Ibn Sīnā's Philosophy of Mathematics

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Abstract We try to find the answers to two main questions of philosophy of mathematics in Ibn Sīnā's philosophy, i.e. what and where are mathematical objects? And how can we know mathematical objects? Ibn Sīnā's ontology implies that mathematical objects are *mental objects*. In his epistemology, Ibn Sīnā emphasises on *intuition* and *thinking* as two main ways of attaining mathematical knowledge. Moreover, Ibn Sīnā's analysis of mathematical propositions *implies* that they are synthetic a priori judgements in the sense of Kant.

1 Introduction

In this chapter we try to find the answers of Ibn Sīnā to two main questions of philosophy of mathematics. These two questions are: (1) what and where are mathematical objects? And (2) how can we know mathematical objects?

Ibn Sīnā elaborated his philosophy in many of his writings without any remarkable change or modification. It is well known that his most detailed book is *al-Shifā'* (*The Book of Healing*). In "theology" or "metaphysics" (*al-Ilāhiyyāt*) of *al-Shifā'*, he discusses mathematics in at least three books (*maqālat*). In book 1, when he tries to characterize *theoretical sciences* by their subject matter, he defines and studies the subject matter of mathematics. In book 3, he puts forward his views on the natures of unity and number. Finally, in book 7, where he criticizes Platonism, he comes back to the subject of mathematical objects, and he criticizes Pythagoras as well.

A natural question related to the arrangement and the order of the topics of *al-Shifā'* may be the following:

Why Ibn Sīnā discusses mathematics in general and, mathematical objects in particular, in metaphysics? And why that is so crucial to metaphysics?

The same question, of course, may be raised for Aristotle's *Metaphysics*. There also we find that discussions about the concept of number are distributed almost everywhere in the book. In the edition that I have now on my desk (Aristotle

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S. Rahman et al. (eds.), *The Unity of Science in the Arabic Tradition*,

1958), the book ends with the section entitled *critique of the other theories of numbers*. Is it not surprising that Metaphysics ends with numbers?

Back to the question, we distinguish three plausible answers:

(1) The concept of number originates from the concept of unity, and this latter is a central notion of metaphysics. So it seems natural a discussion of the concept of number to be included in any book on metaphysics like *al-Ilāhiyyāt* of *al-Shifā'*.
(2) The Pythagorean metaphysics was a dominant philosophy in Aristotle's time and he should have defended his philosophy against Pythagoras. Ibn Sīnā is more or less a follower of Aristotle. So he does the same as Aristotle, even if it is not clear whether Pythagorean philosophy were still active in his time.
(3) There is a third reason that makes it more plausible to me why numbers are discussed in metaphysics, and that comes from a well-known postulate of Aristotle, according to which *the concepts of existence and unity are two universal concepts which have the same extension.*

> Existent and unit are the same and have the same nature, in the sense that they accompany each other... since if we say "one human" and "human", both refer to one thing ... "one human" and "existent human" does not show anything else (Aristotle 1958, book of *Gamma*, chapter 1).

This postulate of Aristotle is accepted by almost all Islamic philosophers. For example, Ibn Sīnā admits it when he discusses about the subjects of philosophy.

> And since unit and existence have the same extension, it is necessary that we study unit as well.[1] (Ibn Sīnā 1997, book 1, chapter 4, p. 36)

And Mullā Ṣadrā says:

> Existence and unity are really the same or it [existence] is a necessary corollary of unity (Mullā Ṣadrā 1981, chapter 4, vol. 3, p. 298).

In other words, whatever really exists is really one, and vice versa, i.e., whatever is really one is really existent. Now, since "existence" is clearly the most crucial notion of metaphysics, this would imply that the notion of "one" is as important as the notion of "existence" in metaphysics.

2 Ontology of Mathematical Objects

In this section we want to find out the place where Ibn Sīnā believes that mathematical objects exist. We believe the ontology of a particular science, like mathematics, is highly related not only to the subject matter of that science, but also to the philosophical tradition in which the latter is characterized. Let us see then how Ibn Sīnā classifies the subject matter of different *theoretical* sciences.

2.1 The Subject Matter of Mathematics

According to Ibn Sīnā (Ibn Sīnā 1997, book 1, chapter 1), philosophical sciences are divided into two categories, *theoretical* sciences and *practical* sciences. Theoretical sciences themselves are subdivided into three parts: physics (*aṭ-Ṭabī'iyyāt*), mathematics (*al-Ta'līmiyyāt*)[2] and metaphysics (*al-Ilāhiyyāt*):

> As we said [in other places], theoretical sciences are of three kinds, physics, mathematics and metaphysics.
>
> And also it was said: the subject matter of physics is bodies, in so far as it is in motion and at rest, and its problems are about accidents that occur to bodies with respects to motion and rest.
>
> The subject matter of mathematics is quantity abstracted from matter or from whatever has quantity. The problems of mathematics are the ones that bear upon the quantity, as quantity. And in the definition of [science of] mathematics, there will not be any reference to a particular kind of matter or power of motion.
>
> Metaphysics is about things that are apart from matter, in both aspects, existence and definition (Ibn Sīnā 1997, book 1, chapter 1, pp. 11–12).

In his classification of philosophical science, Ibn Sīnā follows Aristotle (see also Weber 1984). He argues that the subject matter of sciences is "existent" (*mawjūd*) and an existent (a) may be found combined with matter in both existence and definition (or term) (*ḥadd*), or (b) is combined with matter neither in existence nor in definition, and finally (c) is not combined with matter in definition but with matter in existence. The first kind of existent is the subject matter of physics, the second one is that of metaphysics and the last one is that of mathematics. In the same place, Ibn Sīnā describes the subject matter of mathematics in detail.

> But the subject matter of mathematical sciences is magnitude, either abstracted (*mujarrad*) from matter in the mind, or accompanied by matter in the mind, and is number, either abstracted from matter or accompanied by matter. And mathematics does not discuss the existence of abstract magnitude or abstract number or number with matter, but after accepting the existence of quantity, it is concerned with the accidents of quantity (Ibn Sīnā 1997, book 1, chapter 2, p. 18).

These four kinds of mathematical objects described above correspond to four branches of mathematics, namely (a) geometry, (b) astronomy, (c) arithmetic and (d) music.

