Chapter 5
Time-Varying Electromagnetic Field

5.1 Maxwell's Equations in Differential Form

In general, the presence of a charge distributed with density $\rho\,(\mathrm{C\,m^{-3}})$ and of an impressed current density $\bar{J}_0\,(\mathrm{A\,m^{-2}})$ variable with time gives origin to the electromagnetic field described by the following time-dependent vectors:

$\overline{D}$ electric displacement $(\mathrm{C\,m^{-2}})$
$\overline{E}$ electric field intensity $(\mathrm{V\,m^{-1}})$
$\overline{B}$ magnetic induction (T)
$\overline{H}$ magnetic field intensity $(\mathrm{A\,m^{-1}})$
$\bar{J}$ current density $(\mathrm{A\,m^{-2}})$

As far as the origin of current density is concerned, the following remark can be put forward. In a solid or liquid medium the *conduction current* density is a function of $\overline{E}$

$$\bar{J} = \bar{J}(\overline{E}) \tag{5.1}$$

For a linear medium the above function becomes

$$\bar{J} = \sigma\overline{E} \tag{5.2}$$

Another kind of current is originated by the movement of free ions and electrons (e.g. in gases or vacuum). This *convection current* density is expressed by the formula

$$\bar{J} = \rho_+ u_+ + \rho_- u_- \tag{5.3}$$

where ρ_+ and ρ_- are positive and negative charge densities, respectively, while u_+ and u_- are the relevant velocities of positive and negative free charges.

Finally, the *displacement current* density is defined as

$$\bar{J} = \frac{\partial \overline{D}}{\partial t} \tag{5.4}$$

Considering the principle of charge conservation in any point of the domain, the following equation always holds (charge continuity equation)

$$\overline{\nabla} \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0 \tag{5.5}$$

The coupled electric and magnetic fields influence a charge q (C) by exerting a mechanical force $\overline{F}$ (N) on it (Lorentz's equation)

$$\overline{F} = q\left(\overline{E} + \overline{u} \times \overline{B}\right) \tag{5.6}$$

where $\overline{u}$ is the velocity of the charge with respect to the magnetic field. In particular, the term $q\overline{E}$ modifies the value of velocity, while the term $q\overline{u} \times \overline{B}$ modifies also the direction of velocity.

In a simply-connected domain Ω with boundary Γ filled in by a linear medium characterized by permittivity ε, permeability μ and conductivity σ, the time-varying electromagnetic field is described by the following equations:

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \text{Faraday's equation} \tag{5.7}$$

$$\overline{\nabla} \cdot \overline{D} = \rho \qquad \text{Gauss's electric equation} \tag{5.8}$$

$$\overline{\nabla} \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \text{Ampère's equation} \tag{5.9}$$

$$\overline{\nabla} \cdot \overline{B} = 0 \qquad \text{Gauss's magnetic equation} \tag{5.10}$$

In a three-dimensional domain, the above equations represent a set of eight scalar equations to which constitutive relations (2.65), (2.187), (2.255) must be added.

In total, fifteen scalar unknowns (i.e. field components) have to be determined, subject to suitable boundary conditions.

The system of eight plus nine equations can be solved since there are two relations among the unknowns which are automatically satisfied. In fact, taking the divergence of (5.9) and the time derivative of (5.8), continuity equation (5.5) follows. Similarly, taking the divergence of (5.7) and the time derivative of (5.10), one obtains an identity.

It should be remarked that in (5.9), in general,

$$\overline{J} = \overline{J}_0 + \sigma\overline{E} + \mu\sigma\overline{u} \times \overline{H} \tag{5.11}$$

where $\overline{J}_0$ is the term impressed by an external source, while the last term of the right-hand side takes into account the current density due to motional effect, if any.

In steady conditions all vectors are independent of time. Therefore, the two equations governing the electric field, namely (5.7) and (5.8), are decoupled with respect to the two equations governing the magnetic field, namely (5.9) and (5.10) (see Section 2.2 and Section 2.3).

5.2 Poynting's Vector

Let Maxwell's equations (5.7) and (5.9) be considered. By means of a vector identity (see A.13) one obtains

$$\overline{\nabla}\cdot\left(\overline{E}\times\overline{H}\right)=\overline{H}\cdot\left(\overline{\nabla}\times\overline{E}\right)-\overline{E}\cdot\left(\overline{\nabla}\times\overline{H}\right)=-\overline{H}\cdot\frac{\partial\overline{B}}{\partial t}-\overline{E}\cdot\frac{\partial\overline{D}}{\partial t}-\overline{E}\cdot\overline{J} \quad (5.12)$$

Referring to the specific energy in the electric and magnetic case and under the assumption of linear constitutive relationships, one has

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\overline{H}\cdot\overline{B}+\overline{E}\cdot\overline{D}\right)=\frac{1}{2}\left(\overline{H}\cdot\frac{\partial\overline{B}}{\partial t}+\overline{B}\cdot\frac{\partial\overline{H}}{\partial t}\right)+\frac{1}{2}\left(\overline{E}\cdot\frac{\partial\overline{D}}{\partial t}+\overline{D}\cdot\frac{\partial\overline{E}}{\partial t}\right)=$$

$$=\overline{H}\cdot\frac{\partial\overline{B}}{\partial t}+\overline{E}\cdot\frac{\partial\overline{D}}{\partial t} \quad (5.13)$$

Integrating (5.12) over Ω and using Gauss's theorem (see A.10), it results

$$\int_{\Gamma}\left(\overline{E}\times\overline{H}\right)\cdot\overline{n}d\Gamma=-\frac{\partial}{\partial t}\int_{\Omega}\left(\frac{\overline{H}\cdot\overline{B}}{2}+\frac{\overline{E}\cdot\overline{D}}{2}\right)d\Omega-\int_{\Omega}\overline{E}\cdot\overline{J}d\Omega \quad (5.14)$$

Vector

$$\overline{S}=\overline{E}\times\overline{H} \quad (5.15)$$

is called Poynting's vector (W m^{-2}).

According to (5.14), its flux out of a closed surface Γ is equal to (minus) the sum of the power of the electromagnetic field inside the domain Ω and the power transferred to the current (Poynting's theorem).

5.3 Maxwell's Equations in the Frequency Domain

The most important case of time-varying electromagnetic fields occurs when field sources, namely charge and current densities, vary with sinusoidal law. A given vector

$$\overline{V}(x,y,z,t)=\left[V_{0x}(x,y,z)\cos(\omega t-\varphi),V_{0y}(x,y,z)\cos(\omega t-\varphi),\right.$$
$$\left.V_{0z}(x,y,z)\cos(\omega t-\varphi)\right]=$$
$$=\overline{V}_0\cos(\omega t-\varphi) \quad (5.16)$$

can be expressed as

$$\overline{V}(x,y,z,t)=\left[V_{0x}(x,y,z)\mathrm{Re}\left\{e^{j(\omega t-\varphi)}\right\},V_{0y}(x,y,z)\mathrm{Re}\left\{e^{j(\omega t-\varphi)}\right\},\right.$$
$$\left.V_{0z}(x,y,z)\mathrm{Re}\left\{e^{j(\omega t-\varphi)}\right\}\right]=$$

$$= \overline{V}_0 \mathrm{Re}\left\{e^{j(\omega t-\varphi)}\right\} \tag{5.17}$$

The algebraic quantity $\overline{V} = \overline{V}_0 e^{-j\varphi}$ (phasor) represents the vector $\overline{V}(x, y, z, t)$ in a unique way; moreover, in the frequency domain, since $\frac{d}{dt}\cos(\omega t) = \omega\cos(\omega t + \frac{\pi}{2})$, the differential operator $\frac{\partial}{\partial t}$ is transformed into the complex operator $j\omega$.

Consequently, Maxwell's equation (5.7), (5.8), (5.9) and (5.10) are transformed as follows:

$$\overline{\nabla} \times \overline{E} = -j\omega\overline{B} \tag{5.18}$$

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{5.19}$$

$$\overline{\nabla} \times \overline{H} = \overline{J} + j\omega\overline{D} \tag{5.20}$$

$$\overline{\nabla} \cdot \overline{D} = \rho \tag{5.21}$$

The latter equations are referred to as Helmholtz's equations and are valid at sinusoidal steady state for frequency $f = \frac{\omega}{2\pi}$ (field equations in the frequency domain).

It should be remarked that field quantities in the latter equations are the phasors corresponding to the associated time functions; by definition, the amplitude of the phasor is the maximum value of the corresponding time function.

Considering the constitutive equations, in a non-conducting region free of spatial charges ($\rho = 0$) and impressed currents, from vector identity (A.12), taking into account (5.21) one has

$$\overline{\nabla} \times \overline{\nabla} \times \overline{E} = \overline{\nabla}\left(\overline{\nabla} \cdot \overline{E}\right) - \overline{\nabla}^2\overline{E} = -\overline{\nabla}^2\overline{E} \tag{5.22}$$

From (5.18) and (5.20), if μ is a constant and $\sigma = 0$, $J_0 = 0$, it follows

$$\overline{\nabla} \times \overline{\nabla} \times \overline{E} = \overline{\nabla} \times \left(-j\omega\overline{B}\right) = -j\omega\overline{\nabla} \times \overline{B} =$$
$$= -j\omega\mu\overline{\nabla} \times \overline{H} = -j\omega\mu(j\omega\overline{D}) = \omega^2\mu\varepsilon\overline{E} \tag{5.23}$$

Comparing (5.22) and (5.23), Helmholtz's equation of electric field results

$$\overline{\nabla}^2\overline{E} = -\omega^2\mu\varepsilon\overline{E} = k^2\overline{E} \tag{5.24}$$

with $k = j\omega\sqrt{\mu\varepsilon}$.

If the same procedure is applied to field $\overline{H}$, one obtains

$$\overline{\nabla}^2\overline{H} = -\omega^2\mu\varepsilon\overline{H} = k^2\overline{H} \tag{5.25}$$

At sinusoidal steady state, the Poynting's vector (phasor) resulting from the time average of (5.15), considering the root-mean-square value of each vector, is

$$\overline{S} = \frac{\overline{E} \times \overline{H}^*}{2} \tag{5.26}$$

where the star denotes the conjugate phasor.

