Chapter 2
Unidimensional Problems

2.1 Decisions Under Risk and Uncertainty

From the viewpoint of decision theory, the decision problems including uncertainty can be presented according to a decision table

$$\begin{array}{c} \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{array} \begin{array}{c} \begin{array}{cccc} \omega_1 & \omega_2 & \dots & \omega_m \end{array} \\ \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{bmatrix} \end{array} \tag{2.1}$$

The components of a decision table are the decision alternatives $(d_i,\ i = 1,\dots n)$, the states of nature $(\omega_j,\ j = 1,\dots m)$, and the consequences $(c_{ij},\ i = 1,\dots n,\ j = 1,\dots m)$. The consequence of an action is determined by the action, and by a number of external factors which are beyond the control of the decision maker. A state of nature is a particular set of values that these external factors might assume.

If the state of nature that will actually occur and the consequences associated with the decision alternatives are known, a decision is made under certainty. Under risk and uncertainty, the state of nature that would prevail is not known with certainty. Under risk, the probability of each state of nature occurring and, correspondingly, the probability distribution of consequences are known; otherwise, the decision is made under uncertainty.

The probabilities of the states of nature are rarely known, but it is often possible to estimate these probabilities. If objective probabilities can not be determined, subjective ones, based, for example, on expertise, may be used. This being the case, risk management methods can be applied also to support decision making under uncertainty. Therefore, the distinction between risk and uncertainty is not always clear. In addition to the future states of nature, uncertainty may be related to other elements of the decision making as well. These sources of uncertainty are dealt with in later chapters.

Different decision-makers can take different attitudes towards risk and uncertainty, which may lead to different priority orders among choice alternatives. People can search for good profits, although their probability is not great (risk-seeking person), or settle with lower profits that have great probability if there is risk of large losses (risk-averse person). Decision maker can also be risk-neutral.

Maximization of expected utility is a risk-neutral decision strategy for supporting risky choices (e.g. von Winterfeldt and Edwards 1986). Assume a problem where there are m possible consequences of a decision d_i, $c_{i1}, c_{i2}, \ldots, c_{im}$ that have probabilities of realization $p_1, p_2, \ldots, p_m$. Then, the expected utility from decision d_i is

$$E(u(d_i)) = p_1 u(c_{i1}) + p_2 u(c_{i2}) + \cdots + p_m u(c_{im}) = \sum_{j=1}^{m} p_j u(c_{ij}). \tag{2.2}$$

For decision support under uncertainty, several different decision strategies or rules have been developed (see, e.g. Miller and Starr 1969; Lee and Moore 1975; Cook and Russell 1981). For example, according to Maximin- or Wald-criterion, alternatives are ranked according to the worst possible consequences (risk avoiding behaviour), and the alternative with the best worst-case consequence is chosen. According to Maximax-criterion, alternatives are ranked according to the best possible consequences (risk taking behaviour), and the alternative with the best best-case consequence is chosen. Hurwicz-criterion is a combination of these two categorical rules; the alternative with the greatest weighted mean of the worst and the best possible outcomes is chosen. Here, the weights for the worst and the best possible outcomes reflect the attitude towards risk; e.g. for a risk neutral decision maker the weights are equal.

A more general criterion, which produces the above mentioned criteria as special cases, has been developed by Kangas (1992, 1994). In this approach, the decision-maker determines the importance of three priority measures in decision-making: (i) the worst possible outcome, (ii) the expected outcome, and (iii) the best possible outcome associated with the decision alternative. The alternatives can then be ranked based on the weighted average of these three outcomes. Then, if the weight of the worst outcome, *bw*, is one, one speaks of the maximin-criterion. Correspondingly, if the weight of the best outcome, *bb*, is 1, alternative is chosen according to the maximax-criterion. If the weight of the expected outcome, *be*, is 1 alternative is selected from a risk neutral decision maker's point of view.