Magnitude, which means *quantum continuum* or *continuous quantity* in the above passage, is the subject matter of geometry and astronomy. Geometry studies magnitude and quantity abstracted from matter in the mind, even if they are accompanied by matter in the external world. It means that quantum continuum is accompanied by matter in the real world, but mind can separate it from matter and considers its properties. In astronomy, magnitude is accompanied by matter, both in mind and in the real world. A similar distinction holds between arithmetic and music. In arithmetic, the abstract number is studied, and in music the number and relations between them are discussed when accompanied by sounds.

It is clear that if the subject matter of mathematics is magnitude (quantum continuum) or number (quantum discretum), there is no place in mathematics for questions like "what is the nature of magnitude?" or "is magnitude a substance or an accident?" or "whether quantum discretum must be realized in matter or outside of matter?" etc. Since in mathematics, *the accidents* of magnitude and numbers are studied, not their *essence* and states of existence. These questions lie in the domain of philosophy or metaphysics.

There is a minor point in the above citation that is worth mentioning. The concept of "abstract" used in the above passage is *not* the same as the one Ibn Sīnā uses often elsewhere in his philosophy. In his philosophical research, when Ibn Sīnā uses this concept, it is in opposition with "material". What he means here by "abstraction" is the *possibility* for the estimation (*wahm*) to seize magnitude and number apart from matter. When we discuss the epistemology of Ibn Sīnā, we will come back to this again.

Coming back to the subject matter of mathematics,[3] we can say that according to Ibn Sīnā, it consists of *things that are accompanied by matter in the external world and are abstracted from matter in mind*. He continues:

> But number can be applied to both sensible objects and non-sensible objects, so number, in so far as it is number, does not belong to sensible objects (Ibn Sīnā 1997, book 1, chapter 2, pp. 19–20).

His main point here is that discussion about number and its relations should be understood as abstracted from sensible objects, not when it may belong to sensible objects. So discussion about numbers is not about sensible objects. About "magnitude", the question whether it is a "substance" or an "accident" is less clear:

> But magnitude, [then] is a common name. Some times it is referred to dimension, which is the substratum of natural body, and sometimes, what is meant by it is the quantum continuum, which is referred to line, surface and solid. You have already learned the difference between these two meanings (Ibn Sīnā 1997, book 1, chapter 2, p. 20).

Here Ibn Sīnā recalls his previous discussion of the difference between these two meanings of "magnitude". I believe he refers the logic of *al-Shifā'* (third section, chapter 4), where he says *two bodies that are different with respect to size, are not different in receiving three dimensions;* this is exactly the first meaning of magnitude. What is the source of difference in any two bodies and is subject of change is the quantum continuum susceptible of admitting three directions, that is, length, width and depth.

Then Ibn Sīnā continues his discussion on the subject matter of metaphysics. He says:

> From what is said until now, it became clear that existent, as existent, is the basis of all these subjects, and it must be the subject matter of this science [philosophy], for the reason we mentioned (Ibn Sīnā 1997, book 1, chapter 2, p. 21).

It is a well-known fact in the tradition of Islamic philosophy that the "problems" of every science are "the essential accidents of the subject matter" of that science (see, e.g., Ibn Sīnā 1997, book 1, chapter 2). The essential accidents of "existence" includes, in the first place, among other things, "unit and plural", "potential and actual", "universal and particular" and "necessary and possible" (see Ibn Sīnā 1997, book 1, chapter 1).

It is very interesting to know that when Ibn Sīnā tries to explain the reason for which metaphysics is called *māba'd-aṭ-ṭabī'iyyi* (*whatever is after physics*), he encounters to some difficulties with regard to mathematics.

> And the meaning of ["meta" in]"metaphysics" is relative with respect to our perception, since when we observe the world for the first time, we perceive the natural existence. But if we consider this knowledge in itself [not in relation to us], it is better to be named as "prephysics" (*māqabl-aṭ-ṭabī'iyyi*), since it discusses matter that is prior to physics, in both its substantial (*bi-dhāt*) and conceptual (*bi-al-'umum*) aspects (Ibn Sīnā 1997, book 1, chapter 3, p. 31).

Ibn Sīnā argues that the prefix "meta" in "metaphysics" is related to the stages of our perception. As we will see later in his epistemology, sense perception is the first level in human understanding of the world. This level of perception and some other levels that are closer to sense perception than to intellection, are ways of knowing *physics* or nature. On the other hand, if we consider philosophy in itself, it is prior to physics, since its questions are about matters that have priority relative to natural objects. For example, philosophy discusses the Separate, or abstracts that are the cause of nature, and *every cause is prior to its effect*. Moreover, in philosophy we discuss subjects that are more general than natural objects and *every general matter is prior to a particular one*. He then says:

> But perhaps somebody may claim: the subjects of *pure mathematics* (*riyāḍiyyāt-al-maḥḍa*) which are discussed in arithmetic and geometry, are also before physics and in particular, number, whose existence is not related to physics, since it sometimes exists even in non-physical objects, so the sciences of arithmetic and geometry might be counted as "prephysics" (Ibn Sīnā 1997, book 1, chapter 3, p. 31).

It is worthwhile to note that Ibn Sīnā distinguishes *pure* mathematics from the other parts of mathematics, which nowadays are known as *applied* mathematics. We don't know if that is the first time in the history of science that this distinction is made. However, what interests us here is that he does not count astronomy and music as belonging to pure mathematics. According to Ibn Sīnā, in arithmetic "number" is discussed exactly as "the pure magnitude" is in geometry. In astronomy and music, on the contrary, the subject is quantities and numerical proportions between stars in astronomy, or numerical proportions between sounds in music. The main point of the above critique is that arithmetic and geometry discuss their objects without any relation to external objects exactly as subjects discussed in metaphysics. We count things in the Separate as well.

In answering to the above critique concerning the subject matter of metaphysics, Ibn Sīnā first considers geometry:

> What can be said as answer to this critic is this: the subject of that part of geometry in which lines, surfaces and solids are studied is clearly not separated from physics as regards existence, so the predicates of such subjects are not separated from physics, *a fortiori* (Ibn Sīnā 1997, book 1, chapter 3, p. 32).