Referring to a volume Ω with boundary Γ, (5.14) takes the form

$$\int_\Gamma \left(\frac{\overline{E} \times \overline{H}^*}{2}\right) \cdot \overline{n}\, d\Gamma = -2j\omega \int_\Omega \left(\frac{\overline{H} \cdot \overline{B}^*}{4} + \frac{\overline{E} \cdot \overline{D}^*}{4}\right) d\Omega - \int_\Omega \frac{\overline{E} \cdot \overline{J}^*}{2} d\Omega \tag{5.27}$$

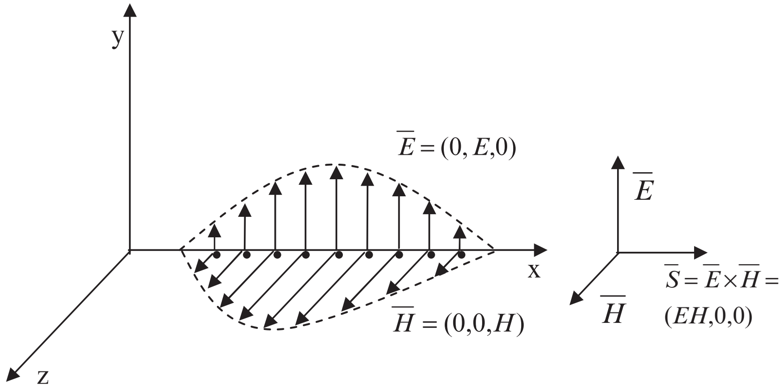

Fig. 5.1 Travelling plane electromagnetic wave

5.4 Plane Waves in an Infinite Domain

Let a simply-connected unbounded domain, filled in by a perfectly insulating medium ($\rho = 0,\ \sigma = 0$), be considered. For the sake of simplicity, let a time-harmonic electric field $E_0 \cos \omega t$ have only a non-zero component in the y-direction and vary only in the x direction (Fig. 5.1).

The Helmholtz's equation (5.24) reduces to

$$\frac{\partial^2 \overline{E}}{\partial x^2} = -\omega^2 \mu\varepsilon \overline{E} \tag{5.28}$$

It can be easily proven that the complex function

$$\overline{E} = \overline{E}_0 e^{j\omega\sqrt{\mu\varepsilon}\left(x - \frac{1}{\sqrt{\mu\varepsilon}} t\right)} \tag{5.29}$$

with $\overline{E}_0$ phasor of the given electric field, is a solution of (5.28).

In the time domain it results

$$E(x, t) = E_0 \cos\left[\frac{\omega}{u}(x - ut)\right] \tag{5.30}$$

with $u = \frac{1}{\sqrt{\mu\varepsilon}}$ (ms^{-1}). It can be verified that also

$$E(x, t) = E_0 \cos\left[\frac{\omega}{u}(x + ut)\right] \tag{5.31}$$

once transformed in its complex form $\overline{E} = \overline{E}_0 e^{j\omega\sqrt{\mu\varepsilon}\left(x + \frac{1}{\sqrt{\mu\varepsilon}} t\right)}$ is a solution of (5.28).

From the physical standpoint, (5.30) and (5.31) represent harmonic waves travelling with velocity u in positive and negative x-direction, respectively.

Owing to (5.18) a time-harmonic field $\overline{B}$ is associated to $\overline{E}$.

In the frequency domain it results

$$\overline{B} = \left(0, 0, \frac{1}{j\omega}\frac{\partial \overline{E}}{\partial x}\right) = \left(0, 0, \sqrt{\mu\varepsilon}\,\overline{E}\right) \tag{5.32}$$

In the time domain one obtains:

$$B(x, t) = \frac{E_0}{u}\cos\left[\frac{\omega}{u}(x \pm ut)\right] \tag{5.33}$$

From (5.30), (5.31) and (5.32), it results that the couple of vectors $\left(\overline{E}, \overline{B}\right)$ defined above is a plane wave; $\overline{E}$ and $\overline{B}$ are orthogonal vectors; the ratio of electric field intensity to induction field intensity is equal to the velocity u of propagation in the dielectric medium.

Moreover, the Poynting's vector $\overline{S}$ in the time domain results

$$\overline{S} = \overline{E} \times \overline{H} = \frac{E_0^2}{u\mu}\left[1 + \cos\left(\frac{2\omega}{u}(x \pm ut)\right)\right]\bar{i}_x \tag{5.34}$$

Therefore, the direction of propagation of the plane wave is orthogonal to both electric and magnetic field (transverse electromagnetic wave, TEM).

5.5 Wave and Diffusion Equations in Terms of Vectors $\overline{E}$ and $\overline{H}$

Considering the constitutive relations (2.65), (2.187), (2.255), Maxwell's equations (5.7)–(5.10) become, in terms of fields $\overline{E}$ and $\overline{H}$,

$$\overline{\nabla} \times \overline{E} = -\mu\frac{\partial \overline{H}}{\partial t} \tag{5.35}$$

$$\overline{\nabla} \times \overline{H} = \overline{J}_0 + \sigma\overline{E} + \varepsilon\frac{\partial \overline{E}}{\partial t} \tag{5.36}$$

$$\overline{\nabla} \cdot \overline{E} = \frac{\rho}{\varepsilon} \tag{5.37}$$

$$\overline{\nabla} \cdot \overline{H} = 0 \tag{5.38}$$

where $\overline{J}_0$ is the impressed current density.

From (5.35) one has

$$\overline{\nabla} \times \overline{\nabla} \times \overline{E} = -\frac{\partial}{\partial t}\left(\overline{\nabla} \times \mu\overline{H}\right) \tag{5.39}$$

Since (see A.12)

$$\overline{\nabla} \times \overline{\nabla} \times \overline{E} = \overline{\nabla}\left(\overline{\nabla} \cdot \overline{E}\right) - \overline{\nabla}^2\overline{E} \tag{5.40}$$

taking into account that, in the absence of free charges (i.e. $\rho = 0$), if ε is a constant $\overline{\nabla} \cdot \overline{E} = \overline{\nabla} \cdot \frac{\overline{D}}{\varepsilon} = 0$, one obtains

$$-\overline{\nabla}^2 \overline{E} = -\frac{\partial}{\partial t}\left(\overline{\nabla} \times \mu \overline{H}\right) \tag{5.41}$$

Then, for a homogeneous medium from (5.41) and (5.36) it results

$$\overline{\nabla}^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} + \mu\sigma \frac{\partial \overline{E}}{\partial t} + \mu \frac{\partial \overline{J}_0}{\partial t} \tag{5.42}$$

namely, the equation governing electric field $\overline{E}$; if $\frac{\partial \overline{J}_0}{\partial t} = 0$, the homogeneous wave equation is obtained.

Similarly, it can be proven that for field H the following equation holds

$$\overline{\nabla}^2 \overline{H} = \mu\varepsilon \frac{\partial^2 \overline{H}}{\partial t^2} + \mu\sigma \frac{\partial \overline{H}}{\partial t} - \overline{\nabla} \times \overline{J}_0 \tag{5.43}$$

If in (5.36) the displacement current density $\varepsilon \frac{\partial \overline{E}}{\partial t}$ can be neglected, then equations (5.42) and (5.43) become

$$\overline{\nabla}^2 \overline{E} = \mu\sigma \frac{\partial \overline{E}}{\partial t} + \mu \frac{\partial \overline{J}_0}{\partial t} \tag{5.44}$$

and

$$\overline{\nabla}^2 \overline{H} = \mu\sigma \frac{\partial \overline{H}}{\partial t} - \overline{\nabla} \times \overline{J}_0 \tag{5.45}$$

respectively; they are the differential equations governing the electromagnetic field under quasi-static conditions (diffusion equations).

In turn, by taking the divergence of both sides of (5.36) and considering (A.8), the equation of charge relaxation follows

$$\overline{\nabla} \cdot \left(\sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t}\right) = -\overline{\nabla} \cdot \overline{J}_0 \tag{5.46}$$

where the driving term is due to the impressed current density. It can be remarked that (5.46) states the current density balance in a dissipative dielectric medium, characterised by both conductivity σ and permittivity ε. In the frequency domain, (5.46) transforms as

$$\overline{\nabla} \cdot \left[(\sigma + j\omega\varepsilon)\,\overline{E}\right] = -\overline{\nabla} \cdot \overline{J}_0 \tag{5.47}$$

where the complex conductivity $\sigma + j\omega\varepsilon$ appears; in (5.47) $\overline{E}$ and $\overline{J}_0$ are the phasors corresponding to the associated time functions.

5.6 Wave and Diffusion Equations in Terms of Scalar and Vector Potentials

In a simply connected domain Ω filled in by a linear and homogeneous medium, the magnetic vector potential $\overline{A}$(Wb m^{-1}) is defined by the equation (see 2.205)

$$\overline{B} = \overline{\nabla} \times \overline{A} \tag{5.48}$$

associated to a suitable gauge condition to be specified later on.

By means of (5.7) one has

$$\overline{\nabla} \times \left(\overline{E} + \frac{\partial \overline{A}}{\partial t} \right) = 0 \tag{5.49}$$

This means that the vector in brackets can be expressed as the gradient of a scalar potential φ(V)

$$\overline{E} + \frac{\partial \overline{A}}{\partial t} = -\overline{\nabla}\varphi \tag{5.50}$$

Hence

$$\overline{E} = -\overline{\nabla}\varphi - \frac{\partial \overline{A}}{\partial t} \tag{5.51}$$

Substituting (5.51) into (5.36) one obtains

$$\overline{\nabla} \times \overline{H} = \overline{J}_0 - \sigma\overline{\nabla}\varphi - \sigma\frac{\partial \overline{A}}{\partial t} - \varepsilon\frac{\partial}{\partial t}\overline{\nabla}\varphi - \varepsilon\frac{\partial^2 \overline{A}}{\partial t^2} \tag{5.52}$$

From (5.48) one has

$$\overline{\nabla} \times \overline{H} = \overline{\nabla} \times \mu^{-1}\overline{\nabla} \times \overline{A} \tag{5.53}$$

and

$$\overline{\nabla} \times \overline{\nabla} \times \overline{A} + \mu\varepsilon\overline{\nabla}\frac{\partial \varphi}{\partial t} + \mu\varepsilon\frac{\partial^2 \overline{A}}{\partial t^2} = \mu\left(\overline{J}_0 - \sigma\overline{\nabla}\varphi - \sigma\frac{\partial \overline{A}}{\partial t} \right) \tag{5.54}$$

In the case of a current-free and charge-free ideal dielectric region ($J_0 = 0$, $\rho = 0$ and $\sigma = 0$) it results

$$\overline{\nabla} \times \overline{\nabla} \times \overline{A} + \mu\varepsilon\overline{\nabla}\frac{\partial \varphi}{\partial t} + \mu\varepsilon\frac{\partial^2 \overline{A}}{\partial t^2} = 0 \tag{5.55}$$

On the other hand, substituting (5.50) into (5.37) gives

$$-\nabla^2\varphi - \frac{\partial}{\partial t}\left(\overline{\nabla} \cdot \overline{A}\right) = 0 \tag{5.56}$$

Equations (5.54) and (5.56) represent the link between the two potentials.

