn Sīnā is going to establish his postulates, which are:

1. There is a First Principle, a necessary Being by essence, one, in no way divisible, which is neither a body nor in a body.
2. The totality of Being emanates from The First Principle.
3. The emanation is not carried out either "according to an intention (على سبيل القصد)" or to achieve any purpose, but by a necessity of the being of the First Principle, that is, His self-intellection.
4. From the One only one proceeds.
5. There is a hierarchy in the emanation, from those whose being is most perfect (الأكملُ وجوداً) to those whose being is least perfect (الأخسُّ وجوداً).

We might think that certain postulates seem to contradict each other, as, for example, 2 and 4, or suspect that some lead to contradictory consequences. To avoid this first impression, Ibn Sīnā introduces further determinations in the course of his deduction. So from 1, 2, 4 and 5 follows that the totality of Being, in addition to the First Principle, is a set ordered by both the logical and axiological predecessor-successor relation, regarding both the priority of the being as well as its excellence. Barring the First Principle, each being can have only one predecessor (as the predecessor of its predecessor, and so on). On the other hand, each being, including the First Principle, can have only one successor (respectively the successor of its successor, and so on). But the philosopher and his commentator know that, taken literally, this order forbids the existence of multiple beings, that is, their independent coexistence, without some having logical priority over others or being more perfect than them; which makes this order clearly false, as al-Ṭūsī says (al-Ṭūsī, 1971, p. 216). Thus it is necessary to introduce further details and intermediary beings.

But 1 and 2 in their turn exclude multiplicity to be a product of the First Principle's "momenta" (نزوعات) and "perspectives" (جهات), since assuming momenta and perspectives in Him amounts to denying His unity and simplicity. Finally, 3, 4, and 5 imply that the emanation as an act of the First Principle is not like a human act, because its Author has neither intention nor purpose. Everything indicates then that intermediary beings (*mutawassiṭa*), hierarchically ordered, no doubt, have to be used to account for the multiplicity-complexity.

Let us begin, as one should, with the First Principle, and designate it, as Ibn Sīnā does in his opuscule *al-Nayrūziyya*, by the first letter of the alphabet—*a.* The First Principle intellects itself by essence. In Its self-intellection, It "intellects" the totality of the being of which It is the very principle (Ibn Sīnā, 1960b, p. 402, *l.* 16), without there being in Itself any obstacle to the emanation of this totality, nor to its rejection. It is in this sense only that the First Principle is said to be the "agent" (فاعل) of the totality of being.

But having admitted this, one has yet to explain how the necessary emanation of the totality of being can be achieved without having to add anything which could be inconsistent with the Unity of the First Principle. Following 1, 4, 5, from the First Principle only one being emanates, a being which necessarily belongs to the second rank in existence and perfection. But, as it is the emanation from a pure and simple unique being, at the same time pure truth, pure power, pure goodness…, with none any of these attributes existing in it independently so as to ensure the unity of the First Principle, this derived being can only be a pure Intellect. This conclusion respects 4, since, if this intellect were not pure, we should conclude that more than one emanates from the One. We have here the first separate Intellect, the first effect (معلول) of the First Principle. Following Ibn Sīnā, let us refer to it as *b*.

Everything is now in place to explain the multiplicity-complexity. By essence, this pure Intellect is an effect: it is therefore contingent. But, as an emanation from the First Principle, it is necessary since it was "intellected" by the latter. This ontological duality is superimposed upon a noetic multiplicity: this pure Intellect knows itself and knows its own being as contingent being, that is, its essence is different from that of the First Principle since the latter is necessary; on the other hand, it knows the First Principle as the necessary Being; and finally it knows the necessity of its own being as an emanation of the First Principle. I have just paraphrased here what Ibn Sīnā writes himself in *al-Shifā'* (ibid., pp. 405-406). He replies in advance to a possible detractor, noting that this multiplicity-complexity is not, if we may say so, a hereditary property: the pure Intellect does not receive it from the First Principle, for two reasons. First the contingency of its being belongs to its own essence, and not to the First Principle, which gave it the necessity of being. On the other hand, the knowledge that it has of itself, as well as the knowledge that it has of the First Principle, is a multiplicity, which is the result of the necessity of its being which derives from the First Principle. In these conditions, Ibn Sīnā can reject the accusation of attributing this multiplicity to the First Principle.