In this part of geometry, he says, line is divided, for example, into straight, curved and other types of lines and surface is divided into affine surface or non-affine surface and also non-affine surface is divided into convex and concave. It is clear that the subject matter of this part of geometry is based on matter, since in the universe of abstracts there does not exist line, surface or solid. We can only have lines, surfaces and solids in physical objects. Line without surface, surface without solid and solid without body is *unrealizable.*[4] For the other part of geometry, whose subject matter is "absolute magnitude", Ibn Sīnā argues:

> In parts of geometry where the subject matter is absolute magnitude [not line, or surface or solid], the absolute magnitude is the subject with respect to its potentiality for every proportion, and this potentiality for proportions is realized not for magnitude that is a form for body or a principle for physics, but for magnitude that is an accident (Ibn Sīnā 1997, book 1, chapter 3, p. 32).

Ibn Sīnā says that we sometimes study "absolute magnitude" in geometry, and not special properties of lines, surfaces or solids. If we consider this part of geometry, the above critique becomes more serious. We did not accept geometry as part of metaphysics, since lines, surfaces and solids are realized only in nature whereas "absolute magnitude" does not depend on physical world and is something abstracted from matter. Does this mean that we should consider this part of geometry as a part of metaphysics? Ibn Sīnā's answer is negative. He argues that when we discuss absolute magnitude in geometry, we mean magnitude in so far as it accepts different relations. That finds determination in lines, surfaces and solids. The other meaning of absolute magnitude, namely, the form of body, is not related to geometry. This meaning of absolute magnitude as the form of body is the principle of natural objects and prior to them, and so discussion about it belongs to metaphysics.

Mullā Ṣadrā does not accept Ibn Sīnā's argument and argues

> Magnitude, as it is, does not exist unless in one of these three species; as every genus is related to its species. How then he [Ibn Sīnā] believes that the absolute magnitude [as a genus] is possible to be apart from physics but apartness of line, surface and solid [as species] from physics is not allowed in physics? Moreover, each of these species is realizable in non-physical world - as it will become clear later -, so the right answer is this: each one of these three species of magnitude is the subject of geometer exactly when it accepts proportions and divisions, like squaring, cubing and other attributions and it accepts these proportions when it belongs to and depends on physical objects (Mullā Ṣadrā 1925, p. 20).

We believe that the above discussion on the meaning of "absolute magnitude" is debatable, and is outside of the scope of this chapter.

Let us see Ibn Sīnā's argument against including arithmetic in metaphysics.

> And as regards number, the critique is more serious and it seems that the science of number [arithmetic] may be counted as [part of] metaphysics... But the reason why arithmetic is not a part of metaphysics will be clear to you soon. The subject matter of arithmetic is not number in all its aspects, since number, sometimes is found in the Separate (*mufāriq*) [like intellect (*'aql*], sometimes in physical objects and sometimes in estimation, in which it is abstracted from every accident [whether physical or abstracted], although it is impossible for number to exist in the external world except in the state of an accident. The number in the Separate is impossible to be the subject of increase or decrease, but it remains only constant. Aye, number should be in such a way that has potentiality of every increase and decrease or every proportion, [and this] is possible only if number be realized in bodies, which has the potentiality of being counted, or number be realized in estimation, and in both cases [realization of number in bodies or in faculty of estimation], number is not out of physics (Ibn Sīnā 1997, book 1, chapter 3, p. 32).

The answer of Ibn Sīnā to the critics that the science of arithmetic is metaphysical may be explained as follows: we may consider number in different ways, asking questions like "has number a real existence?" or "is the existence of number essential or accidental?" etc; and seeking answers for these questions is outside of the science of arithmetic and lies within the sphere of metaphysics.

Number as it is considered in arithmetic is such that it accepts variations and changes. In arithmetic, when we add, subtract or divide two numbers, in fact we look at it as a decrease or an increase of number. Such concept of number is not the same as the one discussed in metaphysics, where it is considered to be *constant*. The reason that such numbers are constant is based on the fact that their *referents*, i.e., separated matters, are *not* subject to any change. So the natural question is this: what kind of number is subject to change, i.e., to be increased or decreased? The answer is: in two cases number has the potency of different proportions. In one case where number is related to physical objects, and since the physical universe is subject to changes, then the number, i.e., the quantity of those objects, will necessarily change. The other case is when number has potency of different proportions in estimation (*wahm*). That means the estimative faculty is able to abstract a number from every numbered object, and to add, subtract and multiply it by other numbers. Even in this case, Ibn Sīnā believes that the source of these changes in the estimation faculty is states and proportions of the physical universe, and if there were no changes in the physical universe, the estimative faculty would not be able to imagine the abstracted numbers with different proportions.[5]

2.2 Ontology of Mathematical Objects

According to the tradition of Islamic philosophy, the objects of mathematics are *quantities* and quantities are *not* substances. In his *al-Shifā'* (*al-Ilāhiyāt*, chapters 3 and 4), Ibn Sīnā argues that both *number*, as a discrete quantity and *magnitude* as a continuous quantity, are *accidents*. That means quantities do not have an *independent existence* in the external world, and so they need some *substrata* to exist.

It is worth knowing that in chapter 2 of the same book, he explains the different meanings of the concept of "unit", which is the basis of his notion of quantity. Despite different meanings, the concept of "unit" shares the property of "not being divisible actually" (see Ibn Sīnā 1997, book 3, chapter 3), and "not accepting plurality" (ibid). He divides the notion of "unit" into "essential unit" and "accidental unit". The notion of "numerical unit" is defined as a kind of "essential unit", which is sometimes called "particular unit".

> And as regards plurality, it is evident that it must be defined in terms of unity, since the unit, is the principle of plurality and existence, and the essence of plurality derives from that [unit]. Besides, in order to give any definition of plurality, we use the term "unit". And it is for this reason that in the definition of plurality, you may say: "plurality is exactly the set (*mujtama'*) of units". You note that in this definition of plurality, "unit" is used, and something else, which is the notion of set (Ibn Sīnā 1997, book 3, chapter 3, p. 112).

Ibn Sīnā then criticizes the people who believe that the above explanation is an *essential* (or real) *definition* of "plurality". By an essential definition, he means, like other Islamic philosophers and logicians, a *definition by terms* (*ḥadd*), which refers to the essence of the defined object (*definiendum*). A common example for an essential definition is the definition of "human" as "rational animal". Ibn Sīnā argues that the concept of "set", which is used in the above definition, conceptually includes "counting" and "repetition". In his argument against any possible essential definition of the concept of "unit", he uses some *epistemological* premises.