If $bb > be > bw$ the decision maker can be classified as a risk seeker. In general, if *bb* is greater than *bw*, one can speak about risk seeking behaviour. Correspondingly, if $bb < be < bw$, or, more generally, if $bb < bw$, the decision maker can be classified as a risk avoider. The decision strategy can be changed flexibly by weighting the coefficients using different weighting schemes. Sensitivity analysis can be made, for example, of the meaning of the attitude towards risk in the choice of the forest plan.

2.2 Measuring Utility and Value

2.2.1 Estimating a Utility Function

All methods for estimating the utility (or value) functions are based on certain axioms. Eilon (1982) presented three basic axioms that are needed in estimation:

2.2.1.1 Connectivity

The decision maker is able to say from two alternative outcomes A and B, if he/she prefers A to B, B to A or if he/she is indifferent between these outcomes.

2.2.1.2 Transitivity

If the decision maker prefers outcome A to B and outcome B to C, then he/she also prefers outcome A to C.

2.2.1.3 Comparability

If the decision maker has three outcomes, A, B and C, and he/she prefers A to B and B to C, he/she can choose a coefficient x such that utility of $xA + (1-x)C = B$.

The first of these axioms may be violated, if the decision maker cannot make his/her mind. In practical work, it has often been noted that axioms 2 and 3 may not hold (e.g. Eilon 1982; Knowles 1984; Bell and Farquhar 1986). Furthermore, the last axiom only makes sense if the outcomes that are compared are quantitative, such as money.

Estimating the utility function at unidimensional scale is typically based on indifference methods (von Winterfeldt and Edwards 1986, p. 217). In practise it means, that decision maker needs to match two outcomes or pairs of outcomes to meet indifference relation. In essence, it means estimating the utility function based on the comparability axiom above.

As the utility is relative, the utility of two consequences can be arbitrarily assigned, and the rest of the utilities are assessed relative to these (Keeney and Raiffa 1976, p. 140). It is assumed that the most preferred consequence is defined as c^* and the least preferred as c^0. The utilities provided by these consequences can be scaled to $u(c^*) = 1$ and $u(c^0) = 0$, respectively. In the next stage, the consequences c_i are compared to lotteries, where the consequence is c^* with probability π and c^0 with probability $(1-\pi)$. Such lotteries are defined with $(c^*,\ \pi,\ c^0)$. The decision-maker is required to state, which is the probability π with which he/she is indifferent between the certain consequence c_i and the lottery. Because of the indifference, the utility $u(c_i)$ must be equal to the expected utility of the lottery.

It follows that

$$u(c_i) = \pi u(c^*) + (1-\pi)u(c^0) = \pi 1 + (1-\pi)0 = \pi. \tag{2.3}$$

When several such comparisons are made, the utility function can be fitted to the expressed probabilities. This approach is called variable probability method (von Winterfeldt and Edwards 1986).

The other possibility is to use a constant probability, say 0.5, and to change the consequence c_i to such amount that the utility it provides would be indifferent with the lottery. This approach is called variable certainty equivalent method.

The well-known von Neumann–Morgestern utility function was already based on such comparisons. In their approach, risk attitudes are dealt with implicitly; form of the utility function describes the decision maker's attitude towards risk (von Neumann and Morgestern 1947). In many cases, the certainty equivalent c_i, which gives the same utility as the lottery, is actually larger or smaller than the expected payoff of the lottery. For instance, the decision-maker may say that 80€ is the certainty equivalent for lottery (200, 0.5, 0). Then, the difference between the expected payoff and certainty equivalent, 100€–80€, is the risk-premium. This is the amount a risk-avert decision maker is willing to "give up" in order to avoid uncertain lottery. If the risk premium is negative, the decision maker is a risk-seeker.