Ibn Sīnā then describes how the other separated Intellects, celestial Orbs, and Souls which enable the Intellects to act, emanate from the Pure Intellect. So, from the pure Intellect *b* emanates, by its intellection of *a,* a second intellect; let it be named *c*; and by its intellection of its own essence, the Soul of the ninth celestial Orb; and by its intellection of its own being as contingent being the body of this ninth Orb. Let us denote the Soul of this Orb and its body as *d.*

Ibn Sīnā thus continues to describe the emanation of Intellects, celestial Orbs with Souls and their bodies. From now on, the matter of sublunary things emanates from every Intellect, the forms of the bodies and human souls. But even if Ibn Sīnā's explanation has the advantage of not separating the question of the multiplicity from that of complexity, that is, the ontological content of the multiplicity, it does not however lead to a rigorous knowledge of the latter, since no general rule is given. Ibn Sīnā does nothing but lead the elements back to the Agent Intellect.

It is precisely here that al-Ṭūsī intervenes. He will actually show that there emanates from the First Principle—following Ibn Sīnā's rule and with the help of a reduced number of intermediaries—a multiplicity such that each effect will have only one cause which exists independently. We shall see that the price of such undeniable progress in knowledge of the multiplicity is impoverishment of the ontological content: from multiplicity-complexity there will in fact remain only the multiplicity.

In his commentary of *al-Ishārāt*, al-Ṭūsī introduces the language and techniques of combinatorial analysis to follow the emanation to the third rank of beings. Here he stops the application of these techniques, to conclude: "if we then go beyond [the first three] ranks, there may exist an uncountable multiplicity (لا يحصى عددها) in only one rank, and go on ad infinitum" (al-Ṭūsī, 1971 pp. 217–218). The intention of al-Ṭūsī is thus clear, and the device applied to the first three ranks leaves no doubt: one must provide the proof and means lacking in Ibn Sīnā. But al-Ṭūsī is at this stage still distant from his goal. Indeed it is one thing to proceed by combinations for a number of objects, and another to introduce a language with its syntax. The language in this case would be that of combinations. It is to this task that al-Ṭūsī applies himself in an independent dissertation (Rashed, 1999), whose title leaves no room for ambiguity: *On the proof of the Mode of Metaphysics: emanation of Things in an Infinite <number> from the Unique First Principle.* In this instance, as we shall see, al-Ṭūsī proceeds in a general way with the help of combinatorial analysis. The text of al-Ṭūsī and its results do not pass away with the death of its author; they are to be found in a later treatise entirely devoted to combinatorial analysis. Thus al-Ṭūsī's solution not only distinguishes a style of research in philosophy, but represents an interesting contribution to the history of mathematics itself.

Al-Ṭūsī's idea is to subject this problem to combinatorial analysis. But, for combinatorics to be used, he has to make sure that the time variable is neutralised, which in the case of the doctrine of emanation involves either discarding Becoming, or, at least, offering a purely logical interpretation of it. This condition has already been suggested by Ibn Sīnā himself, as we have shown. It should rightly be noted that emanation does not take place in time, and anteriority and posteriority have to be understood essentially, not temporally (*al-Shifā'*, VI, 2, p. 266. See Hasnawi 1990, Gardet 1951, Davison 1987, Druart 1987, Morewedge 1972). This interpretation, in our view crucial in the Avicennan system, refers to his own conception of the necessary, the possible and the impossible. Let us recall, briefly, that in *al-Shifā'* (see especially book 3, chapter 4 of *Syllogism*, IV, Ibn Sīnā, 1964), Ibn Sīnā takes up this old problem to reject right from the start all ancient doctrines which are, according to him, circular: they use in the definition of each of the three terms one or the other of the two remaining ones. To break this circularity, Ibn Sīnā intends to restrict the definition of each term by bringing it back to the notion of existence. He distinguishes then what is considered in itself as necessary existence from what, equally considered in itself, can exist and may also not exist. Necessity and contingency are for him inherent in the beings themselves. As for possible being, its existence and non-existence depend on a cause external to it. Contingency does not appear thus as a denied necessity, but as another mode of existence. The possible being might even be, while remaining in itself, of a necessary existence as an effect of another being. Without wanting to follow here the subtleties of Ibn Sīnā's development, it is sufficient to note that, from this particular definition of the necessary and the possible, Ibn Sīnā bases the terms of emanation in the nature of beings, neutralising from the outset—as it has been underlined above—the time variable. From these definitions, he infers some propositions, the majority of which are established by *reductio ad absurdum*. He shows that the necessary cannot but exist, that by essence it cannot have a cause, that its necessity includes all its aspects, that it is one and can in no way admit a multiplicity, that it is simple, without any composition…. On all these points, it is opposed to the possible. Thus it is in the very definition of the necessary and the possible, and in the dialectic in which they enter, that are forever fixed the anteriority of the First Principle and its relation with the Intelligences.