> But it seems that plurality is more known in the imagination (*khayāl*) than unity, as unity is more known in the intellect than plurality, and both are universal matters and are immediately imagined, nevertheless we first imagine "plurality" through sensible things, and we understand unity without any "intellectual" principle, and even if we believe some "intellectual" principle for that, it is an imaginative one. So our definition of plurality in terms of unity is an "intellectual" definition, in the sense that the term "unity" in the definition is immediately imagined (Ibn Sīnā 1997, book 3, chapter 3, p. 113).

Ibn Sīnā admits that our perception of "plurality" happens before "unity", and that is because "plurality" comes directly from sensible objects. The first notion of "plurality" is made when we see something that is not "this". Contrary to "plurality", "unity" takes place in imagination as a *negation* of "plurality", so it will be formed in perception in the next step. Since our perception of "unity" is immediate and non-theoretical, we may define "plurality" in terms of "unity" only through intellection. When it is said that "unity" is something that does not have "plurality", it means that the meaning of "unity", which is self-evident for us, is against the meaning of "plurality". So in this setting, it hints that "unity" is the negation of "plurality".

He finally argues that the above explanation is just a clarification of the concepts "unit" and "plural", and that "plurality" is just another *name* for "a set of units".

He then says:

> Now, we investigate the nature of numbers and its properties... Number exists in the external existent objects, and it also exists in our soul (mind), and this saying: "number has no existence, except in our soul" is not noteworthy. Aye, if he [who holds this view] means that number has no existence apart from any existent object, except in our soul, then he is right, since we have already proved that it is impossible for the unit to have external existence, and naturally, number which its existence is based on unit, is also the like (Ibn Sīnā 1997, book 3, chapter 5, p. 126).

So, according to Ibn Sīnā, we may say that number has two levels of existence. On the first level, it exists only in the mind, that is, abstracted from any existent matter, on the second level, it exists with the external objects. His argument for the existence of the second level is the following: in non-unit objective things, i.e., a collection including more than one object, there are clearly some[6] units. So number necessarily exists, since number is nothing except units that possesses some place in the order of numbers, and each place, itself, is a *unique* species. In this way, each number, as a *species*, is itself a unit, and so it has some special properties attributed to its unity. It is clearly impossible to prove some properties for something that has no reality in the outside world.

> So each number has a particular nature and a form, and the imagination of that in the soul comes from that nature, and that nature is the unity, which constitutes the essence of that

> number. And the meaning of a number is not a plurality without a form of unity. So it is not true to say: "number is the sum of units, not unit", since a number, as it is a sum, is a unit that carries some properties such that a number with other sum and order does not have. And it is not surprising for something to be unit with respect to its kind, like 10 or 3, and also to be plural, with respect to other aspects (Ibn Sīnā 1997, book 3, chapter 5, p. 127).

A number like 10, as a species-unit, has a special form and so possesses properties of being 10, but as a plurality, it may not be the subject of those properties. Instead it has some properties of plurality in opposition to that unity. That is exactly the meaning of "plurality in unity" and "unity in plurality", and there is no contradiction here. We *view* a number in two *different* ways.

Ibn Sīnā believes that the definition of, for example, 10 as the sum of 9 and 1 is *wrong*. His argument is:

> It is not true to say that: the number 10 is 9 and 1, or that is 5 and 5, or that is 1 and 1 and ... until it ends up to 10; since in the statement "ten is nine and one", nine is predicated of ten, and then you conjoin to it the one. This is similar to say "ten is black and sweet", which in this sense, the meaning of the original claim will be: "ten, is both nine and one", and if what you mean by that conjunction is not a definition, but it [has the meaning] like the statement "human is animal and rational", namely, human is such an animal that is rational. In this sense, the meaning of your claim will be: "the number ten, is nine, and it is also one", and this is impossible (Ibn Sīnā 1997, book 3, chapter 5, pp. 127–8).

It is clear that Ibn Sīnā is considering here the case where *conjunctions* play a *logical* role. A statement like "A is B *and* C", logically means that "A is B" *and* "A is C". By this interpretation for "and", it is impossible that both statements "10 is 9" and "10 is 1" be true. But there is a mathematical interpretation for "and" which simply means "addition". That is what Ibn Sīnā ignores. We will explain the origin of his ignorance after the next passage.

Ibn Sīnā then considers other possible meanings that the above definition may have. He concludes that,

> And the number ten is the sum of nine and one, when both are present and the conclusion is something different from each one of them (Ibn Sīnā 1997, book 3, chapter 5, p. 128).

Contrary to modern axiomatic definition of natural numbers, where for example, 10 is defined by 9 + 1, namely, ten is the *successor* of nine, Ibn Sīnā will not accept it as an essential definition. The notion of *addition* for natural numbers as is defined in set theory, is *finally* in term of two primitive (*undefined*) concepts, i.e., *set* and *membership*, with familiar symbol "ε". It may be suggestive to interpret "set" in Ibn Sīnā's philosophy, by "plurality". This interpretation, of course, is debatable. Even if we have some support for the mentioned interpretation, it is too hard to find a similar interpretation for the set theoretical notion of *membership*, which is a *two-place relation*. It is a well-known fact that Ibn Sīnā's logic does not *permit* two-place relations.[7]

He finally goes to the conclusion that we are *not* able to present an *essential definition* for number, i.e., it is *undefinable*.

> And since it is hard to imagine a definition for number in terms of units, its definition necessarily is nothing more than a description (*rasm*) (Ibn Sīnā 1997, book 3, chapter 5, p. 128).

Recall that a *description* (*rasm*) is a definition in terms of *accidental* properties of *definiendum*. Ibn Sīnā then concludes that

> As the great ancient philosopher, the First Teacher [Aristotle] said: "Do not suppose that the number six is three and three, but it is six, at once and immediately" (Ibn Sīnā 1997, book 3, chapter 5, p. 129).

2.3 Existent and Object

In book 1, chapter 5 of *al-Shifā'*, Ibn Sīnā begins a long discussion on the relation between "existent" and "object". His objective is, among other things, to establish the following two facts:

(1) "Existent" and "Object" are self-evident and immediately imagined concepts.
(2) "Existent" and "Object" are conceptually different, but extensionally the same.

