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QUANTIFIERS IN FORMAL AND NATURAL LANGUAGES

For a long time, the word 'quantifier' in linguistics and philosophy simply stood for the universal and existential quantifiers of standard predicate logic. In fact, this use is still prevalent in elementary textbooks. It seems fair to say that the dominance of predicate logic in these fields has obscured the fact that the quantifier expressions form a *syntactic category*, with characteristic interpretations, and with many more members than $\forall$ and $\exists$.

Actually, when Frege discovered predicate logic, it was clear to him that the universal and existential quantifiers were but two instances of a general notion (which he called *second level concept*). That insight, however, was not preserved during the early development of modern logic. It took quite some time before the mathematical machinery behind quantification received, once more, an adequate genera formulation. This time, the notion was called *generalised quantifier*; a first version of it was introduced by Mostowski in the late 1950s. Logicians gradually realised that generalised quantifiers were an extremely versatile syntactic and semantic tool — practically anything one would ever want to say in any logic can be expressed with them. The power of expression, properties and interrelations of various logics with generalised quantifies is now a well established domain of study in mathematical logic.

This is the mathematical side of the coin. The linguistic side looks a bit different. Syntactically, there are many expressions one could place in the same category as *some* and *every*: *no, most, many, at least five, exactly seven, all but three*, ... These expressions — the *determiners* — occur in *noun phrases*, which in turn occur as subjects, objects, etc. in the *NP–VP* analysis of sentences usually preferred by linguists. Logically, however, subject–predicate form had fallen into disrepute since the breakthrough of predicate logic. So it was not obvious how to impose a semantic interpretation on these syntactic forms — except by somehow rewriting them in predicate logic. This may explain why the systematic study of quantifiers in natural language is of a much later date than the one for mathematical language.

The starting-point of this study was when Montague showed that linguistic syntax is, after all, no insurmountable obstacle to systematic and rigorous semantics. Montague did not yet have the quantifiers in a separate category. But in 1981 Barwise and Cooper united Montague's insights with the work on generalised quantifiers in mathematical logic in a study of the characteristics of natural language quantification [Barwise and Cooper, 1981]. At about the same time, but independently and from a slightly different perspective, Keenan and Stavi were investigating the semantic properties of determiner interpretations [Keenan and Stavi, 1986]. It became clear that, in natural language too, the quantifier category is quite rich and semantically powerful. In the few years that have passed since then, the

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subject has developed considerably. In particular, van Benthem has discovered an interesting logical theory behind the mechanisms of natural language quantification — often with no direct counterpart for mathematical language [van Benthem, 1984a].

My main aim in this chapter is to give a comprehensive survey of the logic and semantics of natural language quantification, concentrating on the developments in the last five years or so. the basic tools are the generalised quantifiers from mathematical logic. but it is the *questions* asked about quantifiers, not the methods used, that distinguishes our present perspective on quantifiers from that of mathematical logic.

The basic question facing anyone who studies natural language quantification from a semantic viewpoint can be formulated as follows. Logically, the category of quantifiers is extremely rich. For example, even on a universe with *two* elements, there are $2^{16} = 65536$ possible (binary) quantifiers (the reader who finds this hard to believe may wish to turn directly to Section 4.6 for the explanation). But, in natural languages, just a small portion of these are 'realised' (512, according to Keenan and Stavi). Which ones, and why? What are the *constraints* on determiner interpretations in natural language? what are the *properties* of quantifiers satisfying those constraints.

Most of this paper presents various answers to such questions. But we start, in Section 1, with a selective history of quantifiers: from Aristotle via Frege to modern generalised quantifiers. It will be seen that both Aristotle's and Frege's contributions compare interestingly to the recent developments. That section also gives a thorough introduction to generalised quantifiers, and to some logical issues pertaining to them. In particular, the *logical expressive power* of *monadic* quantifiers is discussed in some detail. Section 2 presents basic ideas of the Montague–Barwise–Cooper–Keenan–Stavi approach to natural language quantification. A number of examples of English quantifier expressions are also collected, as empirical data for alter use. In Section 3, several constraints on quantifiers are formulated and discussed and various properties of quantifiers are introduced. The constraints can also be seen as potential *semantic universals*. Section 4 then presents various results in the *theory* of quantifiers satisfying certain basic constraints; results on how to classify them under various aspects, on how to represent them, on their inferential behaviour and other properties. The paper ends with a brief further outlook and two appendices, one on branching quantification and the other on quantifiers as variables.

This chapter is concerned with the *semantics* of quantification. It examines certain precisely delimited classes of quantifiers that arise naturally in the context of natural language. These classes are related in various ways to the (loosely delimited) class of *natural language quantifiers*, i.e. those that are denotations of natural language determiners. I will make few definite claims about the exact nature of this relationship, but I will discuss several tentative proposals. The idea is to present the *possibilities* for determiner interpretation, and to give a framework sufficiently general for serious discussion of natural language quantifiers, yet restricted in significant ways compared with the generalised quantifier framework of mathematical logic. (I also hope to make it clear that interesting logical issues arise in the restricted framework (and sometimes only in that framework), and thus that logic can fruitfully be inspired by natural language as well as by the language of mathematics.)

So, except for a few rather straightforward things, I shall have little to say about the *syntax* of quantification here. And except for an introductory overview, I will not attempt to survey generalised quantifiers in mathematical logic. For more on quantification and linguistic theory, cf. [Cooper, 1983] or [van Eijck, 1985]. A very comprehensive survey of quantifiers in mathematical logic is given in [Barwise and Feferman, 1985].

The semantic framework used here is that of classical model theory. It is simple, elegant and well known. that it works so well for natural language quantification too is perhaps a bit surprising. However, there are certain things it does not pretend to handle, for example, intensional phenomena, vagueness, collectives, or mass terms. So these subjects will not be taken up here. but then, they receive ample treatment in other parts of this Handbook.

The logical techniques we need are usually quite elementary. the reader should be used to logical and set-theoretic terminology, but, except on a few occasions, there are no other specific prerequisites (the chapter by Hodges in this Handbook gives a suitable background; occasionally, part of the chapter by van Benthem and Doets will be useful). I have intended to make the exposition largely self-contained, in the sense that (a) most proofs and arguments are given explicitly, and (b) when they are not given, references are provided, but he reader should be able to get a feeling for what is going on without going to the references. Naturally, if these intentions turn out not to be realised, it does not follow that the fault lies with the reader.

This is a survey, and most results are from the literature, although several are new, or generalised, or proved differently here. I have tried to give reasonable credit for known results.

1 BACKGROUND FROM ARISTOTELIAN TO GENERALISED QUANTIFIERS

This section gives a condensed account of the development of what can be called *the relational view of quantifiers*. As a chapter in the history of logic, it seems not to be widely known, which is why I have included a subsection on Aristotle and a subsection on Frege. My main purpose, however, is to introduce a precise concept of quantifier sufficiently general to serve as a basis for what will follow. This is the notion of a *generalised quantifier* from mathematical logic. In the last subsections, I will also mention some of the things mathematical logicians do with quantifiers, as a background to what linguistically minded logicians might do with them.

1.1 Aristotle

Aristotle's theory of syllogisms, for ages considered the final system of logic, is not often seen as rather pointless formal exercise, whose main achievement is to have hampered the development of logic for some two thousand years. But to understand Aristotle's contribution to logic one must distinguish his views from those of his followers. It is a fact that most of his followers and commentators were unable, for various reasons, to appreciate his logical insights (to take one simple but important example the point of using *variables*).[1] From the standpoint of modern logic, on the other hand, these insights ought to be easily visible.

There is, however, one obscuring issue. According to widespread opinion, the breakthrough of modern logic rests upon the *rejection* of a basic Aristotelian idea, namely, that sentences have *subject-predicate form*. This was Russell's view, apparently vindicated by the absence of subject-predicate form in today's standard predicate logic. Hence, Aristotle's logic seems to be built on a fundamental mistake.

If we set aside questions concerning the historical causes of the long standstill in logic after Aristotle, there is, however, no necessary incompatibility between modern logic and subject-predicate form.[2] It is quite feasible to give an adequate account of both relations and quantification while preserving subject-predicate form, as we shall see in 2.3. Thus, although it is true that Aristotle's logic could not adequately account for these things, and thus was unable to express many common forms of reasoning, this weakness is not necessarily tied to his use of subject-predicate form.

In addition to matters of syntactic form, however, one ought to consider the *concepts* Aristotle introduced with his logic, the *questions* he raised about it, and the *methods* he used to answer them. Herein lies his greatest contribution.

Thousands of pages have been written on Aristotle's logic, most of them about irrelevant and futile matters (such as the order between the premisses in a syllogism, why he didn't mention the fourth figure, whether a valid syllogism can have a false premiss — Aristotle himself had no doubts about this — , etc.). Readable modern expositions, with references to the older literature, are Łukasiewicz [1957] and Patzig [1959]. Below I wish to point, without (serious) exegetic pretensions, to one important aspect of Aristotle's logic.

The syllogistics is basically a theory of *inference patterns among quantified sentences*. Here a quantified sentence has the form

(1) QXY,

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[1] Actually, contemporaries of Aristotle, like Theophrastus, seem to have understood him rather well. But the medieval reintroduction of Aristotle's logic lost track of many important points. Even 19th century commentators continue in the medieval vein; cf. [Łukasiewicz, 1957].

