Chapter 3
Analytical Methods for Solving Boundary-Value Problems

3.1 Method of Green's Function

The potential of a unit source s, located in Q at a distance r from the field point P in an unbounded homogeneous domain Ω, is called Green's function and is given the symbol G'.

Let $\Omega_Q \subset \Omega$ be the subdomain represented by all source points and let $\Omega_P \subset \Omega$ be the subdomain of all field points, such that $\Omega_Q \cup \Omega_P = \Omega$ and $\Omega_Q \cap \Omega_P = 0$.

In a three-dimensional unbounded domain, after (2.48) for $s = \delta(\bar{r}_Q)$ one has

$$G'(\bar{r}_P, \bar{r}_Q) = \frac{1}{4\pi \left|\bar{r}_P - \bar{r}_Q\right|} = \frac{1}{4\pi r} \tag{3.1}$$

where $r = \left|\bar{r}_P - \bar{r}_Q\right|$ is the distance between source and field point (see Appendix, Fig. A1).

From (2.48) the potential fulfilling Poisson's equation $\nabla^2\phi = -s$ is

$$\phi(\bar{r}_P) = \int_\Omega G'(\bar{r}_P, \bar{r}_Q)\, s(\bar{r}_Q)\, d\Omega \tag{3.2}$$

Therefore, knowing G' and s, by means of (3.2) it is possible to calculate ϕ. The Green's function G' is called the fundamental solution of the Poisson's equation.

For a bounded domain with boundary Γ the modified Green's function G is the potential due to a unit source plus that, g, due to the unit source distributed along the boundary

$$G(\bar{r}_P, \bar{r}_Q) = \frac{1}{4\pi \left|\bar{r}_P - \bar{r}_Q\right|} + g(\bar{r}_P, \bar{r}_Q) \tag{3.3}$$

Knowing g, and so G, and substituting ψ with $4\pi G$ in the Green's formula (2.63), it is possible to calculate ϕ.

By definition, the Green's function is symmetrical, i.e. it is the same, exchanging the source and the field point $G'(\bar{r}_P, \bar{r}_Q) = G'(\bar{r}_Q, \bar{r}_P)$.

Following the same procedure, in a two-dimensional unbounded domain Ω the Green's function G' results

$$G'(\bar{r}_P, \bar{r}_Q) = \frac{1}{2\pi} \ln \left| \bar{r}_P - \bar{r}_Q \right| \tag{3.4}$$

3.1.1 Green's Formula for Electrostatics

In a homogeneous three-dimensional domain Ω with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and permittivity ε, from (2.63) taking $\phi = u$ and $\nabla^2\phi = \nabla^2 u = -\frac{\rho}{\varepsilon}$ and substituting ψ with $4\pi G$, one has

$$u(\bar{x}) = \int_{\Omega} G(\bar{x}, \bar{y}) \frac{\rho(\bar{y})}{\varepsilon} d\Omega + \int_{\Gamma_1} G(\bar{x}, \bar{y}) \frac{\partial u(\bar{y})}{\partial n} d\Gamma + \\ - \int_{\Gamma_2} u(\bar{y}) \frac{\partial G(\bar{x}, \bar{y})}{\partial n} d\Gamma \tag{3.5}$$

where $\bar{n}$ is the normal versor of Γ while the space vectors $\bar{x} \equiv \bar{r}_P$ and $\bar{y} \equiv \bar{r}_Q$ identify field and source point, respectively.

Formula (3.5) is the Green's formula for electrostatics. Using it to determine u, the actual problem is to know the modified Green's function G related to the given field domain Ω.

3.1.2 Green's Functions for Boundary-Value Problems

The particular case of Ω, denoted by $B(0, R)$, representing the region within a sphere ($n = 3$) or a circle ($n = 2$) is considered. The modified Green's function, related to the domain and to Poisson's equation

$$-\nabla^2 u = f \tag{3.6}$$

subject to Dirichlet's or Neumann's conditions, is to be found.

To this end, let the compound function v – called Kelvin's transformation of u – be considered. It is defined by the formula

$$v(\bar{x}) = |\bar{x}|^{n-2} R^{-n+2} u(\bar{a}(\bar{x})), \quad n = 2, 3 \tag{3.7}$$

where the transform $a(\bar{y})$, given by the formula

$$\bar{a}(\bar{y}) = R^2 |\bar{y}|^{-2} \bar{y}, \quad \bar{y} \in R^n \setminus \{0\} \tag{3.8}$$

leaves the boundary $\partial B(0, R)$ invariant. In fact, for $y = R$ it results

$$\bar{a}(\bar{y}) = \bar{y}, \; v(\bar{x}) = |\bar{x}|^{n-2} R^{-n+2} u(\bar{x}) \tag{3.9}$$

The equation satisfied by u is now found; for this task, the radial symmetry is exploited.

In the case n = 3 (sphere), using spherical coordinates (r, φ, ϑ), setting

$$\tilde{u}(r, \varphi, \vartheta) = u(r\cos\varphi\sin\vartheta,\ r\sin\varphi\sin\vartheta, r\cos\vartheta) \tag{3.10}$$

with $r \in (0, R)$; $\varphi \in (0, 2\pi)$, $\vartheta \in (0, \pi)$, and

$$\begin{aligned}\tilde{v}(r, \varphi, \vartheta) &= Rr^{-1}u\left(R^2r^{-1}\cos\varphi\sin\vartheta, R^2r^{-1}\sin\varphi\sin\vartheta, R^2r^{-1}\cos\vartheta\right) = \\ &= Rr^{-1}\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right)\end{aligned} \tag{3.11}$$

with $r \in (R, +\infty)$; $\varphi \in (0, 2\pi)$, $\vartheta \in (0, \pi)$, developing the Laplacian in spherical coordinates (see A.20)

$$\nabla^2 = \frac{1}{r^2}D_r(r^2D_r) + \frac{1}{r^2}D_\vartheta^2 + \frac{\cot(\vartheta)}{r^2}D_\vartheta + \frac{1}{r^2\sin^2\vartheta}D_\varphi^2 \tag{3.12}$$

where $D_r \equiv \frac{\partial}{\partial r}$, $D_\varphi \equiv \frac{\partial}{\partial\varphi}$, $D_\vartheta \equiv \frac{\partial}{\partial\vartheta}$, the following relations hold

$$\begin{aligned}r^2D_r\tilde{v}(r, \varphi, \vartheta) = &-R\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right) + \\ &- R^3r^{-1}D_r\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right)\end{aligned} \tag{3.13}$$

$$\begin{aligned}r^{-2}D_r\left(r^2D_r\tilde{v}\right)(r, \varphi, \vartheta) = &\ 2R^3r^{-4}D_r\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right) + \\ &+ R^5r^{-5}D_r^2\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right)\end{aligned} \tag{3.14}$$

$$D_\vartheta^j\tilde{v}\,(r, \varphi, \vartheta) = Rr^{-1}D_\vartheta^j\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right), \quad j = 1, 2 \tag{3.15}$$

$$D_\varphi^j\tilde{v}\,(r, \varphi, \vartheta) = Rr^{-1}D_\varphi^j\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right), \quad j = 1, 2 \tag{3.16}$$

Consequently, it can be deduced

$$\begin{aligned}\nabla^2\tilde{v}(r, \varphi, \vartheta) = &\ R^5r^{-5}\Big[D_r^2\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right) + 2R^{-2}rD_r\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right) + \\ &+ \left(Rr^{-2}\right)^{-2}D_\vartheta^2\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right)\Big] + \\ &+ R^5r^{-5}\Big[\cot\vartheta\left(Rr^{-2}\right)^{-2}D_\vartheta\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right) + \\ &+ \frac{1}{\sin^2\vartheta}\left(Rr^{-2}\right)^{-2}D_\varphi^2\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right)\Big] = \\ = &\ R^5r^{-5}\nabla^2\tilde{u}\left(R^2r^{-1}, \varphi, \vartheta\right) = \\ = &-R^5r^{-5}\tilde{f}\left(R^2r^{-1}, \varphi, \vartheta\right)\end{aligned} \tag{3.17}$$

with $r \in (R, +\infty)$, $\varphi \in (0, 2\pi)$, $\vartheta \in (0, \pi)$.

If ω represents the region outside the sphere B(0, R), (3.17) proves that function v satisfies the Dirichlet's boundary-value problem

$$\nabla^2 v(\overline{x}) = \hat{f}(\overline{x}), \quad \text{in} \quad \omega = \left\{ \overline{x} \in R^3 : |\overline{x}| > R \right\} \tag{3.18}$$

$$v(x) = g(\overline{x}), \quad \text{on} \quad \partial\omega = \left\{ \overline{x} \in R^3 : |\overline{x}| = R \right\} \tag{3.19}$$

with

$$\hat{f}(\overline{x}) = -R^5 |\overline{x}|^{-5} f\left(R^2 |\overline{x}|^{-2} \overline{x}\right), \quad \overline{x} \in \omega \tag{3.20}$$

and $g(\overline{x})$ given function of the boundary.

