The calculus of variations and the forced pendulum

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Abstract Consider the equation of forced pendulum type:

$$u'' + V_u(t,u) = 0 \quad (*)$$

where $' = d/dt$ and V is smooth and 1-periodic in its arguments. We will show how to use elementary minimization arguments to find a variety of solutions of $(*)$. We begin with periodic solutions of $(*)$ and then find heteroclinic solutions making one transition between a pair of periodics. Then we construct heteroclinics and homoclinics making multiple (even infinitely many) transitions between periodics. If time permits, we may also discuss the construction of related mountain pass orbits of $(*)$.

1 Introduction

The goal of these lectures is to show how elementary variational techniques, in particular minimization arguments, can be used to extract a considerable amount of information about dynamical behavior. We do this for the setting of a forced pendulum model problem. This is a favorite proving ground for many techniques. Among works that are related to ours, we mention in particular [Mor], [A], [Ma82], [Ma93], [B88], [B89], and [Mos86].

The approach taken here uses essentially nothing from the theory of dynamical systems other than the uniqueness of solutions of the initial value problem. Therefore, these techniques can also be used for certain classes of problems for partial differential equations. In part our arguments are simplifications of ones used in [RS]. A disadvantage of our approach is that it does not capture finer dynamical structure that can be obtained using stable and unstable manifolds or notions like hyperbolicity.

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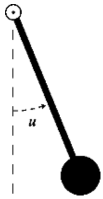

Fig. 1 Schematic of the physical pendulum.

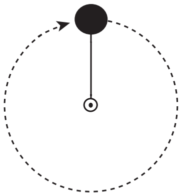

Fig. 2 Schematic of an orbit asymptotic from v to w.

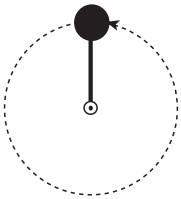

Fig. 3 Schematic of an orbit asymptotic from w to v.

The simple pendulum is modeled by $u'' + \sin u = 0$, u representing the angle made with the vertical direction. More generally we will consider a forced model

$$\text{(DE)} \quad -u'' + V_u(t,u) = 0,$$

where V satisfies

($\mathbf{V_1}$) $V \in C^2(\mathbb{R}^2, \mathbb{R})$ *and is 1-periodic in t and in u.*
Equivalently $V \in C^2(\mathbb{T}^2, \mathbb{R})$, *where* $\mathbb{T}^2$ *is the 2-torus.*

A caveat is in order here: V is the negative of the usual potential energy.

The simplest solutions of (DE) are periodic ones, e.g. if $V_u(t,z) = 0$ for all $t \in \mathbb{R}$ and $z \in \mathbb{Z}$, each such z is an equilibrium, and therefore periodic solution of (DE). By (V_1), if v is a solution of (DE), so is $v + k$ for all $k \in \mathbb{Z}$. Therefore we can seek solutions of (DE) that are asymptotic to a pair of periodics v and w.

We say such a solution is *heteroclinic* from v to w (Fig. 4). Such solutions undergo one ‘transition’. Likewise we can try to find 2, k or infinite transition solutions. Thus a 2-transition solution is *homoclinic* to v or w (see Figs. 2–4). It turns out there are infinitely many solutions of each type, distinguished by the amount of time they spend near v or w.

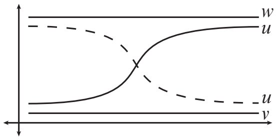

Fig. 4 Graphs of 1-transition orbits between v and w.

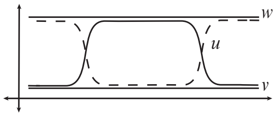

Fig. 5 Graphs of 2-transition orbits between v and w.

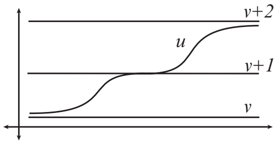

Fig. 6 A monotonic orbit asymptotic to v in the past and to $v+2$ in the future.

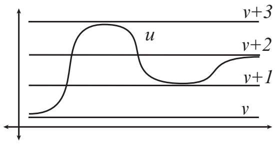

Fig. 7 An orbit which makes several transitions.

There is another kind of 2-transition solution which is monotone: $u(t+1) > u(t)$ (Fig. 1). In the simplest case, such a solution is heteroclinic from v to v + 2. Likewise, there are k and infinite-translation such solutions, and we can concatenate these two types of solutions (Fig. 7).

Within each type of solution as well as for the mixed type, one can seek a so-called symbolic dynamics of solutions that will be described later.

We will show how elementary minimization arguments can be used to find some of these solutions. Unfortunately we will not have enough time to treat the monotone and mixed cases. We begin with the simplest case of periodic solutions and then treat progressively more complex cases.

2 Periodic solutions

Periodic solutions are the easiest to find. We assume V satisfies (V_1). Set $E = W^{1,2}(\mathbb{T}^1)$, the class of 1-periodic functions having square integrable derivatives, i.e.

$$||u||_E^2 = ||u||_{W^{1,2}}^2 = \int_0^1 \left((u')^2 + u^2\right) dt.$$

Note that $u \in E$ implies $u \in C(\mathbb{T}^1)$, in fact $u \in C^{1/2}(\mathbb{T}^1)$, i.e. u is Hölder continuous of order $1/2$. Let

$$L(u) = \frac{1}{2}|u'|^2 + V(t,u),$$

be the Lagrangian associated with (DE) with the corresponding functional

$$I(u) = \int_0^1 L(u)\,dt.$$

Then $I \in C(E,\mathbb{R})$ (even C^2) and for $u,\phi \in E$, the Frechet derivative, $I'(u)\phi$ is given by

$$\begin{aligned} I'(u)\phi &= \lim_{h\to 0}\frac{1}{h}(I(u+h\phi)-I(u)) \\ &= \int_0^1 \left(u'\phi' + V_u(t,u)\phi\right)dt. \end{aligned}$$

If $I'(u)=0$, we say u is a *critical point* of I and $c = I(u)$ is called a *critical value* of I. Note also, if

$$\int_0^1 \left(u'\phi' + V_u(t,u)\phi\right)dt = 0 \tag{1}$$

for all $\phi \in E$, u is called a *weak solution* of (DE). Then we have a “regularity” theorem:

Theorem 2.1. *u is a classical solution of (DE) if and only if $u \in E$ and u is a weak solution of (DE).*

Theorem 2.1 reduces the existence of periodic solutions of (DE) to finding critical points of I in E. In the study of partial differential equations, such regularity theorems are often rather delicate. For the above special case, the proof is quite direct. Since the regularity question will also come up in more complicated settings later, we treat it here for the simplest case.

Proof of Theorem 2.1. If u is a classical solution of (DE), multiplying (DE) by $\phi \in E$ and integrating over $[0,1]$ yields (1). Conversely suppose u is a weak solution of (1). Taking $\phi = 1$ shows

$$\int_0^1 V(t,u)dt \equiv [V(t,u)] = 0,$$

i.e. the constant term in the Fourier expansion of $V(t,u)$ vanishes. It is a calculus exercise to show there is a unique $q \in C^2(\mathbb{T}^1,\mathbb{R})$ solving

$$-\ddot{q} + V_u(t,u) = 0\ \ ,\ \ [q] = 0. \tag{2}$$

Multiplying (2) by $\phi \in E$ and integrating over $[0,1]$ shows

$$\int_0^1 \left(q'\phi' + V_q(t,u)\phi\right)dt = 0. \tag{3}$$

Subtracting (3) from (1) gives

$$\int_0^1 \left(u' - q'\right)\phi' dt = 0 \tag{4}$$

for all $\phi \in E$. Choosing $\phi = u - q$, (4) implies $u' - q' = 0$ and therefore $u = q + const \in C^2(\mathbb{T}^1, \mathbb{R})$. $\square$

How do we find critical points of I? The simplest possibilities are minima. Thus set

$$c = \inf_{u \in E} I(u). \tag{5}$$

Note that I is bounded from below by $V_0 = \min_{\mathbb{R}^2} V$. Let (u_n) be a minimizing sequence for (5), i.e. $I(u_n) \to c$ as $n \to \infty$. Therefore there is an $M > 0$ such that

$$I(u_n) = \int_0^1 \left(\frac{1}{2}(u_n')^2 + V(t, u_n)\right) dt \leq M.$$

Hence

$$||u_n'||_{L^2}^2 \leq 2(M - V_0). \tag{6}$$

Observe that $u_n + j_n$ is also a minimizing sequence for (5) for any choice of $j_n \in \mathbb{Z}$. Therefore u_n may not be bounded. But we can choose j_n so that $[u_n + j_n] \in [0,1]$. Thus without loss of generality, $[u_n] \in [0,1)$. Since

$$u_n(t) - u_n(x) = \int_x^t u_n'(s) ds,$$

one has

$$u_n(t) = [u_n] + \int_0^1 \left(\int_x^t u_n'(s) ds\right) dx,$$

and therefore

$$|u_n(t)| \leq 1 + \int_0^1 ||u_n'||_{L^2} dx = 1 + ||u_n'||_{L^2}. \tag{7}$$

Now (6) and (7) show u_n is bounded in the Hilbert space E. Therefore there is a $v \in E$ such that along a subsequence, $u_n \rightharpoonup v$ (i.e. weakly in E). The functional I is weakly lower semicontinuous. Hence

$$c \leq I(v) \leq \lim_{n \to \infty} I(u_n) = \inf_E I = c. \tag{8}$$

Thus (8) shows $I(v) = c$ and v minimizes I over E. Moreover v is a critical point of I on E. Indeed take $\phi \in E$. Then $\psi(h) \equiv I(v + h\phi) \in C^1(\mathbb{R}, \mathbb{R})$ and has a minimum at $h = 0$. Hence

$$\psi'(0) = 0 = I'(v)\phi \tag{9}$$

for all $\phi \in E$. Thus v is a weak and therefore by Theorem 2.1, a classical solution of (DE).