[2] About the historical causes Russell may well be right. Note that we are also setting aside here the metaphysical claims of Russell's logical atomism, according to which the logical form of sentences mirror the structure of reality.

where X, Y are *universal terms* (roughly 1-place predicate) and Q is one of the quantifiers *all, some, no, not all*. In practice, Aristotle treated these quantifiers as *relations* between the universal terms.[3]

Aristotle chose to study a particular type of inference pattern with sentences of the form (1), the syllogisms. A *syllogism* has two premisses, one conclusion, and three universal terms (variables). Each sentence has two different terms, all three terms occur in the premisses, and one term the 'middle' one, occurs in both premisses but not in the conclusion. It follows that the syllogisms can be grouped into four different 'figures', according to the possible configurations of variables:

$$\frac{\begin{array}{c}Q_1Zy\\ Q_2XZ\end{array}}{Q_3XY} \quad \frac{\begin{array}{c}Q_1YZ\\ Q_2XZ\end{array}}{Q_3XY} \quad \frac{\begin{array}{c}Q_1ZY\\ Q_2ZX\end{array}}{Q_3XY} \quad \frac{\begin{array}{c}Q_1YZ\\ Q_2ZX\end{array}}{Q_3XY}$$

Here the Q_i can be chosen among the above quantifiers, so there are $4^4 = 256$ syllogisms. As a matter of historical fact, Aristotle's specification of the syllogistic form was not quite accurate; he had problems with defining the middle term, and his systematic exposition does not mention the fourth figure (although he in practice admitted syllogisms of this form), but these are minor defects.

Now, the question Aristotle posed — and, in essence, completely answered — can be formulated as follows:

For what choices of quantifiers are the above figures valid?

For example, of we in the first figure let $Q_1 = Q_2 = Q_3 =$ *all*, a valid syllogism results ('Barbara', in the medieval mnemonic); likewise if $Q_1 = Q_2 =$ *no* and $Q_2 =$ *all* ('Celarent'). Note that Aristotle's notion of validity is essentially the modern one: a syllogism is valid if each instantiation of X, Y, Z verifying the premisses also verifies the conclusion (a slight difference is that Aristotle didn't allow the empty or the universal instantiation; this can be ignored here).

There are interesting variants of this type of question. Given some common quantifiers, we can ask for their inference patterns, and try to systematise the answer in some perspicuous way (axiomatically, for example). This is a standard procedure in logic. But we can also turn the question around and ask which quantifiers satisfy the patterns we found: only the ones we started with or others as well? If our common schemes of inference *characterise* our common quantifiers, we have one kind of explanation of the privileged status of the corresponding 'logical constants', and one goal of a theory of quantifiers has been attained.

The latter question is somewhat trivialised in Aristotle's framework, since there were only four quantifiers. For example, the question of which quantifiers satisfy the scheme:

$$\frac{\begin{array}{c}QZY\\ QXZ\end{array}}{QXY}$$

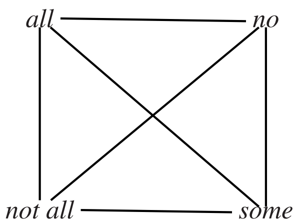
[3]He sometimes comes very close to an explicit statement; cf. the last pages of [Patzig, 1959].

has the obvious answer: just *all*. But the question itself does not depend on the quantifier concept you happen to use. In 4.1 we shall return to it (and in 4.5 to the characterisation of our most common quantifiers), this time with infinitely many quantifiers to choose from, and find some non-trivial answers.

Thus, not only did Aristotle introduce the relational concept of quantifiers, he also asked interesting questions about it. His methods of answering these questions were *axiomatic* (for example, he derived all valid syllogisms from the two syllogisms 'Barbara' and 'Celarent' mentioned above) as well as *model-theoretic* (non-validity was established by means of counter-examples). Even from a modern point of view, his solution leaves only some polishing of detail to be desired. Perhaps this finality of his logic was its greatest 'fault'; it did not encourage applications of the new methods to, say, other inference patterns. Instead, his followers managed to make a sterile church out of his system, forcing logic students to rehearse syllogisms far into our own century. But we can hardly blame Aristotle for that.

It should be noted that outside of logic Aristotle studied quantifiers without restriction to syllogistic form. For example, he made interesting observations on sentences combining negation and quantification (cf. [Geach, 1972]).

We shall not pursue the fate of the relational view of quantifiers between Aristotle and Frege. Medieval logicians spent much time analysing quantified sentences, but they were more or less prevented from having a *concept* of quantifier by their insistence that quantifier words are *syncategorematic*, without independent meaning (this view, incidentally, is still common). Later logicians applied the mathematical theory of relations (converses, relative products, etc.) to give explicit formulations of Aristotle's relational concept, and to facilitate the proofs of his results on syllogisms (cf. [DeMorgan, 1847] or, for a more recent account [Lorenzen, 1958]). These methods were in general only applied to the quantifiers in the traditional *square of opposition* and their converses. A systematic study of quantifiers as binary relations did not appear until the 1980s (cf. Section 4.1).



1.2 Frege

It is undisputed that Frege is the father of modern logic. He invented the language of predicate calculus, and the concept of a formal system with syntactic formation and inference rules. Moreover, his work was characterised by an exceptional theoretical clarity, greatly surpassing that of his contemporaries, and for a long time also his successors, in logic.

There is some difference of opinion, however, as to how 'modern' Frege's conception of logic was. According to Dummett [1973; 1981], we find in Frege, implicitly if not explicitly, just that dualism between a syntactic (proof-theoretic) and a semantic (model-theoretic) viewpoint which is characteristic of modern logic. "Frege would have had within his grasp the concepts necessary to frame the notion of completeness of a formalisation of logic as well as its soundness" [Dummett, 1981, p. 82] Dummett also traces the notion of an interpretation of a sentence, and thereby the semantic notion of logical consequence, to Frege's work.

This evaluation is challenged in [Goldfarb, 1979], a paper on the quantifier linearly (modern) logic. Goldfarb holds the notion of an interpretation to be non-existent in Frege's logic: first, because there are no non-logical symbols to interpret, and second, because the universe is fixed once and for all. The quantifiers range over this universe, and the laws of logic are *about* its objects. furthermore, the logicism of Frege and Russell prevented them, according to Goldfarb, from raising any metalogical questions at all.

Although it takes us a bit beyond a mere presentation of Frege's notion of quantifier, it is worthwhile trying to get clear about this issue. The main point to be made is, I think, that Frege was the only one of the logicians at the time who maintained a sharp distinction between syntax and semantics, i.e. between the expressions themselves and their denotations. This fact alone puts certain metalogical questions 'within the reach' of Frege that would have been meaningless to others. Thus, one cannot treat Frege and Russell on a par here. Moreover, if one loses sight of this, one is also likely to miss the remarkable fact that, while the invention of predicate logic with the universal and existential quantifiers can also be attributed to Peano and Russell, Frege was the only one who had a mathematically precise *concept* of quantifier. This concept seems indeed to have gone largely unnoticed among logicians, at least until the last decade or so; in particular, the inventors of the modern generalised quantifiers do not seem to have been aware of it.

For this reason, Frege, but not Russell, has a prominent place in an historical overview of the relational view of quantifiers — in fact, Russell's explanations of the meaning of the quantifiers are in general quite bewildering (for example, [Russell, 1903, Chapter IV, Sections 59–65], or [Russell, 1956, pp. 64–75 and 230–231]). I will present Frege's concept below, and then return briefly to the issue of how questions of soundness and completeness relate to Frege's logic.

1.2.1 Quantifiers as second level concepts

Let us first recall some familiar facts about Frege's theoretical framework.[4] All entities are either *objects* or *functions*. These categories are primitive and cannot be defined. Functions, however, are distinguished from objects in that they have

[4]More precisely, the system of *Grundgesetze* [1893]. The English translation of the first part of this work by M. Furth is prefaced with an excellent introduction, where more details about Frege's conceptual framework can be found.

(one or more) empty places (they are 'unsaturated'). When the empty places are 'filled' with appropriate *arguments* a *value* is obtained. The value is always an object, but the arguments can either be objects, in the case of *first level* functions, or other functions: *second level* functions take first level functions as arguments, etc. — no mixing of levels is permitted. All functions are total (defined for all arguments of the right sort). They can be called *unary, binary*, etc. depending on the number of arguments.[5]

Concepts are identified with functions whose values are among the two truth values *True* and *False*. Thus they have levels and 'arities' just as other functions.

The meaningful expressions in a logical language ('Begriffsschrift') are simple or complex *names* standing for objects or functions.[6] Names have both a sense ('Sinn') and a denotation ('Bedeutung'); only the denotation matters here. there is a strong parallelism between the syntactic and the semantic level: function names also have empty places (marked by special letters) that can (literally) be filled with appropriate object or function names. In particular, sentences are (complex) object names, denoting truth values.

Complex function names can be obtained from complex object names by deleting simple names, leaving corresponding empty places. For example, from the sentence

23 is greater than 14

we obtain the first level function (concept) names

x is greater than 14,
23 is greater than y,
x is greater than y,

and also the second level

$\Psi(23, 14)$.

Now, suppose the expression

(1) $F(x)$

is a unary first level concept name. Then the following is a sentence.[7]

(2) $\forall x F(x)$.

[5]This notion of 'arity' does not tell us the number of arguments of the arguments, etc; for levels grater than one; we will not need that here.

[6]Actually, Frege did not use "name" for expressions referring to functions. Instead he used "incomplete expression" and the like.

[7]Here I depart from Frege by (i) using modern quantifier notation, and (ii) using the same letter 'x' in (1) and (2). According to Frege, the variable in (1) just marks a place and does not really belong to the concept name, whereas in (2) it is an inseparable part of a function name (cf. below). These distinctions, while interesting, are not essential in the present context.

According to Frege, (2) is obtained by inserting the concept name (1) as an argument in the *second level concept name*

(3) $\forall x \Psi(x)$.