If therefore emanation can be described without appealing to time, it is because its own terms are given in the logic of the necessary and the possible. This doctrine may raise difficulties, but it is not the point here: we know that the conditions for introducing a combinatorics have already been ensured by Ibn Sīnā himself.

We have said that from *a* emanates *b*; the latter is then in the first rank of effects. From *a* and *b* together emanates *c*, that is, the second intellect; from *b* alone emanates *d*, that is, the celestial Orb. We have thus in the second rank two elements *c* and *d* such that each one is not the cause of the other. Up to now we have in all four elements: the first cause *a* and three effects *b*, *c* and *d*. Al-Ṭūsī calls these four elements the *principles.* At this point, let us combine the four elements two by two, then three by three, and finally four by four. We successively get six combinations—*ab*, *ac*, *ad*, *bc*, *bd*, *cd*—, four combinations—*abc*, *abd*, *acd*, *bcd*—, and one combination of four elements—*abcd.* If we take into account the combinations of these four elements 1 by 1, we get a total of 15 elements; of which 12 are in the third rank of effects, without any of them being used as intermediary to obtain the others. That is what al-Ṭūsī sets out in his commentary on the *al-Ishārāt*, as well as in the treatise mentioned above. But as soon as we go beyond the third rank, things quickly get complicated, and al-Ṭūsī has to introduce in his treatise the following lemma:

The number of combinations of n *elements is equal to*

$$\sum_{k=1}^{n} \binom{n}{k}.$$

To calculate this number, al-Ṭūsī uses the equality

$$\binom{n}{k} = \binom{n}{n-k}.$$

So, for $n = 12$, he gets 4095 elements. It should be noted that to deduce these numbers, he gives the expressions of the sum by combining the alphabetical letters.

Al-Ṭūsī returns later to calculate the number of elements of the fourth rank. He then considers the four principles with the twelve beings of the third rank; he gets 16 elements, from which he gets 65520 effects. To reach this number, al-Ṭūsī proceeds with the help of an expression equivalent to

$$(*) \sum_{k=0}^{m} \binom{m}{k} \binom{n}{p-k} \qquad \text{for } 1 \le p \le 16, m = 4, n = 12,$$

the value of which is the binomial coefficient

$$\binom{m+n}{p}.$$

None of these elements—with the exception of *a*, *b*, and *ab*—is an intermediary for the others. Hence al-Ṭūsī's response is general, and (*) gives a rule which permits ascertaining the multiplicity in each rank.

Having established these rules and given the example of the fourth rank, with its 65520 elements, al-Ṭūsī is able to give a definitive answer to the question "of the possibility of the emanation of the accountable multiplicity from the First Principle under the condition that from the One emanates only one and without the effects being successive (in chain). That is what had to be proved."

Al-Ṭūsī's achievement—to make Ibn Sīnā's ontology speak in terms of combinatorial analysis—has driven two important evolutions: both in Ibn Sīnā's doctrine and in combinatorics. It is clear that this time the question of multiplicity is kept at a certain distance from that of the complexity of being. Al-Ṭūsī cares little about the ontological status of each of the thousands of beings which make up, for example, the fourth rank. Even more: metaphysical discourse at this point allows us to speak of a being without allowing us to represent it exactly. This somewhat "formal" evolution of ontology, which is here blatant, does nothing but amplify a trend already present in Ibn Sīnā in his considerations on "the thing (الشَيء)", as we have emphasised above. This "formal" movement is accentuated by the possibility of designating beings by the letters of the alphabet. Even the First Principle is no exception to the rule, since It was denoted by *a.* In this al-Ṭūsī once again amplifies an Avicennan practice while modifying its sense. In the epistle *al-Nayrūziyya*, Ibn Sīnā resorted to this symbolism, but with two differences. On the one hand, he attributed to the succession of the letters of the Arabic alphabet following the order *abjad hawad* the value of a priority order, of logical anteriority; on the other hand, he has used the numerical values of the letters ($a = 1$, $b = 2$, etc.). Although al-Ṭūsī implicitly keeps the order of priority by denoting—as does Ibn Sīnā—the First Principle by *a*, the pure Intellect by *b*, he has dropped the hierarchy in favour of the conventional value of the symbol. And the numerical value has disappeared. This is necessary for the letters to be the objects of a combinatorics. A mathematician and a philosopher, al-Ṭūsī has thought through Ibn Sīnā's doctrine of emanation in a formal sense, thus favouring a trend already present in Ibn Sīnā's ontology.