As was noted above, the minimizing sequence $\{u_n\}$ is bounded in E and therefore in $C^{1/2}(\mathbb{T}^1)$. Hence the subsequence $\{u_n\}$ can be assumed to converge to v in $L^\infty(\mathbb{T}^1)$. Although it is not important here, for future reference, we have a stronger form of convergence:

Proposition 2.1. *$u_n \to v$ in E (i.e. in $W^{1,2}(\mathbb{T}^1)$).*

Proof. If not there is a $\delta > 0$ such that $||u_n' - v'||_{L^2} \geq \delta$. Set $\phi_n = u_n - v$. Then

$$\begin{aligned} I(u_n) &= I(v+\phi_n) \\ &= \int_0^1 \left[\frac{1}{2}|v'|^2 + v'\phi_n' + \frac{1}{2}|\phi_n'|^2 + V(t,v+\phi_n) - V(t,v) + V(t,v)\right] dt \\ &\geq I(v) + \frac{1}{2}\delta^2 + \int_0^1 \left[v'\phi_n' + V(t,v+\phi_n) - V(t,v)\right] dt. \end{aligned} \tag{10}$$

As $n \to \infty$, $I(u_n) \to I(v)$ while the term on the right in (10) approaches zero. Thus $0 \geq 1/2\delta^2$, a contradiction. □

Set $\mathfrak{M}_0 = \{u \in E : I(u) = c\}$. We have shown $\mathfrak{M}_0 \neq \emptyset$.

Example 1: If $V \equiv 0$, then $\mathfrak{M}_0 = \mathbb{R}$.

Example 2: If $V = a(t)(\cos(2\pi u - 1))$, then $\mathfrak{M}_0 = \mathbb{Z}$.

Theorem 2.2. *$\mathfrak{M}_0$ is an ordered set, i.e. $v, w \in \mathfrak{M}_0$ implies $v \equiv w$, $v < w$, or $v > w$.*

Proof. If not, there are points $\xi, \eta \in [0,1]$ such that $v(\xi) = w(\xi)$ and, e.g. $v(\eta) < w(\eta)$. Set $\phi = \max(v,w)$ and $\psi = \min(v,w)$. Then $\phi, \psi \in E$ and

$$2c \leq I(\phi) + I(\psi) = I(v) + I(w) = 2c. \tag{11}$$

Hence by (11), $I(\phi) = c = I(\psi)$ and $\phi, \psi \in \mathfrak{M}_0$. Consequently by Theorem 2.1, ϕ and ψ are classical 1-periodic solutions of (DE). Set $\chi = \phi - \psi$ so $\chi \geq 0$, $\chi(\xi) = 0$ and therefore $\chi'(\xi) = 0$, and $\chi(\eta) > 0$. (DE) implies

$$\chi'' + V_u(t,\phi) - V_u(t,\psi) = 0 = \chi'' + f(t)\chi, \tag{12}$$

where

$$f(t) = \begin{cases} \frac{V_u(t,\phi(t)) - V_u(t,\psi(t)}{\psi(t) - \phi(t)} & \text{if } \phi(t) > \psi(t) \\ V_{uu}(t,\phi(t)) & \text{if } \phi(t) = \psi(t) \end{cases}$$

and $f \in C(\mathbb{T}^1, \mathbb{R})$. Thus χ is a C^2 solution of the linear equation (12) with $\chi(\xi) = 0 = \chi'(\xi)$. Therefore the uniqueness of solutions to the initial value problem for (12) implies $\chi \equiv 0$, contrary to $\chi(\eta) > 0$. Hence $\mathfrak{M}_0$ is ordered. □

Next let $k \in \mathbb{Z}$. Note that V is k-periodic in t so we can seek k-periodic solutions of (DE). Let $u \in W^{1,2}(k\mathbb{T}^1) \equiv E_k$. Set

$$I_k(u) = \int_0^k L(u)\,dt,$$

and

$$\alpha_k = \inf_{u \in E_k} I_k(u).$$

By our above arguments,

$$\mathfrak{M}_k \equiv \{u \in E_k : I_k(u) = \alpha_k\} \neq \emptyset,$$

any $u \in \mathfrak{M}_k$ is a classical k-periodic solution of (DE), and $\mathfrak{M}_k$ is an ordered set.

Surprisingly we gain nothing new by varying k as the next result shows:

Proposition 2.2. $\mathfrak{M}_0 = \mathfrak{M}_k$ *and* $\alpha_k = kc$.

Proof. Let $v \in \mathfrak{M}_k$. Then $v(\cdot + 1) \in \mathfrak{M}_k$. If $v(t) = v(t+1)$ for all $v \in \mathfrak{M}_k$, then $\mathfrak{M}_k = \mathfrak{M}_0$ and $\alpha_k = kc$. Otherwise for some $v \in \mathfrak{M}_k$,

(a) $v(t+1) < v(t)$,

or

(b) $v(t+1) > v(t)$.

If (a) occurs, $v(t) = v(t+k) < \cdots < v(t+1) < v(t)$, a contradiction, and similarly for (b). □

Proposition 2.2 can be used to show that the members of $\mathfrak{M}_0$ possess another minimality property.

Proposition 2.3. *Let* $v \in \mathfrak{M}_0$ *and* $a, b \in \mathbb{R}$ *with* $a < b$. *Set*

$$A = \{w \in W^{1,2}[a,b] : w(a) = v(a), w(b) = v(b)\}$$

and for $w \in A$, *let* $\mathscr{I}(w) = \int_a^b L(w)\,dt$. *Then*

$$\mathscr{I}(v) = \inf_{w \in A} \mathscr{I}(w) \equiv c_A. \tag{13}$$

Proof. $\mathscr{I}$ is weakly lower semi-continuous so as earlier, there is a $u \in A$ such that $\mathscr{I}(u) = c_A$. Choose $\alpha < a$, and $\beta > b$ with $\alpha, \beta \in \mathbb{Z}$. Extend u to $[\alpha, \beta]$ via $u = v$ in $[\alpha, a] \bigcup [b, \beta]$ and further extend u to $\mathbb{R}$ as a $\beta - \alpha$ periodic function. Hence $u \in E_{\beta-\alpha}$ so by Proposition 2.2,

$$I_{\beta-\alpha}(v) \leq I_{\beta-\alpha}(u). \tag{14}$$

But

$$\begin{aligned} I_{\beta-\alpha}(v) &= \int_\alpha^a L(v)\,dt + \mathscr{I}(v) + \int_b^\beta L(v)\,dt \\ &\geq \int_\alpha^a L(u)\,dt + \mathscr{I}(u) + \int_b^\beta L(u)\,dt \\ &= I_{\beta-\alpha}(u). \end{aligned} \tag{15}$$

Thus by (14)–(15), $\mathscr{I}(v) = \mathscr{I}(u) = c_A$. □

Remark: The minimization problem (13) is a special case of

$$\inf_{w\in B} \mathscr{I}(w) \tag{16}$$

where

$$B = \{w \in W^{1,2}[a,b] : w(a) = r, w(b) = s\}.$$

By the argument of (5)–(9), problem (16) has a minimum which is a classical solution of (DE). In several future arguments we will use this observation to establish that the minimizers of certain variational problems are in fact classical solutions of (DE).