(3) is a *simple* name in Frege's logic. It denotes a unary second level concept, namely, the function which, when applied to any unary first level function $f(x)$, gives the value *True* if $f(x)$ has the value *True* for all its arguments, *False* otherwise.[8]

This, of course, is a version of the usual truth condition for universally quantified sentences: (2) is true iff $F(x)$ is true for all objects x. But Frege's formulation makes it clear that (3) denotes just one of many possible second level concepts, for example,

(4) $\neg\forall x \neg\Psi(x)$

(5) $\forall x(\Phi(x) \rightarrow \Psi(x))$.

(4) is the existential quantifier. (5) is the *binary* second level concept of *subordination* between two unary first level concepts. Both can be defined by means of (3) in Frege's logic, and are thus denoted by complex names.

In a similar fashion, quantification over first level functions can be introduced by means of third level concepts, and so on.

Summarising, we find that there is a well defined *Fregean concept of quantifier*:

> *Syntactically, (simple) quantifier names can be seen as variable-binding operators (but see Note 7 on Frege's use of variables). Semantically, quantifiers are second level concepts.*

If we let, in a somewhat un-Fregean way, the *extension of an n-ary first level concept* be the class of n-tuples of objects falling under it, and the *extension of an n-ary second level concept* the class of n-tuples of extensions of first level concepts falling under it, then the extensions of the quantifiers (3)–(5) are

(6) $\forall_u = \{X \subseteq U : X = U\}$

(7) $\exists_u = \{X \subseteq U : X \neq \varnothing\}$

(8) $\mathbf{all}_i = \{\langle X, Y\rangle : X \subseteq U \& Y \subseteq U \& Y \subseteq Y\}$,

where U is the class of all objects. Apart from the fact that the universe is fixed here (and too big to be an element of a class), these extensions are generalised quantifiers in the model-theoretic sense; cf. Section 1.4.

[8]Note that the quantifier (3) must be defined for all unary first level functions (not only for concepts), since functions are total. As we can see, $\forall x \Psi(x)$ is *false* for arguments that ar enot concepts.

1.2.2 Unary vs. binary quantifiers

Frege was well aware that the usual quantifier words in natural language stand for *binary* quantifiers. For example, in 'On Concept and Object' he writes

> . . . the words 'all', 'any', 'no', 'some' are prefixed to concept-words. In universal and particular affirmative and negative sentences, we are expressing *relations between concepts*; we use the words to indicate a special kind of relation ([Frege, 1892, p. 48], my italics).

But he also found that these binary (Aristotelian) quantifiers could be defined by means of the unary (3) and sentential connectives. This was no trivial discovery at the time, and Frege must have been struck by the power and simplicity of the unary universal quantifier. In his logical language he always chose it as the sole primitive quantifier.

The use of unary quantifiers was to become a characteristic of predicate logic, and the success of formalising mathematical reasoning in this logic can certainly be said to have vindicated Frege's choice. It does not follow from this, however, that the same choice is adequate for formalising natural language reasoning. Indeed, we will see later that unary quantifiers are unsuitable as denotations of the usual quantifier words, and that, furthermore, it is simply not the case that all binary natural language quantifiers can be defined by means of unary ones and sentential connectives.

Such a preference for binary quantifiers in a natural language context is, as we can see from the foregoing, in no way inconsistent with Frege's view on quantifiers.[9]

1.2.3 Logical truth and metalogic

Let us return to the DummettGoldfarb dispute about whether metalogical issues such as completeness were in principle available to Frege. The usual notion of completeness of a logic presupposes the notion of logical truth (or consequence),

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[9]There may be deeper reasons for preferring binary quantifiers. For example, [Dummett, 1981] regards Frege's decision to use a unary quantifier as *the* fatal step which eventually led to paradox in his system. This is because in the unary case we quantify over all objects, whereas binary quantifiers can restrict the domain to that part of the universe denoted by the first argument (as we will see in Section 2), thereby avoiding the need to consider a total universe [Dummett, 1981, p. 227].

This argument may point to one cause of Frege's actual choice of an inconsistent system, but it is not by itself conclusive against unary quantifiers. The lesson of the paradoxes is not necessarily that one must not quantify over all objects. Indeed, the Tarskian account of the truth conditions for universally quantified sentences is quite independent of the size of the universe, and logicians often quantify over total domains, e.g. the domain of all sets in Zermelo–Fraenkel set theory, without fearing paradox. (It is another matter that they,for 'practical' reasons, often prefer set domains when this is possible.) So the above argument can only have force, I think, when combined with a general theory of meaning of the type that Dummett advocates (and which in some sense rejects the Tarskian account). These deeper issues in the theory of meaning will not be discussed here.

i.e. truth in all models. But the latter notion was clearly not considered by Frege. As Goldfarb remarks, he had no non-logical constants whose interpretation could vary (it seems that he explicitly rejected the use of such constants; cf. Hodges' chapter, section 17), nor did he consider the idea that the universe could be varied. One universe was enough, namely, the universe U of all objects, and only simple truth in U interested Frege.

However, the notion of truth in U is very close to the notion of logical truth. to fix ideas, consider some standard version of *higher-order logic* (say, the logic L_ω presented in the chapter by van Benthem and Doets, Section 3.1). For the purposes of the present discussion we may identify *Frege's logic* with higher-order logic *without* non-logical symbols.[10] Then we can observe that Frege did not 'miss' any standard logical truths. For, each sentence ψ in L_ω has an obvious translation ψ^* in Frege's logic, obtained by 'quantifying out' the non-logical constants. For example,

$$\forall x Px \rightarrow Pa$$

translates as

$$\forall X \forall y (\forall x Xx \rightarrow Xy),$$

and similarly for higher-order sentences. It is evident that

(9) if ψ is logically true then ψ^* is true in U.

A parenthetical observation is necessary here. Logical truth is often defined as truth in all *set* models, instead of truth in *all* models, whether sets or not. The latter notion is *real* logical truth, and it is with respect to this notion that (9) is evident. As Kreisel has stressed, use of the former notion is only justified for first-order logic, since there the two notions coincide (this follows from the usual completeness proofs). For higher-order sentences, on the other hand, this is open; cf. [Kreisel, 1967].

For first-order logic, there is a converse to (9), provided we disregard sentences such as

$$\exists x \exists y (x \neq y),$$

which have *finite* counter-examples but are still true in the infinite U:

THEOREM 1. *Let M be any infinite class and ψ a first-order sentence. then ψ is true in all infinite models iff ψ^* is true in M.*

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[10] Frege's logic, that is, not his whole system with its (inconsistent) principles of set existence (abstraction). The proposed identification slurs over some details, but is consistent with Frege's idea that logic is about a domain of *objects* (U), upon which a structure of functions of different levels is built, with no mixing between functions and objects, or between functions of different levels.

Proof. (This proof uses some standard techniques of first-order model theory; they can be found in [Chang and Keisler, 1973];; but will not be employed in the sequel.) From left to right this is similar to (9); if only set models are considered we employ Kreisel's observation mentioned above. For the other direction, suppose that $\neg\psi = \neg\psi(P, \ldots)$ has an infinite model $\mathbf{N} = \langle N, R, \ldots \rangle$. Again by Kreisel's observation, we can assume that N is a set. Now distinguish two cases, depending on whether M is a set or not. If M is a set, application of the Löwenheim–Skolem–Tarksi theorem gives us a model $\mathbf{M}_0$ of $\neg\psi$ with the same cardinality as M. Via a bijection from M to $M_0, \mathbf{M}_0$ is isomorphic to a model $\langle M, S, \ldots \rangle$ of $\neg\psi$ with universe M. Thus, $\exists X \ldots \neg\psi(x, \ldots)$ is true in M, i.e. ψ^* is false in M, as was to be proved. Now suppose M is a proper class. Starting with $\mathbf{N} = \mathbf{N}_0$ as before, define uniformly for each ordinal α a model $\mathbf{N}_\alpha$ such that $\mathbf{N}_\alpha$ is a proper elementary extension of $\mathbf{N}_\beta$ when $\beta < \alpha$. The union $\mathbf{M}'$ of all these is then a model of $\neg\psi$ (Tarski's union lemma). Moreover, M' is a proper class, whence there is a bijection from M to M'. It follows as before that ψ^* is false in M. ■

Thus, in a sense it makes no difference for first-order logic if we have, as Frege did, a fixed infinite universe (such as U) and no non-logical constants. More precisely, it follows from the above that the true Π^1_1 sentences of Frege's logic correspond exactly to the standard first-order logical truths on infinite models.

In conclusion, then, we have seen that notions such as completeness and soundness were not directly available to Frege, since they presuppose a notion of logical truth he did not have. But Dummett's position is still essentially correct, I think: Frege's work does introduce a version of the dualism between model theory and proof theory. For, Frege had the notion of *truth*, which he certainly did not confound with *provability*. Clearly he considered all theorems of his system to be true. He did not, as far as we know, raise the converse question of whether all true sentences are provable, but surely it was 'within his grasp'. And for his *logic*, this question turns out to be a version of the completeness question, as noted above. Moreover, the answer is *yes* if we restrict attention to Π^1_1 sentences (by the above result and the completeness of first-order logic), *no* otherwise (higher-order logic is not complete).

1.3 Mostowskian Quantifiers

As we know, Frege's work was neglected in the early phase of modern logic, and the rigor he attained, especially in semantics, was not matched for a long time. But the language of predicate logic was powerful enough to be a success even in the absence of a solid semantic basis. In the history of quantifiers, this period is mainly interesting for its discussions on the role of quantification over infinite domains for the foundation of mathematics, but that is not a subject here.