The case $n = 2$ (circular domain) is now considered. As in the three-dimensional case, the functions

$$\tilde{u}(r, \varphi) = u(r \cos\varphi, r \sin\varphi) \tag{3.21}$$

with $r \in (0, R)$, $\varphi \in (0, 2\pi)$, and

$$\tilde{v}(r, \varphi) = u\left(R^2 r^{-1} \cos\varphi, R^2 r^{-1} \sin\varphi\right) = \tilde{u}\left(R^2 r^{-1}, \varphi\right) \tag{3.22}$$

with $r \in (R, +\infty)$, $\varphi \in (0, 2\pi)$, are introduced. Developing the Laplacian in polar coordinates (see A.18), one has

$$\nabla^2 = D_r^2 + \frac{1}{r} D_r + \frac{1}{r^2} D_\varphi^2 \tag{3.23}$$

Accordingly, the following relations hold

$$D_r \tilde{v}(r, \varphi) = -R^2 r^{-2} D_r \tilde{u}\left(R^2 r^{-1}, \varphi\right) \tag{3.24}$$

$$D_r^2 \tilde{v}(r, \varphi) = R^4 r^{-4} D_r^2 \tilde{u}\left(R^2 r^{-1}, \varphi\right) + 2R^2 r^{-3} D_r \tilde{u}\left(R^2 r^{-1}, \varphi\right) \tag{3.25}$$

$$D_\varphi^j \tilde{v}(r, \varphi) = D_\varphi^j \tilde{u}\left(R^2 r^{-1}, \varphi\right), \quad j = 1, 2 \tag{3.26}$$

Consequently, it results

$$\begin{aligned} \nabla^2 \tilde{v}\,(r, \varphi) = R^4 r^{-4} \Big[& D_r^2 \tilde{u}\left(R^2 r^{-1}, \varphi\right) + \left(R^2 r^{-1}\right)^2 D_r \tilde{u}\left(R^2 r^{-1}, \varphi\right) + \\ & + \left(R r^{-2}\right)^{-2} D_\varphi^2 \tilde{u}\left(R^2 r^{-1}, \varphi\right) \Big] = \\ = {} & R^4 r^{-4} \nabla^2 u\left(R^2 r^{-1}, \varphi\right) = R^4 r^{-4} \tilde{f}\left(R^2 r^{-1}, \varphi\right), \quad r \in (R, +\infty) \end{aligned} \tag{3.27}$$

Therefore, it has been shown that function v fulfils the Dirichlet's boundary-value problem

$$\nabla^2 v(\overline{x}) = \hat{f}(\overline{x}), \text{ in } \quad \omega = \left\{\overline{x} \in R^2 : |\overline{x}| > R\right\} \tag{3.28}$$

$$v(\overline{x}) = g(\overline{x}), \text{ on } \quad \partial\omega = \left\{\overline{x} \in R^2 : |\overline{x}| = R\right\} \tag{3.29}$$

with

$$\hat{f}(x) = R^4 |x|^{-4} f\left(R^2 |x|^{-1} x\right), \quad x \in \omega \tag{3.30}$$

and $g(\overline{x})$ given function of the boundary.

It is now possible to determine the Green's function G_D related to the Dirichlet's condition when $n = 2$ and $n = 3$. It is given by the formula

$$G_D(\overline{x}, \overline{y}) = G'(\overline{x} - \overline{y}) - G'\left(R^{-1} |\overline{y}| (\overline{x} - \overline{a}(\overline{y}))\right), \quad \overline{x}, \overline{y} \in B(0, R) \tag{3.31}$$

where

$$\overline{a}(\overline{y}) = R^2 |\overline{y}|^{-2} \overline{y}, \quad \overline{y} \in R^n \setminus \{0\}, \quad n = 2, 3 \tag{3.32}$$

and G' stands for the fundamental solution of the Laplacian operator, i.e.

$$G'(\overline{x}) = (2\pi)^{-1} \ln |\overline{x}|, \text{ if } n = 2 \tag{3.33}$$

and

$$G'(\overline{x}) = (4\pi)^{-1} |\overline{x}|^{-1}, \text{ if } n = 3 \tag{3.34}$$

assuming that the unit source is located at $y = 0$.

It is necessary to show that

$$G_D(\overline{x}, \overline{y}) = 0, \quad \overline{x} \in \partial B(0, R), \ \ \overline{y} \in B(0, R) \tag{3.35}$$

To this purpose, the identity

$$|\overline{x} - \overline{y}|^2 = |\overline{x}|^2 + |\overline{y}|^2 - 2\overline{x} \cdot \overline{y} \tag{3.36}$$

holding for any pair of vectors $\overline{x}, \overline{y} \in R^n$, is used. In particular, for any $\overline{x} \in \partial B(0, R)$, i.e. $|\overline{x}| = R$, and for any $\overline{y} \in B(0, R)$ it turns out to be

$$\begin{aligned} R^{-2} |\overline{y}|^2 |\overline{x} - \overline{a}(\overline{y})|^2 &= R^{-2} |\overline{y}|^2 \left[|\overline{x}|^2 + |\overline{a}(\overline{y})|^2 - 2\overline{x} \cdot \overline{a}(\overline{y})\right] = \\ &= R^{-2} |\overline{y}|^2 \left[R^2 + R^4 |\overline{y}|^{-2} - 2R^2 |\overline{y}|^{-2} \overline{x} \cdot \overline{y}\right] = \\ &= |\overline{y}|^2 + R^2 - 2\overline{x} \cdot \overline{y} = |\overline{y}|^2 + |\overline{x}|^2 - 2\overline{x} \cdot \overline{y} = |\overline{x} - \overline{y}|^2 \end{aligned} \tag{3.37}$$

implying that (3.36) holds.

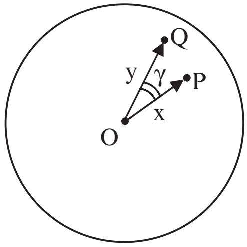

Fig. 3.1 Inner spherical domain Ω

Now, the Green's function related to a sphere B(0, R) when $n = 3$ can be determined. It is given by the formula:

$$G_D(\overline{x}, \overline{y}) = \frac{1}{4\pi\left(r^2 - 2r\rho\overline{u}_x \cdot \overline{u}_y + \rho^2\right)^{\frac{1}{2}}} + \\ - \frac{R}{4\pi\left(r^2\rho^2 - 2rR^2\rho\overline{u}_x \cdot \overline{u}_y + R^4\right)^{\frac{1}{2}}} \tag{3.38}$$

where $\overline{x} = r\overline{u}_x$ and $\overline{y} = \rho\overline{u}_y$. In other words, r and ρ are the Euclidean norms of vectors $\overline{x}$ and $\overline{y}$, respectively, while $\overline{u}_x$ and $\overline{u}_y$ are the unit vectors in the directions of $\overline{x}$ and $\overline{y}$, respectively. The situation is represented in Fig. 3.1, where $\gamma = \cos^{-1}\left(\overline{u}_x \cdot \overline{u}_y\right)$

Comparing (3.3) and (3.38), it follows

$$g(\overline{x}, \overline{y}) = -\frac{R}{4\pi\left(r^2\rho^2 - 2rR^2\rho\overline{u}_x \cdot \overline{u}_y + R^4\right)^{\frac{1}{2}}} \tag{3.39}$$

Moreover, from (3.38), the following relation can be deduced:

$$\begin{aligned} D_{n(\overline{y})}G_D(\overline{x}, \overline{y}) &= D_\rho\left[G_D\left(r\overline{u}_x, \rho\overline{u}_y\right)\right] \\ &= \frac{\rho - r\overline{u}_x \cdot \overline{u}_y}{4\pi\left(r^2 - 2r\rho\overline{u}_x \cdot \overline{u}_y + \rho^2\right)^{\frac{3}{2}}} + \\ &\quad + \frac{R\left(r^2\rho - rR^2\overline{u}_x \cdot \overline{u}_y\right)}{4\pi\left(r^2\rho^2 - 2rR^2\rho\overline{u}_x \cdot \overline{u}_y + R^4\right)^{\frac{3}{2}}}, \quad \overline{x}, \overline{y} \in B(0, R) \end{aligned} \tag{3.40}$$

where the operator $D_n(\overline{y})$ stands for the normal derivative along the direction of vector $\overline{y}$; in particular, the relation

$$\begin{aligned} D_\rho\left[G_D\left(r\overline{u}_x, R\overline{u}_y\right)\right] &= -\frac{R - r\overline{u}_x \cdot \overline{u}_y}{4\pi\left(r^2 - 2rR\overline{u}_x \cdot \overline{u}_y + R^2\right)^{\frac{3}{2}}} + \\ &\quad + \frac{r\left(r - R\overline{u}_x \cdot \overline{u}_y\right)}{4R\pi\left(r^2 - 2rR\overline{u}_x \cdot \overline{u}_y + R^2\right)^{\frac{3}{2}}} = \\ &= \frac{r^2 - R^2}{4R\pi\left(r^2 - 2rR\overline{u}_x \cdot \overline{u}_y + R^2\right)^{\frac{3}{2}}} \end{aligned} \tag{3.41}$$

with $\overline{x} \in B(0,R)$, $\overline{y} \in \partial B(0,R)$, can be obtained. Consequently, the following representation holds:

$$D_{n(\overline{y})}G_D(x,y) = \frac{|\overline{x}|^2 - R^2}{4R\pi\,|\overline{x}-\overline{y}|^3}, \quad \overline{x} \in \partial B(0,R), \quad \overline{y} \in B(0,R) \tag{3.42}$$

Therefore, it can be concluded that the solution to the Dirichlet's problem in $\Omega = B(0,R)$

$$-\nabla^2 u(\overline{x}) = f(\overline{x}) \tag{3.43}$$

with $u(\overline{x}) = g(\overline{x})$ on $\Gamma = \partial B(0,R)$, according to (3.5) is given by the formula

$$u(\overline{x}) = \int_\Omega G_D(\overline{x},\overline{y}) f(\overline{y})\, d\Omega - \int_\Gamma g(\overline{y}) D_{n(\overline{y})} G_D(\overline{x},\overline{y})\, d\Gamma \tag{3.44}$$

where the functions G_D and $D_n(\overline{y})G_D$ have been explicitly computed.

In the case $n = 2$, the Green's function is given by the formula

$$G_D(\overline{x},\overline{y}) = \frac{1}{4\pi}\Big\{\ln(\rho) - \ln(R) + \ln\left(r^2 - 2r\rho\overline{u}_x\cdot\overline{u}_y + \rho^2\right) + \\ - \log\left(r^2\rho^2 - 2rR^2\rho\overline{u}_x\cdot\overline{u}_y + R^4\right)\Big\} \tag{3.45}$$

Then, one gets

$$2\pi D_{n(\overline{y})} G_D(\overline{x},\overline{y}) = 2\pi D_\rho\left[G_D\left(r\overline{u}_x, \rho\overline{u}_y\right)\right] = \\ = \frac{1}{\rho} + \frac{\rho - r\overline{u}_x\cdot\overline{u}_y}{r^2 - 2r\rho\overline{u}_x\cdot\overline{u}_y + \rho^2} - \frac{r^2\rho - R^2 r\overline{u}_x\cdot\overline{u}_y}{r^2\rho^2 - 2rR^2\rho\overline{u}_x\cdot\overline{u}_y + R^4} \tag{3.46}$$

and

$$2\pi D_{n(\overline{y})} G_D\left(\overline{x}, R\overline{u}_y\right) = \frac{1}{R} + \frac{R - r\overline{u}_x\cdot\overline{u}_y}{r^2 - 2Rr\overline{u}_x\cdot\overline{u}_y + R^2} - \frac{r^2 - Rr\overline{u}_x\cdot\overline{u}_y}{R\left(r^2 - 2Rr\overline{u}_x\cdot\overline{u}_y + R^2\right)} = \\ = \frac{1}{R} + \frac{R^2 - r^2}{R\left(r^2 - 2Rr\overline{u}_x\cdot\overline{u}_y + R^2\right)} \tag{3.47}$$

In turn, this gives

$$D_{n(\overline{y})} G_D(\overline{x},\overline{y}) = \frac{1}{2\pi R} + \frac{R^2 - r^2}{2\pi R\left(r^2 - 2Rr\overline{u}_x\cdot\overline{u}_y + R^2\right)} \tag{3.48}$$

with $\overline{x} \in B(0,R)$, $\overline{y} \in \partial B(0,R)$.