Returning to $\mathfrak{M}_0$, since it is ordered, either $\{(t,u(t))\,|\,t\in\mathbb{R}, u\in\mathfrak{M}_0\} = \mathbb{R}^2$, i.e. $\mathfrak{M}_0$ foliates $\mathbb{R}^2$, or there are points $(x,z)\in\mathbb{R}^2$ such that $z\neq u(x)$ for any $u\in\mathfrak{M}_0$, i.e. $\mathfrak{M}_0$ merely laminates $\mathbb{R}^2$. In this latter case there is a smallest $w\in\mathfrak{M}_0$ and largest $v\in\mathfrak{M}_0$ such that $v(x) < z < w(x)$. Hence by Theorem 2.2, $v(t) < w(t)$ for all $t\in\mathbb{R}$. We then call v and w a *gap pair*. It is known that this latter case is generic; indeed given any $v\in\mathfrak{M}_0$, there is a $W\in C^2(\mathbb{T}^1,\mathbb{R})$ such that the $\mathfrak{M}_0$ associated with $V+\varepsilon W$ is $\{v+k\,|\,k\in\mathbb{Z}\}$ for all small $\varepsilon > 0$ [RS].

3 Heteroclinic solutions

Suppose $v,w\in\mathfrak{M}_0$ are a gap pair. We seek solutions of (DE) that are heteroclinic from v to w (or from w to v). A natural approach is to try to find them as minimizers of $\int_{\mathbb{R}} L(u)\,dt$ over a class of functions having the desired asymptotic behavior. However if $\int_0^1 L(v)\,dt = c = \int_0^1 L(w)\,dt \neq 0$, then for each admissible function u, $\int_{\mathbb{R}} L(u)\,dt$ will be infinite. Thus this approach must be modified. The above functional must be "renormalized" so that it is finite on the above class of functions.

This can be done merely assuming (V_1), but it is technically simpler to assume V is also time reversible. Hence suppose

$(\mathbf{V_2})$ $V(-t,z) = V(t,z)$ *for all* $t,z\in\mathbb{R}$

A key consequence of (V_2) is:

Proposition 3.1. *If V satisfies (V_1) and (V_2),*

$$\hat{c} \equiv \inf_{u\in W^{1,2}[0,1]} I(u) = c \tag{1}$$

and if $u\in\mathfrak{M}_0$, then $u(t) = u(-t)$.

Proof. Set $\hat{\mathfrak{M}} = \{u\in W^{1,2}[0,1] : I(u) = \hat{c}\}$. The existence argument of the previous section implies $\hat{\mathfrak{M}}\neq\emptyset$. Clearly $\hat{c}\le c$. To get equality, let $u\in W^{1,2}[0,1]$. Then

$$I(u) = \int_0^{1/2} L(u)\,dt + \int_{1/2}^1 L(u)\,dt \equiv \alpha+\beta.$$

Say $\alpha \leq \beta$. Define $\phi(t) = u(t)$ for $0 \leq t \leq 1/2$, and $\phi(t) = u(1-t)$ for $1/2 \leq t \leq 1$. Then $\phi(0) = \phi(1)$ so ϕ extends naturally to an element of E and by (V_2), $I(\phi) = 2\alpha \leq I(u)$. Therefore

$$c = \inf_E I \leq \inf_{W^{1,2}[0,1]} = \hat{c}$$

so $c = \hat{c}$ and $\mathfrak{M}_0 \subset \hat{\mathfrak{M}}$. But if $u \in \hat{\mathfrak{M}}$, then $I(\phi) = c$ so $\phi \in \mathfrak{M}_0$. Since $\phi \equiv u$ on $[0,1/2]$, uniqueness of solutions of the initial value problem for (DE) implies $u \equiv \phi$ on $[0,1]$, i.e. $u \in \mathfrak{M}_0$. Moreover $u(t) = u(1-t) = u(-t)$ via the 1-periodicity of u. $\square$

With the aid of Proposition 3.1, a renormalized functional can be introduced. For $p \in \mathbb{Z}$ and $u \in W^{1,2}_{loc}(\mathbb{R},\mathbb{R})$, define

$$a_p(u) \equiv \int_p^{p+1} L(u)\,dt - c.$$

By Proposition 3.1, $a_p(u) \geq 0$ for all such p and u. Now we define the renormalized functional:

$$J(u) = \sum_{p\in\mathbb{Z}} a_p(u).$$

Thus $J(u) \geq 0$.

With v, w a gap pair, we define,

$$\begin{aligned}\Gamma_{-\infty} &\equiv \Gamma_{-\infty}(v,w)\\ &\equiv \{u \in W^{1,2}_{loc}(\mathbb{R},\mathbb{R}) : ||u-v||_{L_2[i,i+1]} \to 0, i \to -\infty\}\end{aligned}$$

$$\begin{aligned}\Gamma_{\infty} &\equiv \Gamma_{\infty}(v,w)\\ &\equiv \{u \in W^{1,2}_{loc}(\mathbb{R},\mathbb{R}) : ||u-w||_{L^2[i,i+1]} \to 0, i \to \infty\}\end{aligned}$$

and take as the associated class of admissible functions

$$\Gamma_1 \equiv \Gamma_1(v,w) \equiv \{u \in W^{1,2}_{loc}(\mathbb{R},\mathbb{R}) : v \leq u \leq w\} \cap \Gamma_{-\infty} \cap \Gamma_{\infty}$$

Clearly $\Gamma_1 \neq \emptyset$ and there are u's in Γ_1 such that $J(u) < \infty$. Define

$$c_1 \equiv c_1(v,w) \equiv \inf_{u\in\Gamma_1} J(u). \tag{2}$$

Then we have

Theorem 3.1. *If V satisfies (V_1) - (V_2), and v, w are a gap pair, then*

1. *$\mathfrak{M}_1 \equiv \mathfrak{M}_1(v,w) \equiv \{u \in \Gamma_1 : J(u) = c_1\} \neq \emptyset$.*
2. *Any $U \in \mathfrak{M}_1$ is also a classical solution of (DE).*
3. *$u < U < U(\cdot + 1) < w$.*
4. *$\mathfrak{M}_1$ is an ordered set.*
5. *Any $U \in \mathfrak{M}_1$ is minimal in the sense of Proposition 2.3.*

Proof. Let $\{u_k\}$ be a minimizing sequence for (2). Since we are dealing with an unbounded domain, some extra care must be taken here to ensure that $\{u_k\}$ has a nontrivial limit. E.g. if $\mathfrak{M}_1 \neq \emptyset$ and $U \in \mathfrak{M}_1$, $u_k = U(\cdot - k) \in \Gamma_1$ and u_k converges in C^2_{loc} to $v \notin \Gamma_1$. To avoid such complications, $\{u_k\}$ will be normalized as follows. If $u \in \Gamma_1$ so is $u(\cdot - l)$ for any $l \in \mathbb{Z}$ and $J(u(\cdot - l)) = J(u)$. As $l \to -\infty$, $u|_l^{l+1} \to v$ in L^2 and as $l \to \infty$, $u|_l^{l+1} \to w$ in L^2. Therefore there is a unique $l = l(u) \in \mathbb{Z}$ such that

$$\begin{cases} \int_i^{i+1} (u(t-l) - v(t))\, dt < \frac{1}{2} \int_0^1 (w - v)\, dt, i < 0, i \in \mathbb{Z} \\ \int_0^1 (u(t-l) - v(t))\, dt \geq \frac{1}{2} \int_0^1 (w-v)\, dt. \end{cases} \tag{3}$$

Thus without loss of generality, $\{u_k\}$ can be chosen so that $l(u_k) = 0$.