The idea of a mathematically sharp dividing line between syntax and semantics began to reappear gradually in the 1920s, but not until Tarski's truth definition in 1936 did the notion of truth (in a model) become respectable. Tarski's truth conditions for universally and existentially quantified formulas treat $\forall$ and $\exists$ syncategorematically, but it is natural to try some other quantifiers here, i.e. to consider formulas

$$Qx\psi$$

for Q other than $\forall$ and $\exists$. For example, it is clear what the truth conditions for $\exists_{\geq n}$ and $\exists_{=n}$ should look like. To get a general concept, however, we must treat Q non-syncategorematically, i.e. we must have a *syntactic category* 'quantifier' with a specified range of interpretations. Such a general concept appeared in [Mostowski, 1957].

Recall that Tarski defines the relation

$$\mathbf{M} \vdash \phi[g],$$

('g satisfies ϕ in $\mathbf{M}'$), where $\mathbf{M}$ is model, g an assignment of elements in M to the variables, and ϕ a formula. When ϕ is $\forall x\psi$ or $\exists x\psi$, this can be expressed as a condition on the *set*

$$\psi^{\mathbf{M},g,x} = \{a \in M : \mathbf{M} \vDash \psi[g(a/x)]\}.$$

Thus,

$$\mathbf{M} \vDash \forall x\psi[g] \Leftrightarrow \psi^{\mathbf{M},g,x} = M,$$
$$\mathbf{M} \vDash \exists x\psi[g] \Leftrightarrow \psi^{\mathbf{M},g,x} \neq \varnothing,$$
$$\mathbf{M} \vDash \exists_{\geq n} x\psi[g] \Leftrightarrow \mid \psi^{\mathbf{M},g,x} \mid \geqq n.$$

A condition on subsets of M is, extensionally, just a set of subsets of M. So Mostowski defines a (local) *quantifier on* M to be a set of subsets of M, whereas a (global) *quantifier* is a function(al) $\mathbf{Q}$ assigning to each non-empty set M a quantifier $\mathbf{Q}_M$ on M. Syntactically, a quantifier symbol Q belongs to $\mathbf{Q}$, such that $Qx\psi$ is a formula whenever x is a variable and ψ is a formula, with the truth condition

$$\mathbf{M} \vDash Qx\psi[g] \Leftrightarrow \psi^{\mathbf{M},g,x} \in \mathbf{Q}_M.$$

Examples of such quantifiers are

$$\forall_m = \{M\},$$
$$\exists_M = \{X \subseteq M : X \neg \varnothing\},$$
$$(\exists_{\geq n})_M = \{X \subseteq M : \mid X \mid \geqq n\},$$
$$(\mathbf{Q}_\alpha)_M = \{X \subseteq M : \mid X \mid \geqq \aleph_\alpha\}, \text{ (the cardinality quantifiers)}$$
$$(\mathbf{Q}_C)_M = \{X \subseteq M : \mid X \mid = \mid M \mid\}, \text{ (the Chang quantifier)}$$
$$(\mathbf{Q}_R)_M = \{X \subseteq M : \mid X \mid > \mid M - X \mid\} \text{ (Rescher's 'plurality quantifier').}$$

All of these satisfy the following condition:

ISOM If f is a bijection from M to M' then $X \in \mathbf{Q}_m \Leftrightarrow f[X] \in \mathbf{Q}_{M'}$.

In fact, Mostowski included *ISOM* as a defining condition on quantifiers, expressing the requirement that 'quantifiers should not allow us to distinguish between element/of M/' [1957, p. 13].

1.4 Generalised Quantifiers

Rescher, introducing the quantifier $\mathbf{Q}_R$, noted that $Q_R x\psi(x)$ expresses

(1) Most things (in the universe) are ψ,

but that the related (and more common)

(2) Most ϕs are ψ

cannot be expressed by means of Q_R [Rescher, 1962]. From our discussion of Frege we recognise (2) a *binary* quantifier, **most**, giving, on each M, a binary relation between subsets of M:

$$\mathbf{most}_M = \{\langle X, Y\rangle \in M^2 :\mid X \cap Y \mid > X - Y \mid\}.$$

To account for this, the construction of formulas must be generalised. This was noted by [Lindström, 1966], who introduced the concept of a *generalised quantifier*, defined below.

(2) can be formalised as

$$\textit{most } x, y(\phi(x, \psi(y)).$$

Here the free occurrences of $x(y)$ in $\phi(\psi)$ are bound by the quantifier symbol. In fact, the choice of variables is arbitrary; we can write

$$\textit{most } z, x(\phi(x), \psi(x)),$$

or, more simply,

$$\textit{most } x(\phi(x), \psi(x)).$$

In this way Mostowskian quantifiers on M are generalised to n-ary relations between subsets of M. A further generalisation is to consider relations between *relations* on M. Here is an example:

$$\mathbf{W}^r_M = \{\langle X, R\rangle : X \subseteq M \& R \subseteq M^2 \& R \text{ wellorders } X\}$$

(The name of this quantifier will be explained later). The statement that (the set) ϕ is wellordered by (the relation) ψ is formalised as

$$W^r x, yz(\phi(x), \psi(y, z))$$

(note that y and z are simultaneously bound in ψ).

Quantifiers are associated with *types* (finite sequences of positive numbers; Mostowskian quantifiers have type $\langle 1\rangle$, **most** has type $\langle 1, 1\rangle$, and $\mathbf{W}^r$ has type $\langle 1, 2\rangle$; the principle should be clear. We are now prepared for the following

DEFINITION 2. A (local) *generalised quantifier of type* $\langle k_1, \ldots, k_n\rangle$ *on* M is an n-ary relation between subsets of $M^{k_1}, \ldots, M^{k_n}$, respectively, i.e. a subset of

$P(M^{k_1}) \times \cdots \times P(M^{k_n})$. A (global) *generalised quantifier of type* $\langle k_1, \ldots, k_n \rangle$ is a function(al) **Q** which to each set M assigns a generalised quantifier $\mathbf{Q}_M$ of type $\langle k_1, \ldots, k_n \rangle$ on M. To **Q** belongs a quantifier symbol Q (of the same type) with the following rule: If $\phi_1, \ldots, \phi_n$ are formulas and $\bar{x}_1, \ldots, \bar{x}_n$ are strings of distinct variables of length $k_1, \ldots, k_n$, respectively, then $Q\bar{x}_1, \ldots, \bar{x}_n(\phi_1, \ldots, \phi_n)$ is a formula with the truth condition

$$\mathbf{M} \vDash Q\bar{x}_1 \ldots \bar{x}_n(\phi_1, \ldots, \phi_n)[g] \Leftrightarrow \langle \phi_1^{\mathbf{M},g,\bar{x}_1}, \ldots, \phi_n^{\mathbf{N},g,\bar{x}_n} \rangle \in \mathbf{Q}_M.$$

This definition expresses our final version of the relational view of quantifiers, the one we will use in the sequel. It should be clear that, apart from the relativisation to an arbitrary universe M, the notion of a generalised quantifier (or a *Lindström quantifier* as it is sometimes called) is essentially the same as Frege's notion of a second level concept.[11]

Most of the time we will restrict attention to quantifiers of type $\langle 1, 1, \ldots, 1 \rangle$. These are the *monadic generalised quantifiers*; we will usually call them just *quantifiers*. We can then continue to talk about *unary, binary*, etc. quantifiers, when we mean generalised quantifiers of type $\langle 1 \rangle$, $\langle 1, 1 \rangle$, etc.

Like Mostowski, Lindström included *ISOM* in the definition of generalised quantifiers:

$ISOM$ *If f is a bijection from M to M' then* $\langle R_1, \ldots, R_n \rangle \in \mathbf{Q}_M$
$\Leftrightarrow \langle f[R_1], \ldots, f[R_n] \rangle \in \mathbf{Q}_{M'}$.

(If R is k-ary, $f[R] = \{\langle f(a_1), \ldots, f(a_k) \rangle : \langle a_1, \ldots, ak \rangle \in R\}$.)

Here are some further examples of generalised quantifiers:

$$\begin{aligned}
\mathbf{all}_M &= \{\langle X, Y \rangle \in M^2 : X \subseteq Y\}, \\
\mathbf{some}_M &= \{\langle X, Y \rangle \in M^2 : X \cap Y \neq \varnothing\}, \\
\mathbf{I}_M &= \{\langle X, Y \rangle \in M^2 :\mid X \mid=\mid Y \mid\}, \\
\mathbf{more}_M &= \{\langle X, Y \rangle \in M^2 :\mid X \mid>\mid Y \mid\}, \\
\mathbf{W}_M &= \{R \subseteq M^2 : R \text{ wellorders } M\}.
\end{aligned}$$

I is the *Härtig quantifier*, **more** is sometimes called the *Rescher quantifier* (although Rescher only considered the quantifiers $\mathbf{Q}_R$ and **most** above). **W** is the *wellordering quantifier*. The generalised quantifier $\mathbf{W}^r$ given before is the *relativisation* of **W**. This notion is defined as follows.

DEFINITION 3. If **Q** is of type $\langle k_1, \ldots, k_n \rangle$, the *relativisation of* **Q** is the generalised quantifier $\mathbf{Q}^r$ of type $\langle 1, k_1, \ldots, k_n \rangle$ defined by

$$\langle X, R_1, \ldots, R_n \rangle \in \mathbf{Q}^r_M \Leftrightarrow \langle R_1 \cap X^{k_1}, \ldots, R_n \cap X^{k_n} \rangle \in \mathbf{Q}_X$$

[11] Neither Mostowski nor Lindström seem to have been aware of Frege's concept. there is, however, a tradition within type theory which builds on Frege's work, starting with Church's logic of sense and denotation (cf. [Church, 1951]). More recent works are, e.g. [**?**; Daniels and Freeman, 1978].

(for all $X \subseteq M$ and $R_i \subseteq M^{k_i}$).