Therefore, it can be concluded that, according to (3.5), the solution to the Dirichlet's problem

$$-\nabla^2 u(\overline{x}) = f(\overline{x}) \text{ in } \Omega = B(0,R), \; u(\overline{x}) = g(\overline{x}) \text{ on } \Gamma = \partial B(0,R) \tag{3.49}$$

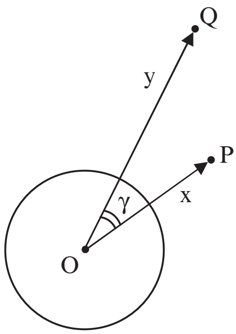

Fig. 3.2 Outer spherical domain ω

is given by the formula

$$u(\overline{x}) = \int_{\Omega} G_D(\overline{x}, \overline{y}) f(\overline{y})\, d\Omega - \int_{\Gamma} g(\overline{y}) D_{n(\overline{y})} G_D(\overline{x}, \overline{y})\, d\Gamma \tag{3.50}$$

Finally the solution to the Dirichlet's problem for the domain ω and its boundary ∂ω outside the domain Ω can now be obtained (Fig. 3.2).

Using (3.43) and (3.20) the representation for the solution v to the problem (3.18)–(3.19) can be deduced from the solution u to the Dirichlet's problem for the sphere, where $B(0, R) = \Omega$ and $B(0, R)^c = \omega$:

$$-\nabla^2 u(\overline{x}) = f(\overline{x}) \text{ in } B(0, R),\ \ u(\overline{x}) = g(\overline{x}) \text{ on } \partial B(0, R) \tag{3.51}$$

where f is defined through the equation

$$\hat{f}(\overline{x}) = -R^5 |\overline{x}|^{-5} f\left(R^2 |\overline{x}|^{-2} \overline{x}\right), \quad \overline{x} \in \omega \tag{3.52}$$

The solution to the equation $R^2|\overline{x}|^{-2}\overline{x} = \overline{y},\ \overline{x} \neq \{0\}$ is $\overline{x} = R^2|\overline{y}|^{-2}\overline{y}$. Indeed, if a solution $\overline{x}$ exists, then $|\overline{x}| = R^2|\overline{y}|^{-1}$, so that $\overline{x} = R^{-2}|\overline{x}|^2\overline{y} = R^2|\overline{y}|^{-2}\overline{y}$. Of course, it is easy to check that $\overline{x} = R^2\, |\overline{y}|^{-2}\, \overline{y}$ solves the given equation. In other words, the inverse transform coincides with the transform itself. Consequently, it is easy to verify that f can be expressed in terms of $\hat{f}$ by the formula

$$f(\overline{y}) = -R^5 |\overline{y}|^{-5} \hat{f}\left(R^2 |\overline{y}|^{-2} \overline{y}\right),\ \overline{y} \in \omega \tag{3.53}$$

Then u can be expressed as follows

$$\begin{aligned} u(\overline{x}) &= R^5 \int_{B(0,R)} G_D(\overline{x}, \overline{y})\, |\overline{y}|^{-5}\, \hat{f}\left(R^2\, |\overline{y}|^{-2}\, \overline{y}\right) d\omega(\overline{y}) + \\ &+ \int_{\partial B(0,R)} g(\overline{y}) D_{n(\overline{y})} G_D(\overline{x}, \overline{y})\, d\sigma(\overline{y}) = \\ &= u_1(\overline{x}) + \int_{\partial B(0,R)} g(\overline{y}) D_{n(\overline{y})} G_D(\overline{x}, \overline{y})\, d\sigma(\overline{y}) \end{aligned} \tag{3.54}$$

A change of variable in the volume integral defining u_1 is convenient. To this purpose, the Jacobian $J(\overline{\eta})$ of the transformation $\overline{y} = R^2 |\overline{\eta}|^{-2} \overline{\eta}$ is to be computed. Assuming

$$y_k = R^2|\overline{\eta}|^{-2}\eta_k, \ D_{\eta_j} y_k = -2R^2|\overline{\eta}|^{-4}\eta_j\eta_k + R^2|\overline{\eta}|^{-2}\delta_{j,k} \tag{3.55}$$

with $j, k = 1, \ldots, n, \ n = 2, 3$, where $\delta_{j,k} = (-1)^{j+k}$ is the Kronecker's index, it turns out to be

$$J(\overline{\eta}) = 2R^2|\overline{\eta}|^{-10} \tag{3.56}$$

whence the formula

$$u_1(\overline{x}) = \frac{1}{2}R^5 \int_{B(0,R)} G_D(\overline{x}, \overline{y}) \, |\overline{y}|^{-5} \, \hat{f}\left(R^2 \, |\overline{y}|^{-2} \, \overline{y}\right) d\omega(\overline{y}) =$$
$$= R^{-5} \int_{B(0,R)^c} G_D\left(\overline{x}, R^2|\overline{\eta}|^{-2}\overline{\eta}\right) |\overline{\eta}|^5 \hat{f}(\overline{\eta}) d\omega(\overline{\eta}) \tag{3.57}$$

is obtained. Since the solution v to the problem (3.28) and (3.29) is related to u by the formula $v(\overline{\xi}) = u\left(R^2|\overline{\xi}|^{-2}\overline{\xi}\right)$, from (3.57) the desired representation

$$v\left(\overline{\xi}\right) = \frac{1}{2}R^{-5} \int_{B(0,R)^c} G_D\left(R^2 \left|\overline{\xi}\right|^{-2} \overline{\xi}, R^2 \, |\overline{\eta}|^{-2} \, \overline{\eta}\right) |\overline{\eta}|^5 \, \hat{f}(\overline{\eta}) \, d\omega(\overline{\eta}) +$$
$$+ \int_{\partial B(0,R)} g(\overline{\eta}) \, D_{n(\overline{\eta})} G_D\left(R^2 \left|\overline{\xi}\right|^{-2} \overline{\xi}, \overline{\eta}\right) d\sigma(\overline{\eta}) \tag{3.58}$$

is derived.

Finally, in the two-dimensional case one has

$$f(\overline{y}) = R^4 \, |\overline{y}|^{-4} \, \hat{f}\left(R^2 \, |\overline{y}|^{-1} \, \overline{y}\right), \ \overline{y} \in B(0, R) \tag{3.59}$$

and

$$J(\overline{\eta}) = R^2 \, |\overline{\eta}|^{-4} \tag{3.60}$$

It follows

$$u_1(\overline{x}) = R^2 \int_{B(0,R)} G_D(\overline{x}, \overline{y}) \, |\overline{y}|^{-4} \, \hat{f}\left(R^2 \, |\overline{y}|^{-2} \, \overline{y}\right) d\omega(\overline{y}) =$$
$$= R^{-4} \int_{B(0,R)^c} G_D\left(\overline{x}, R^2 \, |\overline{\eta}|^{-2} \, \overline{\eta}\right) |\overline{\eta}|^4 \, \hat{f}(\overline{\eta}) \, d\omega(\overline{\eta}) \tag{3.61}$$

and

$$v\left(\overline{\xi}\right) = R^{-4} \int_{B(0,R)^c} G_D\left(R^2 \left|\overline{\xi}\right|^{-2} \overline{\xi}, R^2 \, |\overline{\eta}|^{-2} \, \overline{\eta}\right) |\overline{\eta}|^4 \, \hat{f}(\overline{\eta}) d\omega(\overline{\eta}) +$$
$$+ \int_{\partial B(0,R)} g(\overline{\eta}) D_{n(\overline{\eta})} G_D\left(R^2 \left|\overline{\xi}\right|^{-2} \overline{\xi}, \overline{\eta}\right) d\sigma(\overline{\eta}) \tag{3.62}$$

respectively.

3.1.3 Field of a Point Charge Surrounded by a Spherical Surface at Known Potential

The source charge $+q$ gives rise to an induced charge $-q$ on the sphere; in addition to it, the free charge on the sphere, originating the potential $U = k$, is $q' = 4\pi\varepsilon R k$ (see 2.119).

From Gauss's theorem the field turns out to be

$$E(r) = \frac{q}{4\pi\varepsilon r^2}, \quad 0 < r < R; \quad E(r) = \frac{Rk}{r^2}, \quad r > R \tag{3.63}$$

Independently, the problem can be solved using the method of Green's function. When $f(\overline{y}) = \frac{q}{\varepsilon}\delta(\overline{y})$ and $|\overline{x}| < R$, from (3.44) with $g(\overline{y}) = k$, it turns out to be

$$U(\overline{x}) = \int_\Omega \frac{q}{\varepsilon} G_D(\overline{x}, \overline{y})\delta(\overline{y})d\Omega - \int_\Gamma k D_{n(\overline{y})} G_D(\overline{x}, \overline{y})d\Gamma \tag{3.64}$$

After (3.38) and (3.41) one has

$$\int_\Omega \frac{q}{\varepsilon} G_D(\overline{x}, \overline{y})\delta(\overline{y})d\Omega = \frac{q}{\varepsilon} G_D(\overline{x}, 0) = \frac{q}{\varepsilon}\left(\frac{1}{4\pi r} - \frac{1}{4\pi R}\right), \quad r < R \tag{3.65}$$

and

$$\int_\Gamma k D_{n(\overline{y})} G_D(\overline{x}, \overline{y})d\Gamma = k\int_\Gamma \frac{|\overline{x}|^2 - R^2}{4\pi R\left|\overline{x} - R\overline{u}_y\right|^3}d\Gamma = -k \tag{3.66}$$

respectively.

It can be verified that $U(r) = k$ solves the particular case of $q = 0$, for which

$$\nabla^2 U(r) = D_r^2 U(r) + \frac{2}{r} D_r U(r) = 0, \quad U(R) = k \tag{3.67}$$

holds.