Taking into account that (see A.12)

$$\overline{\nabla} \times \overline{\nabla} \times \overline{A} = -\overline{\nabla}^2 \overline{A} + \overline{\nabla} \left(\overline{\nabla} \cdot \overline{A} \right) \tag{5.57}$$

and by substituting this expression into (5.55) one has

$$-\overline{\nabla}^2 \overline{A} + \overline{\nabla} \left(\overline{\nabla} \cdot \overline{A} \right) + \mu\varepsilon \overline{\nabla} \frac{\partial \varphi}{\partial t} + \mu\varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} = 0 \tag{5.58}$$

or

$$-\overline{\nabla}^2 \overline{A} + \overline{\nabla} \left(\overline{\nabla} \cdot \overline{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} \right) + \mu\varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} = 0 \tag{5.59}$$

If the Lorentz's gauge

$$\overline{\nabla} \cdot \overline{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} = 0 \tag{5.60}$$

is imposed, then from (5.59) one obtains

$$-\overline{\nabla}^2 \overline{A} + \mu\varepsilon \frac{\partial^2 \bar{A}}{\partial t^2} = 0 \tag{5.61}$$

which is the wave equation for the magnetic vector potential $\overline{A}$, subject to boundary and intial conditions. After determining $\overline{A}$, following (5.60), φ is given by

$$\varphi(t) = \varphi_0 - \frac{1}{\mu\varepsilon} \int_0^t \overline{\nabla} \cdot \overline{A}(t')dt' \tag{5.62}$$

with φ_0 to be determined.

Alternatively, imposing gauge (5.60) to equation (5.56), the wave equation for the electric scalar potential is obtained

$$-\nabla^2 \varphi + \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{5.63}$$

After determining φ, $\overline{A}$ can be recovered.

In the case current J_0 and charge ρ are present, (5.61) and (5.63) become

$$-\overline{\nabla}^2 \overline{A} + \mu\varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} = \mu J_0 \tag{5.64}$$

$$-\nabla^2 \varphi + \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} = \frac{\rho}{\varepsilon} \tag{5.65}$$

respectively.

In a three-dimensional unbounded domain Ω, their particular solutions are (see 2.48 and 2.49)

$$\overline{A} = \int_\Omega \frac{\mu \overline{J}'_0}{4\pi r} d\Omega \tag{5.66}$$

$$\varphi = \int_\Omega \frac{\rho'}{4\pi\varepsilon r} d\Omega \tag{5.67}$$

where the values $\overline{J}'_0$ and ρ' are taken at an earlier time $t' = t - r\sqrt{\mu\varepsilon}$ with respect to the time t at which $\overline{A}$ and φ are observed. The latter two potentials are therefore called retarded potentials. Additionally, it can be noted that $\overline{A}$ depends only on $\overline{J}_0$ and φ depends only on ρ. This dependence is, except for the correspondence of time, the same as in magnetostatics and electrostatics, respectively.

In the case of a conductor ($\rho = 0,\ \sigma \neq 0$), by imposing the following gauge

$$\overline{\nabla} \cdot \overline{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} + \mu\sigma\varphi = 0 \tag{5.68}$$

from (5.54) and (5.57) it follows

$$-\overline{\nabla}^2 \overline{A} + \overline{\nabla}\left(\overline{\nabla} \cdot \bar{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} + \mu\sigma\varphi\right) + \mu\varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} + \mu\sigma \frac{\partial \overline{A}}{\partial t} = \mu \bar{J}_0 \tag{5.69}$$

or

$$-\overline{\nabla}^2 \overline{A} + \mu\varepsilon \frac{\partial^2 \bar{A}}{\partial t^2} + \mu\sigma \frac{\partial \overline{A}}{\partial t} = \mu \bar{J}_o \tag{5.70}$$

After determining $\overline{A}$ and so $\overline{\nabla} \cdot \overline{A},\ \varphi$ can be recovered from (5.68).

5.7 Electromagnetic Field Radiated by an Oscillating Dipole

Let a point charge $q(t) = q\sin(\omega t)$ oscillate with angular frequency ω along an element $d\lambda$ of line λ in a three-dimensional domain, so that the resulting current is $i = \omega q \cos \omega t$. If line λ is coincident with the z axis, in the frequency domain the phasor of the elementary vector potential (see 5.66) can be expressed as

$$d\overline{A} = \frac{\mu_0 \bar{I}}{4\pi r} e^{-j\frac{\omega r}{c}} d\lambda \bar{i}_z \tag{5.71}$$

where $\bar{I}$ is the phasor of current i, r is the distance between field point and source point, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is the velocity of the electromagnetic wave in free space and the operator $e^{-j\frac{\omega r}{c}}$ accounts for the phase delay of $d\overline{A}$ with respect to $\bar{I}$.

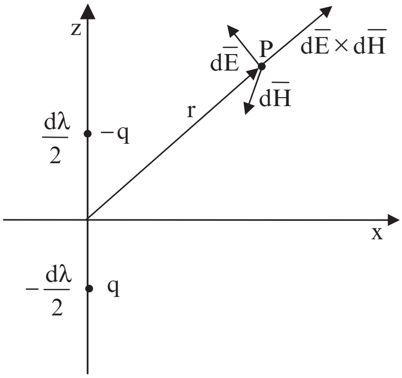

Fig. 5.2 Field radiated at point P by an oscillating dipole

Assuming spherical coordinates with origin at the gravity centre of the dipole (Fig. 5.2), the components of vector potential are

$$\begin{aligned} d\overline{A}_r &= d\overline{A}\cos\vartheta \\ d\overline{A}_\vartheta &= -d\overline{A}\sin\vartheta \\ d\overline{A}_\varphi &= 0 \end{aligned} \tag{5.72}$$

Since $\mu_0\overline{H} = \overline{\nabla} \times \overline{A}$, the components of the elementary magnetic field in the frequency domain are (see A.21–A.23)

$$\begin{aligned} d\overline{H}_r &= d\overline{H}_\vartheta = 0 \\ d\overline{H}_\varphi &= \frac{\overline{I}\sin\vartheta d\lambda}{4\pi r^2}\left(1 + j\frac{\omega r}{c}\right)e^{-j\frac{\omega r}{c}} \end{aligned} \tag{5.73}$$

Thanks to (5.73), it can be noted that lines of magnetic fields are circular and are located on planes normal to the direction of z axis.

According to the Lorentz's gauge (5.60), the phasor of the elementary scalar potential associated to vector potential is

$$d\varphi = j\frac{c^2}{\omega}\overline{\nabla}\cdot\left(d\overline{A}\right) \tag{5.74}$$

Considering (5.51) and (5.74), the relationship between potentials and electric field

$$d\overline{E} = -j\omega d\overline{A} - \overline{\nabla}d\varphi \tag{5.75}$$

becomes

$$d\overline{E} = -j\omega d\overline{A} - j\frac{c^2}{\omega}\overline{\nabla}\left(\overline{\nabla}\cdot d\overline{A}\right) \tag{5.76}$$

After (5.72), (A.19) and (A.24), the components of the elementary electric field follow in phasor form

$$d\overline{E}_r = -j\frac{2\overline{I}\cos\vartheta d\lambda}{4\pi\varepsilon_0 r^3\omega}\left(1 + j\frac{\omega r}{c}\right)e^{-j\frac{\omega r}{c}}$$
$$d\overline{E}_\vartheta = -j\frac{\overline{I}\sin\vartheta d\lambda}{4\pi\varepsilon_0 r^3\omega}\left[1 + j\frac{\omega r}{c} - \left(\frac{\omega r}{c}\right)^2\right]e^{-j\frac{\omega r}{c}}$$
$$d\overline{E}_\varphi = 0 \tag{5.77}$$

The situation is represented in Fig. 5.2.

It is interesting to consider the approximated expressions of field components near the oscillating dipole and far from it, respectively.

Under the approximation $\frac{\omega r}{c} << 1$ of near field, the field components become in phasor form

$$d\overline{H}_\varphi = \frac{\overline{I}\sin\vartheta d\lambda}{4\pi r^2} \tag{5.78}$$

$$d\overline{E}_r = -j\frac{2\overline{I}\cos\vartheta d\lambda}{4\pi\varepsilon_0 r^3\omega} \tag{5.79}$$

$$d\overline{E}_\vartheta = -j\frac{\overline{I}\sin\vartheta d\lambda}{4\pi\varepsilon_0 r^3\omega} \tag{5.80}$$

It can be noted that the magnetic field scales as $\frac{1}{r^2}$ following the Laplace's law of the elementary action valid for a steady current (see 3.79); in turn, the electric field scales as $\frac{1}{r^3}$ according to the static field of a dipole (see Section 2.2.6).

Conversely, under the approximation $\frac{\omega r}{c} >> 1$ of far field, the field components become

$$d\overline{H}_\varphi = j\frac{\overline{I}\sin\vartheta d\lambda}{4\pi c}\left(\frac{\omega}{r}\right)e^{-j\frac{\omega r}{c}} \tag{5.81}$$

$$d\overline{E}_\vartheta = j\frac{\overline{I}\sin\vartheta d\lambda}{4\pi\varepsilon_0 c^2}\left(\frac{\omega}{r}\right)e^{-j\frac{\omega r}{c}} \tag{5.82}$$

The component $d\overline{E}_r$ can be neglected with respect to $d\overline{E}_\vartheta$, apart from points in which $|\sin\vartheta| << 1$. It is important to note that electric and magnetic fields are orthogonal, in phase and tangent to the sphere of radius r; consequently, the Poynting's vector has a radial direction only and the corresponding phasor results

$$d\overline{S} = \frac{d\overline{E}_\vartheta \times d\overline{H}^*_\varphi}{2} = \frac{I^2\sin^2\vartheta\,(d\lambda)^2}{16\pi^2\varepsilon_0 c^3}\left(\frac{\omega}{r}\right)^2\bar{i}_r \tag{5.83}$$

where I is the root-mean-square value of current.

It comes out that the power radiated by the dipole is maximum for $\vartheta = \frac{\pi}{2}$ (equatorial plane) and zero for $\vartheta = 0$ (z axis); furthermore, the average power flowing through a spherical surface is independent of its radius. Finally, the amplitude of fields depends on $\frac{\omega}{r}$; therefore, to make a long-distance transmission, it is necessary to increase the source frequency.

5.8 Diffusion Equation in Terms of Dual Potentials

Let a linear homogeneous isotropic medium, characterized by conductivity σ, permeability μ and permittivity ε be considered, where an impressed current J_0 is present, the time variations of which are small, i.e., if time harmonic variations occur, the angular frequency is much lower than $\frac{\sigma}{\varepsilon}$. Then, displacement current density $\frac{\partial \overline{D}}{\partial t}$ may be neglected with respect to impressed $\overline{J}_0$ and induced $\sigma\overline{E}$ current densities (quasi-static approximation). In this case, Maxwell's equations reduce to

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{5.84}$$

$$\overline{\nabla} \cdot \overline{D} = 0 \tag{5.85}$$

$$\overline{\nabla} \times \overline{H} = \overline{J} = \overline{J}_0 + \sigma\overline{E} \tag{5.86}$$

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{5.87}$$

along with the constitutive relations (5.2) and (2.187).

Given appropriate boundary and initial conditions, vectors $\overline{H}$ (or $\overline{B}$) and $\overline{E}$ (or $\overline{J}$ and $\overline{D}$) are uniquely defined.