Thus for $X \subseteq M$ we can use Q^r to express in M what Q says in X; this will be made precise in 1.6. Note that **all** = $\forall^r$, **some** = $\exists^r$, and **most** = Q^r_R.

1.5 Partially Ordered Prefixes

At this point it is appropriate to mention another generalisation of quantifiers, although not directly related to the relational view. In standard predicate logic each formula can be put in *prenex form*, i.e. with a *linear prefix* $Q_1x_1 \ldots Q_nx_n$, where Q_i is either $\forall$ or $\exists$, in front of a quantifier-free formula. Henkin [1961] suggested a generalisation of this to *partially ordered* or *branching* prefixes, e.g. the following

(1)
$$\begin{array}{l} \forall x - \exists y \\ \qquad\qquad\qquad \searrow \\ \qquad\qquad\qquad\quad \phi(x, y, z, u) \\ \qquad\qquad\qquad \nearrow \\ \forall z - \exists u \end{array}$$

The prefix in (1) is called the *Henkin prefix*. The intended meaning of (1) is that for each x there is a y and for each z there is a u such that $\phi(x, y, z, u)$, where y and u are chosen *independently* of one another. To make this precise one uses *Skolem functions*. (1) can then be written

(1′) $\exists f \exists g \forall x \forall z \phi(x, f(x), z, g(z))$.

The method of Skolem functions works for all prefixes with $\forall$ and $\exists$. For example, the first-order

(2) $\forall x \forall z \exists y \exists u \phi(x, y, z, u)$,

(3) $\forall x \exists y \forall z \exists u \phi(x, y, z, u)$

become

(2′) $\exists f \exists g \forall x \forall z \phi(f(x, z), z, g(x, z))$.

(3′) $\exists f \exists g \forall x \forall z \phi(x, f(x), z, g(x, z))$.

But the dependencies in (1′) cannot be expressed in ordinary predicate logic; somewhat surprisingly, the Henkin prefix greatly increases the expressive power, as we shall see in 1.6.

Although branching quantification generalises another feature of ordinary quantification than the one we have been considering here, it can in fact, be subsumed under the relational view of quantifiers. To the Henkin prefix, for example, corresponds the *Henkin quantifier* **H** of type $\langle 4 \rangle$, defined by

$$\mathbf{H} = \{R \subseteq M^4 : \text{ there are functions } f, g \text{ on } M \text{ such that for all } a, b \in M, \langle a, f(a), b, g(b) \rangle \in R\}.$$

The formula (1) is then written, in the notation of 1.4, (1″) $Hxyzu\phi(x, y, z, u)$.

Observe that branching was only defined for $\forall$ and $\exists$. Can we let other quantifiers branch as well, and consider formulas such as

(4) $$\begin{matrix} Q'x \searrow & \\ & \phi(x, y)? \\ Q''y \nearrow & \end{matrix}$$

It is not immediate what this should mean. Compare the linear

(5) $Q'xQ''y\phi(x, y)$;

this is true in $\mathbf{M}$ iff $X = \{a \in M : \mathbf{M} \vDash Q''y\phi[a, y]\}$ is in $\mathbf{Q}'_M$, and, for each $a \in M, \mathbf{M} \vDash Q''y\phi[a, y]$ iff $Y_a = \{b \in M : \mathbf{M} \vDash \phi[a, b]\}$ is in $\mathbf{Q}''_M$. But the idea with (4) is to evaluate the quantifiers *independently*of each other, and then it is not clear which sets to look for in $\mathbf{Q}'_M$ and $\mathbf{Q}''_M$. Nevertheless, Barwise [1979] shows that for certain Q' and Q'' a reasonable interpretation of (4) can be given, and Westerståhl [1987] extends this to arbitrary Q' and Q''.

Branching quantification is not only of mathematical interest. It can be argued that both the Henkin prefix and the form (4) (for certain non-first-order Q' and Q'') occur essentially in natural languages. Barwise [1979] contains a good presentation of the issues involved here; a brief review will be given in Appendix A.

1.6 Model-Theoretic Logics

The introduction of generalised quantifiers opens up a vast area of logical study. Let *EL* (elementary logic) be standard predicate logic, and, if $\mathbf{Q}^i$ are generalised quantifiers for $i \in I$, let $L(\mathbf{Q}^i)_{i \in I}$ be the logic obtained from *EL* by adding the syntactic and semantic rules for each $\mathbf{Q}^i$ as in Definition 2. The study of such *model-theoretic logics* is sometimes called *abstract model theory*.[12] For a comprehensive survey of this field of mathematical logic the reader is referred to Barwise and Feferman [1985], in particular the chapter [Mundici, 1985]. Below, just a few examples of such logics and their properties will be given.

The expressive power of a logic is most naturally measured by the classes of models its sentences can define. Define $L \leqq L'$ (L' is an *extension* of L) to mean that for each sentence of L there is an equivalent sentence (i.e. one with the same models) of L'. Clearly $\leqq$ is reflexive and transitive, and every logic $L = L(\mathbf{Q}^i)_{i \in I}$ is an extension of *EL*. We write $L \equiv L'$ when $L \leqq L'$ and $L' \leqq L$, and $L < L'$ when $L \leqq L'$ and $L' \nleqq L$.[13]

[12]There are more general concepts of logic, used in abstract model theory. A comparison of various abstract notions of a logic is given in [Westerståhl, 1976].

[13]This partial order concerns *explicit* power of expression, by *single* sentences. One can also consider *implicit* strength (cf. Appendix B.3), or expressibility by *sets* of sentences.

Since formulas are defined inductively, to prove that $L(\mathbf{Q}^i)_{i\in I} \leqq L'$ it suffices to show that each $\mathbf{Q}^i$ is definable in L'. For example, if $\mathbf{Q}^i$ is of type $\langle 2,1\rangle$ it suffices to show that the sentence

$$Q^i xy, z(P_1xy, P_2z)$$

is equivalent to a sentence in L'.

The inductive characterisation of formulas also gives the following result, which explains why *ISOM* is normally assumed for generalised quantifiers in mathematical logic: if each $\mathbf{Q}^i$ satisfies *ISOM*, then truth of sentences in $L(\mathbf{Q}^i)_{i\in I}$ is preserved among isomorphic models. In fact, the inductive proof of this gives slightly more

PROPOSITION 4. *If each* $\mathbf{Q}^i$ *satisfies* ISOM, ϕ *is a formula in* $L(\mathbf{Q}^i)_{i\in I}$, f *an isomorphism from* $\mathbf{M}_1$ *to* $\mathbf{M}_2$, *and* g *an assignment in* M_1, *then*

$$\mathbf{M}_1 \models \phi[g] \Leftrightarrow \mathbf{M}_2 \models \phi[fg].$$

Here is the relative strength of some of the logics we have considered:

THEOREM 5. $EL < L(\mathbf{Q}_0) < L(\mathbf{I}) < L(\mathbf{more}) < L(\mathbf{H})$.

The easiest part of the proof of this theorem is to show that one logic is an extension of the previous one. That $L(\mathbf{Q}_0) \leqq L(\mathbf{I})$ follows from the equivalence

$$Q_0xPx \leftrightarrow \exists y(Py \wedge Ix(Px, Px \wedge x \neq y))$$

(P is infinite iff removal of one element does not change its cardinality). That $L(\mathbf{I}) \leqq L(\mathbf{more})$ is obvious, and that $L(\mathbf{more}) \leqq L(\mathbf{H})$ follows by the following trick (due to Ehrenfeucht):

$$\begin{array}{lcl} \neg \textit{more}\ x(P_1x, P_2x) & \leftrightarrow & \exists f(f \text{is a 1–1 function from } P_1 \text{ to } P_2) \\ & \leftrightarrow & \exists f\forall x\forall z(x = z \leftrightarrow f(x) = f(z)\wedge \\ & & \wedge P_1x \to P_2f(x)) \\ & \leftrightarrow & \exists f\exists g\forall x\forall z(x = \leftrightarrow f(x) = g(z)\wedge \\ & & \wedge P_1x \to P_2f(x)) \\ & \leftrightarrow & Hxyzu(x = z \leftrightarrow y = u \wedge \\ & & \wedge P_1x \to P_2y). \end{array}$$

To prove that one logic is *not* an extension of another, one can either show directly that some sentence in the first is not equivalent to any sentence in the second, or, more indirectly, use *properties* of the two logics to distinguish them. For example, the following well known properties of *EL* can sometimes be used:

1. *The compactness property: If every finite subset of a set of sentences has a model, the whole set has a model.* Consider the following set of $L(\mathbf{Q}_0)$-sentences:

$$\{\neg Q_0x(x = x)\} \cup \{\exists_{\geqq n}x(x = x) : n = 1, 2, 3, \ldots\}.$$

This set has no models, but each finite subset has one. So $L(\mathbf{Q}_0)$ (and all its extensions) is not compact. In particular, $L(\mathbf{Q}_0) \nleqq EL$.

2. *The Tarski property: If a sentence has a denumerable model it has an uncountable model*. Let ϕ be an *EL*-sentence saying that $<$ is a discrete linear ordering with a first element. Then the $L(\mathbf{Q}_0)$-sentence

   (1) $\phi \wedge \forall x \neg Q_0 y(y < x)$

   characterises the natural number ordering $\langle N, < \rangle$ (i.e. $\langle M, R \rangle$ is a model of (1) iff it is isomorphic to $\langle N, < \rangle$). All models of (1) are denumerable, so $L(\mathbf{Q}_0)$ does not have the Tarski property.