The potential is given by

$$U(r) = \frac{q}{\varepsilon}\left(\frac{1}{4\pi r} - \frac{1}{4\pi R}\right) + k, \quad 0 < r < R \tag{3.68}$$

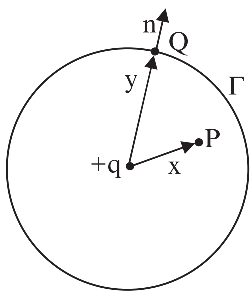

Fig. 3.3 Point charge surrounded by a sphere at potential k

whence (3.63) follows.

In turn, when $R < |\overline{x}|$, from (3.58) with $\hat{f}(\overline{\eta}) = 0$ and $g(\overline{\eta}) = k$ one has

$$v(\overline{\xi}) = \int_{\Gamma} kD_{n(\overline{\eta})}G_D\left(R^2\left|\overline{\xi}\right|^{-2}\overline{\xi}, \overline{\eta}\right)d\Gamma = \frac{Rk}{r}, \quad r > R \tag{3.69}$$

Again, it can be verified that u(r) solves the particular case of $q = 0$, for which

$$\nabla^2 U(r) = D_r^2 U(r) + \frac{2}{r} D_r U(r) = 0, \quad U(R) = k, \quad U(\infty) = 0 \tag{3.70}$$

holds.

Potential and field are represented in Figs. 3.4 and 3.5, respectively.

At $r = R$, U is continuous for any k, while E is not if $k \neq \frac{q}{4\pi\varepsilon R}$. The particular cases of a grounded sphere and a supplied sphere follow, when $k = 0$ with $q \neq 0$ and $k \neq 0$ with $q = 0$, respectively.

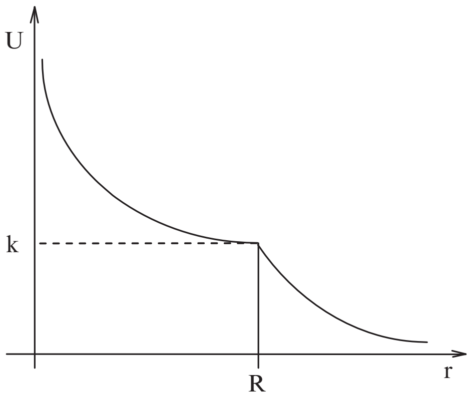

Fig. 3.4 Potential vs position

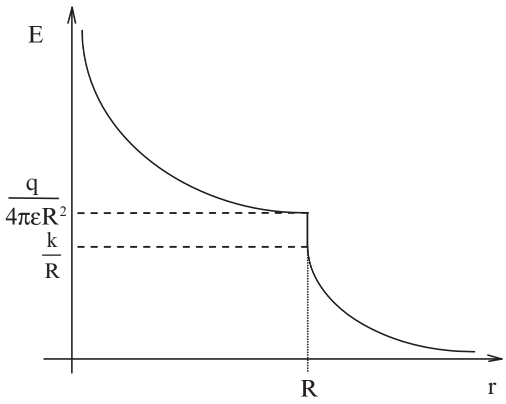

Fig. 3.5 Field vs position ($q > 4\pi\varepsilon Rk$)

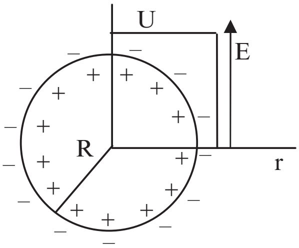

Fig. 3.6 Surface dipole distribution on a sphere

3.1.4 Field of a Surface Dipole Distributed on a Sphere of Radius R

Let a double-layer distribution of charge be considered, characterized by a uniform dipole density $\bar{\tau} = \frac{q}{4\pi R}\bar{i}_r$ (Fig. 3.6). According to (2.124), the potential of a single dipole of moment $\bar{p}$ is given by $\frac{\bar{p}\cdot\bar{i}_r}{4\pi\varepsilon r^2}$, $r \neq 0$; summing elementary contributions, the potential due to the surface dipole distribution is

$$U(r) = \int_\Gamma \frac{\bar{\tau}\cdot\bar{i}_r}{4\pi\varepsilon r^2}\, d\Gamma = \frac{\tau}{4\pi\varepsilon R^2}\int_\Gamma d\Gamma = \frac{\tau}{\varepsilon} = \frac{q}{4\pi\varepsilon R}, \quad 0 < r < R \tag{3.71}$$

The latter holds when the field point P is within the sphere; when P crosses the sphere, the solid angle subtended by the surface has a discontinuity equal to 4π. Consequently, the potential has a discontinuity equal to $\frac{\tau}{\varepsilon}$ determining $U(r) = 0,\ r > R$.

Remarks

The following remark can be put forward. Denoting by $\sigma = \varepsilon\frac{\partial U}{\partial n}$ the charge density ($C\ m^{-2}$) on Γ and with $\tau = \varepsilon U$ the dipole density (($C\ m^{-1}$), see Section 2.2.6) on Γ, respectively, Green's formula (3.5) can be also expressed as

$$U(\bar{x},\bar{y}) = \int_\Omega \frac{\rho(\bar{y})}{\varepsilon}G(\bar{x},\bar{y})\, d\Omega + \int_{\Gamma_1}\frac{\sigma(\bar{y})}{\varepsilon}G(\bar{x},\bar{y})\, d\Gamma + \\ -\int_{\Gamma_2}\frac{\tau(\bar{y})}{\varepsilon}\frac{\partial G(\bar{x},\bar{y})}{\partial n}\, d\Gamma \tag{3.72}$$

Hence, the electrostatic potential U in a domain Ω bounded by Γ is known, knowing G' and ρ in the domain, σ and τ on the boundary. The three terms are called volume term, single-layer term and double-layer term, respectively.

In the case of the surface dipole distributed on a sphere, using (3.72) with $\rho = 0,\ \sigma = 0$ and taking $G = \frac{1}{4\pi r}$ and $U = \frac{\tau}{\varepsilon}$, the potential results

$$U(r) = -\int_\Gamma \frac{\tau}{\varepsilon}\frac{\partial G}{\partial n} d\Gamma = \int_\Gamma \frac{\tau}{\varepsilon}\frac{1}{4\pi r^2} d\Gamma =$$
$$= \frac{q}{4\pi\varepsilon R}\frac{1}{4\pi R^2}\int_\Gamma d\Gamma = \frac{q}{4\pi\varepsilon R}, \quad 0 < r < R \tag{3.73}$$

with $U(r) = 0, \ r > R$.

It can be noted that, at $r = R$, both U and E are not continuous (Fig. 3.6); in particular, the field is singular and can be expressed as $\overline{E}(r) = \frac{q}{4\pi\varepsilon R}\delta(r - R)\overline{i}_r$.

As a final example, let a surface distribution of charges and dipoles be identified, such that the field external to the sphere of radius R is zero, while the inner field is that due to a point charge $+q$ located at the centre.

To this end, forcing a uniform charge distribution of density $\sigma = -\frac{q}{4\pi R^2}$ on the sphere, the relevant potential is $U_\sigma(r) = -\frac{q}{4\pi\varepsilon R}, \ 0 < r < R$ and $U_\sigma(r) = -\frac{q}{4\pi\varepsilon r}$, $R < r$.

Then, adding a surface dipole distribution of density $\tau = \frac{q}{4\pi R}$ the contributions $U_\tau(r) = \frac{q}{4\pi\varepsilon R}, \ 0 < r < R$ and $U_\tau(r) = 0, \ R < r$ to the potential originate.

Since the potential due to the point charge is $U_0(r) = \frac{q}{4\pi\varepsilon r}, \ r \neq 0$, summing the three terms above, one obtains $U(r) = \frac{q}{4\pi\varepsilon r}, \ 0 < r < R$ and $U(r) = 0, \ R < r$.

3.1.5 Green's Formula for Two-Dimensional Magnetostatics

The formula of vector potential A corresponding to (3.5) is

$$A = \int_\Omega G\mu J \, d\Omega + \int_{\Gamma_1} G\frac{\partial A}{\partial n} d\Gamma - \int_{\Gamma_2} A\frac{\partial G}{\partial n} d\Gamma \tag{3.74}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = 0$; in the latter G' is the modified Green's function and μ is the permeability (H m^{-1}) of the material.

3.1.6 Field of a Line Current in a Three-Dimensional Domain: Integral Approach

Let a straight conductor, placed in an unbounded three-dimensional domain and carrying direct current of density J, be considered. Assuming cylindrical coordinates, from (3.74) it follows

$$\overline{A} = A\overline{i}_z, \ \overline{J} = J\overline{i}_z, \ A = \frac{\mu}{4\pi}\int_\Omega \frac{J}{|\overline{r}|} d\Omega \tag{3.75}$$

where $\overline{r}$ is the distance between the fixed field point P and the source point Q (Fig. 3.7) oriented from Q to P and Ω is the conductor volume.

Moving from potential to field, at point P one has

$$\overline{B} = \overline{\nabla}_P \times \overline{A} = \frac{\mu}{4\pi}\int_\Omega \overline{\nabla}_P \times \frac{\overline{J}}{|\overline{r}|} d\Omega \tag{3.76}$$

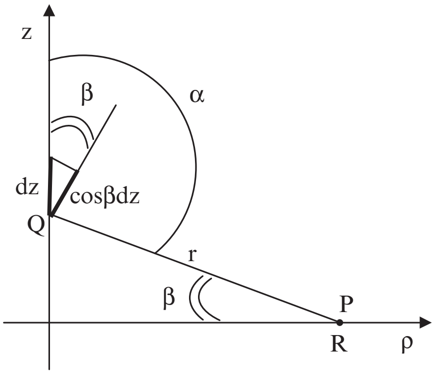

Fig. 3.7 Line current

where the operator $\overline{\nabla}_P$ acts on the coordinates of point P. The following identity holds

$$\overline{\nabla}_P \times \frac{\overline{J}}{|\overline{r}|} = \frac{1}{|\overline{r}|}\overline{\nabla}_P \times \overline{J} - \overline{J} \times \overline{\nabla}_P \frac{1}{|\overline{r}|} \tag{3.77}$$

Since $\overline{J}$ depends on the coordinates of point Q and not on those of point P, it follows that $\overline{\nabla}_P \times \overline{J} = 0$; then, according to (A.3), it comes out that

$$-\overline{J} \times \overline{\nabla}_P \frac{1}{|\overline{r}|} = \frac{\overline{J} \times \overline{r}}{|\overline{r}|^3} \tag{3.78}$$

and therefore

$$\overline{B} = \frac{\mu}{4\pi}\int_\Omega \frac{\overline{J} \times \overline{r}}{|\overline{r}|^3}\, d\Omega \tag{3.79}$$

which is called Laplace's law of the elementary action.