The utility function of risk-neutral person is according to this theory linear, and that of risk-seeker convex. This result has, however, been criticized, since a concave utility function can also be reasoned based on an assumption of decreasing marginal utility (e.g. Sarin 1982; Sheng 1989; Kangas 1992). The estimation of Neumann–Morgenstern type utility function is also considered to be too complicated for practical decision-making processes (Leung 1978; Kirkwood and Sarin 1985; Butler and Loomes 1988). Therefore, risk is not accounted for in most real applications.

Example 2.1. Certainty equivalent method

Assume a game with the best possible outcome being a gain of 10,000€ and the worst outcome a loss of 10,000€. The decision maker is asked what amount of money obtained for certain is indifferent (i.e. giving the same utility) as game

Table 2.1 The obtained utility data

Income	Utility
−10,000	0
−8,400	0.125
−6,400	0.25
−4,400	0.375
−2,000	0.5
400	0.625
3,200	0.75
6,000	0.875
10,000	1

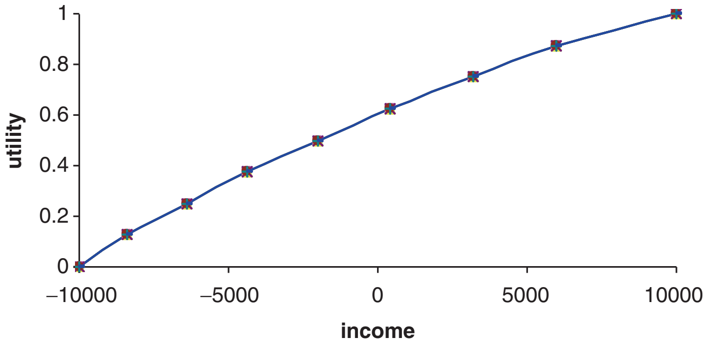

Fig. 2.1 Utility function

(10,000, 0.5, −10,000). If the DM answers −2,000, the risk premium is (0 − −2,000)€ = 2,000€, which means that the decision maker is risk – avoider.

The next question is, what certain outcome is indifferent to game (−10,000, 0.5, −2,000) and game (−2,000, 0.5, 10,000). If the answers were, for instance, −6,400€ and 3,200€, next games to be analyzed are (−10,000, 0.5, −6,400), (−6,400, 0.5, −2,000), (−2,000, 0.5, 3,200) and (3,200, 0.5, 10,000). When the answers to these games are obtained, and u(−10,000) = 0 and u(10,000) = 1, utility of −2,000 is calculated with u(−2,000) = $0.5 \cdot 0 + 0.5 \cdot 1.0 = 0.5$. The rest of the utilities can be calculated in the same way. Finally, the utility function can be fitted to the obtained data (Table 2.1) and the utility of income can be drawn (Fig. 2.1). The function in example 2.1 is concave, which describes a risk-avoider according to theory of von Neumann and Morgestern (1947).

2.2.2 Estimating a Value Function

Unidimensional value functions – or utility functions for no-risk situations – are formed based on comparisons between alternatives, but without lotteries. There exist several methods for estimating value function, of which only a few are presented in this book.

One possibility is to utilize the natural scale with which the performance of alternatives is measured, for instance, money, and scale it to range 0–1 (Keeney 1981). The most popular scaling approach is the maximum score based approach

$$v_i = c_i / \max(c), \tag{2.4}$$

That is, the criterion values c_i are divided with the maximum value among alternatives. The best alternative is assumed to have value one. Rest of the alternatives are relative to that and zero value is only given to an alternative also having zero value in natural scale. Then, the values follow a ratio scale. Another possibility is to scale the natural scale values with score range procedure

$$v_i = (c_i - \min(c)) / (\max(c) - \min(c)) \tag{2.5}$$

The best alternative is assumed to have a value one also in this case, and the worst the value zero. In this case, if $\min(c) > 0$, the alternatives do not follow a ratio scale, but an interval scale. In ratio scale, it is meaningful to compare the ratios between values, in interval scale only the differences. In interval scale, ratio comparisons simply do not make sense: all the alternatives are infinitely better, when compared to the worst alternative. Interval scale can be interpreted as local scale, the length of which depends on the specific planning situation (e.g. Kainulainen et al. 2007). If $\min(c) = 0$, these cases are equal.