3. *The completeness property: The set of valid sentences is recursively enumerable*. Adding to (1) sentences (of *EL*) defining addition and multiplication,and saying that 0 is the least element and $x + 1$ the immediate successor of x, we obtain a sentence θ which characterises the standard model of arithmetic $N = \langle N, <, +, \times, 0, 1 \rangle$. Then, for every $L(\mathbf{Q}_0)$-sentence ψ in this vocabulary,

   $N \vDash \psi \Leftrightarrow \theta \rightarrow \psi$ is valid.

   Thus, since the set of true arithmetical sentences is not recursively enumerable, $L(\mathbf{Q}_0)$ is not complete. This time there is no immediate consequence for extensions of $L(\mathbf{Q}_0)$. For the extensions mentioned in Theorem 5, however, sentences characterising $\boldsymbol{N}$ can be constructed in a similar way, so they are not complete either.

4. *The Löwenheim property: If a sentence has an infinite model it has a denumerable model*. It is not very difficult to show that $L(\mathbf{Q}_0)$ in fact has the Löwenheim property. But $L(\mathbf{I})$ (and its extensions) does not: we can write down a sentence of $L(\mathbf{I})$ saying that $<$ is a dense linear ordering without endpoints, and that there is an element which does not have as many predecessors s it has successors. In a model, the set of predecessors and the set of successors of this element are infinite and of different cardinalities, so the model must be uncountable. It follows, in particular, that $L(\mathbf{I}) \nleqq L(\mathbf{Q}_0)$.

In the proof of Theorem 5, it only remains to show that $L(\mathbf{I})$ is not an extension of $L(\textbf{more})$, and that $L(\textbf{more})$ is not an extension of $L(\mathbf{H})$. A convenient way to prove the former will be given in 1.7. A proof of the latter can be found in [Cowles, 1981].

Recall the definition of relativised quantifiers in Section 1.4.2. We say that $L = L(\mathbf{Q}^i)_{i \in I}$ *relativises*, if

$$L^r = L((\mathbf{Q}^i)^r)_{i \in I} \leqq L,$$

i.e. if the relativisation of each $\mathbf{Q}^i$ is definable in L. $EL, L(\mathbf{Q}_\alpha), L(\mathbf{I}), L(\mathbf{most})$, $L(\mathbf{more})$ and $L(\mathbf{H})$ all relativise. For example,

$$\begin{aligned}
&\forall^r x(Px, P_1x) \leftrightarrow \forall x(Px \to P_1x),\\
&\textit{most}^r\, x(Px, P_1x, P_2x) \leftrightarrow \ \textit{most}\, x(Px \wedge P_1x, P_2x),\\
&H^r v, xyzu(Pv, P_1xyzu) \leftrightarrow Hxyzu((Px \wedge Pz) \to\\
&\quad (Py \wedge Pu \wedge P_1xyzu)).
\end{aligned}$$

$L(\mathbf{Q}_R), L(\mathbf{Q}_C)$ and $L(\mathbf{W})$, on the other hand, do not relativise (cf. Section 1.7).

As the above equivalences show, relativised quantifier symbols are used to make relativised statements. This extends to all L-sentences. Define, for each L-formula ϕ and each unary predicate symbol P, the *relativised formula*

$$\phi^{(P)}$$

In L^r inductively by letting $\phi^{(P)} = \phi$ if ϕ is atomic, $(\neg\psi)^{(P)} = \neg\psi^{(P)}, (\psi \wedge \theta)^{(P)} = \psi^{(P)} \wedge \theta^{(P)}$, and, when ϕ is quantified, beginning with Q^i of type $\langle 2, 1\rangle$, say,

$$Q^i xy, (\psi, \theta)^{(P)} = (Q^i)^r v, xy, z(Pv, \psi^{(P)}, \theta^{(P)}).$$

$\phi^{(P)}$ expresses exactly what ϕ says about the universe restricted to (the denotation of) P. We can formulate this precisely as follows. Call a subset X of the universe of the model $\mathbf{M}$ *universe-like* if $X \neq \varnothing$, the denotations of all individual constants in the vocabulary for $\mathbf{M}$ are in X, and X is closed under the denotations of all function symbols in the vocabulary. In that case, let $\mathbf{M} \mid X$ be the model with universe X, and all the relations etc; in $\mathbf{M}$ restricted to X. Then it can be shown by induction that if X is universe-like and ϕ is an L sentence,

(*REL*) $\quad (\mathbf{M}, X) \models \phi^{(P)} \Leftrightarrow \mathbf{M} \mid X \models \phi$

(here we assume that P does not occur in ϕ and that it denotes X in $(\boldsymbol{M}, X)$).

If L relativises, all this can be done in L, since $\phi^{(P)}$ is then clearly equivalent to an L-sentence.

So far we have only discussed particular logics and their properties. The most exciting part of abstract model theory, however, concerns results relating various properties of logics to each other, and results *characterising* certain logics in terms of their properties. Most famous of these characterisations is still *Lindström's theorem* [1969], which characterises *EL* in terms of the four properties mentioned above (for proofs, cf. [Flum, 1985], van Benthem and Doets or Hodges (both this Handbook series).

THEOREM 6. *If L is compact and has the Löwenheim property, then $L \equiv EL$. Also, of L relativises, then* (a) *if L is complete and has the Löwenheim property then $L \equiv EL$;* (b) *if L has the L'owenheim and Tarski property then $L \equiv EL$.*

1.7 The Strength of Monadic Quantifiers

In general, it may be quite difficult to determine whether $L \leqq L'$ or not, where L and L' are logics with generalised quantifiers. In the case of *monadic* quantifiers, however, things become much easier. Since this case is what we shall mainly be dealing with, I will devote the present subsection to developing some machinery for comparing the expressive power of logics with monadic quantifiers. The machinery will be applied in particular to the quantifiers **more** and **most**. I use these quantifiers later to illustrate some important points concerning natural language quantification, and it will then be instructive to have established their logical properties.

This subsection is a bit more technical than the previous ones; I have written out proofs of results that are new or not easily found in the literature (cf. the bibliographical note at the end). The reader can skip or glance through it now, and return to it for a definition or a result that is used later.

From now on, when $\mathbf{Q}$ is an m-ary monadic quantifier, we will write simply

$$\mathbf{Q}_M X_1 \ldots X_m,$$

instead of $\langle X_1, \ldots, X_m \rangle \in \mathbf{Q}_M$. Thus,

$$\mathbf{all}_M AB \Leftrightarrow A \subseteq B,$$
$$\mathbf{most}_M AB \Leftrightarrow |A \cap B| > |A - B|,$$
$$\mathbf{more}_M AB \Leftrightarrow |A| > |B|,$$

etc.

Let $\mathbf{M} = \langle M, A_0, \ldots, A_{k-1} \rangle$ be a K-*ary monadic structure* (i.e. the A_i are subsets of M, and the vocabulary consists of k unary predicate symbols). The following terminology will be used her an in later sections. If $X \subseteq M$, let $X^0 = X$ and $X^1 = M - X$. If s is a function from $\{0, \ldots, k-1\}$ to $\{0, 1\}$, i.e. if $s \in 2^k$, let

$$P_s^M = A_0^{s(0)} \cap \ldots \cap A_{k-1}^{s(k-1)}.$$

$\{P_s^M\}_{s \in 2^k}$ is a *partition* of M, and, up to isomorphism, the number of elements in these partition sets is all there is to say about $\mathbf{M}$. In other words, if $|P_s^{\mathbf{M}}| = |P_s^{\mathbf{M}'}|$ for all $s \in 2^k$, then $\mathbf{M}$ and $\mathbf{M}'$ are isomorphic. Finally, let

$$U_i^{\mathbf{M}},$$

for $1 \leqq i \leqq 2^{2^k}$, be all possible *unions* of the partition sets (including $\varnothing$), in some fixed order.

If L is a logic, $\mathbf{M}$ a structure (not necessarily monadic), $X \subseteq \mathbf{M}$, and $a_1, \ldots, a_n \in M$, X is said to be L-*definable in* $\mathbf{M}$ *with parameters* $a_1, \ldots, a_n$, if there is an L-formula ϕ in the vocabulary of $\mathbf{M}$ such that

$$a \in X \Leftrightarrow \mathbf{M} \vDash \phi[a, a_1, \ldots, a_n].$$

The following is an almost immediate consequence of this definition and Proposition 4:

LEMMA 7. *If L satisfies ISOM, X is L-definable in* $\mathbf{M}$ *with parameters* $a_1, \ldots,$ a_n*, and f is an automorphism on* $\mathbf{M}$ *(i.e. an isomorphism from* $\mathbf{M}$ *to* $\mathbf{M}$*) with* $f(a_i) = a_i$*, then* $f[X] = X$.

If A, B are sets, $A \oplus B$, the *symmetric difference* between A and B, is $(A - B) \cup (B - A)$. We say that B is an X*-variant* of A, if $A \oplus B \subseteq X$.

LEMMA 8. *Suppose that L satisfies ISOM and that* $\mathbf{M}$ *is a monadic structure. Then the L-definable sets in* $\mathbf{M}$ *with parameters* $a_1, \ldots, a_n$ *are precisely the* $\{a_1,$ $\ldots, a_n\}$*-variants of the unions* $U_i^{\mathbf{M}}$.