3 From *ars inveniendi* to *ars analytica*

Due to reasons internal to the evolution of the discipline, mathematicians of the ninth century confronted the problem of the duality of order: is the order of exposition identical to the order of discovery? Naturally, this question was raised concerning the very model of the mathematical composition at that time and for many centuries to come, namely Euclid's *Elements.* Thābit ibn Qurra devotes a treatise to this problem in which he claims that Euclid's order of exposition is just the logical order of proofs, and differs from the order of discovery. To characterise the latter, Thābit develops a psychological doctrine of mathematical invention. We are already in a sense within the philosophy of mathematics.

This question of order was soon to be included in a problematic of a more general nature, that of analysis and synthesis, profoundly transformed. Mentioned by Galen, Pappus, and occasionally Proclus, this topic had never assumed the dimension that it took on in the tenth century. The development of mathematics and the conception of new chapters from the ninth century were enormously significant for the breadth and understanding of this subject, giving rise to the development of a real philosophy of mathematics. Indeed we witness in succession the elaboration of a philosophical logic of mathematics, then the project of an *ars inveniendi* and, finally, of an *ars analytica.*

Everything began, apparently, with Ibrāhīm ibn Sinān (909–946). He wrote a book devoted entirely and uniquely to analysis and synthesis, entitled *On the Method of Analysis and Synthesis in the Problems of Geometry* (Rashed and Bellosta, 2000, chapter I). The importance of this is clear. From now on analysis and synthesis constitute a domain which the mathematician can occupy both as a geometer and as a logician-philosopher. Here is how Ibn Sinān describes his enterprise and his intention:

> I have then, exhaustively, established in this book a method designed for students, which contains all that is necessary to resolve the problems of geometry. I have exposed in general terms the various classes of geometric problems; I have then subdivided these classes and illustrated each of them by an example; I have afterwards shown the student the way thanks to which he will be able to know in which of these classes to put the problems which will be posed to him, by which he will know how to analyse the problems — as well as the subdivisions and conditions necessary to that purpose —, and to carry out their synthesis — as well as the necessary conditions for that —, then how he will know whether the problem is among those which are solvable in one or several trials, and more generally, all that he must know in these matters. I have pointed out the kind of errors committed by the geometers when analysing because of a habit they have acquired: excessive abbreviation. I have also indicated for what reason there may seemingly be for the geometers, in the propositions and the problems, a difference between analysis and synthesis, and I have shown that their analysis is different from synthesis only due to abbreviations, and that, if they had completed their analysis as it should be, it would have been identical to the synthesis; the doubt would then have left the hearts of those who suspect them of producing in the synthesis things which had not been mentioned previously in the analysis — the things, lines, surfaces, and such, which are seen in their synthesis, without having been mentioned in the analysis; I have shown that and I have illustrated it by examples. I have presented a method thanks to which analysis is such that it coincides with synthesis; I have warned against the things which are tolerated by the geometers in analysis, and I have shown what kind of errors attach to them if they are tolerated (Rashed and Bellosta, 2000, pp. 96–98).

The intention of Ibn Sinān is clear, and his project is well articulated: to classify the geometric problems according to different criteria in order to show how to carry on, in each class, by analysis and synthesis, and to point to the occurrence of errors so that they can be avoided. Here is a broad outline of his classification.

1. The problems the assumptions of which are completely given
    1.1. The true problems
    1.2. The impossible problems
2. The problems for which it is necessary to modify some assumptions
    2.1. The problems with discussion (diorism)

- 2.2. The indeterminate problems
  - 2.2.1. The indeterminate problems strictly speaking
  - 2.2.2. The indeterminate problems with discussion
- 2.3. The overabundant problems
  - 2.3.1. The indeterminate problems to which an addition is made
  - 2.3.2. The problems with discussion to which an addition is made
  - 2.3.3. The true problems to which an addition is made

To this may further added the modal classification of propositions.

