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Interactive Continuation Tools

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Systematic bifurcation analysis requires the repeated continuation of different phase objects in free parameters, the detection and analysis of their bifurcations, and branch switching. Such computations produce a lot of numerical data that must be analyzed and, finally, presented in graphical form. Thus, continuation programs should not only be efficient numerically but should allow for interactive management and have a user-friendly graphics interface. The development of such programs is progressing rapidly. Here we make an attempt to survey existing interactive continuation and bifurcation tools and outline their history and perspectives. This is followed by the presentation of a framework that organizes the different types of objects and bifurcating branches. We give a brief overview of how such a framework is implemented in the recent software environment MATCONT. In the final two sections we give a few examples that illustrate the use of MATCONT and indicate directions of future developments.

2.1 Overview of Existing Software

During the last decades, considerable efforts have been made to develop general-purpose software tools for bifurcation analysis. One may distinguish at least three types (generations) of such software:

1. Noninteractive packages and codes,
2. Interactive programs,
3. Software environments.

Since the development of numerical algorithms advances with each generation, these tools also differ in supported computations. We give here an overview for each of the above types, with emphasis on the best known packages.

2.1.1 Noninteractive Packages and Codes

Noninteractive packages and codes first appeared in the beginning of the 1980s and were written in FORTRAN. They allowed one to continue equilibria and limit cycles of ODEs, as well as detect and subsequently continue their basic bifurcations: limit point (saddle-node) bifurcation, Hopf bifurcation, and period-doubling bifurcation. The most widely used packages of this generation are AUTO86 [12] and LINLBF [24]. Although these two packages supported a similar level of bifurcation analysis, they employed very different numerical algorithms. For example, LINLBF bases its test functions to locate Hopf and Neimark-Sacker bifurcation points on Hurwitz determinants, while AUTO86 bases the detection and location of all local bifurcations on monitoring the eigenvalues (multipliers). Moreover, the continuation of equilibrium (fixed point) bifurcations in AUTO86 is done using extended augmented systems that include eigenvectors, while in LINLBF minimally augmented systems are used. The most essential difference, however, lies in the continuation of limit cycles and their codimension-one bifurcations. In LINLBF these tasks are performed via numerically constructed and differentiated Poincaré maps, while AUTO86 employs the discretization of the corresponding boundary value problems using piecewise-polynomial approximations and orthogonal collocation. The latter proved its superiority for more complex multi-dimensional ODEs.

Bifurcation theory relies on center manifold reduction, followed by transformation to a normal form. The computation of normal-form coefficients is an important aspect of bifurcation software; for background we refer to [32]. The explicit computation of normal-form coefficients is not supported by AUTO86, and LINLBF only computes normal-form coefficients for local codimension-one bifurcations. There were a few codes available in the 1980s for simple numerical normal-form and branching analysis, e.g. STUFF [5] and BIFOR2 [22], but switching to the computation of different bifurcating objects required manual restarting of the code with new initial data.

Several other noninteractive packages for the continuation of simplest bifurcations in ODEs also appeared around this time, e.g. BIFPACK by Seydel [38, 39], and development continues to date, in particular for large-scale dynamical systems with packages such as LOCA[3] by Salinger et al. [35, 36].

2.1.2 Interactive Programs

Interactive programs for the bifurcation analysis of ODEs appeared at the end of the 1980s, when workstations and IBM-PC compatible computers became widely available at universities and general research institutes. All programs of this generation have a simple Graphical User Interface (GUI) with buttons,

[3] LOCA is available via `http://www.cs.sandia.gov/loca/`.

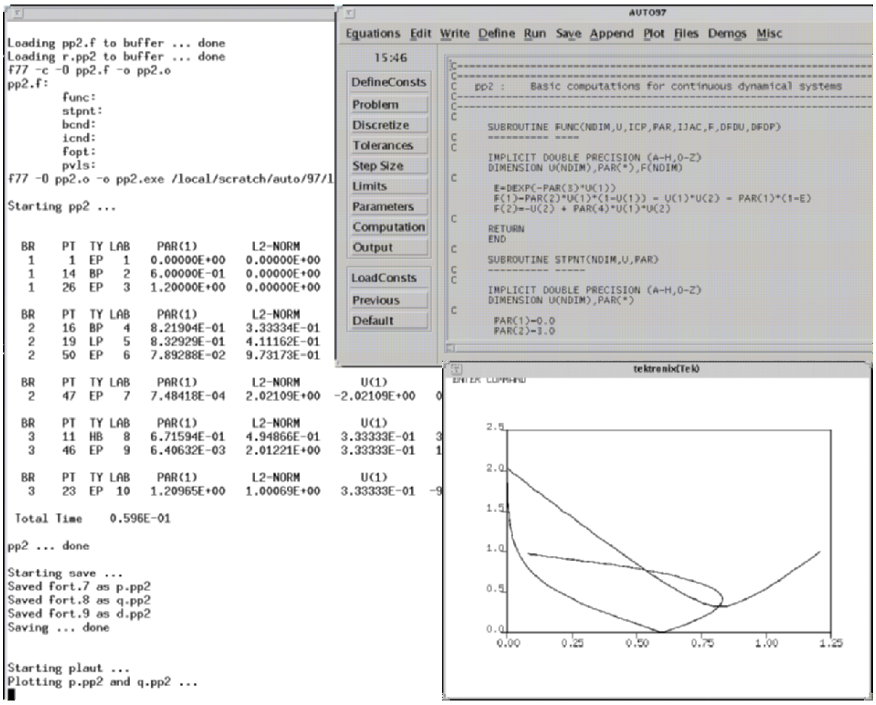

Fig. 2.1. Screen snapshot of AUTO94/97.

windows, and pull-down menus, and support the on-line input of the right-hand side of ODEs (through compilation with a FORTRAN or C compiler). Computed curves could now be plotted directly in a graphics window.

The continuation code AUTO86 mentioned above already comes with a simple interactive graphics program called `plaut` that allows for graphical presentation of computed data. There are versions of `plaut` for most of the widespread workstations, as well as a Matlab version `mplaut`[4], written by De Feo. There have been several attempts to improve the user interface of AUTO86 and later versions of AUTO. The first interactive version of AUTO86 was developed at Princeton University by Taylor and Kevrekidis [41] for SGI workstations. Another example is XPPAUT[5] by Ermentrout [15] for workstations and PCs, which developed from combining the MS-DOS program PHASEPLANE with AUTO. XPPAUT is also capable of simple phase-plane analysis, including the computation of one-dimensional global invariant manifolds of equilibria, as is SCIGMA [40]. Note that XPPAUT is still widely used and includes tools for the analysis of delay equations, functional equations, and stochastic equations. Doedel, Wang, and Fairgrieve also designed the interac-

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[4] `mplaut` is available via `http://www.math.uu.nl/people/kuznet/cm`.

[5] XPPAUT is available via `http://www.math.pitt.edu/~bard/xpp/xpp.html`.