The scaled scores obtained with (2.4) and (2.5) are often interpreted as value function. Such interpretation is quite common in practical decision support tools, as it is not necessary to ask decision makers any questions to form this kind of value function. Another interpretation is that the different variables and measurement scales are just standardized to the same scale.

If the scaled values are interpreted as a value function, it means that the analysis is based on an assumption of a linear value function. The value function may, however, be assumed to follow a certain type of non-linear function. In such a case, the decision-maker can choose the shape from a few predefined ones (e.g. exponential function). Then, a function of that type is fitted to the values presented in natural scale, but no more questions concerning the value function are asked from the decision-maker.

A group of methods useful for estimating value function are the so-called direct rating methods. In these methods, the decision maker is assumed to be able to rank the alternatives from best to worst. The best alternative and/or the worst alternative are given some arbitrary number of points, for instance 100 points for the best alternative and 0 for the worst alternative. Decision maker is then asked to give the rest of the alternatives points, related to the best and worst alternatives (von Winterfeldt and Edwards 1986, p. 229). These points are then scaled to 0–1 interval.

Example 2.2. Assume five alternatives, which all produce different amounts of money. It is assumed that the natural scale is in linear relationship with the value scale. The alternatives are scaled to value scale both utilising a ratio scale and an interval scale. The alternatives and different versions of scaling are presented in Table 2.2. In Fig. 2.2, the ratio scale value function is presented with diamonds and the interval version with squares.

Example 2.3. The decision maker was first asked to rank the alternatives from best (1) to worst (5). After that, decision maker was asked to give points between

Table 2.2 Scaling from original scale to utility scale

Alternative	Money	Ratio scale	Interval scale
1	250	1	1
2	124	0.496	0.427
3	76	0.304	0.209
4	55	0.22	0.114
5	30	0.12	0

Table 2.3 Points given to the alternatives

Alternative	Money	Order	Points	Value
1	250	1	100	1
2	124	2	60	0.6
3	76	3	35	0.35
4	55	4	20	0.2
5	30	5	0	0