Since $\{u_k\}$ is a minimizing sequence, there is an $M > 0$ such that for all $k \in \mathbb{N}$,

$$J(u_k) \leq M. \tag{4}$$

Hence for all $p \in \mathbb{N}$,

$$\sum_{-p}^{p} a_i(u_k) = \int_{-p}^{p+1} L(u_k)\, dt - (2p+1)c \leq M \tag{5}$$

and (5) implies

$$\int_{-p}^{p+1} |u_k'|^2\, dt \leq M_1, \tag{6}$$

where M_1 depends on p but not k. Since $v \leq u_k \leq w$, $\{u_k\}$ is bounded in $W^{1,2}_{loc}(\mathbb{R}, \mathbb{R})$. Consequently there is a $U \in W^{1,2}_{loc}$ such that along a subsequence $u_k \to U$ weakly in $W^{1,2}_{loc}$ and in L^∞_{loc}. (In fact in the spirit of Proposition 2.1, $u_k \to U$ in $W^{1,2}_{loc}$ along a subsequence, but we do not need this additional information). Since $\int_{-p}^{p+1} L(u)\, dt$ is weakly lower semi-continuous,

$$\sum_{i=-p}^{p} a_i(U) \leq M$$

for all $p \in \mathbb{N}$ and hence $J(U) \leq M$. Moreover by (3),

$$\begin{cases} \int_i^{i+1} (U - v)\, dt \leq \frac{1}{2} \int_0^1 (w - v)\, dt, i < 0, i \in \mathbb{Z} \\ \int_0^1 (U - v)\, dt \geq \frac{1}{2} \int_0^1 (w-v)\, dt. \end{cases} \tag{7}$$

We claim $U \in \Gamma_1$. The L^∞_{loc} convergence of $\{u_k\}$ implies $v \leq U \leq w$. Thus we need only show U satisfies the asymptotic requirements of Γ_1. To do so, note first that since $J(U) < \infty$, $a_p(U) \to 0$ as $|p| \to \infty$, i.e. $\int_p^{p+1} L(U)\, dt \to c$ as $|p| \to \infty$. Set $U_p(t) = U(t+p)$ for $t \in [0,1]$. Then $U_p \in W^{1,2}[0,1]$ and $I(U_p) \to c$ as $|p| \to \infty$. Hence as $|p| \to \infty$, $\{U_p\}$ is a minimizing sequence for (1). Consequently along a subsequence $\{U_p\}$ converges weakly in $W^{1,2}$ and strongly in L^∞ to $u^\pm \in \mathfrak{M}_0$. But $v \leq U_p \leq w$ implies either $u^\pm = v$ or $u^\pm = w$. By (7), as $p \to -\infty$,

$$\frac{1}{2}\int_0^1 (w-v)\,dt \geq \int_p^{p+1} (U-v)\,dt = \int_0^1 (U_p - v)\,dt \to \int_0^1 (u^- - v)\,dt.$$

Therefore $u^- = v$ and since v is the only possible limit of a subsequence of $\{U_p\}$ as $p \to -\infty$, the full sequence $U_p \to v$ as $p \to -\infty$.

It remains to prove that $U_p \to w$ as $p \to \infty$. For this, we no longer have (7) to help as for $p \to -\infty$, so more work is required. Following the argument of Proposition 2.1, we can assume $U_p \to u^+$ in $W^{1,2}[0,1]$ along our subsequence. In fact, $U_p \to u^+$ along the full sequence as $p \to \infty$ for otherwise there are a pair of subsequences such that $U_p \to v$ in $W^{1,2}$ along the first, and $U_p \to w$ in $W^{1,2}$ along the second as $p \to \infty$. But U_p cannot only be close (in $W^{1,2}[0,1]$ and therefore in $L^\infty[0,1]$) to both v and w. Therefore there is an $\varepsilon > 0$ and a third subsequence such that along it, $||U_p - \phi||_{W^{1,2}[0,1]} \geq \varepsilon$ as $p \to \infty$ for $\phi = v$ and $\phi = w$.

Now we have

Lemma 3.1. *For any $\varepsilon > 0$, there is a $\gamma(\varepsilon) > 0$ such that $||U_p - \phi||_{W^{1,2}[0,1]} \geq \varepsilon$ implies $I(U_p) \geq c + \gamma(\varepsilon)$*

Proof. Otherwise, there is a sequence of p's going to infinity such that $I(U_p) \to c$ while $||U_p - \phi||_{W^{1,2}[0,1]} \geq \varepsilon$. As above along a subsequence, $U_p \to v$ or w in $W^{1,2}[0,1]$, a contradiction. □

Completion of the Proof of Theorem 3.1. Let $S = \{p \in \mathbb{N} : ||U_p - u^+||_{W^{1,2}}[0,1] \geq \varepsilon\}$. Then by Lemma 3.1,

$$J(U) \geq \sum_{p \in S} a_p(U) \geq \sum_{p \in S} \gamma(\varepsilon) = \infty,$$

contrary to $J(U) \leq M$. Thus $U_p \to u^+$ in $W^{1,2}[0,1]$ as $p \to \infty$.

Now finally to show that $u^+ = w$, suppose $u^+ = v$. By the reasoning just used and (7), there is an $i \in \mathbb{Z}$, $i \leq 0$, and $\varepsilon > 0$ such that $||U_i - \phi||_{W^{1,2}[0,1]} \geq \varepsilon$ with $\phi = v$ and $\phi = w$. Hence by Lemma 3.1

$$a_i(U) \geq \gamma(\varepsilon).$$

Therefore for large k,

$$a_i(u_k) \geq \frac{1}{2}\gamma(\varepsilon). \tag{8}$$

Choose $\delta > 0$ and free for the moment. Since $u^+ = v$, there is a $q > 0$ such that $||U_q - v||_{L^\infty[0,1]} \leq \delta/2$. Hence along our subsequence for all large k, $||u_k - U||_{L^\infty[q,q+1]} \leq \delta/2$. Thus $||v - u_k||_{L^\infty[q,q+1]} \leq \delta$ for large k. Define u_k^* to be equal to v for $t \leq q$, equal to ϕ_k for $q \leq t \leq q+1$, and equal to u_k for $q+1 \leq t$, where ϕ_k is a minimizer of the variational problem

$$\inf \int_q^{q+1} L(u)\,dt$$

over

$$K = \{u \in W^{1,2}[q,q+1] : u(q) = v(q), u(q+1) = u_k(q+1)\}.$$

The minimality properties of v and w imply $v \le \phi_k \le w$ and therefore $u_k^* \in \Gamma_1$. Set $u(t) = v(t) + (t-q)(u_k(q+1) - v(q+1))$ so $u \in K$.

Moreover

$$a_q(\phi_k) \le a_q(u) \le \beta(\delta) \tag{9}$$

where $\beta(\delta) \to 0$ as $\delta \to 0$. Now by (8)–(9)

$$\begin{aligned} J(u_k^*) - J(u_k) &= \sum_{-\infty}^{\infty} [a_p(u_k^*) - a_p(u_k)] \\ &= a_q(u_k^*) - \sum_{-\infty}^{q} a_p(u_k) \\ &\le \beta(\delta) - \frac{\gamma(\varepsilon)}{2}. \end{aligned} \tag{10}$$

Choosing δ so small that $\beta(\delta) \le \frac{1}{4}\gamma(\varepsilon)$, (10) contradicts that $\{u_k\}$ is a minimizing sequence for (2). Thus $U \in \Gamma_1$, and $J(U) \ge c_1$. On the other hand,

$$\sum_{-p}^{p} a_i(U) \le \liminf_{k\to\infty} \sum_{-p}^{p} a_i(u_k) \le \liminf_{k\to\infty} J(u_k) = c_1$$

so letting $p \to \infty$, we conclude $J(U) = c_1$. This establishes statement 1 of Theorem 3.1.

To prove statement 2 of Theorem 3.1, first we will obtain the minimality property 5. If it is not true, there are numbers $r < s$ and a function

$$\phi \in \{u \in W^{1,2}[r,s] : u(r) = U(r) \text{ and } u(s) = U(s)\}$$

such that

$$\int_r^s L(\phi)\,dt < \int_r^s L(U)\,dt.$$

Since v and w satisfy the minimality property, we can assume $v \le \phi \le w$. But then replacing $U|_r^s$ by $\phi|_r^s$ gives $U^* \in \Gamma_1$ with $J(U^*) < J(U)$, contrary to Theorem 3.1, part 1. Therefore U satisfies the minimality property and by the remark following Proposition 2.3, U is a solution or (DE).

Next, statement 4 of Theorem 3.1 follows by a mild variant of the proof of Theorem 2.2; Suppose $U, W \in \mathfrak{M}_1$. Thus $\phi = \max(U,W)$ and $\psi = \min(U,W) \in \Gamma_1$ so for all $p \in \mathbb{N}$,

$$\sum_{-p}^{p} [a_i(\phi) + a_i(\psi)] = \sum_{-p}^{p} [a_i(U) + a_i(W)].$$

Letting $p \to \infty$, this shows

$$2c_1 \le J(\phi) + J(\psi) = J(U) + J(W) = 2c_1$$

Therefore $\phi, \psi \in \mathfrak{M}_1$, so by what has already been shown, ϕ and ψ are solutions of (DE) with $\phi \geq \psi$. The proof then concludes as for Theorem 2.2.

Lastly to verify statement 3 of Theorem 3.1, note that $v \leq U, U(\cdot+1) \leq w$ with equality impossible by the argument of Theorem 2.2 again. Moreover since $U, U(\cdot+1) \in \mathfrak{M}_1$, which is ordered, either; (i) $U(t) \equiv U(t+1)$, (ii) $U(t) > U(t+1)$, or (iii) $U(t) < U(t+1)$. If alternative (i) holds, U is 1-periodic and therefore $U \notin \Gamma_1$ while (ii) implies $U(t) > U(t+k) \to w(t)$ as $k \to \infty$. Thus $U > w$ and again $U \notin \Gamma_1$. Thus (iii) holds. □

We conclude this section with a result that shows the gap condition is not only sufficient for there to exist minimizing heteroclinics from v to w, but also is necessary.