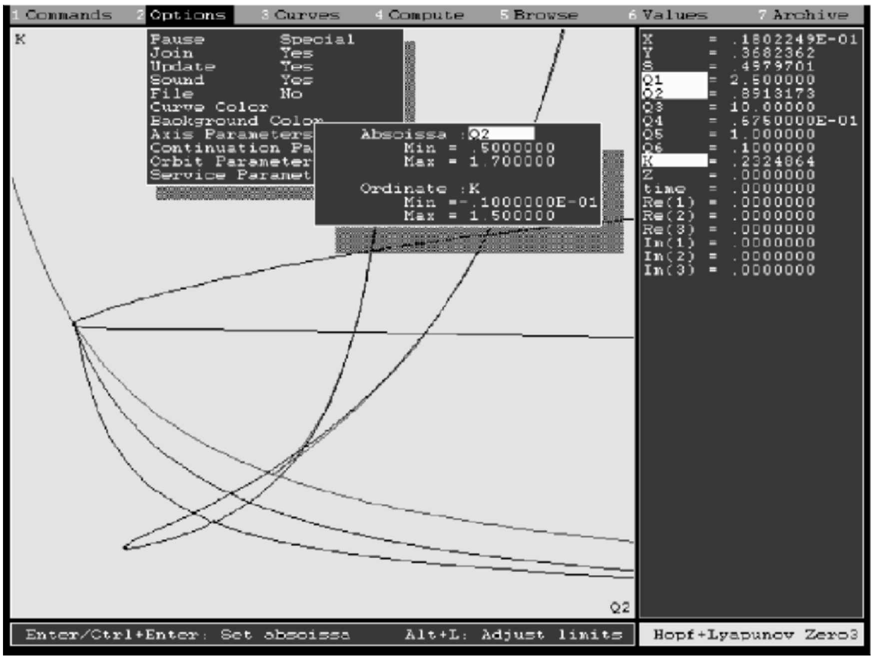

Fig. 2.2. Screen snapshot of LocBif 2.0.

tive version Auto94 for UNIX workstations with X-Windows; see Fig. 2.1. This version has extended numerical capabilities, including the continuation of all codimension-one bifurcations of limit cycles and fixed points. The software was upgraded in 1997 to support the continuation of homoclinic orbits using HomCont [9, 13]. This version is called Auto97[6] and there is also the C-version Auto2000 which has a new interactive graphics browser.

The major difficulty in using all versions of Auto is the analysis of detected bifurcation points and switching at these points to the continuation of other bifurcation curves, which requires browsing of several output files and a good understanding of their formats. However, due to the exceptional numerical efficiency of Auto, attempts to provide a better GUI for it continue to date; see, for example, Oscill8[7].

The first user-friendly interactive bifurcation program for bifurcation analysis was LocBif[8] developed for PCs under MS-DOS by Khibnik, Kuznetsov, Levitin, and Nikolaev [25]; a screen snapshot of version 2.0 is shown in Fig. 2.2. The numerical part of the program is based on the non-interactive code

[6] Auto is available via `http://cmvl.cs.concordia.ca/`.

[7] Oscill8 is available via `http://oscill8.sourceforge.net/doc/`.

[8] LocBif is available via `http://www.math.uu.nl/people/kuznet/LOCBIF`; LocBif works as a DOS-application under MS-Windows, but is no longer supported.

LINLBF and allows for continuation of equilibrium, fixed-point, and limit cycle bifurcations up to codimension three. The program allows for easy switching between the computation of various curves at detected bifurcation points. The user can manipulate individual computed bifurcation curves, which are stored separately in an archive. Version 1.0 of LOCBIF uses an external FORTRAN compiler and has a very simple keyboard-based interface, but version 2.0 can be driven by a mouse and has a special built-in compiler for the right-hand side. Neither version, however, has special tools to output the computed curves in a graphic format.

The program CANDYS/QA[9] by Feudel and Jansen [16] also belongs to this generation but is less widely used. All programs mentioned so far have closed architecture.

2.1.3 Software Environments

The first *software environments* for bifurcation analysis were DSTOOL[10] and CONTENT[11], developed in the 1990s. Both programs support the simulation of ODEs. The user can define/modify a dynamical model, perform a rather complete analysis, and export the results in a graphical form, all without leaving the program. Though hard, it is possible to extend them. The programs have an elaborate GUI and provide off- or on-line help and extensive documentation for users and developers.

DSTOOL [3] runs under UNIX or Linux. It performs simple phase-plane analysis and includes the computation of equilibria and associated one-dimensional stable and unstable manifolds, along with the continuation of equilibria and their codimension-one bifurcations, which is done by using parts of the LINLBF code.

The interactive software CONTENT was developed by Kuznetsov and Levitin with contributions by De Feo, Sijnave, Govaerts, Doedel, and Skovoroda; a screen snapshot of CONTENT 1.5 is shown in Fig. 2.3. The software runs on most popular workstations under UNIX and on PCs under Linux or MS-Windows and supports the continuation of equilibria and their bifurcations of codimension up to two. CONTENT uses minimal and extended augmented systems, as described in [19, 20], as well as the continuation of limit cycles using AUTO-like algorithms. Moreover, CONTENT supports the normal-form computations for many equilibrium bifurcations, taking advantage of internally generated symbolic derivatives of order up to three, and allows for branch switching by using algebraic branching equations. The software provides extensive storage, export and import facilities for computed curves and diagrams, including their numerical and PostScript formats. Switching between

[9] CANDYS/QA is available via `http://www.agnld.uni-potsdam.de/~wolfgang/candys.html`.
[10] DSTOOL is available via `http://www.cam.cornell.edu/~gucken/dstool`.
[11] CONTENTis available via `http://www.math.uu.nl/people/kuznet/CONTENT/`.

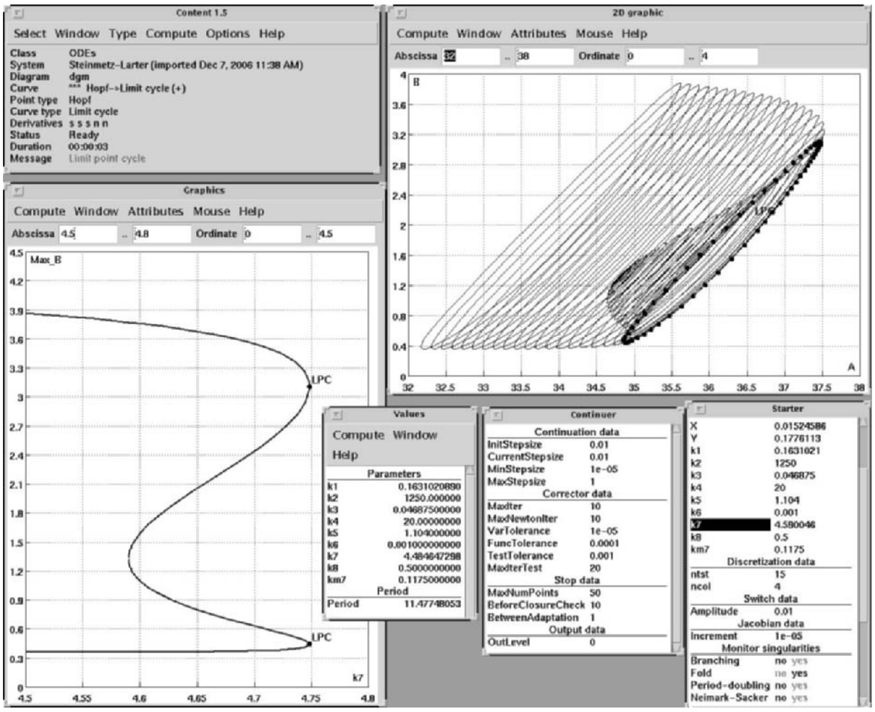

Fig. 2.3. Screen snapshot of CONTENT 1.5.

various bifurcating objects at special points is very easy and flexible in CONTENT. The latest software project MATCONT[12] is a Matlab interactive toolbox for the continuation and bifurcation analysis of ODEs [11] that is based on experience in developing and using CONTENT.