This is a special case of Section 5.1 and is particularly important in low-frequency applications (eddy current problem).

The electromagnetic field can be also described in terms of potentials in two different ways.

According to the $\overline{A} - \phi$ method (see Section 5.6) a magnetic vector potential $\overline{A}$(Wb m^{-1}) is introduced by (5.48); moreover, an electric scalar potential ϕ(V) is defined according to (5.50).

In order to specify $\overline{A}$ uniquely, a further condition must be introduced: this may be the Coulomb's gauge (2.206) or the Lorentz's gauge (5.60).

This way $\overline{E}$ and $\overline{H}$ can be expressed by means of two potentials (see Section 2.1.4), namely $\overline{A}$ and ϕ.

From (5.86) taking into account (5.48) and (5.51) one has

$$\overline{\nabla} \times \mu^{-1}\overline{\nabla} \times \overline{A} = \overline{J}_0 - \sigma\frac{\partial \overline{A}}{\partial t} - \sigma\overline{\nabla}\phi \tag{5.88}$$

From (5.5), taking into account (5.51), it follows

$$\overline{\nabla} \cdot \left(\overline{J}_0 - \sigma\frac{\partial \overline{A}}{\partial t} - \sigma\overline{\nabla}\phi\right) = 0 \tag{5.89}$$

Equations (5.88) and (5.89) with appropriate boundary and initial conditions solve the electromagnetic problem in terms of $\overline{A}$ and ϕ. In a region where $\sigma = 0$ (eddy-current free) the latter reduce to the classical equations of magnetostatics (see Section 2.3.1). On the other hand, (5.88) is a special case of (5.54).

Moreover, imposing the gauge $\overline{\nabla} \cdot \overline{A} + \mu\sigma\phi = 0$, from (5.88) one obtains

$$-\overline{\nabla}^2\overline{A} + \mu\sigma\frac{\partial\overline{A}}{\partial t} = \mu\overline{J}_0 \tag{5.90}$$

that represents the diffusion equation in terms of vector potential; it is an approximation of equation (5.70) in the quasi-static state. After determining $\overline{A}$, scalar potential $\varphi = -(\mu\sigma)^{-1}\,\overline{\nabla} \cdot \overline{A}$ can be recovered.

Alternatively, following the $\overline{T} - \Omega$ method, in regions free of impressed current ($J_0 = 0$) an electric vector potential T ($A\ m^{-1}$) can be defined as

$$\overline{\nabla} \times \overline{T} = \overline{J} \tag{5.91}$$

Comparing (5.91) and (5.86) it turns out that $\overline{H}$ and $\overline{T}$, which have the same curl, must differ by the gradient of a function Ω(A) (magnetic scalar potential)

$$\overline{H} = \overline{T} - \overline{\nabla}\Omega \tag{5.92}$$

The electric and magnetic vectors, $\overline{J}$ and $\overline{H}$, have been so expressed in terms of two potentials.

In order to define $\overline{T}$ uniquely, a gauge must be introduced.

The equation governing the electromagnetic field can be now expressed in terms of $\overline{T}$ and Ω. In fact, from (5.86) taking the curl of both members and taking into account (5.84) and (5.92), one has

$$\overline{\nabla} \times \left(\sigma^{-1}\overline{\nabla} \times \overline{T}\right) = \overline{\nabla} \times \sigma^{-1}\overline{J}_0 - \frac{\partial}{\partial t}\mu\left(\overline{T} - \overline{\nabla}\Omega\right) \tag{5.93}$$

and from (5.87)

$$\overline{\nabla} \cdot \mu\left(\overline{T} - \overline{\nabla}\Omega\right) = 0 \tag{5.94}$$

In regions where $\sigma = 0$ one has $\overline{J} = 0$ and therefore, from (5.91), $\overline{\nabla} \times \overline{T} = 0$.

Moreover, imposing the gauge $\overline{\nabla} \cdot \overline{T} = \mu\sigma\frac{\partial\Omega}{\partial t}$, from (5.93) and (5.94) one obtains two independent equations for T and Ω, namely

$$\overline{\nabla}^2\overline{T} - \mu\sigma\frac{\partial\overline{T}}{\partial t} = -\overline{\nabla} \times \overline{J}_0 \tag{5.95}$$

and

$$\nabla^2\Omega - \mu\sigma\frac{\partial\Omega}{\partial t} = 0 \tag{5.96}$$

subject to appropriate boundary conditions. They are

$$\overline{n} \times \overline{T} = 0,\ \ \Omega = 0 \tag{5.97}$$

or

$$\overline{n} \cdot \overline{T} = 0,\ \ \frac{\partial\Omega}{\partial n} = 0 \tag{5.98}$$

if the boundary is normal to a flux line (i.e. $\overline{n} \times \overline{B} = 0$) or it is parallel to a flux line (i.e. $\overline{n} \cdot \overline{B} = 0$), respectively.

After determining $\overline{T}$, Ω is given by

$$\Omega(t) = \Omega_0 + (\mu\sigma)^{-1} \int_0^t \overline{\nabla} \cdot \overline{T}(t')\, dt' \tag{5.99}$$

with Ω_0 to be determined.

5.9 Weak Eddy Current in a Conducting Plane under AC Conditions

Let a conducting plane of thickness b and infinite extension, as shown in Fig. 5.3, be considered.

A time-varying magnetic field $\overline{H} = (H,\ 0,\ 0)$, with $H = H_0 \sin \omega t$, is impressed to the conductor characterized by conductivity σ.

Thanks to symmetry, all variables depend merely on y coordinate. From (5.35), $\overline{\nabla} \times \overline{E}$ turns out to be directed along the x axis and to depend merely on the z component of $\overline{E}$. It follows that the electric field $\overline{E}$ induced within the conductor is $\overline{E} = (0, 0, E)$; the same holds for induced current density $\overline{J} = (0, 0, J)$. Therefore

$$\overline{\nabla} \times \overline{E} = \left(\frac{\partial E}{\partial y}, 0, 0 \right) \tag{5.100}$$

From (5.35), neglecting the magnetic field created by $\overline{J} = \sigma\overline{E}$, one has

$$\frac{\partial E}{\partial y} = -\mu_0 \frac{\partial H}{\partial t} \tag{5.101}$$

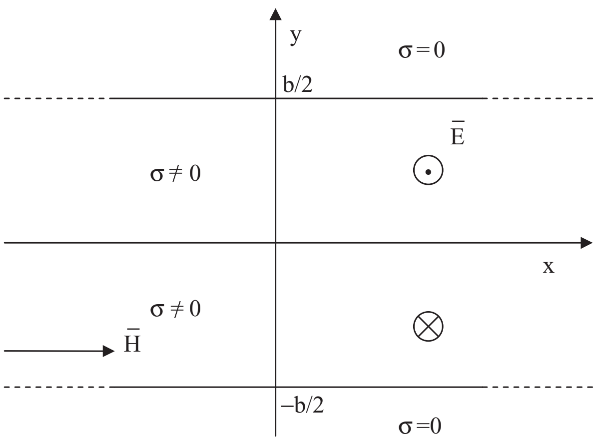

Fig. 5.3 Conducting plane in a magnetic field

or

$$\frac{\partial E}{\partial y} = -\omega\mu_0 H_0 \cos \omega t \tag{5.102}$$

Therefore:

$$E(y, t) = E(y) \cos \omega t \tag{5.103}$$

$$E(y) = -\omega\mu_0 H_0 y + k \tag{5.104}$$

with k to be determined.

Following (5.86) with $\overline{J}_0 = 0$ (solenoidality of the specific current $\sigma\overline{E}$), the boundary condition is

$$E\left(-\frac{b}{2}\right) = -E\left(\frac{b}{2}\right) \tag{5.105}$$

It follows

$$\frac{1}{2}\omega\mu_0 H_0 b + k = \frac{1}{2}\omega\mu_0 H_0 b - k \tag{5.106}$$

Therefore, it results that $k = 0$ and

$$E(y, t) = -\omega\mu_0 H_0 y \cos \omega t \tag{5.107}$$

In terms of eddy current density one has

$$\overline{J}(y, t) = -\sigma\omega\mu_0 H_0 y \cos \omega t \overline{i}_z, \; -\frac{b}{2} < y < \frac{b}{2} \tag{5.108}$$

and $\overline{J}(y, t) = 0$ elsewhere.

By assuming $b = 2\,\text{cm}$, $H_0 = 10^4\ \text{A m}^{-1}$, $\sigma = 5.93\ 10^7 \Omega^{-1}\text{m}^{-1}$, $f = 50\,\text{Hz}$, the amplitude of induced electric field is shown in Fig. 5.4.

5.10 Strong Eddy Current in a Conducting Plane under AC Conditions

Unlike the previous example, if the magnetic field due to the induced current dominates over the impressed field within the conductor, then the governing equations become

$$\overline{\nabla} \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t} \tag{5.109}$$

$$\overline{\nabla} \times \overline{H} = \sigma\overline{E} \tag{5.110}$$

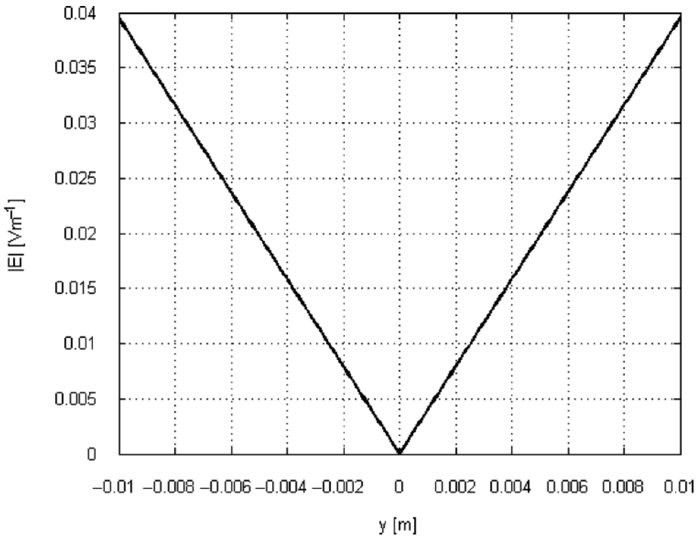

Fig. 5.4 Induced electric field in the conductor cross-section (weak reaction)

Thanks to symmetry

$$\overline{\nabla} \times \overline{E} = \left(\frac{\partial E}{\partial y}, 0, 0\right); \; \overline{\nabla} \times \overline{H} = \left(0, 0, -\frac{\partial H}{\partial y}\right) \tag{5.111}$$

From (5.109) one has

$$\frac{\partial E}{\partial y} = -\mu_0 \frac{\partial H}{\partial t} \tag{5.112}$$

and from (5.110)

$$-\frac{\partial H}{\partial y} = \sigma E \tag{5.113}$$

so that differentiating the latter with respect to y and substituting into the former yields

$$\frac{1}{\sigma}\frac{\partial^2 H}{\partial y^2} = \mu_0 \frac{\partial H}{\partial t} \tag{5.114}$$

Likewise, after differentiating (5.112) with respect to y and (5.113) with respect to t, one has

$$\frac{\partial^2 E}{\partial y^2} = -\mu_0 \frac{\partial}{\partial y}\frac{\partial H}{\partial t} \quad \text{and} \quad -\frac{\partial}{\partial t}\frac{\partial H}{\partial y} = \sigma \frac{\partial E}{\partial t} \tag{5.115}$$

respectively. Therefore

$$\mu_0^{-1}\frac{\partial^2 E}{\partial y^2} = \sigma\frac{\partial E}{\partial t} \tag{5.116}$$

results.