On the first fact, Ibn Sīnā emphasizes the *epistemological* value of the two concepts, in the sense that they are the most general concepts.

> So we say that the meanings of existent, object and necessity are immediately pictured in soul, namely, their imagination do not need any more known objects to be imagined (Ibn Sīnā 1997, book 1, chapter 5, p. 39).

In this section we are more interested in the second fact. He asserts:

> We say: the meanings of "existent" and "object" are both imagined in soul and they have two different meanings. So "existent" and "affirmed" (*positive*), (*al-muthbat*) and "realized" (*al-muḥaṣṣal*), are [different] names with the same meaning, and we are confident that the meanings of these words are present to the soul of anyone who reads this book. Sometimes "object" and all its synonyms in all other languages refer to another meaning [other than "existent"], since everything has a reality [an essence], which is due to that reality [essence], that the object is what it should be. So a triangle has the reality [essence] of being a triangle, and whiteness has the reality [essence] of being white, and this is a meaning of "object" which sometimes we call as "specific existent", and by "specific existent", we do not mean an affirmative existent. The word "existent" also refers to several meanings, one of which is the reality [essence] an object has, as if the specific existent of a thing is exactly the reality [essence] that the thing were based on it (Ibn Sīnā 1997, book 1, chapter 5, pp. 41–2).

We may summarize Ibn Sīnā's view in this way: "object" is the same as "specific existent", and they both refer to the essence of things. So, in some sense, "object" or "specific existent" refer to the essence of things, and "existent" refers to the things themselves.

Ibn Sīnā continues his argument to establish that "object" and "existent" are conceptually different. He then provides an argument to show that these two concepts have the same extension, i.e., we can consider something as the extension of the concept of "object", if and only if it is an extension of the concept of "existent", i.e., *"object" and "existent" are extensionally equivalent.*

> And the necessity of the meaning of existence cannot be separated from "object", namely, the meaning of existent is always necessary for an object, since an object either exists in the external world, or in estimation or in the intellect (Ibn Sīnā 1997, book 1, chapter 5, p. 43).

As Ibn Sīnā says, if an object is in the external world, it satisfies the predicate "existent", and if it is not in the external world, it can be considered as existent only when we imagine it in the mind, and if something exists neither in the external world nor in the mind, it is *not* an object at all. So the necessary condition for something, which does not exist in the external world, to be an object is to exist in the mind (estimation or intellect).

Before closing this section, we would like to consider another ontology of mathematical objects that we believe is *wrongly* attributed to Ibn Sīnā. In Rashed 1984, R. Rashed argued that Ibn Sīnā codified a comprehensive doctrine of philosophy such that embraces al-Fārābī's admission of irrational numbers as mathematical objects. In Rashed 1984, it is argued that since algebra is the intersection of arithmetic and geometry, its tool, i.e., "an algebraic unknown", can be read as an "object", which represents a number or a geometric magnitude. Something more can be said, since a number may be irrational, so an object can represent a quantity that can only be known by approximations. Although this algebraists' tool must be universal enough to cover different contents, it also must exist independent of what is determined so that getting better approximations be possible.

Roshdi Rashed rightly believes that an Aristotelian theory is not able to have such ontology, so it is necessary to suggest a new ontology by which we are able to speak of an object without any specific property, an ontology that permits us to know an object without being able to represent it in an exact way. He finally claims:

> Ainsi, tout existent est une chose, mais la réciproque n'est pas exacte, bien qu'il soit impossible qu'une chose n'existe ni comme sujet concret, ni dans l'esprit (Rashed 1984, p. 35).

But Ibn Sīnā argues against people who claim: "the concept of object is more general than of existent" (Ibn Sīnā 1997, book 1, chapter 5). His argument is based on this fact that in predication in statements and in our knowledge of objects, knowledge is about concepts that are *in our mind*, so they have *mental existence*, although they may not exist in the external world. In fact, as already mentioned, a necessary condition for something to be an object is that it should exist *mentally*.

> And this happened to them, because of their ignorance to this truth that [the subject of] predication is something that has existence in soul, although they may be nothing in the external world, and the meaning of predication of such things is that they have some relations with the [external] existent (Ibn Sīnā 1997, Article 1, book 5, p. 45).

3 Epistemology of Mathematical Sciences

In this section we briefly explain Ibn Sīnā's epistemology to find the answer to our second question, i.e., "how can we know mathematical objects"? As D. Gutas pointed out (Gutas 2001), Ibn Sīnā's epistemology was under "inevitable shifts of emphasis and terminology over the years". These modifications culminated in the writings of the final period of Ibn Sīnā's philosophical activity, especially in *Pointers and Reminders* (*al-Ishārāt wat-tanbīhāt*). In description of his epistemological theory, we essentially make use of this book.

In this book, as elsewhere, Ibn Sīnā identifies the mental faculties of the soul in terms of their epistemological function. According to Ibn Sīnā, knowledge begins with *abstraction*. The concept of "abstraction" (*tajrīd*) in Islamic Philosophy, and in epistemology in particular, is a significant notion. In fact what distinguishes the levels of perception boils down to the degree of "abstraction" (*tajrīd*). We briefly mention that contrary to Arabic word "tajrīd", the word "abstraction" loses the sense of the intensification of existence and reality that takes place as the degree of *tajrīd* increases. So a better translation for the Arabic word "tajrīd" may be the English word "disengagement", or "detachment". Nevertheless, in this chapter we use the common word "abstraction" and its derivatives for "tajrīd" and its corresponding derivatives. There are vast investigations on different possible meanings of "abstraction" in the Islamic philosophy, and in Ibn Sīnā's philosophy in particular. We refer the reader to a survey of this topic in Hasse 2001.

In the purest sense, "abstract" (*mujarrad*) is an attribute of God, the Necessary Existence in itself, since the Necessary Existence has no attachment to or dependence upon anything other than itself.

More specifically, "abstract" is the attribute of the intellect that is able to see things as they actually are, that is, without their entanglement in the obscurities of imagination and sense perception. It is also the essential attribute of the forms or quiddities that the intellect perceives (In this final use, it comes close to the term "abstracted").