To conclude this section, we mention that numerical bifurcation analysis of smooth iterated maps is also supported by existing software. Location, analysis, and continuation of fixed-point bifurcations are very similar to those for equilibria of ODEs, and are supported, for example, by AUTO, LOCBIF, and CONTENT [18]. Other problems, particularly the analysis of global bifurcations, require special algorithms. For example, one needs special algorithms for the computation of the one-dimensional stable and unstable invariant manifolds of fixed points of maps; implementations for one-dimensional manifolds already exist in DSTOOL [29, 30] and DYNAMICS [34, 43, 44]. Such algorithms are necessary for the continuation of homoclinic orbits and their tangencies [4], which is also implemented as an AUTO-driver by Yagasaki [42]. The computation of two- or higher-dimensional invariant manifolds, for example, global stable and unstable invariant manifolds and invariant tori, and their bifurcations both for ODEs and maps is much more difficult and only a few algorithms

[12] MATCONT is available via `http://www.matcont.ugent.be/`.

are available; see Chap. 4, and [28, 31] for global manifolds and [6, 14, 37] for invariant tori.

2.2 Bifurcation Objects and Their Relations

The application of bifurcation theory to multi-parameter dynamical systems requires a clear *continuation strategy* that should provide rules on how to increase and decrease the number of control parameters while studying objects of different codimension. In addition, this strategy should suggest how to switch between different bifurcations of the same codimension, keeping the number of control parameters constant. As was first mentioned in [25] from a theoretical point of view, this strategy must be based on *graphs of adjacency* [1] that describe relationships between bifurcations. Below, we present two graphs of adjacency, describing *detection* relationships and *branching* relationships. The next section identifies the bifurcation objects for ODEs; the equivalent for maps is done in Sect. 2.2.2.

2.2.1 Bifurcation Objects in ODEs

Tables 2.1 and 2.2 list the codimension-zero, -one, and -two objects that can be found in generic continuous dynamical systems, along with associated labels based on standard terminology [32]. Table 2.1 lists all objects related to equilibria and limit cycles, while Table 2.2 focusses on objects related to homoclinic orbits of equilibria. The relationships between these objects are complicated.

The detection relationships between the objects in Tables 2.1 and 2.2 are presented in Figs. 2.4 and 2.5, respectively. For example, the arrows from O to EP and LC mean that it is generically possible for a computed orbit (O) to converge to a (stable) equilibrium (EP) or to a (stable) limit cycle (LC). An arrow from an object A different from O to an object B means that the continuation of a one-parameter family of objects of type A can generically lead to the detection of an object of type B, either because object B is a special case of object A or because it is a limiting case when the parameter tends to a special value. An example of the first situation is a Hopf bifurcation point (H) on a curve of equilibria (EP); an example of the second situation is a homoclinic orbit of a hyperbolic saddle (HHS), because it is the limit of a branch of periodic orbits (LC) when the period tends to infinity. We do not distinguish between the two situations, because the difference depends somewhat on the definition of a family of objects and it may depend on the implementation of the defining system that is used in the computation of the branch (e.g. an H point on a family of LC objects). Note that any computation normally starts with a point P as an initial condition to generate an orbit O; this first step does not feature in Fig. 2.4. We are interested in generic detection relationships, which is why the arrows always connect objects from

Table 2.1. Objects and associated labels related to equilibria and limit cycles of ODEs

Type of object	Label
Point	P
Orbit	O
Equilibrium	EP
Limit cycle	LC
Limit Point (fold) bifurcation	LP
Hopf bifurcation	H
Limit Point bifurcation of cycles	LPC
Neimark-Sacker (torus) bifurcation	NS
Period Doubling (flip) bifurcation	PD
Branch Point	BP
Cusp bifurcation	CP
Bogdanov-Takens bifurcation	BT
Zero-Hopf bifurcation	ZH
Double Hopf bifurcation	HH
Generalized Hopf (Bautin) bifurcation	GH
Branch Point of Cycles	BPC
Cusp bifurcation of Cycles	CPC
Generalized Period Doubling	GPD
Chenciner (generalized Neimark-Sacker) bifurcation	CH
1:1 Resonance	R1
1:2 Resonance	R2
1:3 Resonance	R3
1:4 Resonance	R4
Fold–Neimark-Sacker bifurcation	LPNS
Flip–Neimark-Sacker bifurcation	PDNS
Fold-flip	LPPD
Double Neimark-Sacker	NSNS

one codimension level down to objects on the next codimension level. The only two exceptions are the arrows from EP to BP and from LC to BPC, which jump over two codimension levels. In fact, these situations are non-generic, but they are so often found in systems with equivariance or invariant subspaces that most software packages support their detection.

The branching relationships between the objects in Tables 2.1 and 2.2 can be obtained directly from Figs. 2.4 and 2.5, respectively. In general, if there is an arrow in Fig. 2.4 or 2.5 from an object A different from O to an object B then for each object of type B there is a unique one-parameter family of objects of type A that branches off B, provided a total of $k+1$ free variables is chosen, where k is the codimension level of A. There are only four exceptions:

1. The arrows from EP to BP and from LC to BPC: there are generically two codimension-zero curves emanating from the codimension-two points.