It is assumed that both H and E are time-harmonic functions

$$\begin{aligned} H(y,t) &= H(y)\cos(\omega t + \varphi_H) \\ E(y,t) &= E(y)\cos(\omega t + \varphi_E) \end{aligned} \tag{5.117}$$

or

$$\begin{aligned} H(y,t) &= \Re e\left\{H(y)e^{j\varphi_H}e^{j\omega t}\right\} = \Re e\left\{\overline{H}e^{j\omega t}\right\} \\ E(y,t) &= \Re e\left\{E(y)e^{j\varphi_E}e^{j\omega t}\right\} = \Re e\left\{\overline{E}e^{j\omega t}\right\} \end{aligned} \tag{5.118}$$

Combining (5.118) with (5.114) gives

$$\frac{1}{\sigma}\frac{\partial^2\overline{H}}{\partial y^2} = \mu_0 j\omega\overline{H} \quad \text{and} \quad \frac{\partial^2\overline{H}}{\partial y^2} - j\omega\sigma\mu_0\overline{H} = 0 \tag{5.119}$$

where $\overline{H}$ now denotes the phasor corresponding to H(y, t).

Similarly

$$\mu_0^{-1}\frac{\partial^2\overline{E}}{\partial y^2} = \sigma j\omega\overline{E} \quad \text{and} \quad \frac{\partial^2\overline{E}}{\partial y^2} - j\omega\sigma\mu_0\overline{E} = 0 \tag{5.120}$$

holds.

Let quantities

$$\alpha^2 = j\omega\sigma\mu_0;\ k = \sqrt{\frac{\omega\sigma\mu_0}{2}}(m^{-1});\ \delta = \frac{1}{k}(m) \tag{5.121}$$

be defined, where δ is called penetration depth or skin depth. It results $(1+j)^2k^2 = \alpha^2 = (1+j)^2\delta^{-2}$. The general solution to the equation

$$\frac{\partial^2\overline{H}}{\partial y^2} - \alpha^2\overline{H} = 0 \tag{5.122}$$

is

$$\overline{H} = C_1e^{\alpha y} + C_2e^{-\alpha y} \tag{5.123}$$

The application of the boundary conditions

$$y = \pm\frac{b}{2};\quad \overline{H} = \overline{H}_0 \tag{5.124}$$

gives

$$\overline{H}_0 = C_1 e^{-\alpha\frac{b}{2}} + C_2 e^{+\alpha\frac{b}{2}}$$
$$\overline{H}_0 = C_1 e^{\alpha\frac{b}{2}} + C_2 e^{-\alpha\frac{b}{2}} \tag{5.125}$$

This implies

$$e^{-\alpha\frac{b}{2}}(C_1 - C_2) = e^{+\alpha\frac{b}{2}}(C_1 - C_2)$$
$$C_1 = C_2 \tag{5.126}$$

and

$$\overline{H}_0 = C_1\left(e^{-\alpha\frac{b}{2}} + e^{\alpha\frac{b}{2}}\right) = C_1 2\mathrm{ch}\left(\alpha\frac{b}{2}\right)$$
$$C_1 = C_2 = \frac{\overline{H}_0}{2\mathrm{ch}\left(\alpha\frac{b}{2}\right)} \tag{5.127}$$

Finally, from (5.123) it follows

$$\overline{H} = \overline{H}_0 \frac{e^{\alpha y} + e^{-\alpha y}}{2\mathrm{ch}\left(\alpha\frac{b}{2}\right)} = \overline{H}_0 \frac{\mathrm{ch}(\alpha y)}{\mathrm{ch}\left(\alpha\frac{b}{2}\right)} \tag{5.128}$$

Because of (5.113), one has

$$\bar{J} = -\frac{\partial \overline{H}}{\partial y} = -\alpha \overline{H}_0 \frac{\mathrm{sh}(\alpha y)}{\mathrm{ch}\left(\alpha\frac{b}{2}\right)} \tag{5.129}$$

Returning to the time domain, the amplitude of time-varying field H inside the conductor is given by the norm of (5.128). Using the identity $|\mathrm{ch}(u + jv)| = \sqrt{\cos^2 u + \mathrm{sh}^2 v}$ with u and v real numbers, after (5.121) and (5.128) it follows

$$\left|\overline{H}(y)\right| = \left|\overline{H}_0\right| \frac{\sqrt{\cos^2\frac{y}{\delta} + \mathrm{sh}^2\frac{y}{\delta}}}{\sqrt{\cos^2\frac{b}{2\delta} + \mathrm{sh}^2\frac{b}{2\delta}}}; \quad -\frac{b}{2} < y < \frac{b}{2} \tag{5.130}$$

while $\left|\overline{H}(y)\right| = \left|\overline{H}_0\right|$ elsewhere.

By assuming $b = 2\,\mathrm{cm},\ H_0 = 10^4\ \mathrm{A\ m^{-1}},\ \sigma = 5.93\ 10^7 \Omega^{-1}\mathrm{m}^{-1},\ f = 10^3\,\mathrm{Hz}$, the distribution of magnetic field shown in Fig. 5.5 is obtained. When frequency f decreases, the magnetic field tends to become constant and equal to H_0.

In turn, after (5.129) the amplitude J of the eddy current density is given by:

$$\left|\bar{J}(y)\right| = \beta\frac{\left|\overline{H}_0\right|}{\delta}\sqrt{\cos^2\frac{y}{\delta}\mathrm{sh}^2\frac{y}{\delta} + \sin^2\frac{y}{\delta}\mathrm{ch}^2\frac{y}{\delta}} = \beta\frac{\left|\overline{H}_0\right|}{\delta}\sqrt{\sin^2\frac{y}{\delta} + \mathrm{sh}^2\frac{y}{\delta}} \tag{5.131}$$

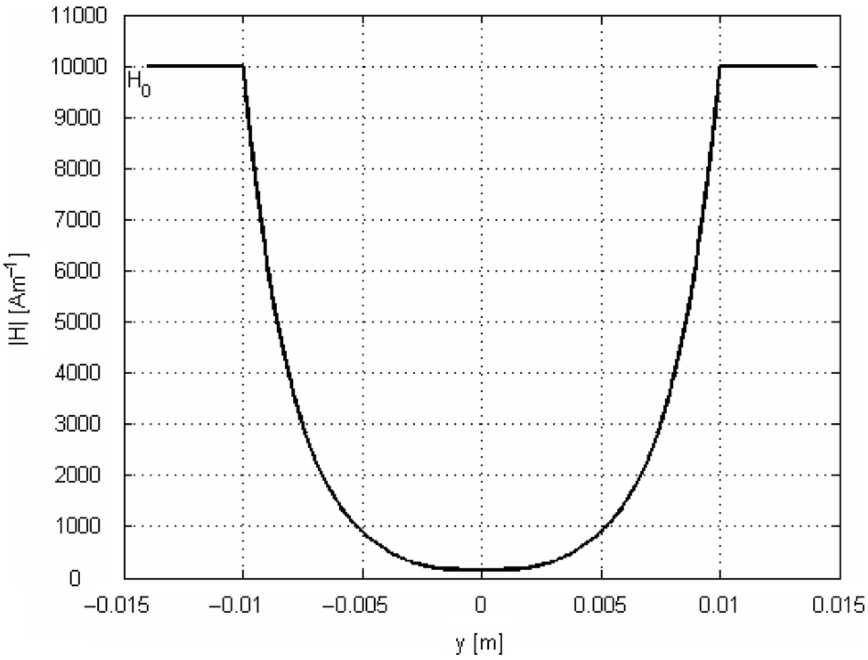

Fig. 5.5 Magnetic field in the conductor cross section

with $\beta^{-1} = \frac{1}{\sqrt{2}}\sqrt{\cos^2\frac{b}{2\delta} + \mathrm{sh}^2\frac{b}{2\delta}}$, $\delta = \sqrt{\frac{2}{\omega\sigma\mu_0}}$, $-\frac{b}{2} < y < \frac{b}{2}$ and $J(y) = 0$ elsewhere.

The maximum value J_m of the induced current density is

$$J_m = J\left(-\frac{b}{2}\right) = J\left(\frac{b}{2}\right) = \sqrt{2}\frac{|\overline{H}_0|}{\delta}\sqrt{\frac{\sin^2\frac{b}{2\delta} + \mathrm{sh}^2\frac{b}{2\delta}}{\cos^2\frac{b}{2\delta} + \mathrm{sh}^2\frac{b}{2\delta}}} \tag{5.132}$$

By assuming the previous data, the distribution of electric field $E(y) = \sigma^{-1}J(y)$ shown in Fig. 5.6 is obtained. When frequency f decreases, the maximum value of electric field decreases and its distribution tends to become linear. In turn, when f increases, the magnetic field decreases in the limits

$$|\overline{H}| \to 0, \quad -\frac{b}{2} < y < \frac{b}{2} \tag{5.133}$$

and

$$|\overline{H}| \to |\overline{H}_0|, \quad y = \pm\frac{b}{2} \tag{5.134}$$

when f tends to infinity.