Theorem 3.2. *Let V satisfy (V_1) - (V_2), and further let $v, w \in \mathfrak{M}_0$ with $v < w$. Suppose there is a $U \in \Gamma_1(v,w)$ such that*

$$J(U) = \inf_{u \in \Gamma_1(v,w)} J(u).$$

Then v and w are a gap pair.

Proof. Otherwise there is a $\phi \in \mathfrak{M}_0$ such that $v < \phi < w$. There is a smallest $\alpha \in \mathbb{R}$ such $\phi(\alpha) = U(\alpha)$. Define $W(t) = U(t)$, for $t \leq \alpha$, $W(t) = \phi(t)$ when $\alpha \leq t \leq \alpha+1$, and $W(t) = U(t-1)$ when $\alpha+1 \leq t$. Then $W \in \Gamma_1(v,w)$ and $J(W) = J(U)$. Set

$$S = \{u \in W^{1,2}[\alpha - 1/2, \alpha + 1/2] : u(\alpha \pm 1/2) = W(\alpha \pm 1/2)\}.$$

The remark following Proposition 2.3 shows there is a $\psi \in S$ such that ψ is a solution of (DE) and

$$\int_{\alpha-1/2}^{\alpha+1/2} L(\psi)\, dt = \inf_{u \in S} \int_{\alpha-1/2}^{\alpha+1/2} L(u)\, dt.$$

We claim

$$A \equiv \int_{\alpha-1/2}^{\alpha+1/2} L(\psi)\, dt < \int_{\alpha-1/2}^{\alpha+1/2} L(W)\, dt \equiv B.$$

Indeed if $A = B$, W is a solution of (DE) in $(\alpha - 1/2, \alpha + 1/2)$. But $W = \phi$ in $[\alpha, \alpha + 1/2]$. Since ϕ is a solution of (DE) for all t, uniqueness for solutions of the initial value problem for (DE) imply $W = \phi$ in $(\alpha - 1/2, \alpha + 1/2)$. Since U minimizes J in $\Gamma_1(v,w)$, as in Theorem 3.1, U is a solution of (DE) on $\mathbb{R}$. But $U = W = \phi$ in $(\alpha - 1/2, \alpha)$. Therefore $U \equiv \phi$, contrary to $||U - v||_{L^2[i,i+1]} \to 0$ as $i \to -\infty$. Thus $A < B$. But then gluing $W|_{-\infty}^{\alpha-1/2}$ to ψ to $W|_{\alpha+1/2}^{\infty}$ produces $\Phi \in \Gamma_1(v,w)$ with $J(\Phi) < J(W)$ contrary to the minimality of W. □

Remark: Theorem 3.2 does not exclude the possibility of there being a heteroclinic solution of (DE) in $\Gamma_1(v,w)$. If there is one, it cannot be a minimizer. In fact if v, f, and g, w are gap pairs with $f \leq g$, there is a monotone heteroclinic U from v to w.

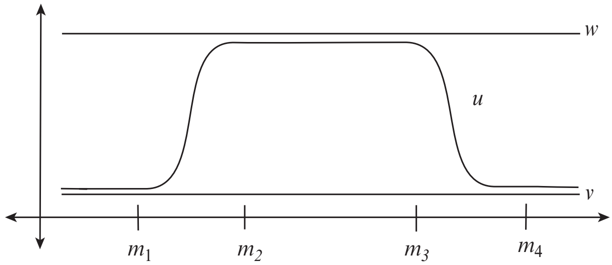

Fig. 8 An admissible u.

4 Multitransition solutions: the simplest case

Suppose v, w are a gap pair for (DE). In Section 3 we showed there are heteroclinic solutions of (DE) in $\mathfrak{M}_1(v,w)$. The same argument gives heteroclinic solutions in $\mathfrak{M}_1(w,v)$. The goal of this section is to find solutions of (DE) which lie between v and w, undergo two transitions, and are homoclinic to v or to w.

We will show there are infinitely many such solutions provided that $\mathfrak{M}_1(v,w)$ and $\mathfrak{M}_1(w,v)$have gaps. The solutions are obtained as local minima of J on appropriate classes of functions. To introduce a suitable class of admissible functions, let $m = (m_1,\dots,m_4) \in \mathbb{Z}^4$ and $\rho = (\rho_1,\dots,\rho_4) \in \mathbb{R}^4$ with $m_i < m_{i+1}$ and $0 < \rho_i << 1$.

Define

$$Y_{1,2} \equiv Y_{1,2,m,\rho} \equiv \{u : u(m_1) - v(m_1) \le \rho_1, w(m_2) - u(m_2) \le \rho_2\}$$

$$Y_{3,4} \equiv Y_{3,4,m,\rho} \equiv \{u : w(m_3) - u(m_3) \le \rho_3, u(m_4) - v(m_4) \le \rho_4\}$$

$$Y \equiv Y_{m,\rho} \equiv \{u \in W^{1,2}_{loc}(\mathbb{R},\mathbb{R}) : v \le u \le w\} \cap Y_{1,2} \cap Y_{3,4}.$$

The numbers ρ_i have to be chosen in a special way which we postpone until needed.

Set

$$b \equiv b_{m,\rho} \equiv \inf_{u\in Y} J(u) \tag{1}$$

Proposition 4.1. *For all (m,ρ), there is a $U = U_{m,\rho} \in Y$ such that $J(U) = b$.*

Proof. It is straightforward to show there is a $\bar{u} \in Y$ such that $J(\bar{u}) < \infty$. Let $\{u_n\}$ be a minimizing sequence for (1). We can assume $J(u_n) \le J(\bar{u})$. As for Theorem 3.1, this implies $\{u_n\}$ is bounded in $W^{1,2}_{loc}$ and there is a $U \in W^{1,2}_{loc}$ such that along a subsequence, $u_n \to U$ weakly in both $W^{1,2}_{loc}$ and L^∞_{loc}. This latter convergence implies U satisfies the pointwise constraints of Y, so $U \in Y$. As in Section 3, $J(U) = b$. □

Proposition 4.2. *U satisfies (DE) except possibly at $t = m_i$, $1 \le i \le 4$ (independently of ρ and m).*

Proof. This follows since U possesses a minimality property for each interval in the complement of the m_i. Eg. For $r \le s \le m_1$, U minimizes $\int_r^s L(u)\,dt$ over

$$\{u \in W^{1,2}[r,s] : u(r) = U(r), u(s) = U(s)\}$$

Hence by the remark following Proposition 2.3, U satisfies (DE) in (r,s). $\square$

Next we will show that U is asymptotic to v as $|t| \to \infty$. For this we require that ρ_1 and ρ_4 be small.

Proposition 4.3. *For ρ_1 (resp. ρ_4) sufficently small, $||U - v||_{W^{1,2}[i,i+1]} \to 0$ as $i \to -\infty$ (resp. $i \to \infty$).*

Proof. We treat the ρ_1 case. Since $J(U) = b < \infty$, $||U - \phi||_{W^{1,2}[i,i+1]} \to 0$ as $i \to -\infty$, where $\phi \in [v,w]$ via the proof of Theorem 3.1. If $\phi = w$, for any $\delta > 0$, there is an $l \in \mathbb{Z}, l < m_1$ such that $||U - w||_{W^{1,2}[l,l+1]} \le \delta$.

Let ψ_l be a minimizer of the problem:

$$\inf \int_{l-1}^{l} L(u)\, dt$$

over

$$\{u \in W^{1,2}[l-1,l] : u(l-1) = w(l-1), u(l) = U(l)\}.$$

As in (9)

$$a_{l-1}(\psi_l) \le \beta(\delta) \tag{2}$$

with $\beta(\delta) \to 0$ as $\delta \to 0$. Similarly let f be a minimizer of the problem

$$\inf \int_{m_1}^{m_1+1} L(u)\, dt$$

over

$$\{u \in W^{1,2}[m_1, m_1+1] : u(m_1) = U(m_1), u(m_1+1) = v(m_1+1)\}$$

and again as in (9),

$$a_{m_1}(f) \le \beta(\rho_1). \tag{3}$$

Set $\bar{U}$ be equal to w for $t \le l-1$, equal to ψ_l for $l-1 \le t \le l$, equal to U for $l \le t \le m_1$, equal to f for $m_1 \le t \le m_1+1$, and equal to v for $m_1+1 \le t$.
Then $\bar{U} \in \Gamma_1(w,v)$ and

$$\begin{aligned} \sum_{i=l}^{m_1-1} a_i(U) &= \sum_{i=l-1}^{m_1} a_i(\bar{U}) - a_{l-1}(\bar{U}) - a_{m_1}(\bar{U}) \\ &= J(\bar{U}) - a_{l-1}(\psi_l) - a_{m_1}(f) \\ &\ge c_1(w,v) - \beta(\delta) - \beta(\rho_1) \end{aligned} \tag{4}$$

via (2)–(3).