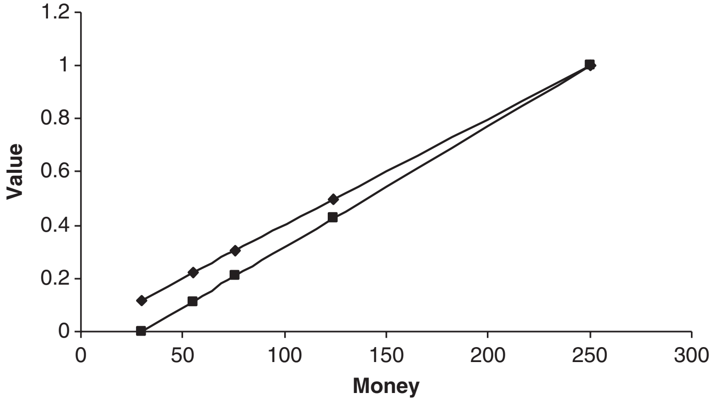

Fig. 2.2 Scaled utilities

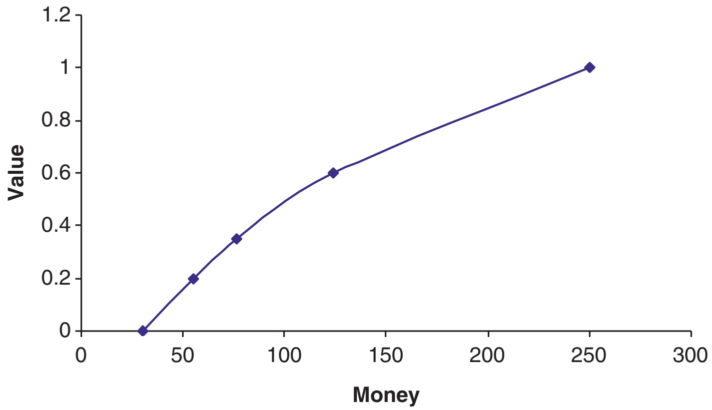

Fig. 2.3 Value function obtained from given points

Table 2.4 Indifferent changes

i	x_i	x_{i+1}	z_{i+1}	Points
0	0	30	30	1
1	30	70	40	1
2	70	120	50	1
3	120	180	60	1
4	180	250	70	1

0 and 100 to the three middle alternatives of the example 2.2. The obtained data is presented in Table 2.3 and the resulting value function is presented in Fig. 2.3.

Methods based on indifference are also used for estimating value functions, not only utility functions. In the case of value functions, decision makers are not asked to select a certain amount of money equal in utility to some lottery, but for selecting an outcome or difference in outcome that is indifferent to another outcome or difference in outcomes. One example of the indifference methods is the difference standard sequence method. In this approach, first a zero level x_0 is defined, i.e. the level which is least preferred. Then, a small but meaningful improvement z_1 from this level x_0 to a better level $x_1 = x_0 + z_1$ is selected. Then, the decision maker is asked, which improvement z_2 from level x_1 to level $x_2 = x_1 + z_2$ is equally preferred to the improvement z_1 from level x_0. After that, the decision-maker is asked which improvement z_3 from level x_2 is equally preferred to the improvement z_2 from level x_1 and so on. Thus, decision maker has to compare changes, the utility of which is assumed to depend on the level achieved so far. Since all these improvements $z_1 \ldots z_n$ are equally preferred, they can all be given same amount of points, say 1. Then, the points at each level can be calculated, so that $v(x_0) = 0,\ v(x_1) = 1,\ v(x_n) = n$, and the value function can be calculated by dividing the points at each level by n.

If the change z_i is smaller than the equally preferred change z_{i+1}, the value function is concave and shows marginally decreasing value (von Winterfeldt and Edwards 1986, p. 233). Thus, a concave utility function can be due to decreasing marginal value, not just the sign of risk-aversion.

Table 2.5 Resulting values

Money	Points	Value
0	0	0
30	1	0.20
70	2	0.40
120	3	0.60
180	4	0.80
250	5	1.00

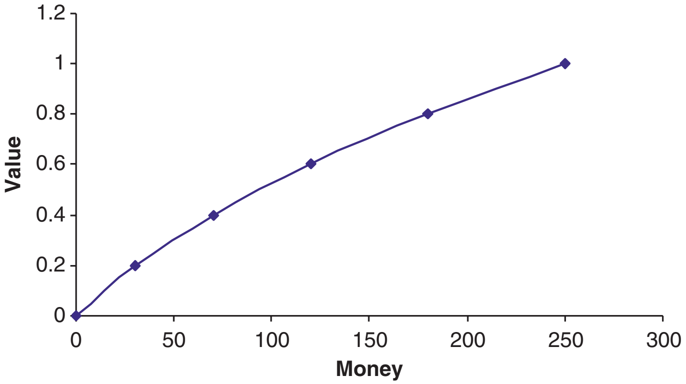

Fig. 2.4 Utility function resulting from indifferent changes

Example 2.4. The situation is the same as in the above examples. In this case the zero level x_0 is set to 0€, and the first improvement step z_1 is 30€. Then the decision maker evaluates that a change $0 \rightarrow 30$ is equally preferred to a change $30 \rightarrow 70$, and a change $30 \rightarrow 70$ is equally preferred to change $70 \rightarrow 120$ and so on. All the equally preferred changes are given in Table 2.4., the resulting values in Table 2.5., and the obtained value function is presented in Fig. 2.4.

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