> Perception (*idrāk*) of an object consists in attaining a true image (idea) (*mithāl*) of the object by the one who perceives (*mudrik*) [subject], and the *mudrik* observes that. So either when that is perceived object is exactly the same object outside of the *mudrik*, which possibility is incorrect; since then something which does not exist outside [of the *mudrik*], should necessarily exist [outside]. The examples are many geometrical figures, or many impossible hypotheses - things that do not exist. Or the perception [of an object] is a true image of the object pictured on the *mudrik* himself in such a way that it has no difference in quiddity (*māhiyya*) with that [the object], and it is the form, which remains (Ibn Sīnā 1960, physics, chapter 3, p. 33).

What Ibn Sīnā intends here is an *explanation* of the concept of perception (*idrāk*), not presenting a (real) *definition*. He explains that perception of an object may be described in two ways. He argues against the possibility that perception consists in *transferring* of things outside of the *mudrik* to *mudrik*. His argument is based on some evident *counterexamples*, e.g., *assumptions* in geometry, which we *know* that they do *not* exist in the outside world, but we *know* them. He then accepts the other possibility, according to which perception is to attain the form and the quiddity of the object. An immediate consequence of his description for perception is that we may perceive objects that may not exist outside us, and only the form of the objects are perceived by *mudrik*.

Now let us see how Ibn Sīnā stratifies the levels of perception:

> Sometimes an object is sensible, and that is when it is seen; [and] sometimes it is imagined, and that is when the object itself is absent and its form is present in *mudrik*; as when you have seen Zayd; and then when he is absent from you, you imagine him. Sometimes an object is intelligible (*ma'qul*), and that is [like] when you understand the meaning of human from Zayd, a meaning that holds for other things as well. When Zayd is sensible, it is with appearances [which are different from his quiddity] which do not affect his quiddity, whenever they [appearances] have been disappeared; like to have place, position, quality and determined quantity, such that if they are replaced by something else, the reality of human's quiddity would not have been changed. Sense will perceive Zayd in a state that has these appearances, namely, appearances which are interconnected with him, because of the matter from which he was created. Sense will not remove those appearances from Zayd and will not perceive him unless by the connection that exists between sense and its matter; for this reason, whenever this connection is lost, the form of that [Zayd] will not be present to the sense. But imaginative faculty imagines Zayd with all these appearances and cannot abstract him absolutely from these appearances. But it can abstract him from the positional relation upon which sense was dependent; so Zayd is present in imagination even when his positional relation is absent. But intellect can abstract a quiddity from all its personal appearances and establish it in such a way as if it [intellect] manipulates the sensible to a form of intelligible. But an object without these appearances - appearances not necessary for its quiddity — is intelligible in itself and does not need any manipulation to be prepared to be intelligible; but it may need to be abstracted by a faculty that is responsible for intellection (Ibn Sīnā 1960, physics, chapter 3, p. 34).

In this passage, Ibn Sīnā is going to describe different stages of perception. In his description, perception is classified into four stages based essentially on a hierarchy of abstractions of objects. These four stages are: (1) sense perception (*ḥiss*), (2) imagination (*takhayyul*), (3) estimation (*wahm*) and (4) intellection (*ta'aqqul*). A natural objection may be the absence of the third type, i.e., estimation, in the passage quoted from *al-Ishārāt wat-tanbīhāt*. The answer is this: in the above passage, Ibn Sīnā, as a definite example, considers "Zayd", that cannot be perceived by estimation, as we will see the reason when we explain the meaning of the term "estimation" according to Ibn Sīnā. In his other books, like *al-Shifā'*, he does not take any definite example, and so he is able to distinguish all four types of perception. Now we explain these four types of perception in more details:

(1) **Sense-perception** (*ḥiss*): Sense perception responds to the particular with its given form and material accidents, such as place, time, position, quality, etc. As a mental event, being a perception of an object rather than the object itself, perception occurs in the particular. So a sensible object has three conditions: (i) presence of the object, (ii) with material accidents, and (iii) particularity. All these conditions hold for a definite object like Zayd, when we see him. Then such a concept is definite and does not hold for more than a person. Now let us analyze this activity of the soul in details. To classify the formal features in abstraction from material accidents, we must retain the images given by sensation and also manipulate them by disconnecting parts and aligning them according to their formal and other properties. However, retention and manipulation are distinct epistemological functions, and cannot depend on the same psychological faculty; therefore Ibn Sīnā distinguishes faculties of relation and manipulation as appropriate to those diverse epistemological functions.

(2) **Imagination** (*khayāl*): In imagination, among three above conditions, the first condition does not hold, i.e., the same Zayd is absent. Ibn Sīnā identifies the retentive faculty as 'representation' and charges the imagination with the task of reproducing and manipulating images. To conceptualize our sense perception and to order it according to its quiddities, we must have and be able to re-invoke images of what we experienced but is now absent. For this we need sensation and imagination; in addition, to order and classify the content of representation, we must be able to discriminate, separate out and re-combine parts of images, and therefore must possess imagination and reason. By carrying out this manipulation, imagination allows us to produce images of objects we have not already seen out of the images of things we have experienced. So imagination can also generate images of intelligibles.

(3) **Estimation** (*wahm*): That is to perceive the particular meaning of non-sensible, like perception of kindness a father has for his child. So estimation is also of particular concepts, which have not been perceived by any senses. Among the conditions necessary for sense perception, only the condition (iii) holds for estimation. This is a faculty for perceiving non-sensible "intentions that exist in the individual sensible objects". A sheep fears a wolf because it estimates that the animal may do it harm; this estimation is more than representation and imagination, since it includes an intention that is additional to the perceived and abstracted form and concept of the animal.

(4) **Intellection** (*ta'aqqul*): Intellection is the final stage of perception in which none of the three conditions hold. It is to perceive the universal concepts, e.g., "human" by abstraction of Zayd, removing all material appearances that will not change the quiddity of "human".

According to Ibn Sīnā, intelligibles divide into two kinds, material intelligibles and immaterial intelligibles. So we have two kinds of intellection, depending on the corresponding objects. In material intellection, we first perceive a particular human, like Zayd, by sense perception in the presence of the material object, as described in the stage of sense perception. Then it is understood by common sense while the object is absent; and then the object will be abstracted from all material features such that it will be prepared to be understood by intellection. In immaterial intellection, the object itself is intelligible such that it does not need to be abstracted from material accidents, like abstract realities, souls, etc.