Table 2.2. Objects and associated labels related to homoclinic orbits of equilibria of ODEs

Type of object	Label
Limit cycle	LC
Homoclinic orbit of a Hyperbolic Saddle	HHS
Homoclinic orbit of a Saddle-Node	HSN
Neutral saddle	NSS
Neutral saddle-focus	NSF
Neutral Bi-Focus	NFF
Shilnikov-Hopf	SH
Double Real Stable leading eigenvalue	DRS
Double Real Unstable leading eigenvalue	DRU
Neutrally-Divergent saddle-focus (Stable)	NDS
Neutrally-Divergent saddle-focus (Unstable)	NDU
Three Leading eigenvalues (Stable)	TLS
Three Leading eigenvalues (Unstable)	TLU
Orbit-Flip with respect to the Stable manifold	OFS
Orbit-Flip with respect to the Unstable manifold	OFU
Inclination-Flip with respect to the Stable manifold	IFS
Inclination-Flip with respect to the Unstable manifold	IFU
Non-Central Homoclinic to saddle-node	NCH

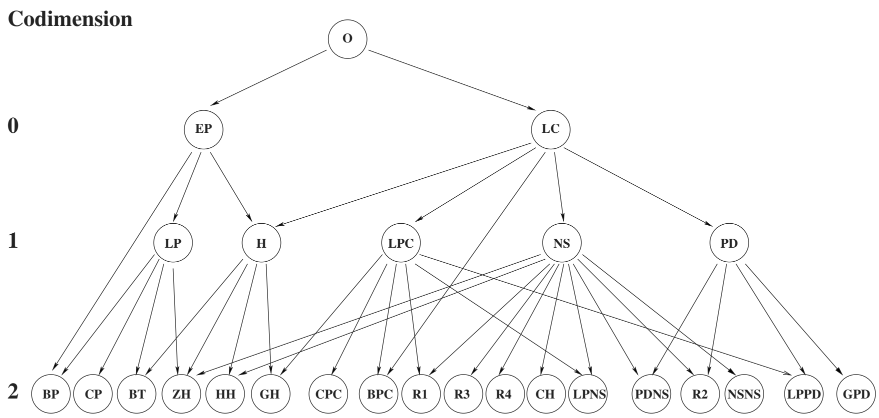

Fig. 2.4. Detection relationships between bifurcations of equilibria and limit cycles of ODEs; the branching relationships are found by following the arrows in the opposite directions, with four exceptions as discussed in the text.

2. The arrows from H to HH and from NS to NSNS: there are generically two codimension-one curves emanating from the codimension-two points.
3. The arrow from NS to ZH: the existence of the NS curve rooted in the ZH point is subject to an inequality constraint.

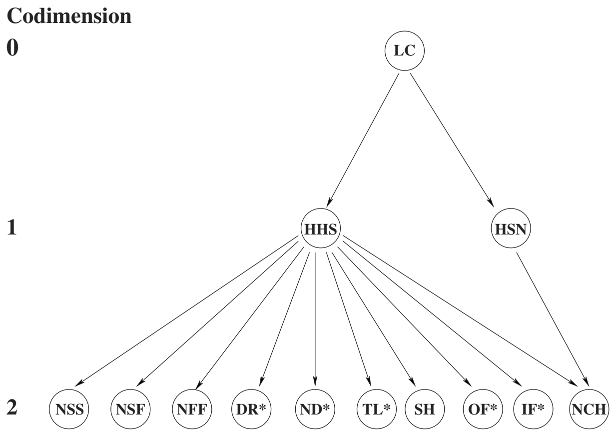

Fig. 2.5. Detection relationships between homoclinic bifurcations of ODEs; here * stands for S or U; the branching relationships are found by following the arrows in the opposite directions.

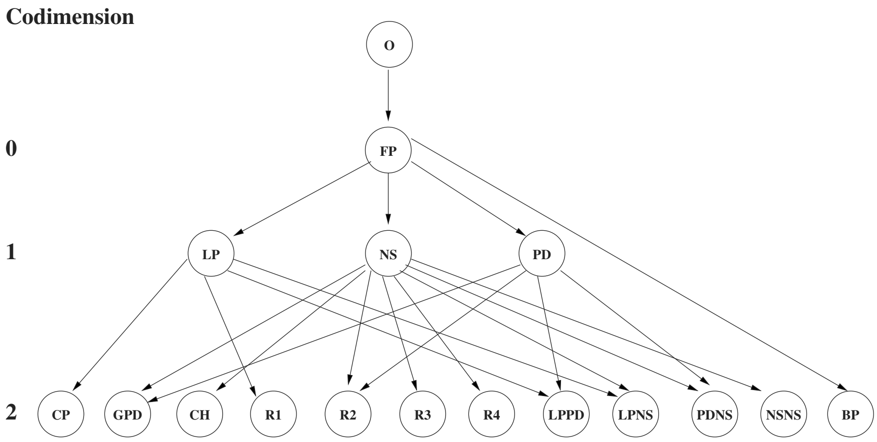

Fig. 2.6. Detection relationships between dynamical objects for maps.

4. The arrow from NS to HH: there are generically two NS curves emanating from an HH point.

We note that generically a curve of HHS orbits emanates from a Bogdanov Takens point (BT), as well as two such curves from a Zero-Hopf bifurcation point (ZH). These are not indicated in Figs. 2.4 and 2.5.

2.2.2 Bifurcation Objects for Cycles of Maps

In this section we present the equivalent objects and relationships for maps. Table 2.3 lists the codimension-zero, -one, and -two objects that can be found

Table 2.3. Objects and associated labels related to equilibria and cycles of maps

Type of object	Label
Point	P
Orbit	O
Fixed Point	FP
Limit Point of cycle bifurcation	LP
Period Doubling Point of cycles	PD
Neimark-Sacker bifurcation	NS
Branch Point	BP
Cusp bifurcation	CP
Generalized Period Doubling	GPD
Chenciner (generalized Neimark-Sacker) bifurcation	CH
1:1 Resonance	R1
1:2 Resonance	R2
1:3 Resonance	R3
1:4 Resonance	R4
Fold–Neimark-Sacker bifurcation	LPNS
Flip–Neimark-Sacker bifurcation	PDNS
Fold-flip	LPPD
Double Neimark-Sacker	NSNS

in generic maps, together with the associated labels [32]. The detection relationships between them are presented in Fig. 2.6. The precise meaning of the arrows is simpler than in the case of ODEs: if we exclude O then an arrow from an object A to an object B indicates that object B can generically be found as a regular point on a branch of objects of type A. The only exception is the arrow from FP to BP which is again not generic but found in many examples that exhibit a form of equivariance or have invariant subspaces.

The branching diagram for maps, on the other hand, is far more complicated than for ODEs; this is largely due to the fact that one needs to consider different iterates of the underlying maps, which causes an additional complication. For reasons of clarity we, therefore, present two branching diagrams; see Figs. 2.7 and 2.8. As before, the arrows indicate the type of object to which one can generically switch from a given codimension-one or -two bifurcation point. If the arrow is dashed then this switching is subject to additional constraints. Furthermore, several switches to branches of lower codimension lead to curves with double, triple or quadruple iteration number, which is indicated by the symbols $\times 2$, $\times 3$, and $\times 4$, respectively.