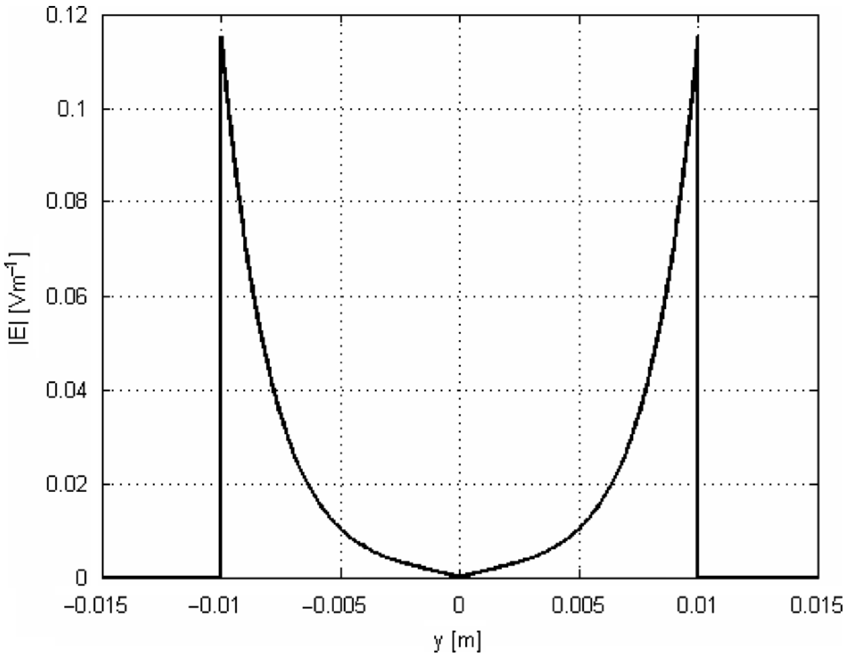

Fig. 5.6 Induced electric field in the conductor cross section (strong reaction)

Resorting to the definition of power loss given in Section 2.4.3, the surface power density $P(W\ m^{-2})$ dissipated in the conductor is

$$P = \frac{1}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{1}{\sigma}|\bar{J}(y)|^2 dy = \frac{H_0^2}{\sigma\delta^2\left(\cos^2\frac{b}{2\delta} + sh^2\frac{b}{2\delta}\right)}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left(\sin^2\frac{y}{\delta} + sh^2\frac{y}{\delta}\right)dy =$$

$$= \frac{H_0^2}{2\sigma\delta}\frac{sh\frac{b}{\delta} - \sin\frac{b}{\delta}}{\cos^2\frac{b}{2\delta} + sh^2\frac{b}{2\delta}} = \frac{H_0^2}{2\sigma\delta}\frac{sh\frac{b}{\delta} - \sin\frac{b}{\delta}}{\frac{\cos\frac{b}{\delta}+1}{2} + \frac{\left(e^{\frac{b}{2\delta}} - e^{-\frac{b}{2\delta}}\right)^2}{4}} =$$

$$= \frac{H_0^2}{\sigma\delta}\frac{sh\frac{b}{\delta} - \sin\frac{b}{\delta}}{\cos\frac{b}{\delta} + ch\frac{b}{\delta}} = H_0^2\sqrt{\frac{\omega\mu_0}{2\sigma}}\frac{sh\,kb - \sin kb}{ch\,kb + \cos kb} \tag{5.135}$$

where $k = \delta^{-1}$.

Alternatively, in the frequency domain, since $\bar{E}$ and $\bar{H}$ are orthogonal vectors, the phasor of the Poynting's vector (5.26) is

$$\bar{S} = \frac{\bar{E}\,\bar{H}^*}{2} = -\frac{\alpha H_0^2}{2\sigma}\frac{sh(\alpha y)ch^*(\alpha y)}{\left|ch\left(\frac{\alpha b}{2}\right)\right|^2} \tag{5.136}$$

It results

$$\mathrm{sh}(\alpha y)\mathrm{ch}^{*}(\alpha y) = \frac{\mathrm{sh}(2ky) + j\sin(2ky)}{2} \quad (5.137)$$

where the star denotes the complex conjugate and

$$\left|\mathrm{ch}\left(\frac{\alpha b}{2}\right)\right|^2 = \mathrm{sh}^2\left(\frac{kb}{2}\right) + \cos^2\left(\frac{kb}{2}\right) \quad (5.138)$$

respectively. After substitution, it comes out

$$\overline{S} = -\frac{kH_0^2}{4\sigma}\frac{[\mathrm{sh}\,(2ky) - \sin\,(2ky)] + j\,[\mathrm{sh}\,(2ky) + \sin\,(2ky)]}{\mathrm{sh}^2\left(\frac{kb}{2}\right) + \cos^2\left(\frac{kb}{2}\right)} \quad (5.139)$$

Therefore, the total power dissipated per unit section of the conductor is

$$P = \mathrm{Re}\left\{\overline{S}\left(-\frac{b}{2}\right)\right\} - \mathrm{Re}\left\{\overline{S}\left(\frac{b}{2}\right)\right\} = \frac{k\left|\overline{H}_0\right|^2}{2\sigma}\frac{\mathrm{sh}\,kb - \sin\,kb}{\mathrm{sh}^2\frac{kb}{2} + \cos^2\frac{kb}{2}}$$
$$= \left|\overline{H}_0\right|^2\sqrt{\frac{\omega\mu_0}{2\sigma}}\frac{\mathrm{sh}\,kb - \sin\,kb}{\mathrm{ch}\,kb + \cos\,kb} \quad (5.140)$$

coincident with (5.135).

5.11 Eddy Current in a Cylindrical Conductor under Step Excitation Current

The problem is that of searching for the current density distribution J(r, t) in a conductor of infinite length and circular cross-section of radius R carrying current i(t) defined as a step function: $i(t) = 0$ when $t < 0$ and $i(t) = I$ when $t \geq 0$.

According to (5.2) and (5.44), assuming cylindrical coordinates, the following equation holds

$$\frac{\partial^2 J}{\partial r^2} + \frac{1}{r}\frac{\partial J}{\partial r} - \mu\sigma\frac{\partial J}{\partial t} = 0, \quad 0 \leq r \leq R \quad (5.141)$$

subject to the boundary condition

$$\frac{\partial J}{\partial r} = 0, \quad r = 0 \quad (5.142)$$

the integral condition

$$2\pi\int_0^R J(r, t)r\,dr = I \quad (5.143)$$

and the initial condition

$$J(r, 0^+) = \frac{I}{2\pi R}\delta^-(r - R) \tag{5.144}$$

where $\delta^-(r - R) = \lim_{r_0 \to R^-} \delta(r - r_0)$. At $t = 0^+$ a step of current is applied; in the frequency domain, it corresponds to a vanishing penetration depth (see Section 5.10); accordingly, current I is concentrated at $r = R$. The laminar current density $J_S = J(r, 0^+)$ determines the magnetic field $\overline{H}$ such that $\overline{n} \times \overline{H} = \overline{J}_S$.

The solution to (5.141) can be obtained by means of the separation of variables

$$J(r, t) = J_m + \sum_{k=1}^{\infty} R_k(r) T_k(t), \quad 0 \le r \le R \tag{5.145}$$

Substituting (5.145) into (5.141) gives

$$\sum_{k=1}^{\infty} \frac{d^2 R_k}{dr^2} T_k + \frac{1}{r} \sum_{k=1}^{\infty} \frac{dR_k}{dr} T_k - \mu_0 \sigma \sum_{k=1}^{\infty} R_k \frac{dT_k}{dt} = 0 \tag{5.146}$$

Let one assume that for any $k \ge 1$

$$R_k'' T_k + \frac{1}{r} R_k' T_k - \mu_0 \sigma R_k T_k' = 0 \tag{5.147}$$

where $R_k'' \equiv \frac{d^2 R_k}{dr^2}$, $R_k' \equiv \frac{dR_k}{dr}$ and $T_k' \equiv \frac{dT_k}{dt}$ respectively.

After dividing each term of (5.147) by $R_k T_k$, it results

$$\frac{R_k''}{R_k} + \frac{1}{r} \frac{R_k'}{R_k} - \mu_0 \sigma \frac{T_k'}{T_k} = 0 \tag{5.148}$$

The latter transforms into the following pair of ordinary differential equations (see Section 3.3)

$$\mu_0 \sigma \frac{T_k'}{T_k} \equiv -\lambda_k^2 \tag{5.149}$$

$$\frac{R_k''}{R_k} + \frac{1}{r} \frac{R_k'}{R_k} \equiv -\lambda_k^2 \tag{5.150}$$

namely

$$T_k' + \frac{\lambda_k^2}{\mu_0 \sigma} T_k = 0 \tag{5.151}$$

$$R_k'' + \frac{1}{r} R_k' + \lambda_k^2 R_k = 0 \tag{5.152}$$

where $\lambda_k^2 \ne 0$ is the separation constant.

The solution to (5.151) is of the type

$$T_k(t) = \alpha_k e^{-\frac{\lambda_k^2 t}{\mu_0 \sigma}} \tag{5.153}$$

where α_k is a coefficient to be determined.

In turn, from (5.152) it results

$$R_k(r) = \beta_k J_0(\lambda_k r) + \gamma_k Y_0(\lambda_k r) \tag{5.154}$$

where β_k and γ_k are coefficients to be determined, while $J_0(\lambda_k r)$ and $Y_0(\lambda_k r)$ are the zero-order Bessel's functions of first and second kind, respectively. It is to be noted that $J_0(\lambda_k r)$ and $Y_0(\lambda_k r)$ tend to 1 and to minus infinity, respectively, when r approaches zero. Since at $r = 0$ current density should take a finite value at any time, constant γ_k must be zero. Therefore, it results

$$J(r,t) = J_m + \sum_{k=1}^{\infty} c_k J_0(\lambda_k r) e^{-\frac{\lambda_k^2 t}{\mu_0 \sigma}}, \quad 0 \le r \le R \tag{5.155}$$

where constants J_m, $c_k \equiv \alpha_k \beta_k$ and λ_k are to be determined by imposing boundary and initial conditions.

If R is very large and so the term $\frac{1}{r}\frac{\partial J}{\partial r}$ in (5.141) is neglected, a closed form of (5.155) can be determined in a straightforward way. In fact, under this assumption (5.152) and (5.154) become

$$R_k'' + \lambda_k^2 R_k = 0 \tag{5.156}$$

and

$$R_k(r) = a_k \cos(\lambda_k r) + b_k \sin(\lambda_k r) \tag{5.157}$$

respectively, where a_k and b_k are coefficients to be determined. As a consequence, (5.145) becomes

$$J(r,t) = J_m + \sum_{k=1}^{\infty} e^{-\frac{\lambda_k^2 t}{\mu_0 \sigma}} [a_k \cos(\lambda_k r) + b_k \sin(\lambda_k r)] \tag{5.158}$$

From (5.142) it follows that:

$$\lambda_k b_k e^{-\frac{\lambda_k^2 t}{\mu \sigma}} = 0 \tag{5.159}$$

Therefore, $b_k = 0$. In turn, from (5.143) and (5.158) it results

$$J_m \pi R^2 + \sum_{k=1}^{\infty} \frac{2\pi}{\lambda_k^2} d_k e^{-\frac{\lambda_k^2 t}{\mu_0 \sigma}} [\lambda_k R \sin(\lambda_k R) + \cos(\lambda_k R) - 1] = I \tag{5.160}$$

where $d_k \equiv a_k$; (5.160) is fulfilled if $J_m = \frac{I}{\pi R^2}$, which represents the impressed current density when $t \to \infty$, and if either $d_k = 0,\ k \geq 1$ or

$$[\lambda_k R \sin(\lambda_k R) + \cos(\lambda_k R) - 1] = 0, \quad k \geq 1 \tag{5.161}$$

Two solutions to (5.161) exist, namely $\lambda_k = \frac{2k\pi}{R}$ with $k \geq 1$ integer number and $\{\gamma_k\}$ such that $\lambda_k < \gamma_k < \lambda_{k+1}$ with $k \geq 1$ integer number.