Now our main question reduces itself to "*on what stage of perception we perceive Mathematical objects, in particular, number and magnitude*?"

Corresponding to the classification of philosophical sciences in terms of their subject matter, intellection is divided into two kinds, *theoretical* intellection and *practical* intellection. Theoretical intellection is responsible for knowledge and perception of intelligibles, and practical intellection is like a ladder to attain moral values, …. Ibn Sīnā distinguish four levels or layers for theoretical intellection, (a) *potential* intellection (*al-'aql al-hayoulāni*), a stage where no intelligible is perceived yet, but it has the capacity or potentiality to accept *primary* intelligibles, (b) *dispositional* intellection (*al-'aql bi-al-malaka*), a stage where intellection has passed from the pure potentiality and has perceived the primary intelligibles, and is prepared to acquire the *secondary* intelligibles,[8] either by *thinking* or by *intuition* (*ḥads*), (c) *actual* intellection (*al-ʿaql bi-al-fiʿl*), a stage where the intellection *knows* that he has acquired the secondary intelligibles, and finally, (d) *active* intellection (*ʿaql muṭlaq*), where the inellection *observes* the secondary intelligibles (see Ibn Sīnā 1960, physics, chapter 3, section 10).

So in the levels (a) and (b), intellection has only the potentiality to acquire the secondary intelligibles whereas in the levels (c) and (d), it can present, observe and study them. For acquiring the secondary intelligibles in the stage (b), Ibn Sīnā mentions *two ways*, first *thinking*, and the second *intuition*. In the next chapter of the same book, he describes four differences between "thinking" and "intuition". These characteristics may be summarized as the following: (i) contrary to "thinking", there is no *search* in "intuition", and that is when, without enough background or premises, we sometimes acquire the middle term [of a syllogism], intentionally, or unintentionally, (and in both cases, without any movement of the mind), (ii) in contrast to "thinking" which may be unsuccessful in its search, "intuition" hits spontaneously the middle term and comes to the point immediately, (iii) "thinking" is often about particulars, since it searches by the assistance of the imaginative faculty, and (iv) "thinking" takes place in time but "intuition" is immediate and spontaneous.

According to Ibn Sīnā, *knowledge* is acquired in the second level of theoretical intellection, which is in *contact* with the third level, i.e., the active intellect through thinking *and* intuition. For the mathematical sciences, the meaning of thinking is not much debatable. Ibn Sīnā himself was a well-known logician of his time and also knew the *Elements* of Euclid. So it is clear that, for him, "thinking" in mathematical sciences is nothing else than *deductions* or *proofs* of mathematical propositions by means of axioms and rules. This *simple* or *formal* picture of mathematical thinking does *not* explain what really is going on in the mathematician's mind. The process of catching the *middle terms*, as the medium or means, to prove the main claim or proposition, is not explained by this simple picture of the mathematical thinking. In the mathematical science, *lemmata* play the same role as *middle terms* in a syllogism. Here Ibn Sīnā introduces a new way or method to fill the gap. That is called "intuition". His description of the concept of intuition establishes a crucial element in the process of mathematical discovery. It is worth mentioning that, as D. Gutas interprets in Gutas 2001, Ibn Sīnā probably came up to his theory of intuition by his own experience as a mathematician. His example to explain different ways where intuition occurs is the states of problems solving in geometry. As D. Gutas explained in Gutas 2001, in standard version of Ibn Sīnā's theory of intuition, *all* intelligible knowledge is acquired *only* through intuition. In his "revised" version, which is met with in the writings of the later period of Ibn Sīnā's philosophical activity, "a second way of acquiring the middle terms and the intelligibles is introduced. This is thinking, which is now defined as a movement of the soul in search of the middle terms, thus taking over a large part of the former definition of intuition." (See Gutas 2001, for details)

The theory of intuition in epistemology of mathematical sciences is very involved and complicated. In modern epistemology of mathematical sciences, there are different and various interpretations and explanations for the concept of "intuition". The most attractive ones are Gödel's and Brouwer's concepts of intuition. Each one of these concepts of intuition has been interpreted in different ways. For the Gödel's notion of intuition, see, e.g., Maddy 1996 and Parsons 1996, and for the Brouwer's concept of intuition, see, e.g., van Stigt 1990. To locate the place of Ibn Sīnā's theory of intuition in this complicated geography of theories of intuitions needs a separate chapter.

Before closing this chapter, we will briefly investigate the status of mathematical propositions in Ibn Sīnā's philosophy. According to his logic, universals, as *predicates* in propositions, are either essences or accidentals. Note that the concept of "accidental" here is different from the concept of "accident" (*ʿaraḍ*), which is against the concept of "substance" (*jawhar*). The concept of "accidental" (*ʿaraḍī*) is in opposition to the concept of "essential" (*dhātī*). *Accidentals* are divided into two types, *necessary* and *unnecessary* accidentals. A necessary accidental is defined as an accidental which is impossible to be separated from the essence. In fact, every science discusses the necessary accidentals of its subject matter. A necessary accidental is necessary either for *existence* or for *quiddity*. For example, "heat" is a necessary accidental for the *existence* of the "fire". On the other hand, the necessary accidentals for quiddity are divided again into two types, *self-evident* necessary and *non-self-evident* necessary. A self-evident necessary accidental itself is divided into two smaller types, it may be *strictly* self-evident or *non-strictly* self-evident. Instead of giving definitions of theses nested terms, let us look at some examples (Ibn Sīnā 1960, logic, chapter 2):

1. In the proposition "A tri*angle* has angles", the predicate "angle" is a strictly self-evident necessary accidental for the subject "triangle".
2. In the proposition "The number four is even", the predicate "even" is a non-strictly self-evident necessary accidental for the subject "the number four".
3. In the proposition "The sum of angles of a triangle is equal to two right angles", the predicate "the sum of angles being equal to two right angles" is a non-self-evident necessary accidental for the subject "triangle".