This classification is made from several criteria: the number of solutions, the number of assumptions, their compatibility and their possible independence.

A little over two centuries later, al-Samaw'al takes up this classification, still starting from the number of solutions and the number of assumptions (Ahmad and Rashed, 1972). He further refines the classification. He distinguishes identities from the problems which have an infinite number of solutions without being identities. He furthermore introduces the notion of undecidable problems, for which no proof, either of existence or of impossibility, can be found (Rashed, 1984b, p. 52). Unfortunately the author gives no example. The least to be said however is that the mathematician was able to shift the Aristotelian notions of the necessary, possible and impossible to those of computability and semantic undecidability.

In his book, Ibn Sinān discusses other logical problems such as the place of auxiliary constructions, the reversibility of analysis, and apagogic reasoning. Analysis and synthesis thus appear in his book both as a discipline and as a method. The former is in fact a philosophical and pragmatic logic, since it makes possible the combination of an *ars inveniendi* and an *ars demonstrandi*, the latter is a technique founded on a proof theory that Ibn Sinān endeavoured to elaborate.

One generation after Ibn Sinān, the mathematician al-Sijzī (last third of the tenth century) designed a different project, that of an *ars inveniendi* which meets both logical and didactic requirements. Al-Sijzī begins by enumerating certain methods aimed at facilitating mathematical invention—at least seven. Among them, "analysis and synthesis" figures as the principal method, which are provided with effective means of discovery by several specific methods such as the method of punctual transformations and the method of ingenious devices. All these specific methods share the idea of transforming and varying the figures as well as the propositions and solution techniques. Summarising his project, al-Sijzī writes

> As the examination of the nature of propositions (الأشكال) and of their properties in themselves is surely carried out following one of these two ways: either we imagine the necessity of their properties by having their species vary, an imagining which draws on sensation or what is common to the senses; or by setting these properties and also the lemmas they necessitate, successively, by a geometric necessity […] (Rashed, 2002, p. 818).

For al-Sijzī, the *ars inveniendi* consists mainly of two ways. All specific methods are put together in the first way, while the second is nothing but "analysis and synthesis". It is this distinction, the nature of the first way and this close relationship between the two, which single out al-Sijzī's conception and reflect the originality of his contribution.

It remains to be noted that the first of the two ways is divided into two, according to the two senses of the term *shakl* (شكل). This term, chosen to render διάγραμμα by the translators of Greek mathematical writings, designates as the latter indifferently both the figure and the proposition. This double meaning is not too fraught with ambiguity as long as the figure graphically translates—in a static manner, if I may say so—the proposition; in other words, as long as geometry remains mainly the study of figures. But complications arise when the figures are subjected both to transformations and variations, as is already the case in certain branches of geometry at the time of al-Sijzī. The double reference then requires clarification. Let us begin with the first sense, that of "figure".

In this treatise, al-Sijzī recommends, on three occasions, proceeding by variation of the figure: when a punctual transformation is carried out; when one element of the figure is changed, all others remaining fixed; finally, when an auxiliary construction is chosen. But several elements are common to these different techniques. Firstly the goal: we always try to reach, thanks to transformation and variation, invariable properties of the figure associated with the proposition, those which characterise it specifically. It is precisely these invariable properties which are stated in the figure as a proposition. The second element is also related to the goal: variation and transformation are means of discovery since they lead to invariable properties. The imagination takes over at this stage, a power of the soul capable of drawing upon the multiplicity suggested by the senses, through the variable properties of the figures, the invariable properties, and the essence of things. The third element concerns the particular role of the figure, as a representation this time: the role, mentioned by al-Sijzī, of fixing the imagination, of helping it in its task when it draws upon the sensation. And the last element, but not the least, deals with the duality figure-proposition: there is no one-to-one relation. To the same and sole proposition can be related a variety of figures; just as to one sole figure can be related a whole family of propositions. Al-Sijzī chose to deal at length with the last case. These new connections between figure and proposition that al-Sijzī was the first to point out, so far as I know, require that a new chapter of *ars inveniendi* be thought through: the analysis of figures and their connections to propositions. This is precisely what seems to have been inaugurated by al-Sijzī.