Accordingly, a particular solution is

$$J(r, 0^+) = \frac{I}{2\pi R}\delta^-(r - R) \tag{5.162}$$

and

$$J(r, t) = \frac{I}{\pi R^2} + \sum_{k=1}^{\infty} d_k e^{-\frac{4k^2\pi^2 t}{\mu_0 \sigma R^2}} \cos\left(2k\pi\frac{r}{R}\right) \tag{5.163}$$

where coefficient d_k is determined for $r = 0$ by means of (5.144); this implies

$$\sum_{k=1}^{\infty} d_k = -\frac{I}{\pi R^2}, \quad r \in [0, R) \tag{5.164}$$

Considering the kth contribution to (5.163), it can be noted that the current carried by the round conductor is distributed sinusoidally in space and diffuses exponentially with a time constant $\tau_k = \frac{\mu_0 \sigma R^2}{4k^2\pi^2}$. At time t the penetration depth can be defined as $\delta_k = 2\pi\sqrt{\frac{t}{\mu_0\sigma}}$ and the minimum value of current density is located on the axis of the conductor; $t = 0^+$ is the critical instant, when a laminar current density J_S is originated at $r = R$ such that $\int_\Gamma J_S d\Gamma = \int_\Omega J_m d\Omega$ fulfilling condition (5.143).

In the insulating medium surrounding the conductor, the current density is zero, while the induced electric field $\overline{E} = E\overline{i}_z$ fulfils the equation

$$\frac{\partial E}{\partial r} = \frac{\mu_0 I}{2\pi r}\delta(t), \quad r > R \tag{5.165}$$

By integrating the latter with respect to r, the finite variation ΔE of the field between any position $r > R$ and the boundary $r = R$ of the conductor results

$$\Delta E = E(r) - E(R) = \frac{\mu_0 I}{2\pi} \ln\frac{r}{R}\delta(t), \quad r \geq R \tag{5.166}$$

The field is impulsive, i.e. $E = 0,\ t \neq 0$.

5.12 Electromagnetic Field Equations in Different Reference Frames

In free space, let us consider an inertial frame of reference $O = (x,\ y,\ z)$ in which the observer perceives an electric field of intensity $\overline{E}$ and a magnetic field of induction $\overline{B}$ at time t.

Let a second frame of reference $O' = (x', y', z')$ move at a constant velocity $\overline{u} = (u, 0, 0)$ with respect to O at time t'.

Lorentz's transformation of coordinates, in which any observer measures the same velocity c of light in the free space, i.e. defines the same wave equation, can be obtained in the following way. Let time t and time t' be initialised so that at $t = t' = 0$ the axes of the two frames are coincident, namely $x' = x,\ y' = y,\ z' = z$. Owing to symmetry, it can be stated that O' moves at a velocity u with respect to O and, conversely, O moves at a speed $-u$ with respect to O'. This implies

$$x' = \gamma(x - ut) \tag{5.167}$$

$$x = \gamma(x' + ut') \tag{5.168}$$

where γ is a dimensionless coefficient to be determined. To this end, let a light flash, originated at the origin of both systems at $t = t' = 0$, be considered. The light travels as a spherical wave in both frames with the same speed c; therefore, the equation of the wavefront is

$$x^2 + y^2 + z^2 = c^2t^2 \tag{5.169}$$

in frame O and

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \tag{5.170}$$

in frame O'. Since $y' = y$ and $z' = z$, it follows

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \tag{5.171}$$

and then

$$t'^2 = \frac{x'^2}{c^2} - \frac{x^2}{c^2} + t^2 \tag{5.172}$$

Replacing x' by means of (5.167) one obtains

$$t'^2 = \frac{\gamma^2 - 1}{c^2}x^2 - 2\frac{\gamma^2 u}{c^2}xt + \left(\frac{\gamma^2 u^2}{c^2} + 1\right)t^2 \tag{5.173}$$

Independently, taking t' from (5.168) and using (5.167) one has

$$t'^2 = \left(\frac{1 - \gamma^2}{\gamma u}\right)^2 x^2 - 2\frac{\gamma^2 - 1}{u}xt + \gamma^2 t^2 \tag{5.174}$$

By equating the corresponding coefficients of (5.173) and (5.174), it results

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{5.175}$$

and then

$$t' = \frac{t - \frac{u}{c^2}x}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{5.176}$$

Finally, the Lorentz's transformation from O to O′ results

$$x' = \gamma(x - ut);\; y' = y;\; z' = z;\; t' = \gamma\left(t - \frac{u}{c^2}x\right) \tag{5.177}$$

The inverse transformation can be obtained by changing the sign of velocity u, namely

$$x = \gamma(x' + ut');\; y = y';\; z = z;\; t = \gamma\left(t' + \frac{u}{c^2}x'\right) \tag{5.178}$$

Galilean transformation, on the contrary, is:

$$x' = x - ut;\; y' = y;\; z' = z;\; t' = t \tag{5.179}$$

and

$$x = x' + ut';\; y = y';\; z = z';\; t = t' \tag{5.180}$$

Using Lorentz's transformation, Maxwell's equations remain the same, if electric field intensity $\overline{E}$ transforms as follows:

$$E'_x = E_x;\; E'_y = \gamma\left(E_y - uB_z\right);\; E'_z = \gamma\left(E_z + uB_y\right) \tag{5.181}$$

If $u << c$, then $\gamma \cong 1$ and

$$E'_x = E_x;\; E'_y = E_y - uB_z;\; E'_z = E_z + uB_y \tag{5.182}$$

For vector $\overline{B}$, the transformation is

$$B'_x = B_x;\; B'_y = \gamma\left(B_y + u\mu_0\varepsilon_0 E_z\right);\; B'_z = \gamma\left(B_z + u\mu_0\varepsilon_0 E_y\right) \tag{5.183}$$

If $u << c$, then

$$B'_x = B_x;\; B'_y = B_y + u\mu_0\varepsilon_0 E_z;\; B'_z = B_z + u\mu_0\varepsilon_0 E_y \tag{5.184}$$

For the sake of some examples, let us first focus on a point charge q moving at a constant velocity $\overline{u}$ with $u << c$ in free space with respect to a fixed frame (Fig. 5.7).

If the observer moves together with the charge, he/she just observes

$$\overline{E}' = \frac{q}{4\pi\varepsilon_0 r^2}\bar{i}_r$$
$$\overline{B}' = 0 \tag{5.185}$$

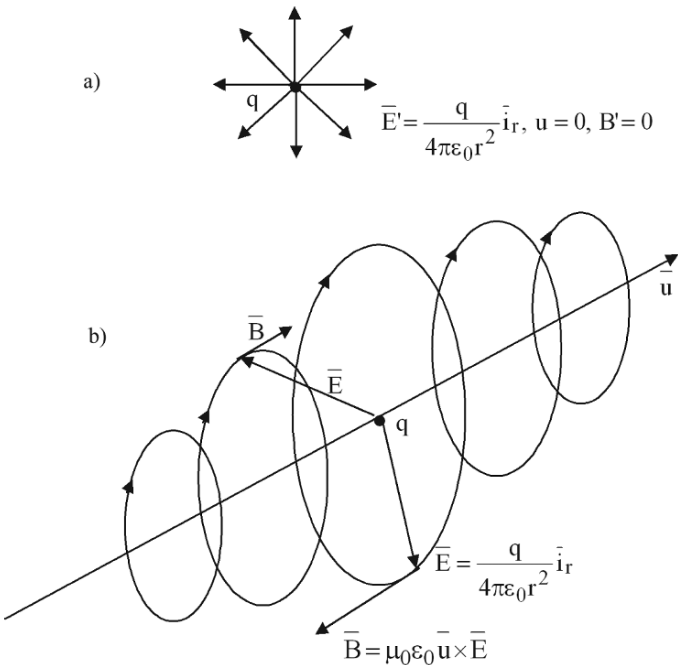

Fig. 5.7 Fields of a moving charge in different frames: (a) moving observer; (b) fixed observer

where r is the radial coordinate. An observer in the fixed frame sees

$$\overline{E} = \overline{E}'$$
$$\overline{B} = \mu_0 \varepsilon_0 \overline{u} \times \overline{E} \tag{5.186}$$

Therefore, it is reasonable to state that magnetism is a relativistic aspect of electricity; in other words, a magnetic field is given, if a relative motion between charge and observer is established.

In particular, (5.186) gives the field of a single travelling charge; the induction field can be expressed as

$$\overline{B} = \frac{\mu_0}{4\pi} \frac{q\overline{u} \times \overline{i_r}}{r^2} \tag{5.187}$$

If $nd\Omega$ travelling charges of value q are available in the elementary volume $d\Omega$, the elementary field is

$$d\overline{B} = \frac{\mu_0}{4\pi} \frac{nq\overline{u} \times \overline{i_r}}{r^2} d\Omega = \frac{\mu_0}{4\pi} \frac{\overline{J} \times \overline{i_r}}{r^2} d\Omega \tag{5.188}$$

where $\overline{J} = nq\overline{u}$ is the current density. If the direction of $\overline{u}$, and so $\overline{J}$, is coincident with the z axis, then (5.188) corresponds to (3.79) and by integration the Biot-Savart law follows (see 3.81).

As a second example, let a rectangular coil placed in a uniform induction field $\overline{B}$ orthogonal to it be considered (Fig. 5.8). It is assumed that one of the four sides of the coil is movable with a constant velocity $\overline{u}$ with u << c. While an observer at rest with respect to a fixed frame measures the magnetic field of induction $\overline{B}$ and no electric field, a second observer, located on the movable side, observes an electric field $\overline{E}$ parallel to the movable side, the magnitude of which is E = uB.

5.12.1 *A Relativistic Example: Steady Motion and Magnetic Diffusion*

Let a pair of plane and parallel electrodes be considered; they are supposed to have infinite extension in the x direction, along which a finite portion of width w is taken into account. The length of the electrodes in the z direction is finite and equal to λ while the distance between them is equal to d (Fig. 5.9). An external circuit forces a constant current I through a conductive strip, filling the region between the two electrodes, such that current lines are normal to the electrodes; assuming d << λ, end effects in the current distribution are neglected.

Current lines when the strip speed is zero are also shown.