2.3 The Implementation in MATCONT

The framework described in the previous section has been implemented in the recent Matlab-based software environment MATCONT [11]. At present

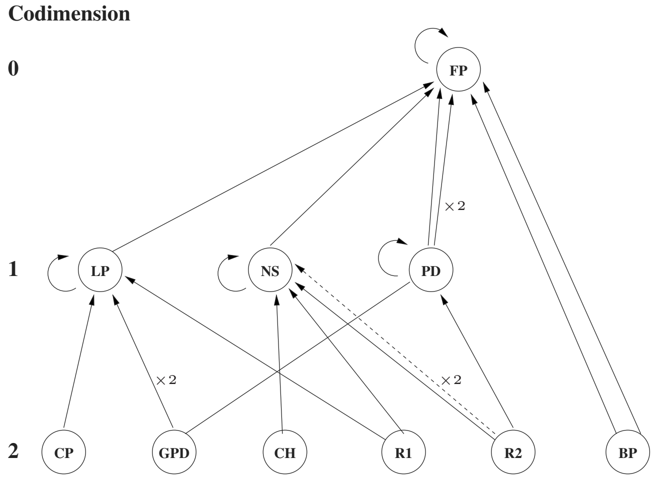

Fig. 2.7. Partial branching relationships for maps; see also Fig. 2.8. Dashed lines indicate switching subject to constraints and ×2 indicates switching to a curve with twice the period.

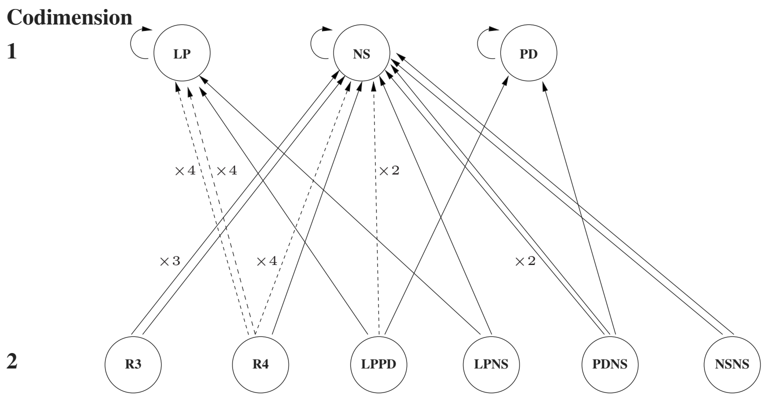

Fig. 2.8. Partial branching relationships for maps; see also Fig. 2.7. Dashed lines indicate switching subject to constraints and ×2 (×3, ×4) indicates switching to a curve with twice (three times, four times) the period.

three related Matlab packages are distributed, namely a command-line version Cl_Matcont and a GUI version Matcont for ODEs, and a command-line version Cl_MatcontM for Maps[13]. As in Auto and Content, limit cycles are computed by an approach based on the discretization via piecewise-polynomial approximation with orthogonal collocation of the corresponding boundary value problem. However, Matcont uses sparse Matlab solvers

[13] Cl_Matcont, Matcont, and Cl_MatcontM are all available via `http://www.matcont.ugent.be/`.

instead of the original AUTO algorithm (a special block elimination; see Chap. 1). The same approach is applied for homoclinic orbits, in combination with the continuation of invariant subspaces for the equilibrium end point of the homoclinic orbit; for details on this method we refer to [10] and for its implementation in MATCONT to [17].

Nearly all functionalities described in Sect. 2.2 are supported. Remaining functionalities (now under construction) are:

- Branch switching at HH (equilibria), NSNS (limit cycles) and NSNS (cycle of maps) to the secondary branch of type H, NS (limit cycle) or NS (cycle of maps), respectively.
- Branch switching from ZH (equilibria) to NS (limit cycles).
- Branch switching from HH (equilibria) to NS (limit cycles).

A computationally more difficult problem is branch switching from ZH to HHS [8]. It is also planned to have a GUI version for CL_MATCONTM and to introduce automatic differentiation routines for the computation of the normal-form coefficients, which are now computed either via numerical directional derivatives or using a user-supplied code. Preliminary evidence indicates that finite difference approximations are not reliable for these computations. Also, for high-order iterates of maps the normal-form computations are much faster when using automatic differentiation compared to symbolically generated derivatives. The computation of normal-form coefficients for codimension-two bifurcations of limit cycles is not yet supported in MATCONT and is another topic for further development.

In some cases normal-form coefficients are not very informative. For example, for a limit point of equilibria (LP) the quadratic normal-form coefficient a is defined up to a nonzero multiple and the LP point is nondegenerate if $a \neq 0$ and degenerate (CP) if $a = 0$. However, because of truncation and round-off errors the value computed for a will always be nonzero. Therefore, the value of a reported at LP points is not very useful. However, provided its continuity along the LP-branch is ensured, this value is important for the detection of the CP-points.

In other cases the normal-form coefficients are very useful for the user, because their values determine the number and type of branches of new objects that emanate from the bifurcation points and whether these objects are stable or not. In the following two sections we discuss the cases that are of particular interest.

2.3.1 Normal-Form Coefficients for Bifurcations of ODEs as Given in MATCONT

The implementation in MATCONT provides normal-form coefficients for all codimension-one and -two bifurcations of equilibria, and periodic normal-form coefficients for all codimension-one bifurcations of limit cycles; see [32] and [33] for further details and notation used. We discuss here four cases:

1. The Hopf bifurcation H of an equilibrium, where the equilibrium has a pair of purely imaginary eigenvalues, is determined by the first Lyapunov coefficient l_1, which is the real part of the third-order coefficient in the complex normal form. If $l_1 < 0$ then the Hopf bifurcation is supercritical, i.e., unstable fixed points coexist with stable periodic orbits on one side of the bifurcation point in the center manifold. If $l_1 > 0$ then the Hopf bifurcation is subcritical, i.e., stable fixed points coexist with unstable periodic orbits on one side of the bifurcation point in the center manifold.
2. The Zero-Hopf bifurcation ZH, also called saddle-node Hopf, fold-Hopf or zero-pair bifurcation [32], is a codimension-two bifurcation where an equilibrium has one zero eigenvalue together with a pair of purely imaginary eigenvalues. The normal form involves quadratic coefficients denoted s and θ; see [32, Lemma 8.11]. An NS curve emanates from the ZH point only if $s\,\theta < 0$. The implementation in MATCONT also computes a relatively technical term $E(0)$. If $E(0) < 0$ then time has to be reversed in the unfolding analysis in [32], i.e., stable becomes unstable, and vice versa.

The above two cases deal with bifurcation of equilibria. For limit cycles we have:

3. The period-doubling or flip bifurcation PD, where the limit cycle has one Floquet multiplier at -1, involves the coefficient c in the periodic normal form that determines the bifurcation. If $c < 0$ then the flip bifurcation is supercritical, i.e., unstable periodic orbits coexist with stable double-period orbits on one side of the bifurcation point in the center manifold. If $c > 0$ then the flip bifurcation is subcritical, i.e. stable periodic orbits coexist with unstable double-period orbits on one side of the bifurcation point in the center manifold.
4. At a Neimark-Sacker bifurcation NS the limit cycle has a pair of complex conjugate Floquet multipliers on the unit circle. The bifurcation is determined by the cubic coefficient $\mathrm{Re}(d)$ of the periodic normal form. If $\mathrm{Re}(d) < 0$ then the NS bifurcation is supercritical, i.e., unstable limit cycles coexist with stable invariant tori on one side of the bifurcation point in the center manifold. If $\mathrm{Re}(d) > 0$ then the NS bifurcation is subcritical, i.e., stable limit cycles coexist with unstable invariant tori on one side of the bifurcation point in the center manifold.