Since the conductor is cylindrical and the current density is uniform, $J\, d\Omega = I\, dz$ and the volume integral becomes a line integral

$$\overline{B} = \frac{\mu}{4\pi}\int_{-\infty}^{+\infty} \frac{I\overline{i}_z \times \overline{r}}{|\overline{r}|^3}\, dz = \frac{\mu I}{4\pi}\int_{-\infty}^{+\infty} \frac{\overline{i}_\vartheta\, |\overline{r}| \sin\alpha}{|\overline{r}|^3}\, dz = \overline{i}_\vartheta \frac{\mu I}{4\pi}\int_{-\infty}^{+\infty} \frac{\sin\alpha}{|\overline{r}|^2}\, dz \tag{3.80}$$

Substituting $r\cos\beta = R$ and $\cos\beta\, dz = r\, d\beta$, since $\sin\alpha = \cos\beta$, finally it results

$$\overline{B} = \overline{i}_\vartheta \frac{\mu I}{4\pi R}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\beta\, d\beta = \overline{i}_\vartheta \frac{\mu I}{2\pi R} \tag{3.81}$$

and

$$\overline{H} = \overline{i}_\vartheta \frac{I}{2\pi R} \tag{3.82}$$

coincident with (2.227) when $\beta = 0$, so that $r \equiv R$.

3.1.7 Field of a Current-Carrying Conductor of Rectangular Cross-Section

In a two-dimensional unbounded domain, which is supposed to be homogeneous and free of ferromagnetic material, after (3.4) the Green's function for a cylindrical conductor carrying a constant current I is given by

$$A(r) = \frac{\mu_0 I}{2\pi} \ln r, \quad r > 0 \tag{3.83}$$

Let a conductor of rectangular cross-section, having width 2a and height 2b and carrying a constant current distributed with uniform density J, be considered (Fig. 3.8). At the gravity centre of the conductor cross-section the origin of a rectangular system of coordinates is placed.

After integrating the elementary contributions, the potential of the rectangular conductor is given by

$$\begin{aligned} A(x, y) &= \frac{\mu_0 J}{2\pi} \int_{-a}^{a} \int_{-b}^{b} \ln\left[r\left(x, y, x', y'\right)\right] dy' dx' = \\ &= \frac{\mu_0 J}{4\pi} \int_{-a}^{a} \int_{-b}^{b} \ln\left[\left(x - x'\right)^2 + \left(y - y'\right)^2\right] dy' dx' \end{aligned} \tag{3.84}$$

where (x', y') and (x, y) are source point Q and field point P, respectively. It turns out to be

$$\begin{aligned} A(x, y) = \frac{\mu_0 J}{4\pi} \Big\{ & (a - x)(b - y) \ln\left[(a - x)^2 + (b - y)^2\right] + \\ & + (a + x)(b - y) \ln\left[(a + x)^2 + (b - y)^2\right] + \\ & + (a - x)(b + y) \ln\left[(a - x)^2 + (b + y)^2\right] + \\ & + (a + x)(b + y) \ln\left[(a + x)^2 + (b + y)^2\right] + \end{aligned}$$

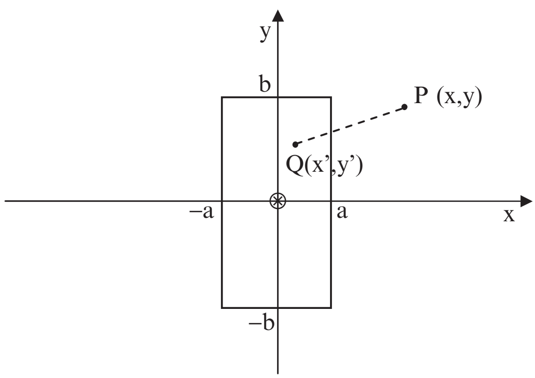

Fig. 3.8 Conductor of rectangular cross-section

$$
\begin{aligned}
&+ (a - x)^2 \left(\mathrm{arctg}\frac{b - y}{a - x} + \mathrm{arctg}\frac{b + y}{a - x} \right) + \\
&+ (a + x)^2 \left(\mathrm{arctg}\frac{b - y}{a + x} + \mathrm{arctg}\frac{b + y}{a + x} \right) + \\
&+ (b - y)^2 \left(\mathrm{arctg}\frac{a - x}{b - y} + \mathrm{arctg}\frac{a + x}{b + y} \right) + \\
&+ (b + y)^2 \left(\mathrm{arctg}\frac{a - x}{b + y} + \mathrm{arctg}\frac{a + x}{b + y} \right) \Bigg\}
\end{aligned} \tag{3.85}
$$

for $x \neq \pm a,\ y \neq \pm b$. If the assumption $b << a$ holds, the model of the current sheet follows. It turns out to be

$$
\begin{aligned}
A(x, y) &= \frac{\mu_0 J}{4\pi} \int_{-a}^{a} \ln\left[\left(x - x'\right)^2 + y^2 \right] dx' = \\
&= \frac{\mu_0 J}{4\pi} \Bigg\{ (a + x) \ln\left[(a + x)^2 + y^2 \right] + (a - x) \ln\left[(a - x)^2 + y^2 \right] + \\
&\quad + 2y \left(\mathrm{arctg}\frac{a + x}{y} + \mathrm{arctg}\frac{a - x}{y} \right) - 4a \Bigg\}, \quad y \neq 0
\end{aligned} \tag{3.86}
$$

where J is the line current density ($A\ m^{-1}$). After (3.85) or (3.86), from (2.205) the components of induction field can be obtained.

3.2 Method of Images

Field problems characterized by concentrated sources in non-homogeneous domains with simple boundaries can be solved by the method of images.

Electrostatic images

Let a dielectric half-space Ω of permittivity ε with a point charge q at a distance h from a conducting half-space be considered. The field in the dielectric region Ω is uniquely specified by the charge q and the boundary condition of the region, where $\overline{E} \cdot \overline{t} = 0$ holds. Comparing this field with that produced in an unbounded dielectric domain of permittivity ε by two point charges q and −q at a distance 2h, one can conclude that in the dielectric region Ω the fields are the same. Therefore in this region the field is equal to that produced by charge q and its image −q placed at distance 2h in a homogeneous domain of permittivity ε (Fig. 3.9).

Therefore, according to (2.118) the electric field is expressed by

$$
\overline{E} = \frac{1}{4\pi\varepsilon} q \frac{1}{x^2 + (y - h)^2} \overline{i}_{r_1} - \frac{1}{4\pi\varepsilon} q \frac{1}{x^2 + (y + h)^2} \overline{i}_{r_2}, \quad y > 0 \tag{3.87}
$$

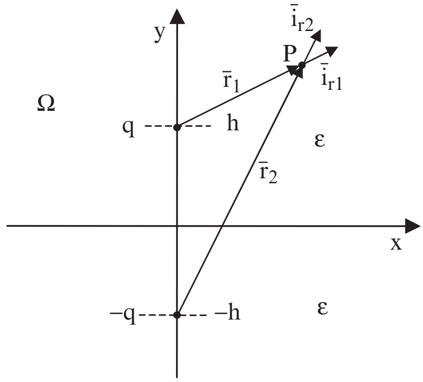

Fig. 3.9 Point charge near the boundary with a conducting half space: source and image charge

where the radial unit vectors are defined as follows

$$\bar{i}_{r_1} = \left(\frac{x}{\sqrt{x^2 + (y-h)^2}}, \frac{y-h}{\sqrt{x^2 + (y-h)^2}} \right), \quad y > 0 \tag{3.88}$$

$$\bar{i}_{r_2} = \left(\frac{x}{\sqrt{x^2 + (y+h)^2}}, \frac{y+h}{\sqrt{x^2 + (y+h)^2}} \right), \quad y > 0 \tag{3.89}$$

In turn, the electric field for $y < 0$ is given by $\bar{E} = 0$.

Field lines are plotted in Fig. 3.10.

More generally, let a dielectric medium of permittivity ε_1 be considered, filling the upper half-space where a point charge q is located; at a distance h from the charge, let another dielectric medium of permittivity ε_2 fill the lower half-space (Fig. 3.11).

In this case, the field in the upper half-space is equivalent to that produced in a homogeneous region of permittivity ε_1 by both source charge q and image charge $q' = -\alpha q$ with $0 \le \alpha < 1$, placed at a distance 2h from q.

In an analogous way, the field in the lower half-space is equivalent to that produced by a second image charge $q'' = \beta q$ with $0 \le \beta < 1$, placed instead of q in a homogeneous region of permittivity ε_2.

In fact, at the interface $y = 0$, the transmission conditions for tangential component of electric field and normal component of induction (2.78) and (2.77) imply

$$E_x - \alpha E_x = \beta E_x, \quad y = 0^+ \tag{3.90}$$

$$\varepsilon_1 E_y + \varepsilon_1 \alpha E_y = \varepsilon_2 \beta E_y, \quad y = 0^- \tag{3.91}$$

respectively. It is easily found that

$$\alpha = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2}; \ \beta = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \tag{3.92}$$

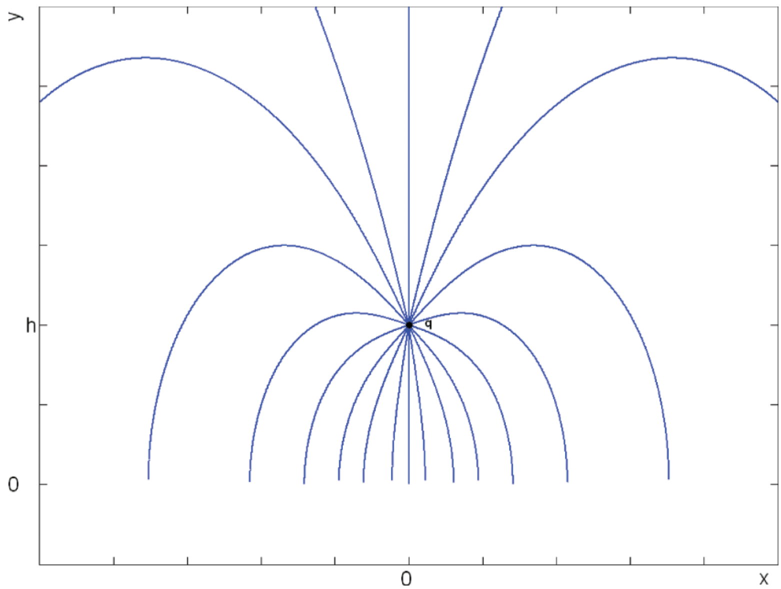

Fig. 3.10 Electrostatic images: field lines for $y > 0$

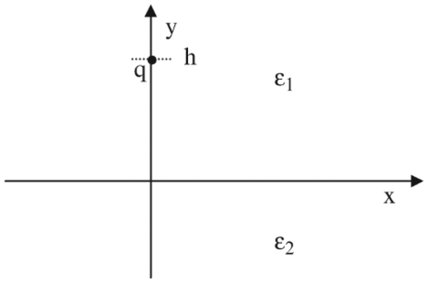

Fig. 3.11 Point charge near the boundary of two dielectric half-spaces

Therefore, the electric field for $y > 0$ is expressed by

$$\bar{E} = \frac{1}{4\pi\varepsilon_1} q \frac{1}{x^2 + (y-h)^2} \bar{i}_{r1} - \frac{1}{4\pi\varepsilon_1} \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2} q \frac{1}{x^2 + (y+h)^2} \bar{i}_{r2} \tag{3.93}$$

$$\bar{i}_{r1} = \left(\frac{x}{\sqrt{x^2 + (y-h)^2}}, \frac{y-h}{\sqrt{x^2 + (y-h)^2}} \right) \tag{3.94}$$

$$\bar{i}_{r2} = \left(\frac{x}{\sqrt{x^2 + (y+h)^2}}, \frac{y+h}{\sqrt{x^2 + (y+h)^2}} \right) \tag{3.95}$$