One generation later, Ibn al-Haytham (d. after 1040) conceives another project: founding a scientific art, with its rules and vocabulary. Ibn al-Haytham begins by recalling that mathematics is founded on proofs. By proof, he means "the syllogism which necessarily indicates the truth of its own conclusion" (Rashed, 1991, p. 36; 2002, p. 162 sqq.). This syllogism is made up in its turn "of premises whose truth and validity are recognised by the understanding, without its being troubled by any doubt about them; and of an order and arrangement of these premises such that they compel the listener to be convinced of their necessary consequences and to believe in the validity of what follows on their arrangement" (ibid.). The Art of analysis (صناعة التحليل) offers the method to obtain these syllogisms, that is, "to pursue the research of their premises, to contrive to find them, and to try to find their arrangement" (ibid.). In this sense, the Art of analysis is an *ars demonstrandi*. It is also an *ars inveniendi*, since it is because of this art that we are led to "discover the unknowns of mathematical science and how to carry on seeking the premises (literally 'to hunt (تصيد) for the proofs'), which are the material of proofs indicating the validity of what is discovered from the unknowns of these sciences, and the method to reach the arrangement of these premises and the figure of the combination" (ibid., p. 38).

For Ibn al-Haytham, it is indeed an *Ars* (τέχνη, صناعة) *Analytica*, which has to be conceived and constructed. But to my knowledge nobody before him considered analysis and synthesis as an art or, more precisely, as a double art, of proof and discovery. In the former, the analyst (المحلل) has to know the principles (أصول) of mathematics. This knowledge has to be backed both by an "ingenuity" and an "intuition formed by the art" (حدس صناعي). Indispensable for discovery, this intuition is equally proved to be necessary when the synthesis is not the strict reversal of the analysis, but requires further data and properties which have to be discovered. That the knowledge of principles, ingenuity and intuition are numerous means that the analyst must have at his disposal the ability to discover mathematical unknowns. The "laws" and "principles" of this analytical art remain yet to be ascertained. This necessary knowledge is the subject of a discipline which bears on the foundations of mathematics, and which deals with the "knowns". It must itself be constructed. The latter is a feature peculiar to Ibn al-Haytham, since nobody before him, not even Ibn Sinān, had considered elaborating an analytical art founded on a specific mathematical discipline. To this Ibn al-Haytham devotes a second treatise, *The Knowns* (Rashed, 1993c), one that he had promised in his treatise on *Analysis and Synthesis* (Rashed, 1991, p. 68). He himself presents this new discipline as that which offers the analyst the "laws" of this art and the "foundations" in which discovery of properties and apprehension of premises are brought to completion; in other words, it reaches the basis of mathematics, the prior knowledge of which is in fact, as we have said, necessary to the completion of the art of analysis: these are the notions called the "knowns" (ibid., p. 58). It should be observed that whenever he deals with a foundational problem, as in his treatise *On Squaring the Circle* (Rashed, 1993b, pp. 91–95), Ibn al-Haytham comes back to the "knowns".

According to Ibn al-Haytham, a notion is said to be "known" when it remains invariable and admits no change, whether or not it is thought by a knowing subject. The "knowns" refer to the invariable properties, independent of the knowledge that we have of them, and remain unchanged even though the other elements of the mathematical object vary. The aim of the analyst, according to Ibn al-Haytham, is precisely to lead to these invariable properties. Once these fixed elements have been reached, his task ends, and the synthesis can then start. The *Ars inveniendi* is neither mechanical nor blind, it should lead to the "knowns" through sustained ingenuity.

The analytical art thus requires for its construction a mathematical discipline, itself to be constructed. The latter contains the "laws" and the "principles" of the former. According to this conception, the analytical art cannot be reduced to any logic, but its own logical component is immersed in the mathematical discipline. We immediately discover the limit of the range of this art.

Conclusion

The contributions briefly sketched here indicate several situations where mathematicians deal with the philosophy of mathematics. We have previously examined other situations where philosopher-mathematicians and mathematician-philosophers contribute to the philosophy of mathematics. These contributions are obviously part of the history of philosophy and the history of the sciences, the history of the mathematical thought of classical Islam. To neglect these contributions is both to impoverish of the history of philosophy and to cut short the history of mathematics.

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