2.3.2 Normal-Form Coefficients for Bifurcations of Maps as Given in MATCONTM

The implementation in MATCONTM provides normal-form coefficients for all codimension-one and -two bifurcations of fixed points; see [32] for details. The codimension-one cases are very similar to the corresponding bifurcations of limit cycles listed in the previous section. The user should, in particular, be aware of:

1. The period-doubling or flip bifurcation PD, where the fixed point has an eigenvalue -1. The sign of the cubic normal-form coefficient b_1 determines whether the bifurcation is supercritical ($b_>0$) or subcritical ($b_1 < 0$) as before.
2. The Neimark–Sacker bifurcation NS, where the fixed point has a pair of complex conjugate eigenvalues on the unit circle. As before, the NS bifurcation is supercritical (subcritical) if the cubic normal-form coefficient $c_1 = \mathrm{Re}(d_1)$ is negative (positive) and no strong resonances (1:1, 1:2, 1:3, 1:4) are present.

For codimension-two bifurcation points the user should pay particular attention to:

1. At a 1:2 Resonance point R2 the fixed point has a pair of complex conjugate eigenvalues on the unit circle that are both at -1. The normal form contains two cubic coefficients C_1 and D_1 that determine this bifurcation. If $C_1 < 0$, then an NS curve of double-period cycles emanates from the R2 point. The MatcontM output is $[c, d] = [4C_1, -2D_1 - 6C_1]$.
2. At a 1:4 Resonance point R4 the fixed point has eigenvalues $\pm i$. This bifurcation is determined by a complex normal-form coefficient $A_0 = a+ib$. If $a^2+b^2-1 > 0$ then two half lines $l_{1,2}$ of limit points of quadruple-period cycles emanate from the R4 point. If $|b| > (1 + a^2)/\sqrt{1 - a^2}$ then there is a curve of quadruple-period cycles that contains an NS bifurcation point.
3. At a fold-flip bifurcation LPPD the fixed point has eigenvalues 1 and -1. MatcontM computes normal-form coefficients $\frac{a}{2e}$ and $\frac{be}{2}$; see [32] for details. If $be > 0$ then an NS curve of double period emanates from the LPPD point. In this case, MatcontM also reports an approximation of the corresponding first Lyapunov coefficient. The NS points of the second iterate are stable in the center manifold if this coefficient is negative; they are unstable if it is positive.

2.4 Examples and Applications

We end this chapter with two examples that illustrate how Matcont is used in practice. In the next section we describe the process of a continuation strategy for a vector field. We use the model of a Van der Pol–Duffing oscillator that is also used in Chap. 4. Section 2.4.2 illustrates the use of MatcontM for a discrete model of a production strategy involving two competing firms.

2.4.1 The Koper Model

In [27] Koper introduced the following model to describe a three-dimensional Van der Pol–Duffing oscillator:

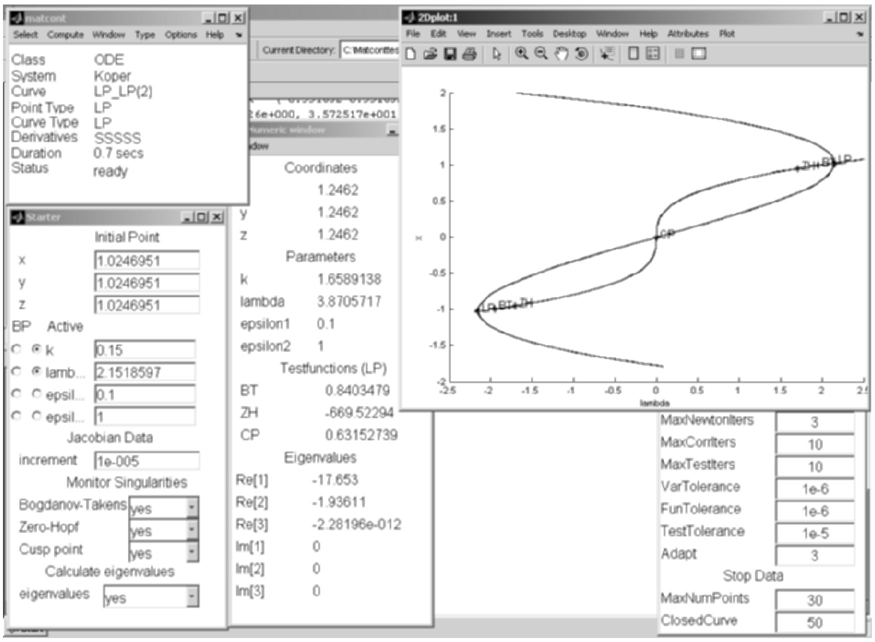

Fig. 2.9. Screen snapshot of MATCONT with the computed equilibrium and LP curves of the Van der Pol–Duffing oscillator (2.1).

$$\begin{cases} \dot{x} = (ky - x^3 + 3x - \lambda)/\varepsilon_1, \\ \dot{y} = x - 2y + z, \\ \dot{z} = \varepsilon_2(y - z). \end{cases} \tag{2.1}$$

As in [27] we use $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 1$. We note that if $(x(t), y(t), z(t))$ is a solution of (2.1) for a particular value of λ, then $(-x(t), -y(t), -z(t))$ is a solution for $-\lambda$. Therefore, bifurcation diagrams in which λ is represented usually have some symmetry.

We begin the analysis of (2.1) by determining the equilibria. Note that an equilibrium solution (x_0, y_0, z_0) satisfies $x_0 = y_0 = z_0$, which must be a solution of

$$kx - x^3 + 3x - \lambda = 0. \tag{2.2}$$

In particular, for $\lambda = 0$ and $k = 0.15$ the equilibria are $(0, 0, 0)$ and (x_0, y_0, z_0) with $x_0 = y_0 = z_0 = \pm\sqrt{3.15} \approx \pm 1.77482393492988$. By selecting one of these latter two points in MATCONT we compute by numerical continuation the solution of (2.2) as a function of λ; the cubic solution curve is visualized in the two-dimensional graphics window of the screen snapshot of MATCONT in Fig. 2.9. On the equilibrium curve MATCONT detects two limit points LP at $\lambda = \pm 2.151860$ and reports for both points the critical normal-form coefficient $a = -4.437060$. We select one of the LP points, set both k and λ as free parameters, and compute a curve of LP points that connects