On the other hand, let g be a minimizer of

$$\inf \int_{m_1-1}^{m_1} L(u)\, dt$$

over

$$\{u \in W^{1,2}[m_1-1,m_1] : u(m_1-1) = v(m_1-1), u(m_1) = U(m_1)\}$$

Then as for (2)–(3),

$$a_{m_1-1}(g) \leq \beta(\rho_1). \tag{5}$$

Set U^* equal to v for $t \leq m_1-1$, and equal to g for $m_1-1 \leq t \leq m_1$. By the minimality property of U in $(-\infty,m_1]$, and (5)

$$\sum_{-\infty}^{m_1-1} a_i(U) \leq \sum_{-\infty}^{m_1-1} a_i(U^*) \leq \beta(\rho_1). \tag{6}$$

Since

$$\sum_{l}^{m_1-1} a_i(U) \leq \sum_{-\infty}^{m_1-1} a_i(U),$$

(4)–(6) imply

$$c_1(w,v) \leq \beta(\delta) + 2\beta(\rho_1) \tag{7}$$

which is impossible for δ and ρ_1 small. Thus U is asymptotic to v as $t \to -\infty$ and similarly as $t \to \infty$. □

Next we will obtain an upper bound for $b = b_{m,\rho}$.

Proposition 4.4. *Let $\varepsilon > 0$. Then there is an $m_0(\varepsilon)$ such that if $m_2 - m_1, m_4 - m_3 \geq m_0(\varepsilon)$,*

$$b_{m,\rho} \leq c_1(v,w) + c_1(w,v) + \varepsilon$$

Proof. Let $\bar{U} \in \mathfrak{M}_1(v,w)$. Then there are $\alpha, \beta \in \mathbb{Z}$ with $\alpha \leq \beta$ such that if $\bar{f}, \bar{g}$ are respectively minimizers of

$$\int_{\alpha-1}^{\alpha} L(u)\,dt, \qquad \int_{\beta}^{\beta+1} L(u)\,dt$$

over

$$\{u \in W^{1,2}[\alpha-1,\alpha] : u(\alpha-1) = v(\alpha-1), u(\alpha) = \bar{U}(\alpha)\},$$
$$\{u \in W^{1,2}[\beta,\beta+1] : u(\beta) = \bar{U}(\beta), u(\beta+1) = w(\beta+1)\}.$$

Then

$$a_{\alpha-1}(\bar{f}), \quad a_\beta(\bar{g}) \leq \frac{\varepsilon}{4}. \tag{8}$$

Gluing $v|_{-\infty}^{\alpha-1}$ to $\bar{f}$ to $\bar{U}|_{\alpha}^{\beta}$ to $\bar{g}$ to $w|_{\beta+1}^{\infty}$ defines $U^* \in \Gamma_1(v,w)$ (Fig. 9). Since $J(\bar{U}) = c_1(v,w)$ by (8),

$$\begin{aligned} J(U^*) &= a_{\alpha-1}(\bar{f}) + \sum_{\alpha}^{\beta-1} a_i(\bar{U}) + a_\beta(\bar{g}) \\ &\leq c_1(v,w) + \frac{\varepsilon}{2} \end{aligned} \tag{9}$$

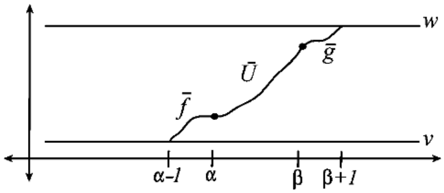

Fig. 9 The construction of U^* in Proposition 8.

Similarly let $\underline{U} \in \mathfrak{M}_1(w,v)$. As above there are $r,s \in \mathbb{Z}$ with $r < s$ such that if $\underline{f}$, respectively $\underline{g}$ are the minimizers of

$$\int_{r-1}^{r} L(u)\,dt, \qquad \int_{s}^{s+1} L(u)\,dt$$

over

$$\{u \in W^{1,2}[r-1,r] : u(r-1) = w(r-1), u(r) = \underline{U}(r)\},$$

$$\{u \in W^{1,2}[s,s+1] : u(s) = \underline{U}(s), u(s+1) = v(s+1)\}.$$

then

$$a_{r-1}(\underline{f}), \quad a_s(\underline{g}) \le \frac{\varepsilon}{4}. \tag{10}$$

and gluing $w|_{-\infty}^{r-1}$ to $\underline{f}$ to $\underline{U}|_r^s$ to $\underline{g}$ to $v|_{s+1}^{\infty}$ produces $U_* \in \Gamma_1(w,v)$ with

$$J(U_*) \le c_1(w,v) + \frac{\varepsilon}{2}. \tag{11}$$

Finally set $U^{**}(t)$ equal to $U^*(t - m_2 + \beta + 1)$ for $t \le m_2$, and equal to $U_*(t - m_3 + r - 1)$ for $m_2 \le t$. By construction U^{**} satisfies the constraints of $Y_{m,\rho}$ at $t = m_2$ and m_3. For $m_2 - m_1 \ge \beta - \alpha + 2$, $U^{**}(m_1) = U^*(m_1 - m_2 + \beta + 1) = v(\alpha - 1) = v(m_1)$ so U^{**} satisfies the constraint at $t = m_1$. Similarly the constraint at $t = m_4$ holds if $m_4 - m_3 \ge s - r + 2$. Therefore $U^{**} \in Y_{m,\rho}$ and by (9) and (11),

$$b_{m,\rho} \le J(U^{**}) \le c_1(v,w) + c_1(w,v) + \varepsilon \qquad \square$$

Next we will refine our choice of ρ. Recall $\mathfrak{M}_1(v,w)$ and $\mathfrak{M}_1(w,v)$ have gaps. Define $\rho_- : \mathfrak{M}_1(v,w) \to (0, w(0) - v(0))$ via $\rho_-(u) = u(0) - v(0)$. Therefore ρ_- is a monotone function of u and $\rho_-(\mathfrak{M}_1(v,w))$ has gaps. Choose ρ_1 to lie in such a gap, i.e.

$$\rho_1 \in (0, w(0) - v(0)) \backslash \rho_-(\mathfrak{M}_1(v,w)).$$

Note that ρ_1 can be chosen as small as desired since $f \in \mathfrak{M}_1$ implies $f(\cdot - l) \in \mathfrak{M}_1(v,w)$ for any $l \in \mathbb{Z}$ so for large l, $\rho_-(f(\cdot - l))$ is near 0.

Similarly define $\rho_+ : \mathfrak{M}_1(v,w) \to (0, w(0) - v(0))$ via $\rho_+(u) = w(0) - u(0)$ so ρ_+ is also monotone and $\rho_+(\mathfrak{M}_1(v,w))$ has gaps. Choose ρ_2 in such a gap. Likewise $\rho_-, \rho_+ : \mathfrak{M}_1(w,v) \to (0, w(0) - v(0))$ as above. Choose ρ_3 and ρ_4 in associated gaps. An important consequence of this choice of the ρ_i is:

Proposition 4.5. *Let*

$$\Lambda_1(v,w) = \{u \in \Gamma_1 : u(0) - v(0) = \rho_1 \ or \ w(0) - u(0) = \rho_2\}.$$

Set

$$d_1(v,w) = \inf_{u \in \Lambda_1(v,w)} J(u) \tag{12}$$

Then for $|\rho|$ small, $d_1(v,w) > c_1(v,w)$.

Remark Defining $\Lambda_1(w,v)$ and $d_1(w,v)$ in the obvious way, we also have $d_1(w,v) > c_1(w,v)$.

Proof of Proposition 4.5. Let $\{u_n\}$ be a minimizing sequence for (12). As in the proof of Theorem 3.1, there is a $P \in W^{1,2}_{loc}$ such that along a subsequence $u_n \to P$ weakly in $W^{1,2}_{loc}$ and also in L^∞_{loc}. This latter convergence implies $v \le P \le w$ and P satisfies one of the constraints at $t = 0$.

Also, as earlier $J(P) < \infty$ and therefore P asymptotes to v or w as $t \to -\infty$ and as $t \to \infty$. If (a) $P(0) = v(0) + \rho_1$, since ρ_1 is small, the argument of Proposition 4.3 shows $||P - v||_{W^{1,2}[i,i+1]} \to 0$ as $i \to -\infty$, while if (b) $w(0) = P(0) + \rho_2$, similarly $||P - w||_{W^{1,2}[i,i+1]} \to 0$ as $i \to \infty$.