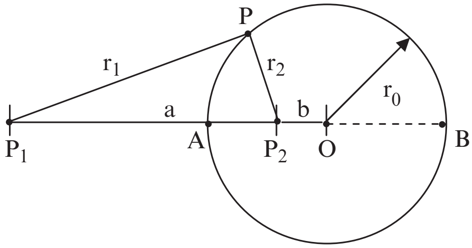

Fig. 3.12 Image of a charge with respect to a grounded sphere

In turn, the electric field for $y < 0$ is given by

$$\overline{E} = \frac{1}{4\pi\varepsilon_2}\frac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}q\frac{1}{x^2+(y-h)^2}\overline{i}_{r1} \tag{3.96}$$

Now, let a point charge q_1 be located at point P_1 externally to a conducting sphere, of radius r_0 and centre O, having potential $U = 0$; the distance between P_1 and the centre of the sphere is d.

The field distribution does not change if the spherical surface is replaced by an equivalent point-charge q_2, located at a suitable point P_2, such that the sphere represents its zero-potential surface; in that case q_2 is called the image charge of q_1 with respect to the sphere. The problem is that of identifying: (i) distance a between P_1 and P_2; (ii) displacement b of P_2 with respect to the sphere centre O; (iii) value q_2 of the image charge, knowing the value q_1 of source charge and the distance $d = a + b$ (Fig. 3.12).

From (2.119) the potential at point P is

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}\right) \tag{3.97}$$

where r_1 and r_2 are the distances of P from P_1 and P_2, respectively.

All points of the domain for which $U = 0$ should fulfil

$$\frac{q_1}{r_1}+\frac{q_2}{r_2}=0 \tag{3.98}$$

or

$$\frac{r_1}{r_2}=-\frac{q_1}{q_2}=k \tag{3.99}$$

In a two-dimensional domain they belong to a circle surrounding the point P_2 where the image charge should be located. Equation (3.99) for points A and B gives

$$\frac{a+b-r_0}{r_0-b}=\frac{a+b+r_0}{r_0+b}=k \tag{3.100}$$

The last two equations yield

$$k=\frac{d}{r_0} \tag{3.101}$$

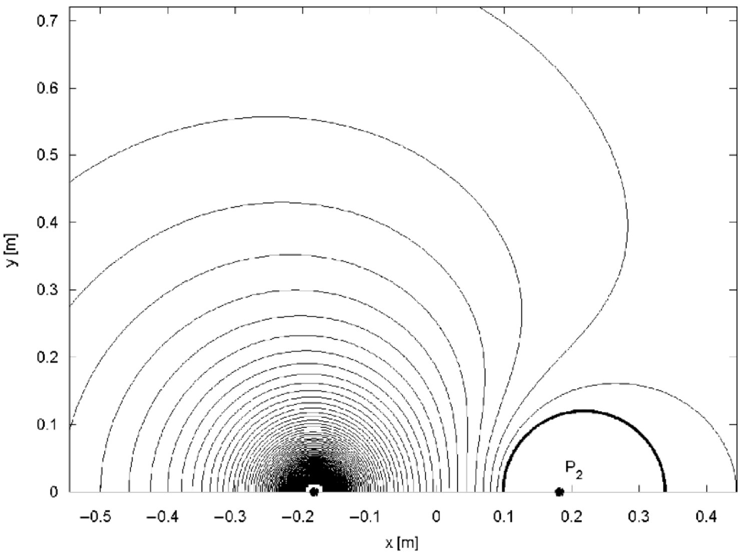

Fig. 3.13 Contour plot of potential for two unlike charges of different magnitude

$$a = \frac{d^2 - r_0^2}{d} \tag{3.102}$$

$$b = \frac{r_0^2}{d} \tag{3.103}$$

$$q_2 = -\frac{q_1}{k} \tag{3.104}$$

It should be noted that source and image charge are unlike and have different magnitude.

Assuming $q_1 = 4\,\mu C$, $d = 40\,cm$, $r_0 = 12\,cm$, one has $\frac{1}{k} = 0.3$, $q_2 = -1.2\,\mu C$, $a = 36.4\,cm$ and $b = 3.6\,cm$; the corresponding potential lines are shown in Fig. 3.13.

Finally, if the potential of the sphere is $U = U_0 \neq 0$, the problem can be solved as above with the addition of a second image charge $q_3 = 4\pi\varepsilon_0 r_0 U_0$ placed at the centre of the sphere.

Magnetostatic images

A magnetic region Ω of permeability μ with a line current I, located at a distance h from a half-space of infinite permeability and parallel to the space itself, is considered.

The field in the magnetic region Ω is uniquely specified by the current I and the boundary condition, where $\overline{H} \cdot \overline{t} = 0$ holds. Comparing this field with that produced in an unbounded magnetic domain by two line currents of equal magnitude and equal sign at a distance 2h, one can conclude that in the magnetic region the fields are the same. Therefore in this region the field is equal to that of current I and its image I at distance 2h placed in a homogeneous domain of permeability μ (Fig. 3.14).

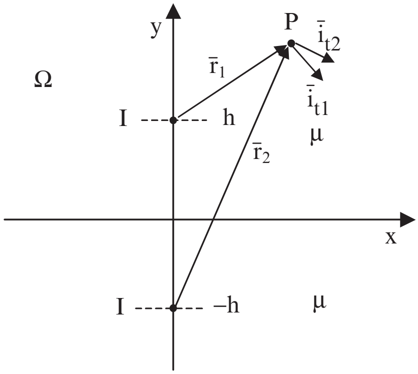

Fig. 3.14 Source and image currents

Therefore, the flux density for $y > 0$ is expressed by

$$\bar{B} = \frac{\mu}{2\pi} I \frac{1}{\sqrt{x^2 + (y-h)^2}} \bar{i}_{t1} + \frac{\mu}{2\pi} I \frac{1}{\sqrt{x^2 + (y+h)^2}} \bar{i}_{t2} \tag{3.105}$$

where the tangential unit vectors are defined as follows

$$\bar{i}_{t1} = \left(-\frac{y-h}{\sqrt{x^2 + (y-h)^2}}, \frac{x}{\sqrt{x^2 + (y-h)^2}} \right) \tag{3.106}$$

$$\bar{i}_{t2} = \left(-\frac{y+h}{\sqrt{x^2 + (y+h)^2}}, \frac{x}{\sqrt{x^2 + (y+h)^2}} \right) \tag{3.107}$$

In turn, the flux density for $y < 0$ is given by $\bar{B} = 0$.

Field lines for $y > 0$ are plotted in Fig. 3.15.

More generally, let a line current I, placed in a half-space of permeability μ_1 at a distance h from the boundary of a half-space of permeability μ_2, be considered (Fig. 3.16).

In analogy to the electrostatic case, the magnetic field in the upper half-space is equivalent to that produced, in a homogeneous region of permeability μ_1, by both source current I and image current $I' = \alpha I$ with $0 \le \alpha < 1$ placed at a distance 2h from I.

In a similar way, the field in the lower half-space is equivalent to that produced by a second image current $I'' = \beta I$ with $0 \le \beta < 1$ placed instead of I in a homogeneous region of permeability μ_2.

The transmission conditions for tangential component of magnetic field and normal component of induction at $y = 0$ imply

$$H_x - \alpha H_x = \beta H_x, \quad y = 0^+ \tag{3.108}$$

$$\mu_1 H_y + \mu_1 \alpha H_y = \mu_2 \beta H_y, \quad y = 0^- \tag{3.109}$$

$$0 \le \alpha < 1 \quad 0 \le \beta < 1 \tag{3.110}$$

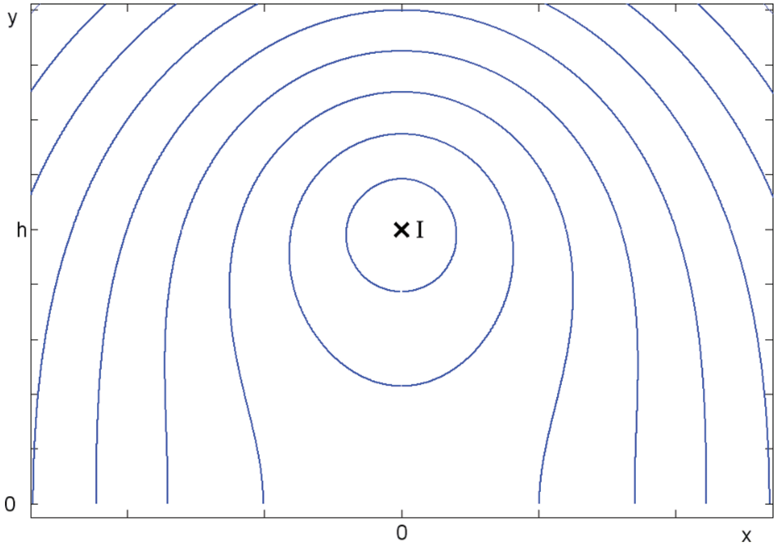

Fig. 3.15 Magnetostatic images: vector plot for $y > 0$

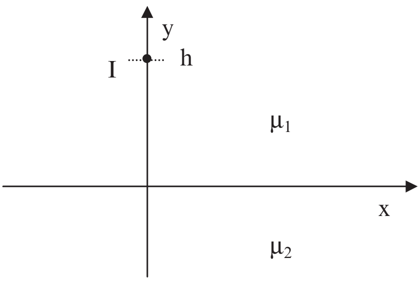

Fig. 3.16 Line current near the boundary of two magnetic half-spaces

respectively. It is easily found that

$$\alpha = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}; \quad \beta = \frac{2\mu_1}{\mu_1 + \mu_2} \tag{3.111}$$