Proof. Clearly all these sets are also definable. Now suppose X is L-definable in $\mathbf{M}$ from $a_1, a \ldots, a_n$. Then so is $X' = X = \{a_1, \ldots, a_n\}$. It suffices to show that X' has the desired form. Let $s_1, \ldots, s_p$ be those $s \in 2^k$ for which $X' \cap P_s^m \neq \varnothing$. Thus,

$$X' \subseteq P_{s_1}^M \cup \ldots \cup P_{s_p}^M.$$

Suppose X' is *not* and $\{a_1, \ldots, a_n\}$-variant of $P_{s_1}^M \cup \ldots \cup P_{s_P}^M$. Then, for some i, there is $a \in P_{s_i}^M - X'$ such that $a \neq a_1, \ldots, a_n$. But, by the construction, there is $b \in P_{s_i}^M \cap X$; such that $b \neq a_1, \ldots, a_n$. let $f(a) = b, f(b) = a$, and $f(x) = x$ when $x \neq a, b$. Then f is an automorphism on $\mathbf{M}$ leaving $a_1, \ldots, a_n$ fixed, so $f[X'] = X'$, by Lemma 7. But this contradicts the fact that $a \in f[X'] - X'$. ■

Now we restrict attention to logics with monadic quantifiers satisfying ISOM. For simplicity, assume that $L = L(\mathbf{Q})$, where $\mathbf{Q}$ is *binary*; the results below extend immediately to logics $L(\mathbf{Q}^o \mapsto \mathbf{Q}^i)_{i \in I}$, with monadic $\mathbf{Q}^i$.

The *quantifier rank* of L-formulas is defined inductively as follows:

$$\begin{aligned}
&qr(\phi = 0, \text{ if } \phi \text{ is atomic,}\\
&qr(\neg\phi) = qr(\phi)\\
&qr(\phi \wedge \psi) = \mathit{max}(qr(\phi), qr(\psi)),\\
&qr(\exists x\phi) = qr(\phi + 1)\\
&qr(Qx(\phi, \psi)) = \mathit{max}(qr(\phi), qr(\psi)) + 1.
\end{aligned}$$

we write

$$\mathbf{M} \equiv_{n,Q} \mathbf{M}'$$

to mean the same $L(\mathbf{Q})$-sentences of quantifier rank at most n are true in $\mathbf{M}$ and $\mathbf{M}'$. $\mathbf{M} = \equiv_Q \mathbf{M}'$ ($\mathbf{M}$ and $\mathbf{M}'$ are $L(\mathbf{Q})$*-equivalent*) if, for all n, $\mathbf{M} \equiv_{n,Q} \mathbf{M}'$. Our main tool will be an equivalent but more workable formulation of the relation $\equiv_{n,Q}$. This is accomplished in the next definition. If $a_1, \ldots, a_n \in M$ and $b_1, \ldots, b_n \in M'$ we write $(a_1, \ldots, a_n) \simeq_p (b_1, \ldots, b_n)$ to mean that $\{\langle a_i, b_i \rangle :$

$1 \leqq i \leqq n\}$ is a *partial isomorphism* from $\mathbf{M}$ to $\mathbf{M}'$ (i.e. $a_i = a_j$ iff $b_i = b_j$, and $a_i \in A_m$ iff $b_i \in A'_m$).

In what follows, $\mathbf{M}$ and $\mathbf{M}'$ are k-ary monadic structures.

DEFINITION 9.

(a) $X \approx_n Y$ iff either $|X| = |Y| < n$ or $|X|, |Y| \geqq n$.

(b) $\mathbf{M} \approx_n \mathbf{M}'$iff $P_s^{\mathbf{M}} \approx_n P_s^{\mathbf{M}'}$ for all $s \in 2^k$

(c) $\mathbf{M} \approx_{n,Q} M'$ iff

(i) $\mathbf{M} \approx_n \mathbf{M}'$

(ii) If $(a_1, \ldots, a_{n-1}) \simeq_p (b_1, \ldots, b_{n-1}), X_i, X_j$ are $\{a_1, \ldots, a_{n-1}\}$-variants of $U_i^{\mathbf{M}}, U_j^{\mathbf{M}}$, an Y_i, Y_j the *corresponding* $\{b_1, \ldots, b_{n-1}\}$-variants of $U_i^{\mathbf{M}'}, U_j^{\mathbf{M}'}$, then

$$\mathbf{Q}_M X_i X_j \Leftrightarrow \mathbf{Q}_{M'} Y_i Y_j.$$

THEOREM 10. $\mathbf{M} \equiv_{n\mathbf{Q}} \mathbf{M}' \Leftrightarrow \mathbf{M} \approx_{n,\mathbf{Q}} \mathbf{M}'$.

Proof. $\Rightarrow$: It is clear that (i) holds. As for (ii), let $\psi_i(y, x_1, \ldots, x_{n-1}), \psi_j(y, x_1, \ldots, x_{n-1})$ be formulas which L-define X_i, X_j in $\mathbf{M}$ with parameters $a_1, \ldots, a_{n-1})$. Each a_p belongs to exactly one $P_{s_p}^{\mathbf{M}}$; let this set be defined by $\theta_p(x)$. If $\mathbf{Q_M} X_i, X_j$, then

$$\begin{aligned}\mathbf{M} \vDash \exists x_1, \ldots, x_{n-1}(\theta_1(x_1) \wedge \ldots \wedge \theta_{n-1}(x_{n-1}) \wedge \\ \wedge Qy(\psi_i(y, x_1, \ldots, x_{n-1}), \psi_j(y, x_1, \ldots, x_{n-1}))).\end{aligned}$$

This sentence has quantifier rank n. Thus, it is also true in $\mathbf{M}'$, whence there are $b'_1, \ldots, b'_{n-1} \in M'$ such that $b'_p \in P_{s'_p}^{\mathbf{M}'}$ and

$$\mathbf{M}' \vDash Qy(\psi_i, \psi_j)[b', \ldots, b'_{n-1}].$$

Let f map b'_p on b_p and leave everything else in M' as it is. It follows that f is an automorphism on $\mathbf{M}'$, so

$$\mathbf{M}' \vDash Qy(\psi_i, \psi_j)[b_1, \ldots, b_{n-1}].$$

but this means that $\mathbf{Q}_{M'} Y_i, Y_j$. The converse is similar.

$\Leftarrow$: We prove by (downward) induction over $p \leqq n$ that

($*$) If $(a_1, \ldots, a_p) \simeq_p (b_1, \ldots, b_p)$ and $qr(\phi) \leqq n-p$, then $\mathbf{M} \vDash \phi[a_i, \ldots, a_p] \Leftrightarrow \mathbf{M}' \vDash \phi[b_1, \ldots, b_p]$.

The case $p = 0$ gives the result. $(*)$ is clear for $p = n$. So suppose $(*)$ holds for $p, (a_1, \ldots, a_{p-1}) \simeq_p (b_1, \ldots, b_{p-1})$ and $qr(\phi) = n - p + 1$. We may suppose that ϕ begins with a quantifier symbol. If this symbol is $\exists$, the result follows easily from the induction hypothesis and the fact that $\mathbf{M} \approx_n \mathbf{M}'$. So suppose ϕ is $Qx(\psi_1, \psi_2)$. Let $\psi_i^{\mathbf{M}} = \{a \in M : \mathbf{M} \vDash \psi_i[a, a_1, \ldots, a_{p-1}]\}, i = 1, 2$, and similarly for $\psi_i^{\mathbf{M}'}$. By Lemma 8, each ψ_i^M is an $\{a_1, \ldots, a_{p-1}\}$-variant of some union $U_{j_i}^{\mathbf{M}}$ of partition sets.

CLAIM: $\psi_i^{\mathbf{M}'}$ is the corresponding $\{b_1, \ldots, b_{p-1}\}$-variant of $U_{j_i}^{\mathbf{M}'}$.

The result follows immediately from the chain and (ii) above. The proof of the claim is straightforward, using the induction hypothesis together with the fact that $\mathbf{M} \approx_n \mathbf{M}'$. ■

As noted, the theorem extends to logics with several monadic quantifiers (satisfying ISOM). We use this in the next corollary. A k-ary quantifier $\mathbf{Q}$ is said to be *closed under* $\approx_{n,\mathbf{Q}^1\ldots\mathbf{Q}^m}$ if $\mathbf{Q}_M A_0 \ldots A_{k-1}$ and $\langle M, A_0, \ldots, A_{k-1}\rangle \approx_{n,\mathbf{Q}^1,\ldots,\mathbf{Q}^m} \langle M', A_0', \ldots, A_{k-1}'\rangle$ implies $\mathbf{Q}_{\mathbf{M}'} A_0', \ldots A_{k-1}'$.

COROLLARY 11. *A monadic quantifier* $\mathbf{Q}$ *is definable in* $L(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$ *if and only if, for some natural number* n, $\mathbf{Q}$ *is closed under* $\approx_{n,\mathbf{Q}^1\ldots\mathbf{Q}^m}$.

Proof.[outline] If $\mathbf{Q}$is defined by a sentence ϕ in $L(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$, i.e. if

$$\mathbf{Q}_M A_0, \ldots, A_{k-1} \Leftrightarrow \langle M, A_0, \ldots, A_{k-1}\rangle \vDash \phi,$$

just let n be the quantifier rank of ϕ and use the theorem. Conversely, note that, with a fixed finite vocabulary there are, up to logical equivalence, only finitely many $L(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$-sentences of quantifier rank at most n. Now take the disjunction of all such sentences which are 1-complete n-descriptions of the models $\langle M, A_0, \ldots, A_{k-1}\rangle$ for which $\mathbf{Q}_M A_0, \ldots, A_{k-1}$; this disjunction defines $\mathbf{Q}$. ■

We will now apply these results to some particular monadic quantifiers. First, note the following special cases of Theorem 10:

1. $\mathbf{M} \equiv_n \mathbf{M}' \Leftrightarrow \mathbf{M} \approx_n \mathbf{M}'$,

2. If $\mathbf{Q}$ is first-order definable, then $\mathbf{M} \equiv_{n,\mathbf{Q}} \mathbf{M}' \Leftrightarrow \mathbf{M} \approx_n \mathbf{M}'$.