Suppose (a) holds. Then either (c) $||P - v||_{W^{1,2}[j,j+1]} \to 0$ as $j \to \infty$ or (d) $||P - w||_{W^{1,2}[j,j+1]} \to 0$ as $j \to \infty$. If (c) occurs, $u_n(0)$ is near $v(0) + \rho_1$ along a subsequence as $n \to \infty$. Hence as in the proof of Lemma 3.1, there is a $\gamma(\rho_1) > 0$ (independent of n) such that

$$a_0(u_n) \ge \gamma(\rho_1) \tag{13}$$

for large n. Moreover for any $\delta > 0$, there is an $l = l(\delta) \in \mathbb{N}$ such that $||u_n - v||_{L^\infty[l,l+1]} \le \delta$ for large n along the subsequence.

Now in the spirit of the proof of Proposition 4.3, set u_n^* equal to v, on $t \le l$, equal to g_n on $l \le t \le l+1$, and equal to u_n on $t \ge l+1$ where g_n minimizes

$$\int_l^{l+1} L(u)\, dt$$

over

$$\{u \in W^{1,2}[l,l+1] : u(l) = v(l),\, u(l+1) = u_n(l+1)\}.$$

Thus as in (9) again,

$$a_l(u_n) \le \beta(\delta). \tag{14}$$

Now by (13)–(14),

$$\begin{aligned} J(u_n) &\ge a_0(u_n) + \sum_{l+1}^{\infty} a_i(u_n) \qquad (15)\\ &\ge \gamma(\rho_1) + \sum_{l}^{\infty} a_i(u_n^*) - a_l(g_n)\\ &\ge \gamma(\rho_1) + J(u_n^*) - \beta(\delta). \end{aligned}$$

Choosing δ so that $\beta(\delta) \le 1/2\gamma(\rho_1)$ and noting that $u_n^* \in \Gamma_1(v,w)$, (15) yields

$$J(u_n) \ge c_1(v,w) + \frac{1}{2}\gamma(\rho_1). \tag{16}$$

Thus $d_1(v,w) \ge c_1(v,w) + \frac{1}{2}\gamma(\rho_1)$ for this case.

On the other hand, if (a) and (d) occur, $P \in \Lambda_1(v,w)$ and by earlier arguments $J(P) = d_1(v,w)$. Since $\Lambda_1(v,w) \subset \Gamma_1(v,w)$, $d_1(v,w) \ge c_1(v,w)$. If $d_1 = c_1$, then $P \in \mathfrak{M}_1(v,w)$ and by Theorem 3.1, P is a solution of (DE). Consequently $P(0) - v(0) = \rho_1 = \rho_-(P) \in \rho_-(\mathfrak{M}_1(v,w))$ contrary to the choice of ρ_1. Thus $d_1 > c_1$. The remaining cases are treated in the same fashion as above. □

With the aid of Proposition 4.5, we have:

Proposition 4.6. *Set*

$$\mu = \frac{1}{2}\min(d_1(v,w) - c_1(v,w), d_1(w,v) - c_1(w,v)).$$

If $|\rho|$ is small and U satisfies an m_i constraint with equality, then for $m_3 - m_2 >> 1$,

$$b_{m,\rho} \ge c_1(v,w) + c_1(w,v) + \mu. \tag{17}$$

Assuming Proposition 4.6 for the moment, combining Proposition 4.5 and Proposition 4.6 we have

$$\mu \le b_{m,\rho} - c_1(v,w) - c_1(w,v) \le \varepsilon \tag{18}$$

provided that an m_i constraint holds with equality. Here μ depends only on ρ and $m_3 - m_2$ while $m_2 - m_1$, $m_4 - m_3 \ge m_0(\varepsilon)$. Thus choosing $\varepsilon < \mu$, (18) yields a contradiction. Therefore U satisfies (DE) for all t and we have

Theorem 4.1. *If (V_1)–(V_2) hold, v and w are a gap pair, and in addition $\mathfrak{M}_1(v,w)$ and $\mathfrak{M}_1(w,v)$ have gaps, then for $|\rho|$ small and $m_{i+1} - m_i$ large, there is a $U = U_{m,\rho} \in Y_{m,\rho}$ which is a solution of (DE) with $J(U) = b_{m,\rho}$.*

Remark: That U satisfies the constraints with strict inequality implies U has a local minimization property.

Corollary 4.1. *There are infinitely many distinct 2-transition solutions of (DE).*

Proof. Simply take different sets of (m_i)'s with $m_{i+1} - m_i$ larger and larger. □

To complete the proof of Theorem 4.1, we give the

Proof of Proposition 4.6. By the minimality property of $U|_{m_2}^{m_3}$,

$$\int_{m_2}^{m_3} L(U)\,dt = \inf_{u \in A} \int_{m_2}^{m_3} L(u)\,dt \tag{19}$$

where

$$A = \{u \in W^{1,2}[m_2,m_3] : u(m_2) = U(m_2),\, u(m_3) = U(m_3)\}.$$

Since ρ_2, ρ_3 are small and $w(m_2) - U(m_2) \le \rho_2$, $w(m_3) - U(m_3) \le \rho_3$, as in (9), (19) implies

$$\sum_{i=m_2}^{m_3-1} a_i(U) \le \beta(\rho_2) + \beta(\rho_3). \tag{20}$$

We claim that given any $\sigma > 0$, there is an $\alpha(\sigma) > 0$ such that for $m_3 - m_2 \ge \alpha(\sigma)$, $||U - w||_{W^{1,2}[i,i+1]} \le \sigma$ for some $q \in [m_2, m_3 - 1]$. Otherwise by Lemma 3.1,

$$\sum_{j=m_2}^{m_3-1} a_j(U) \ge (m_3 - m_2 - 1)\gamma(\sigma) \tag{21}$$

which goes to infinity as $m_3 - m_2 \to \infty$. But this is contrary to (20) which shows that the left hand side of (21) is small.

Now suppose for convenience that we have equality at an m_1 or m_2 constraint point. Set $\Phi(t)$ equal to $U(t)$ for $t \le q$, equal to $f(t)$ for $q \le t \le q+1$, equal to $w(t)$ for $t \ge q+1$, where f minimizes

$$\int_q^{q+1} L(u)\, dt$$

over

$$\{u \in W^{1,2}[q, q+1] : u(q) = U(q), u(q+1) = w(q+1)\}.$$

Therefore $\Phi \in \Gamma_1(v, w)$.

Similarly set $\Psi(t)$ equal to $w(t)$ for $t \le q$, equal to $g(t)$ for $q \le t \le q+1$, and equal to $U(t)$ for $t \ge q+1$, where g minimizes

$$\int_q^{q+1} L(u)\, dt$$

over

$$\{u \in W^{1,2}[q, q+1] : u(q) = w(q), u(q+1) = U(q+1)\}.$$

Therefore $\Phi \in \Gamma_1(w, v)$ and

$$d_1(v, w) + c_1(w, v) \le J(\Phi) + J(\Psi) \le J(U) - a_q(U) + a_q(f) + a_q(g) \tag{22}$$

Since $||U - w||_{W^{1,2}[q,q+1]} \le \sigma$ it follows as in (9) again that $a_q(f) + a_q(g) \le 2\beta(\sigma) \to 0$ as $\sigma \to 0$. Then for σ so small that

$$2\beta(\sigma) \le \mu, \tag{23}$$

$J(U) = b$ and (22)–(23) imply (17). $\square$

5 Multitransition solutions: general case

The ideas used in proving Theorem 4.1 work equally well to get k transition solutions of (DE) and then even infinite transition solutions via a limit argument, provided that the construction does not depend on k. However, given Theorem 4.1, there is a simpler geometrical argument giving the k and infinite transition cases, as well as an associated symbolic dynamics of solutions. We will illustrate with the case of $k = 3$ and then discuss the general case.

Choose $\rho, r \in \mathbb{R}^4$ and $m, n \in \mathbb{Z}^4$ such that there are associated solutions U and W of (DE) with $U \in Y_{m,\rho}(v,w)$, and $W \in Y_{n,r}(w,v)$. We seek a 3-transition solution heteroclinic from v to w. For $j \in \mathbb{Z}$, set $\tau_j u(t) = u(t-j)$. Because of their asymptotic properties for $j_1 >> 1$, $\tau_{j_1}U(t) < W(t)$ for all $t \in \mathbb{R}$.