Therefore, the flux density for $y > 0$ is expressed by

$$\bar{B} = \frac{\mu_1}{2\pi} I \frac{1}{\sqrt{x^2 + (y-h)^2}} \bar{i}_{t1} + \frac{\mu_1}{2\pi} \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} I \frac{1}{\sqrt{x^2 + (y+h)^2}} \bar{i}_{t2} \tag{3.112}$$

$$\bar{i}_{t1} = \left(-\frac{y-h}{\sqrt{x^2 + (y-h)^2}}, \frac{x}{\sqrt{x^2 + (y-h)^2}} \right) \tag{3.113}$$

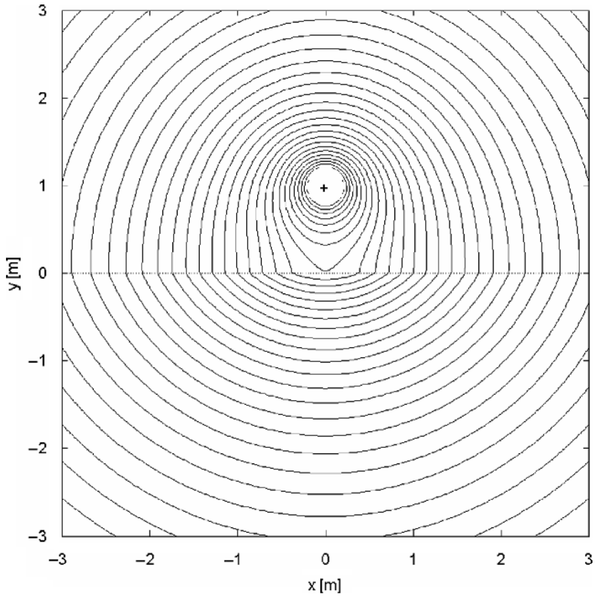

Fig. 3.17 Line current: field of source and image current ($\mu_2 = 20\mu_1$)

$$\bar{i}_{t2} = \left(-\frac{y+h}{\sqrt{x^2+(y+h)^2}}, \frac{x}{\sqrt{x^2+(y+h)^2}} \right) \tag{3.114}$$

In turn, the flux density for $y < 0$ is given by

$$\overline{B} = \frac{\mu_2}{2\pi}\frac{2\mu_1}{\mu_1+\mu_2} I \frac{1}{\sqrt{x^2+(y-h)^2}} \bar{i}_{t1} \tag{3.115}$$

In Fig. 3.17 the contour plot of flux lines is reported in the case $\mu_2 = 20\mu_1$.

3.2.1 Magnetic Field of a Line Current in a Slot

Let the test case shown in Fig. 2.13b be considered. A rectangular slot, having width a and height b is surrounded by ferromagnetic material of infinite permeability (closed slot). A constant line current I is concentrated at the gravity centre of the slot where the origin of a system of rectangular coordinates is placed (Fig. 3.18). Due to the presence of ferromagnetic material, the following boundary conditions hold:

$$B_x = 0 \quad \text{for} \quad y = \left(+\frac{b}{2}\right)^- \quad \text{and} \quad y = \left(-\frac{b}{2}\right)^+ \tag{3.116}$$

$$B_y = 0 \quad \text{for} \quad x = \left(+\frac{a}{2}\right)^- \quad \text{and} \quad x = \left(-\frac{a}{2}\right)^+ \tag{3.117}$$

for an observer located in the slot (flux lines orthogonal to the air/iron boundary).

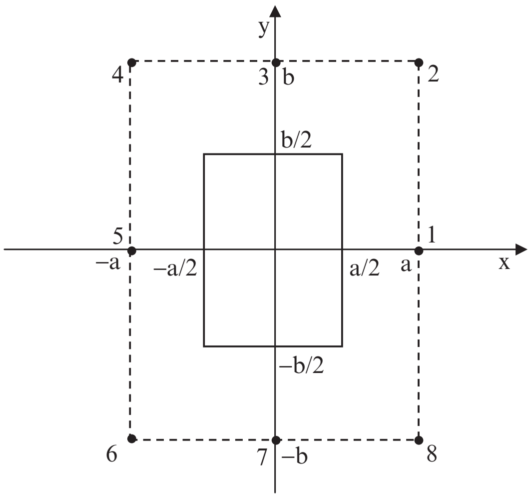

Fig. 3.18 Closed slot and images

Table 3.1 Location of images for the closed slot (single images)

Image k	x_k	y_k
1	a	0
2	a	b
3	0	b
4	−a	b
5	−a	0
6	−a	−b
7	0	−b
8	a	−b

If images due to multiple reflections in x and y directions are neglected, then eight equivalent currents $I_k = I$, $k = 1, 8$ approximate the effect of the slot boundary; they have to be placed symmetrically according to Table 3.1.

The closed slot is characterized by a double air/iron boundary in both x and y directions: in principle, the images form an infinite series, because each image current gives rise to a new reflection with respect to the boundaries. A better approximation of the field in the slot is obtained if a second layer of images is taken into account; then, twenty-four sources are originated, according to Table 3.2.

The total field is thus given by the superposition of the fields due to source current and like image currents, all of them being located in an unbounded domain of permeability μ_0, namely

$$\overline{B} = \frac{\mu_0 I}{2\pi\sqrt{x^2 + y^2}}\bar{i}_t + \sum_k \frac{\mu_0 I_k}{2\pi\sqrt{(x - x_k)^2 + (y - y_k)^2}}\bar{i}_{t,k}$$
$$-\frac{a}{2} \le x \le \frac{a}{2},\ -\frac{b}{2} \le y \le \frac{b}{2},\ x \ne x_k,\ y \ne y_k \tag{3.118}$$

Table 3.2 Location of images for the closed slot (double images)

Image k	x_k	y_k
1	a	0
2	a	b
3	0	b
4	−a	b
5	−a	0
6	−a	−b
7	0	−b
8	a	−b
9	2a	0
10	2a	b
11	2a	2b
12	a	2b
13	0	2b
14	−a	2b
15	−2a	2b
16	−2a	b
17	−2a	0
18	−2a	−b
19	−2a	−2b
20	−a	−2b
21	0	−2b
22	a	−2b
23	2a	−2b
24	2a	−b

with

$$\bar{i}_t = \left(-\frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right) \tag{3.119}$$

and

$$\bar{i}_{t,k} = \left(-\frac{y - y_k}{\sqrt{(x - x_k)^2 + (y - y_k)^2}}, \frac{x - x_k}{\sqrt{(x - x_k)^2 + (y - y_k)^2}} \right) \tag{3.120}$$

where (x_k, y_k) are the coordinates of kth image, while the summation index is $k = 1, \ldots, 8$ and $k = 1, \ldots, 24$ when single and double images are considered, respectively.

The case of a magnetically open slot, the height of which is assumed to be much greater than its width a, accommodating a conductor of height b, can be easily treated; boundary conditions become

$$B_x = 0, \quad y = \left(-\frac{b}{2}\right)^+ \tag{3.121}$$

$$B_y = 0, \quad x = \left(-\frac{a}{2}\right)^+, \left(\frac{a}{2}\right)^- \tag{3.122}$$

Again, the field is given by (3.118)–(3.120), this time taking the summation index $k = \{1, 5, 6, 7, 8\}$ and $k = \{1, 5, 6, 7, 8, 9, 17, 18, 19, 20, 21, 22, 23, 24\}$ when single and double images are considered, respectively.

The following remark can be put forward. Expanding the field in terms of current images is equivalent to set a particular solution fulfilling the given boundary conditions. The fact that boundary conditions (3.116) and (3.117) are not fully satisfied is due to the truncation of multiple images.

3.2.2 *Magnetic Field of a Line AC Current over a Conducting Half-Space*

The case of an AC line current located in a magnetic region of permeability μ at a distance h from a conducting half-space of infinite conductivity is here discussed (Fig. 3.19). The effect of induced currents in the conducting space gives rise to a flux barrier, i.e. the conducting plane can be treated as a space of zero permeability located at a distance h from the current; therefore, the field for $y > 0$ can be expressed as

$$\begin{aligned}\overline{B} = {} & \frac{\mu}{2\pi} I \frac{1}{\sqrt{x^2 + (y-h)^2}} \bar{i}_{t1} + \\ & - \frac{\mu}{2\pi} I \frac{1}{\sqrt{x^2 + (y+h)^2}} \bar{i}_{t2}, \quad y > 0, \ y \neq h \qquad (3.123)\end{aligned}$$

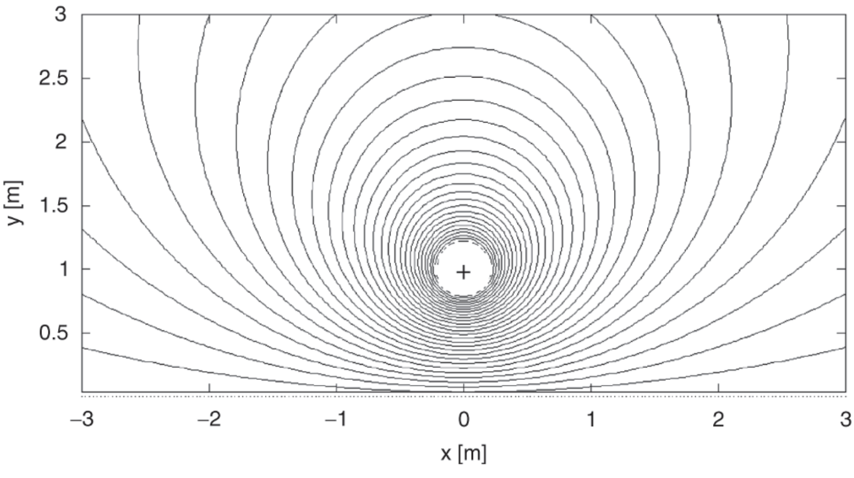

Fig. 3.19 AC line current near a conducting half-space of infinite permeability: field lines

with

$$\bar{i}_{t1} = \left(-\frac{y-h}{\sqrt{x^2+(y-h)^2}}, \frac{x}{\sqrt{x^2+(y-h)^2}} \right) \tag{3.124}$$

and

$$\bar{i}_{t2} = \left(-\frac{y+h}{\sqrt{x^2+(y+h)^2}}, \frac{x}{\sqrt{x^2+(y+h)^2}} \right) \tag{3.125}$$

In other words, the field for $y > 0$, is given by the superposition of source current I and image current $-I$ located at a distance 2h in an unbounded homogeneous region of permeability μ; the boundary between magnetic and conducting regions is a flux line.