According to Ibn Sīnā, a demonstration transfers *truth, certainty and necessity* from the premises to the conclusions. Premises or *first principles* are generally divided into two parts, the first principles for *all sciences* are called *common principles* (*al-uṣūl al-mutaʿārafa*), and the first principles for every special science called *postulates* (*al-uṣūl al-mawḍūʿa*). For example, "whole is bigger than [its] part" or "contradiction is impossible", etc are common principles, and "the shortest line between two points is a straight line" is a postulate for the science of geometry. Ibn Sīnā has a vast investigation in his different writings on the *ways* the common principles are acquired by the mind. A class of these common principles called as *awwaliyyāt*, are acquired *only* through the intellective faculty. These are propositions that are *obvious* for the intellective faculty and accepting them is *necessary*. The above two examples of the common principles are of this category. Contrary to the common principles, which are certain, the postulates are *susceptible of doubt* (*mashkūk*).

Mathematical science is one of the main parts of the demonstrative sciences, which is based on *the certain premises* and demonstrations or proofs which transfer certainty from premises to conclusions. Mathematical premises are either the common principles (*awwaliyyāt*), like the proposition "the whole is bigger than [its] part", or innates (*fiṭrī*), like the proposition "the number four is even" (see Ibn Sīnā 1960, logic, chapter 9). According to Ibn Sīnā, mathematical propositions are *certain*, *necessary* and have *essential truth*.

A natural question for a philosopher of mathematics is:

What relations may exist between Ibn Sīnā's characterization of mathematical propositions and mathematical knowledge, on the one hand, and Kant's classification of propositions into analytic and synthetic propositions and mathematical knowledge into a priori and a posteriori knowledge, on the other hand?

The following quotation gives a *partial* answer to the above question:

> It is not the case that every science uses postulates, but in some sciences only definitions and *awwaliyyāt* are used, for example in arithmetic. But in geometry, all kinds of principles [definitions, common principles and postulates] are used (Ibn Sīnā 1956, chapter 12).

The immediate conclusion is, according to Ibn Sīnā, that *arithmetical knowledge is a priori* and *geometrical propositions are synthetic* in the sense of Kant. Moreover we can conclude that, by Ibn Sīnā's analysis, arithmetical propositions are *not* analytic, since the negations of arithmetical propositions are *not* self-contradictory. So according to Ibn Sīnā, *arithmetical propositions are synthetic* in the sense of Kant as well. We admit that our conclusion about non-analyticity of arithmetical propositions is debatable. One may argue that Ibn Sīnā's concept of the common principles (*awwaliyyāt*) is wider than the *usual set of the logical* axioms. That would imply that arithmetical propositions are analytic. We leave *open* this problem.[9]

We have not found any *explicit* claim of Ibn Sīnā on priority or posteriority of geometrical postulates. However, based on his writings, in particular his discussion on the difference between common principles and postulates, and an example from geometry in Ibn Sīnā 1956, chapter 12, we believe that, most probably, he will admit geometrical knowledge as *a priori knowledge*.[10]

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Notes

1. All translations from Arabic into Farsi are my translations, and all terms and expressions inside [ ] are my *interpretations*.
2. Mathematics is translated into "*al-Ta'līmiyyāt*", which literally means "what is related to "*ta'līm*", and "*ta'līm*" itself means "teaching and learning". This translation of "mathematics" into Arabic is very close to the original meaning of the word *"mathēma"*.
3. It should be mentioned that after having presented his definitions of the subject matter of three branches of philosophical sciences, i.e., physics, mathematics and metaphysics, Ibn Sīnā immediately discusses the subject matter of logic. Apart from how he describes that, the point is that he includes "Logic" in theoretical philosophy, at least as far as the description of

its subject matter is concerned. That is, somehow implausible in his doctrine. Note that metaphysics of *al-Shifā'* is as called "thirteenth art" (*fann*). The best way to justify counting the different parts of *al-Shifā'*, is to start with physics including 8 arts, and then mathematics including 4 arts, and finally metaphysics starts from thirteenth art. In this way, metaphysics matches with the overall plan of the book. There is no place for logic in metaphysics of *al-Shifā'*. That means, according to Ibn Sīnā, logic is *not* a branch of theoretical *sciences* (See also Sabra 1980 for more details).

4. According to Mullā Ṣadrā the estimative faculty abstracts line, surface and solid from matter, but these are not separated from matter in external existence (see Mullā Ṣadrā 1925, p. 20). He then concludes that, at least, this part of geometry cannot be counted as a part of metaphysics.
5. Here Mullā Ṣadrā has a third reason for not considering arithmetic as part of metaphysics. He says that number, which is the subject of arithmetic, and the unity, which is the principle of arithmetical numbers, is different from the unity that exists in the Separate and, moreover, the Separate does have numbers constructed of units. A number, which is a quantity, may have proportions and such a number can be only found in matter, since such a number is an accident of physics, not something as a principle of the physical objects (see Mullā Ṣadrā 1925, p. 20).
6. Here "some" means "at least two". It is worth knowing that according to Ibn Sīnā, "number" is another name for "plurality" and this concept is applied only for sets with *at least two elements*, so "numbers" starts from 2, i.e., zero and one are *not* numbers. Unit *is* the building block of all numbers, but it is *not* a number itself (see Ibn Sīnā 1997, book 3, chapter 3). So *empty set* and *singletons* do *not* exist even in the mind.
7. See, for example, the *admissible* syllogisms in Ibn Sīnā 1956.
8. A natural question may arise here: Is there any relation between "secondary intelligibles" and "immaterial intellection"? It is plausible to assume that the objects of immaterial intellection that can be perceived *through* "forms" of objects are *necessarily* secondary intelligibles. However, there are also objects of immaterial intellection that are perceived without having "forms", like ego (See also Sabra 1980).
9. Kant's notion of intuition is *interpretable* in the concept of construction, and his conclusion on the synthetic (a priori) property of mathematical statements is based on his notion of intuition. The term "construction" in Kant's time had an established use in at least one part of mathematics, i.e., in geometry. It is natural to assume that what Kant primarily has in mind are constructions in geometry (see Hintikka 1992 for more details). As is mentioned before, Ibn Sīnā came up to his notion of intuition mainly through his experiences in geometry. So Ibn Sīnā's notion of intuition *may* have relation to what is called *construction of middle terms*.
10. There are many other important questions in philosophy of mathematics that are not considered in this chapter. One of the most controversial is the concept of "infinity". Ibn Sīnā's theory of infinity is very similar to Aristotle's, in the sense that he does *not* believe in *actual infinity*, and he believes in potential infinity as a procedural character, see, e.g., Ibn Sīnā 1960, *aṭ-Ṭabī'iyyāt*.

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