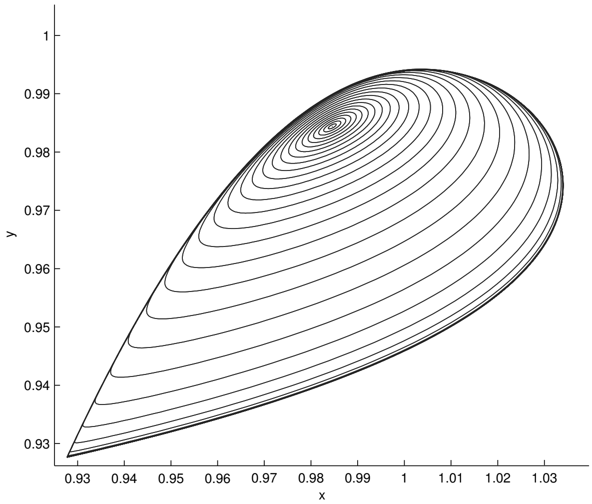

Fig. 2.10. Limit cycles of (2.1) started from a Hopf point and converging to a homoclinic orbit.

the two LP points for $k = 0.15$; this curve is also shown in Fig. 2.9. During the computation of the LP curve MATCONT detects two BT points at $(k, \lambda) = (-0.050000, 1.950209)$ and $(k, \lambda) = (-0.050000, -1.950209)$, with normal-form coefficients $(a, b) = (6.870226e + 000, 3.572517e + 001)$ and $(a, b) = (-6.870226e+000, -3.572517e+001)$, respectively; two Zero-Neutral Saddle points (formally ZH) at $(k, \lambda) = (-0.300000, 1.707630)$ and $(k, \lambda) = (-0.300000, -1.707630)$; and a cusp point CP at $(k, \lambda) = (3.000000, 0.000000)$ with normal-form coefficient $c = 5.649718e - 002$. These codimension-two points are also shown in Fig. 2.9.

Starting from the BT point at $\lambda = 1.950209$ we can compute a Hopf curve in the two free parameters k and λ. We stop, fairly arbitrarily, at the Hopf point with $x_0 = y_0 = z_0 = 0.98460576$, $k = -0.25185549$, and $\lambda = 1.7513143$. Starting from this point we keep k fixed and compute a curve of limit cycles (LC) as a function of λ; see Fig. 2.10. It is visually clear that the limit cycles converge to a homoclinic orbit; this can also be inferred from the fact that the parameters change very slowly at the end of the continuation, while the period increases rapidly.

When computing limit cycles, MATCONT allows for the computation and visualization of their *phase response curves* (PRC) [21] as well as the time derivatives of these phase response curves (dPRC). The study of such curves is an important subject in the theory of weakly connected neural networks [23]. In particular, it is well known that they take very specific shapes in the neighborhoods of bifurcations of limit cycles [7]. We demonstrate this by presenting

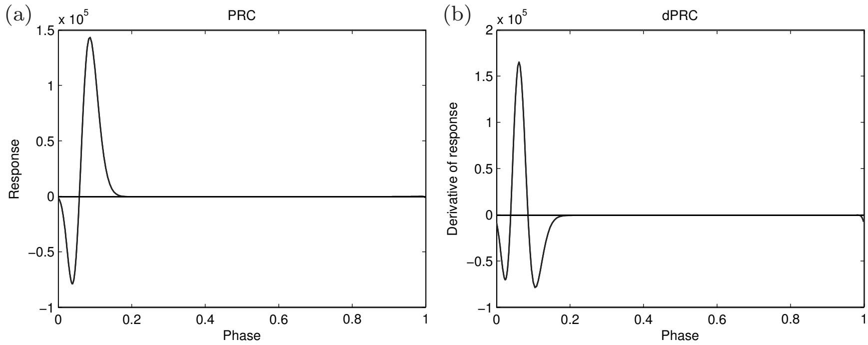

Fig. 2.11. The phase response curve PRC (a) and its derivative dPRC (b) of a limit cycle of (2.1) close to a homoclinic orbit.

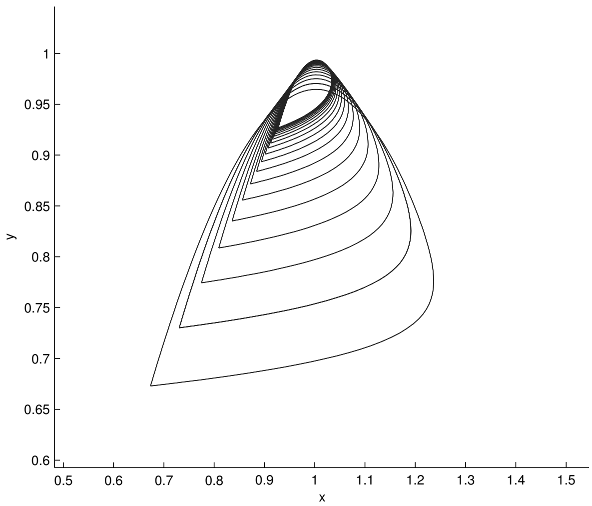

Fig. 2.12. Continuation of an orbit of (2.1) that is homoclinic to a hyperbolic saddle, starting from a limit cycle with large period.

the curves PRC and dPRC in Fig. 2.11 for the limit cycle of the above continuation at $\lambda = 1.7510571$. This limit cycle has period 46.799011, that is, it is close to a homoclinic orbit.

It is possible in MATCONT to start a continuation of homoclinic orbits in two parameters from a limit cycle close to a homoclinic orbit; see Fig. 2.5. An example is presented in Fig. 2.12, where we start from the last limit cycle computed in the previous run, declare it to be of type HHS, and choose k and λ as the two free parameters. The time length of the discretized part of the

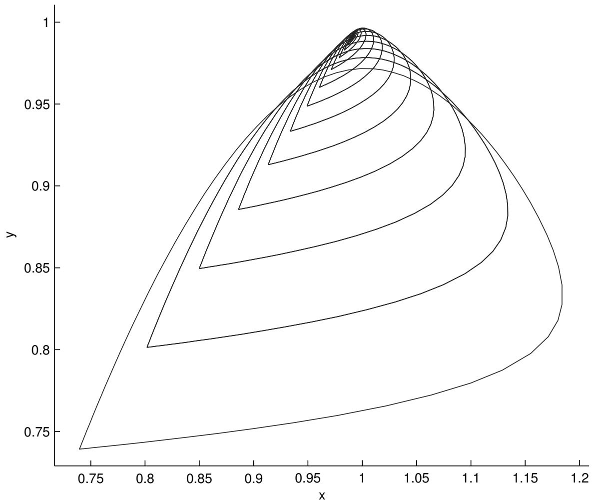

Fig. 2.13. Continuation of an orbit of (2.1) that is homoclinic to a hyperbolic saddle, starting from a Bogdanov-Takens point.

orbit is kept fixed at the period of the original limit cycle while the distances from the end points in the stable and unstable directions are free.

It is also possible to start the continuation of a curve of homoclinic orbits from a Bogdanov-Takens point BT; cf. Sect. 2.2.1. An example of such a continuation is presented in Fig. 2.13. Here we started from the BT point at $\lambda = 1.950209$ that is shown in Fig. 2.9. In this case the distance from the end point in the unstable direction was fixed.