3.3 Method of Separation of Variables

In a two-dimensional homogeneous domain Ω, with constant permeability μ and no current, using rectangular coordinates, Laplace's equation of magnetic vector potential A is from (2.208)

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = 0 \tag{3.126}$$

If the domain boundaries lay along constant x or constant y lines, then the following solution can be tried

$$A(x, y) = X(x)Y(y) \tag{3.127}$$

where X and Y depend on x only and y only, respectively. Consequently, substituting (3.127) into (3.126), it turns out to be

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0, \quad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \tag{3.128}$$

if $X(x) \neq 0$ for any x and $Y(y) \neq 0$ for any y.

The only way (3.128) are true is that separately

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \tag{3.129}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = k^2 \tag{3.130}$$

where $k^2 \neq 0$ is called the separation constant; it is assumed that both X(x) and Y(y) are non-constant functions.

Then, the partial differential equations (3.128) reduce to a pair of ordinary differential equations

$$\frac{d^2X}{dx^2} + k^2X = 0 \tag{3.131}$$

$$\frac{d^2Y}{dy^2} - k^2Y = 0 \tag{3.132}$$

For $k^2 \neq 0$, the two general solutions are given by

$$X(x) = \alpha_k \sin(|k|\,x) + \beta_k \cos(|k|\,x) \tag{3.133}$$

$$Y(y) = \gamma_k \mathrm{sh}(|k|\,y) + \delta_k \mathrm{ch}(|k|\,y) \tag{3.134}$$

If $k^2 = 0$, it results

$$X(x) = \alpha_0 + \alpha_1 x, \quad Y(y) = \beta_0 + \beta_1 y \tag{3.135}$$

The most general solution of (3.126) is then given by

$$A(x, y) = c_1 + c_2x + c_3y + c_4xy + \\ + \sum_{n=1}^{\infty} \left[\alpha_n \sin(n\,|k|\,x) + \beta_n \cos(n\,|k|\,x)\right]\left[\gamma_n \mathrm{sh}(n\,|k|\,y) + \delta_n \mathrm{ch}(n\,|k|\,y)\right] \tag{3.136}$$

In principle, the separation constant and all unknown coefficients can be determined by imposing the boundary conditions. The actual problem is to fit the latter; although there is an infinite number of solutions to Laplace's equation, it is often impossible to identify analytically the right set of constants fulfilling field conditions along the boundary.

In the case of a current source in the field domain, the Poisson's equation holds

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = -\mu J \tag{3.137}$$

and a particular solution should be added to the general one in order to take the source term J into account. In that case, boundary conditions are then imposed on the whole solution.

Finally, the following remark can be put forward. If the case

$$\frac{d^2X}{dx^2} - h^2X = 0 \tag{3.138}$$

$$\frac{d^2Y}{dy^2} + h^2Y = 0 \tag{3.139}$$

with $h^2 \neq 0$ is considered, the behaviour of X(x) and Y(y) is found to be symmetrical with respect to the corresponding solutions of the case (3.131)–(3.132).

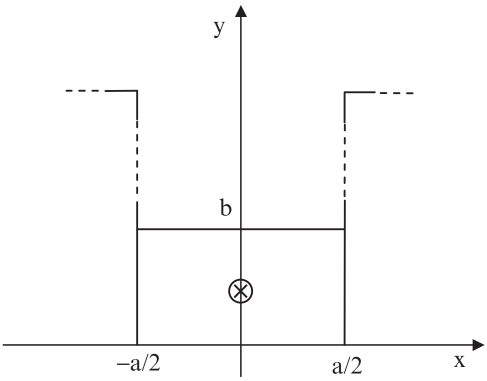

Fig. 3.20 Conductor in the slot

3.3.1 Magnetic Field of a Current Uniformly Distributed in a Slot

Let the test problem shown in Fig. 2.13a be considered; the height of the slot (Fig. 3.20) is assumed to be much greater than its width a.

A conductor of rectangular cross section, having width a and height b, is located at the bottom of the slot. The conductor carries a constant current I,supposed to be uniformly distributed inside the cross section. Due to the presence of ferromagnetic material of infinite permeability, the following boundary conditions hold

$$A = 0, \quad \text{for} \quad y = 0^+ \tag{3.140}$$

$$\frac{\partial A}{\partial x} = 0, \quad \text{for} \quad x = \left(+\frac{a}{2}\right)^- \quad \text{and} \quad x = \left(-\frac{a}{2}\right)^+ \tag{3.141}$$

Because of symmetry, the potential should be an even function of x; taking this into account, after (3.136) the general solution to the Laplace's equation in a rectangular domain can be expressed as

$$A_L = \sum_{n=1}^{\infty} c_n \mathrm{sh}(nky)\cos(nkx) \tag{3.142}$$

with k and c_n to be determined. A particular solution A_P to Poisson's equation, after integrating twice the right-hand side of (3.137) with respect to y, is

$$A_P(y) = -\frac{1}{2}\mu J y^2 \quad \text{with } J = \frac{I}{ab} \tag{3.143}$$

Consequently, the solution for the potential is

$$A = A_P + A_L = -\frac{1}{2}\mu J y^2 + \sum_{n=1}^{\infty} c_n \mathrm{sh}(nky)\cos(nkx) \tag{3.144}$$

with $-\frac{a}{2} \le x \le \frac{a}{2},\ 0 \le y \le b.$

In the region $y > b,\ J = 0$ and the field tends to be uniform such that $\left(B_x,\ B_y\right) = (B_0,\ 0)$ at least for $y >> b$; accordingly, a suitable expression of the potential in this region is

$$\tilde{A}_L = \alpha + \beta y + \sum_{n=1}^{\infty} \gamma_n e^{-nky} \cos(nkx),\ -\frac{a}{2} < x < \frac{a}{2}, \quad b \le y \tag{3.145}$$

with $\alpha,\ \beta,\ \gamma_n$ to be determined.

Boundary conditions are now imposed. It follows

$$\frac{\partial A}{\partial x} = -\sum_{n=1}^{\infty} nkc_n \mathrm{sh}(nky) \sin(nkx) \tag{3.146}$$

and

$$\left.\frac{\partial A}{\partial x}\right|_{x=\pm\frac{a}{2}} = \mu \sum_{n=1}^{\infty} nkc_n \mathrm{sh}(nky) \sin\left(nk\frac{a}{2}\right) = 0 \tag{3.147}$$

which is true if $k = \frac{2\pi}{a},\ c_n \neq 0$.

Moreover $A(x, 0) = 0$ is automatically fulfilled.

Finally, the asymptotic boundary condition states

$$\lim_{y\to\infty} B_x = B_0 \tag{3.148}$$

$$\lim_{y\to\infty} B_y = 0 \tag{3.149}$$

From (3.145), for $y > b$ and $-\frac{a}{2} < x < \frac{a}{2}$, one has

$$B_x = \frac{\partial \tilde{A}_L}{\partial y} = \beta - \sum_{n=1}^{\infty} nk\gamma_n\, e^{-nky} \cos(nkx) \tag{3.150}$$

and, because of (3.148), it results $\beta = B_0$.

In turn, one has

$$B_y = -\frac{\partial \tilde{A}_L}{\partial x} = \sum_{n=1}^{\infty} nk\gamma_n e^{-nky} \sin(nkx) \tag{3.151}$$

and (3.149) is always fulfilled, for $y >> b$ and $-\frac{a}{2} < x < \frac{a}{2}$.

At $y = b$, the continuity of potential requires $A_L + A_P = \tilde{A}_L$ i.e.

$$-\frac{1}{2}\mu J b^2 + \sum_{n=1}^{\infty} c_n \mathrm{sh}(nkb) \cos(nkx) = \alpha + B_0 b + \sum_{n=1}^{\infty} \gamma_n e^{-nkb} \cos(nkx) \tag{3.152}$$

Moreover, the continuity of field components B_x and B_y requires

$$-\mu Jb + \sum_{n=1}^{\infty} nkc_n ch(nkb)\cos(nkx) = B_0 - \sum_{n=1}^{\infty} nk\gamma_n e^{-nkb}\cos(nkx) \qquad (3.153)$$

and

$$\sum_{n=1}^{\infty} nkc_n sh(nkb)\sin(nkx) = \sum_{n=1}^{\infty} nk\gamma_n e^{-nkb}\sin(nkx) \qquad (3.154)$$

respectively.

It follows

$$B_0 = -\mu Jb \qquad (3.155)$$

$$\alpha = \frac{1}{2}\mu Jb^2 \qquad (3.156)$$

$$\gamma_n = -c_n e^{nkb} ch(nkb) = c_n e^{nkb} sh(nkb) \qquad (3.157)$$

One has $\gamma_n = c_n = 0$ if $nkb \neq 0$.

Therefore, one obtains

$$A = -\frac{1}{2}\mu Jy^2, \quad 0 \le y \le b \qquad (3.158)$$

$$A = \mu Jb\left(\frac{b}{2} - y\right), \quad b \le y \qquad (3.159)$$

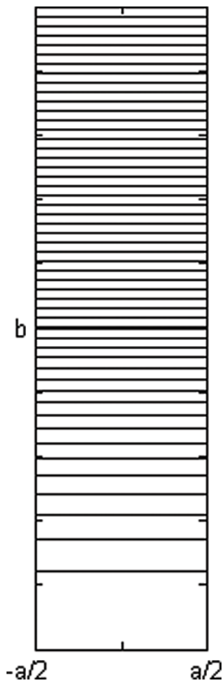

Fig. 3.21 Conductor in the slot: plot of flux lines

Correspondingly, the field components are

$$B_x = -\mu J y, \quad 0 \le y \le b \tag{3.160}$$

$$B_x = -\mu J b, \quad b \le y \tag{3.161}$$

and $B_y = 0$ everywhere. Fig. 3.21 shows the plot of flux lines.