Take $j_2 >> j_1$. Then $\tau_{j_1}U < \tau_{j_2}W$. Finally take $j_3 >> j_2$. Then $\tau_{j_3}U < \tau_{j_2}W$. For simplicity, we will take $j_1 = j, j_2 = 2j, j_3 = 3j$ for sufficiently large j. Consider $\{\tau_{-lj}W : l \in \mathbb{N}\}$ and $\{\tau_{(3+i)j}U : i \in \mathbb{N}\}$.

Delete from the region between the graphs of v and w the set of points above all of the shifted W's we have mentioned and below the shifted U's. Denote the remaining region by R and set

$$Y(R) \equiv \{u \in W^{1,2}_{loc} : (t,u(t)) \in \bar{R}\}.$$

(See Fig. 10). Define

$$c(R) = \inf_{u \in Y(R)} J(u).$$

Then we have:

Theorem 5.1. *Under the hypothesis of Theorem 4.1*

1. $\mathfrak{M}(R) = \{u \in Y(R) : J(u) = c(R)\} \neq \emptyset$.
2. *Any* $U \in \mathfrak{M}(R)$ *is a classical solution of (DE) and is interior to R.*
3. $||U - v||_{L^2[i,i+1]} \to 0$, $i \to -\infty$, *and* $||U - w||_{L^2[i,i+1]} \to 0$, $i \to \infty$.
4. *U has a local minimization property: for any* $r < s$, *U minimizes* $\int_r^s L(u)\,dt$ *over the class of* $W^{1,2}[r,s]$ *functions with* $u(r) = U(r)$, *and* $u(s) = U(s)$ *provided that* $s - r$ *sufficiently small.*

Proof. We will sketch the proof. A minimizing sequence converges as earlier to U lying in $\bar{R}$ with $J(U) = c(R)$. Since $J(\bar{U}) < \infty$, (3) of the theorem holds due

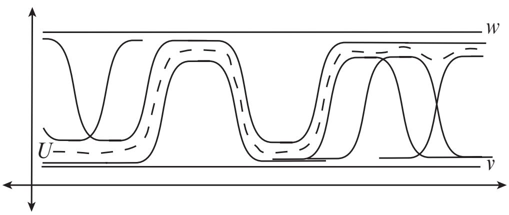

Fig. 10 A U in $Y(R)$.

to the form of R. The boundary of R consists of curves possessing local or global minimization properties and this readily implies (4), which in turn gives the first part of (2). Lastly the basic existence and uniqueness theorem for ordinary differential equations implies U cannot touch ∂R as in the proof of Theorem 2.2. $\square$

Next we will show how to generalize Theorem 5.1 and at the same time get a symbolic dynamics of solutions (Fig. 10). Choose U, W, and j as above so in particular the graphs of $\tau_{\pm j}U$ and W do not intersect. This implies the same is true of the graphs of $\tau_{ij}U$ and $\tau_{lj}W$ for all $i,l \in \mathbb{Z}$. Define

$$\Sigma \equiv \{\sigma = \{\sigma_i\}_{i\in\mathbb{Z}} : \sigma_i \in \{+,-\}\}.$$

For each $\sigma \in \Sigma$, we define a region $R(\sigma)$ lying between the graphs of v and w as follows. Set

$$S = \{(t,z) : t \in \mathbb{R}, v(t) \leq z \leq w(t)\}.$$

If $\sigma_i = +$, remove the region below $\tau_{ji}U$ from S; if $\sigma_i = -$, remove the region above $\tau_{ji}W$ from S. $R(\sigma)$ is what remains after carrying out this excision process for all $i \in \mathbb{Z}$. Then we have;

Theorem 5.2. *For each $\sigma \in \Sigma$, there is a solution $U_{R(\sigma)}$ of (DE) with the graph of $U_{R(\sigma)}$ lying in $R(\sigma)$. Moreover $U_{R(\sigma)}$ has the local minimization property of Theorem 5.2.*

Remark: If $\sigma_i = -$, $U_{R(\sigma)}$ will be L^∞ close to v on a large interval while if $\sigma_i = +$, $U_{R(\sigma)}$ will be L^∞ close to w on a large interval. In particular if $\sigma_i = -$ for all i near $-\infty$, $U_{R(\sigma)}$ asymptotes to v as $t \to -\infty$, while if $\sigma_i = +$ for all i near ∞, $U_{R(\sigma)}$ asymptotes to w as $t \to \infty$. The dynamics of the symbol σ reflect the dynamics of the solution $U_{R(\sigma)}$.

Proof of Theorem 5.2. We will sketch the proof. First we introduce four subsets of Σ:

$$\Sigma^{++} \equiv \{\sigma \in \Sigma : \sigma_i = + \text{ for all large } |i|\}$$

$$\Sigma^{--} \equiv \{\sigma \in \Sigma : \sigma_i = - \text{ for all large } |i|\}$$

$$\Sigma^{+-} \equiv \{\sigma \in \Sigma : \sigma_i = + \text{ for all large negative } i, \text{ and } \sigma_i = - \text{ for all large positive } i\}$$

$$\Sigma^{-+} \equiv \{\sigma \in \Sigma : \sigma_i = - \text{ for all large negative } i, \text{ and } \sigma_i = + \text{ for all large positive } i\}$$

Let Σ^* be the union of these four sets. Any $\sigma \in \Sigma^*$ has a finite number of changes of σ_i as i increases. For $\sigma \in \Sigma^*$, set

$$Y(\sigma) = \{u \in W^{1,2}_{loc} : (t,u(t)) \in \bar{R}(\sigma) \text{ for all } t \in \mathbb{R}\},$$

and define

$$c(\sigma) = \inf_{u\in Y(\sigma)} J(u).$$

Then $c(\sigma) < \infty$ and the proof of Theorem 5.1 shows there is a $U_{R(\sigma)} \in Y(\sigma)$ such that $U_{R(\sigma)}$ satisfies (2) and (4) of Theorem 5.1 and also possess the asymptotics associated with σ.

Next suppose $\sigma = \{\sigma_i\}_{\in\mathbb{Z}} \in \Sigma \backslash \Sigma^*$. For each $n \in \mathbb{N}$, define $f_n(\sigma) \in \Sigma^*$ via $f_n(\sigma)$ equal to σ_i, $|i| \leq n$, equal to σ_n, $i > n$, and equal to σ_{-n} when $i < -n$. Therefore by what was previously shown, there is a $U_n \in Y(f_n(\sigma))$ such that $J(U_n) = c(f_n(\sigma))$.

Since $v \leq U_n \leq w$, the functions U_n are uniformly bounded. By (DE), they are also bounded in C^2. Therefore using (DE), as $n \to \infty$, U_n converges along a subsequence in C^2 to $U(\sigma)$, a solution of (DE). Moreover for any $l \in \mathbb{N}$, if $n \geq l$, for $|t| \leq l$, the graph of U_n lies in

$$R(f_n(\sigma)) \cap \{(t,z) : |t| \leq l, v(t) < z < w(t)\}$$

$$= R(\sigma) \cap \{(t,z) : |t| \leq l, v(t) < z < w(t)\}$$

from which it follows that the graph of U lies in $R(\sigma)$. Finally the local minimality property is preserved by the L^∞_{loc} convergence of the U_n. $\square$

We conclude this section with some open questions. First, is it possible to give a variational characterization of $U(\sigma)$ for $\sigma \in \Sigma \backslash \Sigma^*$? The difficulty is that for such σ, $J(U(\sigma)) = \infty$. We suspect that a second renormalization of J can be made which allows for a direct variational characterization of $U(\sigma)$. A second question is whether it is possible to classify these multi-transition solutions. How many parameters do they really depend on?

6 The tip of the iceberg

In a sense the class of solutions of (DE) we have studied in these lectures merely represent the tip of the iceberg. All of these solutions lie between a gap pair. Even if we had had time to study the monotone solutions of (DE) mentioned in the introduction that cross a finite number of gaps, we are still only dealing with bounded solutions which therefore have rotation number 0.

For $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ it is straightforward to find minimal solutions of (DE) satisfying $u(t+q) = u(t) + p$. In terms of the pendulum, they make p rotations in time q and have rotation number p/q. Thus replacing $\mathfrak{M}_0$ by such a class of minimizers, there are analogues of the results of the previous sections. There are also minimal solutions with an irrational rotation number which can be obtained as limits of the rational ones.

In addition to these minimal solutions there are nonminimal solutions that can be obtained variationally. E.g. there are mountain pass solutions lying between a gap pair v, w. In fact there is a sequence $\{u_n\}$ of such solutions with periods which go to infinity as $n \to \infty$. Likewise there are mountain pass heteroclinics between a gap pair in $\mathfrak{M}_1(v,w)$. These facts can be proven using versions of the mountain pass theorem.

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