2.4.2 The Duopoly Model

We demonstrate the use of MatcontM for an example of two competing firms that decide on annual production quantities in a duopoly environment. The two firms are homogeneous with regard to forming their expectation and the action effect on each other. The model that we use is the two-dimensional map

$$F : \begin{cases} x_1(t+1) = (1-\rho)x_1(t) + \rho\mu x_2(t)(1-x_2(t)), \\ x_2(t+1) = (1-\rho)x_2(t) + \rho\mu x_1(t)(1-x_1(t)), \end{cases} \tag{2.3}$$

described in [2, 26]. The duopoly model assumes that at each discrete time t the two firms produce the quantities $x_1(t)$ and $x_2(t)$, respectively, and decide their productions $x_1(t+1)$ and $x_2(t+1)$ for the next period. The parameter $\mu > 0$ measures the intensity of the effect that one firm's actions has on the other firm. The parameter ρ, which is typically in $[0, 1]$, has an averaging effect.

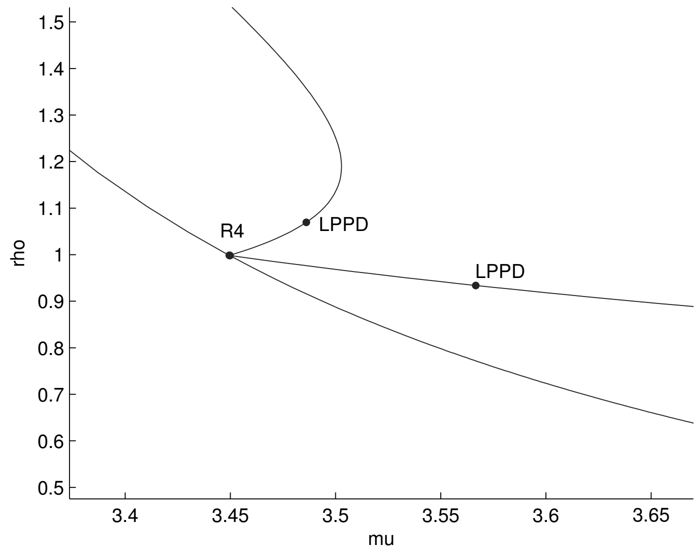

Fig. 2.14. An R4 point on an NS curve of (2.3) with emanating branches of fold curves of period-four cycles.

We start with parameter values $\mu = 3.5$ and $\rho = 0.1$. It is checked easily that F has a fixed point

$$(x_1, x_2) = \left(\frac{\mu + 1 + \sqrt{(\mu+1)(\mu-3)}}{2\mu}, \frac{\mu + 1 - \sqrt{(\mu+1)(\mu-3)}}{2\mu} \right).$$

We now perform a continuation of fixed points of F with free parameter ρ and find an NS point at $(x_1, x_2) = (0.857143, 0.428571)$ for $\rho = 0.888889$. The normal-form coefficient is $-6.273434\text{e}{+}001$. Since it is negative, the NS bifurcation is supercritical.

Starting from this NS point we can now compute a curve of NS points in the two free parameters μ and ρ. On this curve we find a 1:4 resonance point R4 at $(x_1, x_2) = (0.849938, 0.439960)$ for $\rho = 1.000000$ and $\mu = 3.449490$. It is worthwhile to note that this R4 point lies precisely on the boundary of the region where $\rho \leq 1$, i.e., the region that is relevant from the application's point of view. The normal-form coefficient is $A_0 = (-3.000000\text{e}{+}000 - 9.231411\text{e}{-}017\, i)$. Since $|A_0| > 1$, two cycles of period four are born at the R4 point. Furthermore, near the R4 point their region of existence is bounded by two fold curves of period-four cycles that emanate from the R4 point. We can start the continuation of these curves from the codimension-two point R4 in MatcontM. Interestingly, on each of these two curves an LPPD point (of the fourth iterate) is found. A picture of this situa-

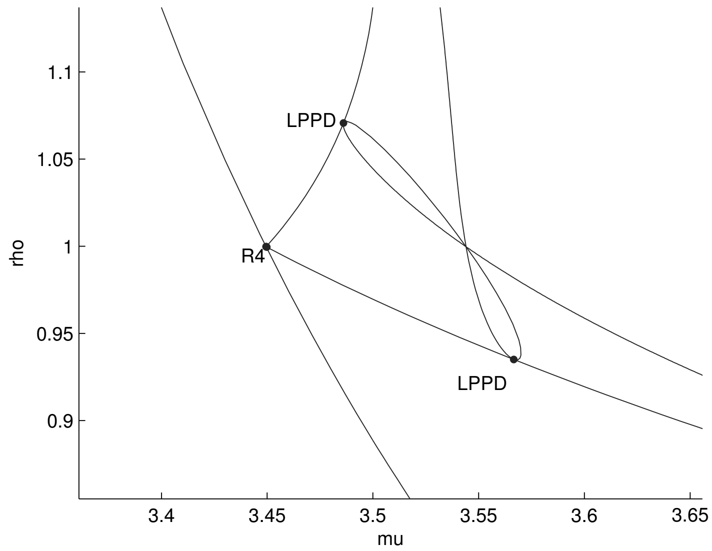

Fig. 2.15. The diamond-shaped region bounded by period-four LP and PD curves near the R4 point on the NS curve of (2.3) contains stable period-four cycles.

tion is presented in Fig. 2.14. The lower LPPD point is detected at $(x_1, x_2) = (0.841586, 0.354516)$ for $\rho = 0.935299$ and $\mu = 3.566686$; its normal-form coefficients are $a/(2e) = 2.574002\text{e}{+}000$ and $be/2 = -5.829597\text{e}{+}001$. The upper LPPD point is detected at $(x_1, x_2) = (0.836428, 0.522216)$ for $\rho = 1.071080$ and $\mu = 3.486079$, and has normal-form coefficients $a/(2e) = 1.733856\text{e}{+}000$ and $be/2 = -2.471512\text{e}{+}001$. We note that the lower LPPD point is in the region relevant to applications while the upper one is not.

It is further interesting to compute the PD curves that emanate from the LPPD points and they are presented in Fig. 2.15. The stable period-four cycles exist in the diamond-shaped region bounded by curves LP and PD of the fourth iterate F^4.

2.5 Directions for Future Development

We presented an overview of software tools for bifurcation analysis. At present, the state of the art is a software environment that provides a clear continuation strategy as implemented in Matcont. We showed two examples of how to use Matcont for ODEs and maps. Matcont has the advantage that it is implemented in Matlab, which is standard in many applied fields, particularly in engineering. Furthermore, its numerical capabilities include the computation of normal-form coefficients and automatic branch switching.

The development of MATCONT is ongoing. In the near future, we hope to implement the remaining functionalities listed in Sect. 2.3. Furthermore, we plan to include algorithms for the computation of invariant manifolds and develop facilities to analyze global bifurcations. Other directions for further development would be the provision of higher-codimension bifurcations and the corresponding detection and branching relationships. Moreover, it would be of interest to generalize the functionalities to other classes of systems, for example, systems with symmetry or preserved quantities; see also Chap. 9.

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