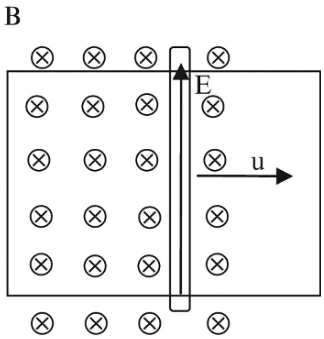

Fig. 5.8 Rectangular coil with a movable side in an induction field

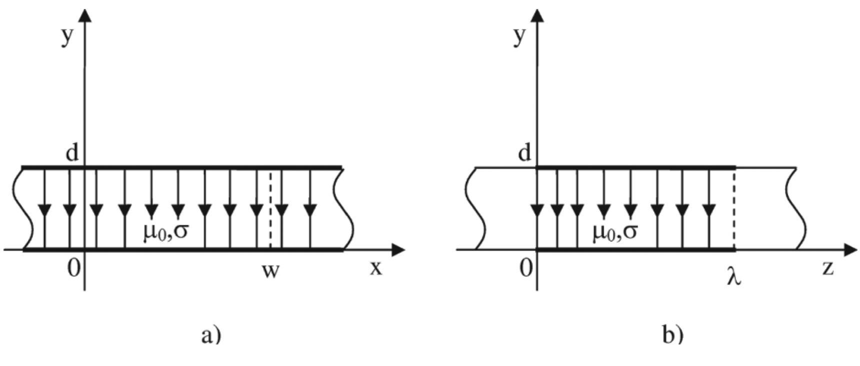

Fig. 5.9 Parallel electrodes with a conducting strip: (a) x − y cross-section, (b) z − y cross-section

The strip of infinite extension, which exhibits conductivity σ and permeability μ_0, is free to slide at a constant speed $\overline{u} = u\overline{i}_z$. After (5.45) and (5.11) the induction field in the strip is governed by the following equation

$$\mu_0\sigma\frac{\partial\overline{B}}{\partial t} - \overline{\nabla}^2\overline{B} = \mu_0\sigma\overline{\nabla} \times \left(\overline{u} \times \overline{B}\right) \tag{5.189}$$

subject to appropriate boundary and initial conditions; thanks to the assumptions made on the geometry of the electrodes, $\overline{J} = (0, J, 0)$ and $\overline{B} = (B, 0, 0)$. The problem can be tackled in either of two ways.

Steady State in the Fixed Frame

This viewpoint implies that

- the observer is at rest with respect to the electrodes
- the strip slides at a speed $\overline{u}$ with respect to the observer
- the field is steady, i.e. $\frac{\partial B}{\partial t} = 0$

The governing equation (5.189) reduces to

$$\frac{\partial^2 B}{\partial z^2} = \mu_0\sigma u\frac{\partial B}{\partial z} \tag{5.190}$$

As far as the boundary conditions are concerned, the following remark can be put forward. At $z = \lambda$, the Ampère's law gives $-H(0)w + H(\lambda)w = I$. Assuming that the field is zero at $z = 0$, it turns out to be

$$B(0) = 0, \quad B(\lambda) = \frac{\mu_0 I}{w} \tag{5.191}$$

Consequently, the solution to (5.190) is

$$B(z) = \sum_{i=1}^{2} k_i e^{\lambda_i z} \tag{5.192}$$

with λ_i such that $\lambda_i^2 - \mu_0\sigma u\lambda_i = 0$, giving $\lambda_1 = 0$ and $\lambda_2 = \mu_0\sigma u$, respectively. Applying boundary conditions (5.191), it follows

$$k_1 + k_2 = 0, \quad k_1 + k_2 e^{\mu_0\sigma u\lambda} = \frac{\mu_0 I}{w} \tag{5.193}$$

namely

$$k_1 = \frac{\mu_0 I}{w}\frac{1}{1 - e^{\mu_0\sigma u\lambda}}, \quad k_2 = -k_1 \tag{5.194}$$

Therefore, it results

$$B(z) = \frac{\mu_0 I}{w} \frac{1 - e^{\mu_0 \sigma u z}}{1 - e^{\mu_0 \sigma u \lambda}}, \quad u \neq 0, \quad 0 < z < \lambda \tag{5.195}$$

The distribution of flux lines is non-linear with z.

The associated current density $\overline{J} = \mu_0^{-1} \overline{\nabla} \times \overline{B}$ is

$$J(z) = \mu_0^{-1} \frac{\partial B}{\partial z} = \frac{\mu_0 I}{w} \frac{\sigma u e^{\mu_0 \sigma u z}}{e^{\mu_0 \sigma u \lambda} - 1}, \quad u \neq 0, \quad 0 \leq z \leq \lambda \tag{5.196}$$

which is non-uniform with z.

In the case u = 0 (strip at rest), after (5.195) and (5.196) it follows

$$B(z) = \frac{\mu_0 I}{w} \lim_{u \to 0} \frac{\mu_0 \sigma z e^{\mu_0 \sigma u z}}{\mu_0 \sigma \lambda e^{\mu_0 \sigma u \lambda}} = \frac{\mu_0 I}{w} \frac{z}{\lambda} \tag{5.197}$$

and

$$J(z) = \frac{\mu_0 I}{w} \lim_{u \to 0} \frac{\sigma e^{\mu_0 \sigma u z} + \mu_0 \sigma^2 u z e^{\mu_0 \sigma u z}}{\mu_0 \sigma \lambda e^{\mu_0 \sigma u \lambda}} = \frac{I}{w \lambda} \tag{5.198}$$

The distribution of flux lines is linear with z, while the current density is uniform.

Transient State in the Moving Frame

This viewpoint implies that the observer travels at the same speed as the field; therefore, in (5.189) u = 0. In order to have a non-uniform field in the strip equal to (5.195), an appropriate value of the time derivative of B should be prescribed; this way, a problem of transient magnetic diffusion is set up. In particular, comparing (5.189) with u = 0 and (5.190), it turns out to be

$$\mu_0 \sigma \frac{\partial B}{\partial t} - u \frac{\partial^2 B}{\partial z^2} = 0 \tag{5.199}$$

with

$$\frac{\partial B}{\partial t} = u \frac{\partial B}{\partial z} = \frac{\mu_0^2 I}{w} \frac{\sigma u^2 z e^{\mu_0 \sigma u z}}{e^{\mu_0 \sigma u \lambda} - 1} \tag{5.200}$$

It can be noted that the time derivative varies with coordinates and is constant in time. Accordingly, the initial condition is

$$B(z) = \frac{\mu_0 I}{w} \frac{z}{\lambda}, \quad t = 0 \tag{5.201}$$

Boundary conditions are the same as in the previous case.

5.12.2 Galileian and Lorentzian Transformations in Electromagnetism

It can be shown that equations of electromagnetism are invariant with respect to the Lorentzian transformation, not to the Galilean transformation. In particular, referring to the one-dimensional wave equation (see Section 5.6), it can be proven that if $\phi(x, t)$ fulfils the equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{5.202}$$

then $\phi(x', t')$ fulfils the equation

$$\frac{\partial^2 \phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t'^2} = 0 \tag{5.203}$$

where (x,t) is related to (x', t') through (5.177), in which u is now called v.

In fact, using the chain derivation rule with respect to x', one has

$$\frac{\partial \phi}{\partial x'} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial \phi}{\partial t}\frac{\partial t}{\partial x'} = \frac{\partial \phi}{\partial x}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \\ + \frac{\partial \phi}{\partial t}\frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \tag{5.204}$$

at the first order, and

$$\frac{\partial^2 \phi}{\partial x'^2} = \frac{\partial^2 \phi}{\partial x^2}\left(1 - \frac{v^2}{c^2}\right)^{-1} + \frac{\partial^2 \phi}{\partial x \partial t}\frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\frac{\partial x}{\partial x'} + \\ + \frac{\partial^2 \phi}{\partial t \partial x}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\frac{\partial t}{\partial x'} + \frac{\partial^2 \phi}{\partial t^2}\frac{v^2}{c^4}\left(1 - \frac{v^2}{c^2}\right)^{-1} \tag{5.205}$$

at the second order. Since $\frac{\partial^2}{\partial x \partial t} = \frac{\partial^2}{\partial t \partial x}$, it follows that

$$\frac{\partial^2 \phi}{\partial x'^2} = \frac{\partial^2 \phi}{\partial x^2}\left(1 - \frac{v^2}{c^2}\right)^{-1} + 2\frac{\partial^2 \phi}{\partial x \partial t}\frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1}\frac{\partial x}{\partial x'} + \\ + \frac{\partial^2 \phi}{\partial t^2}\frac{v^2}{c^4}\left(1 - \frac{v^2}{c^2}\right)^{-1} \tag{5.206}$$

In turn, by deriving with respect to t', one obtains

$$\frac{\partial \phi}{\partial t'} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial \phi}{\partial t}\frac{\partial t}{\partial t'} = \frac{\partial \phi}{\partial x}v\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \\ + \frac{\partial \phi}{\partial t}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \tag{5.207}$$

at the first order, and

$$\frac{\partial^2\phi}{\partial t'^2} = \frac{\partial^2\phi}{\partial x^2} v\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\frac{\partial x}{\partial t'} + \frac{\partial^2\phi}{\partial x\partial t}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\frac{\partial x}{\partial t'} + \\ + \frac{\partial^2\phi}{\partial t\partial x} v\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\frac{\partial t}{\partial t'} + \frac{\partial^2\phi}{\partial t^2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\frac{\partial t}{\partial t'} \tag{5.208}$$

at the second order. It follows that

$$\frac{\partial^2\phi}{\partial t'^2} = \frac{\partial^2\phi}{\partial x^2} v^2\left(1-\frac{v^2}{c^2}\right)^{-1} + 2\frac{\partial^2\phi}{\partial x\partial t} v\left(1-\frac{v^2}{c^2}\right)^{-1} + \\ + \frac{\partial^2\phi}{\partial t^2}\left(1-\frac{v^2}{c^2}\right)^{-1} \tag{5.209}$$

As a result, it turns out to be

$$\frac{\partial^2\phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} = \frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{5.210}$$

Conversely, using Galileian transformations (5.179) one has

$$\frac{\partial^2\phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} = \frac{\partial^2\phi}{\partial x^2}\frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} + 2\frac{\partial^2\phi}{\partial x\partial t}\frac{\frac{v}{c^2}-\frac{v}{c^2}}{1-\frac{v^2}{c^2}} + \\ + \frac{\partial^2\phi}{\partial t^2}\frac{\frac{v^2}{c^4}-\frac{1}{c^2}}{1-\frac{v^2}{c^2}} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} \tag{5.211}$$

It follows that

$$\frac{\partial^2\phi}{\partial x'^2} = \frac{\partial^2\phi}{\partial x^2} \tag{5.212}$$

and

$$\frac{\partial^2\phi}{\partial t'^2} = v^2\frac{\partial^2\phi}{\partial x^2} + 2v\frac{\partial^2\phi}{\partial x\partial t} + \frac{\partial^2\phi}{\partial t^2} \tag{5.213}$$

One obtains

$$\frac{\partial^2\phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} = \frac{\partial^2\phi}{\partial x^2} - \frac{v^2}{c^2}\frac{\partial^2\phi}{\partial x^2} - 2\frac{v}{c^2}\frac{\partial^2\phi}{\partial x\partial t} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \\ = \left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\phi}{\partial x^2} - 2\frac{v}{c^2}\frac{\partial^2\phi}{\partial x\partial t} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{5.214}$$

Finally, it results

$$\frac{\partial^2\phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} \neq \frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{5.215}$$