Using this, one easily shows that quantifiers such as $\mathbf{Q}_\alpha, \mathbf{Q}_C, \mathbf{Q}_R$ are *not* first-order definable. Next, note that an $\{a_1, \ldots, a_{n-1}\}$-variant of $U_i^{\mathbf{M}}$ has cardinality $\geqq \aleph_\alpha$ iff $U_i^{\mathbf{M}}$ has cardinality $\geqq \aleph_\alpha$ iff one of the partition sets in U_i^M has cardinality $\geqq \aleph_\alpha$. Thus, when $\mathbf{Q} = \mathbf{Q}_\alpha$, we need only consider the partition sets (not variants of unions of them) in Definition 9(c). This makes it easy to show,for example, that if $\alpha \neq \beta$, $L(\mathbf{Q}_\alpha)$ and $L(\mathbf{Q}_\beta)$ have *incomparable* expressive power.

3. $L(\mathbf{Q}_R \nleqq L(\mathbf{I})$.

Proof. By the theorem, it suffices to find, for each n structures $\langle M, A\rangle$ and$\langle M', A'\rangle$ such that $\langle M, A\rangle \equiv_{n,I} \langle M', A'\rangle, (\mathbf{Q}_R)_M A$, and $\neg(\mathbf{Q}_R)_{M'} A'$. But this is easy. For example, let $|A| = 4n, |M - A| = 2n, |A'| = 2n, |M' - A;| = -4n$. There are just four unions of partition sets to consider in each structure, and it is easy to verify that the conditions in Definition 9(c) are satisfied. ■

4. "$|A|$ is even" is not expressible in L(**more**).

Proof. For each n, choose $M, M', A \subseteq M, A' \subseteq M'$ such that $|A| = 4n, |M - A| = |M' - A'| = n, |A'| = 4n + 1$. Then $\langle M, A\rangle \approx_{n,\mathbf{more}} \langle M', a'\rangle$, so $\langle M, A\rangle \equiv_{n,\mathbf{more}} \langle M', a'\rangle$ by the theorem, but $|A|$ is even and $|A'|$ is odd. ■

The following result is from Barwise and Cooper [1981]:

5. $L(\mathbf{most}) \nleq L(\mathbf{Q}_R)$, i.e. $L(\mathbf{Q}_R)$ does not relativise.

Proof. Given n, choose $\langle M, A_0, A_1\rangle, \langle M', A'_0, A'_1\rangle$ such that $A_0 \cap A_1 = \varnothing, A'_0 \cap A'_1 = \varnothing, |A_0| = |A_1| = n, |M| = 6n, |A'_0| = n, |A'_1| = n + 1, |M'| = 6n + 2$. So $\varnothing, A_0, A_1, A_0 \cup A_1$ all have cardinalities less than their complements, and this continues to hold if $n - 1$ elements are 'moved around' in the model. The same holds for $\mathbf{M}'$, and it is then easy to see that $M \equiv_{n,Q_R} M'$. However, $\neg$**most**$_M, A_0 \cup A_1 A_1$ and **most**$_{M'} A'_0 \cup A'_1 A'_1$. ■

Similarly, we can prove that $\mathbf{Q}_C$ does not relativise. Note that only finite structures have been used so far. The next and final application involves infinite structures.

6. $L(\mathbf{Q}_0) \nleq L(\mathbf{most})$.

Proof. This time, choose $\langle M, A\rangle, \langle M', A'\rangle$ such that $|M - A| = |M' - A'| = n, |A| = \aleph_0$, and $|A'| = 3n$. Again, it is not hard to see that $\langle M, A\rangle \approx_{n,\mathbf{most}} \langle M', A'\rangle$ (especially if we use the characterisation of $\approx_{n,\mathbf{most}}$ given in Theorem 12 below), but A is infinite and A' is finite. ■

Finally, we shall consider more closely the relative expressive power of **most** and **more**. Note first that the four properties of logics mentioned in Section 1.6 do not enable us to distinguish between these two quantifiers: we saw that $L(\mathbf{more})$ does not have any of these properties, and similar arguments establish that neither does $L(\mathbf{most})$. For example, if we replace the second conjunct in the sentence (1) in Section 1.6 by a sentence saying that, for each x (except the first) there is a *greatest* element $y < x$ with the property that most of the x-predecessors are not predecessors of y, then we again obtain a characterisation of the natural number ordering.

The next result characterises the relations $\equiv_{n,\mathbf{Q}}$ and $\equiv_{\mathbf{Q}}$ for monadic structures, when **Q** is **most** or **more**.

THEOREM 12.

(a) $\mathbf{M} \equiv_{n,\mathbf{more}} \mathbf{M}'$ *iff, whenever* $(a_1, \ldots, a_{n-1}) \simeq_p (b_1, \ldots, b_{n-1}), X_i, X_j$ *are* $\{a_1, \ldots, a_{n-1}\}$*-variants of* $U_i^{\mathbf{M}}, U_j^{\mathbf{M}}$ *and* Y_i, Y_j *the corresponding* $\{b_1, \ldots, b_{n-1}\}$*-variants of* $U_i^{\mathbf{M}'}, U_j^{\mathbf{M}'}$*, we have* $|X_i| > |X_j| \Leftrightarrow |Y_i| > |Y_j|$.

(b) For $\equiv_{n,\mathbf{most}}$ *we have the same condition, except that* $X_i, X_j (Y_i, Y_j)$ *are required to be disjoint.*

(c) $\mathbf{M} =_{\mathbf{more}} \mathbf{M}'$ *iff* $\mathbf{M} \equiv_{\mathbf{most}} \mathbf{M}'$ *iff* $\mathbf{M} \equiv_{\aleph_0} \mathbf{M}'$ *and, for all* $s, t \in 2^k$, $|P_s^{\mathbf{M}}| > |P_t^{\mathbf{M}}$ *od* $\Leftrightarrow |P_s^{\mathbf{M}'}| > |P_t^{\mathbf{M}'}|$.

Proof.

(a) This is Theorem 10, except that we must show that the condition on the right hand side of the equivalence ((ii) in Definition 9 (c)) implies that $\mathbf{M} \approx_n \mathbf{M}'$. So suppose first $|P_s^{\mathbf{M}}| < n$. Suppose also that$|P_s^{\mathbf{M}'}| \neq |P_s^{\mathbf{M}}|$, say $|P_s^{\mathbf{M}'}| < |P_s^{\mathbf{M}}|$ (the other case is similar). If $P_s^{\mathbf{M}'} = \{b_1, \ldots, b_r\}$, choose $a_1, \ldots, a_r \in P_s^{\mathbf{M}}$ and let $Y_i = \varnothing = P_s^{\mathbf{M}'} - \{b_1, \ldots, b_r\}$ and $Y_j = \varnothing$. It follows from the condition that $X_i = P_s^{\mathbf{M}} - \{a_1, \ldots, a_r\}$ is empty, contradicting our assumption. The case when $|P_s^{\mathbf{M}}| \geqq n$ is similar.

(b) From left to right, note that **most** allows us to compare the cardinalities of *disjoint* sets $X, Y \subseteq M$: then $|X| < |Y|$ iff $\mathbf{most}_M X \cup Y X$. In the other direction, observe first that the argument in (a) above goes through under the disjointness requirement. Moreover, the proof of Theorem 10 ($\Leftarrow$:) also goes through under this requirement, since the formula *most* $x(\psi_1, \psi_2)$ only 'compares' disjoint sets.

(c) Clearly $\mathbf{M} \equiv_{\mathbf{more}} \mathbf{M}'$ implies $\mathbf{M} \equiv_{\mathbf{most}} \mathbf{M}'$, which in turn implies the rightmost condition in (c). Now suppose that condition holds; we must show that, for all $n, \mathbf{M} \approx_{n,\mathbf{more}} \mathbf{M}'$. So take n, and suppose $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}, X_i, X_j, Y_i, Y_j$ are as in (a) above. We assume $|X_i| > |X_j|$ and show that , in this case, $|Y_i| > |Y_j|$; the other direction is similar.

Case 1: X_i and X_j are both finite. Then the partition sets in $U_i^{\mathbf{M}}$ are finite and thus have the same cardinality as the corresponding partition sets in $U_i^{\mathbf{M}'}$, since $\mathbf{M} \approx_n \mathbf{M}'$ for all n. X_i differs from $U_i^{\mathbf{M}'}$ only by certain of the $a_1, \ldots, a_{n-1}$, and Y_i differs in the same way from $U_i^{\mathbf{M}'}$. Therefore, $|X_i| = |Y_i|$ and $|X_j| = |Y_j|$, and the conclusion follows.

Case 2: X_i and X_j are both infinite. Then $|X_i|$ is the *max* of the cardinalities of the partition sets making up $U_i^{\mathbf{M}}$; say, $|X_i| = |P_s^{\mathbf{M}}|$, and similarly $|X_j| = |P_t^{\mathbf{M}}|$. It then follows from the condition in (c) that $|Y_i| = |P_s^{\mathbf{M}'}|$ and $|Y_j| = P_t^{\mathbf{M}'}|$. Since $|P_s^{\mathbf{M}}| > |P_t^{\mathbf{M}}|$ we have, again by the condition, $|P_s^{\mathbf{M}'}| > |P_t^{\mathbf{M}'}|$.

Case 3: X_i is infinite and X_j is finite. Arguing as in Cases 1 ad 2, we see that Y_i is infinite and Y_j is finite. ■

Thus, the relations $\equiv_{\mathbf{most}}$ and $\equiv_{\mathbf{more}}$ coincide on monadic structures (but *not* the relations $\equiv_{n,\mathbf{most}}$ and $\equiv_{n,\mathbf{more}}$). Nevertheless, $L(\mathbf{more})$ is more expressive than $L(\mathbf{most})$, even if we restrict attention to monadic structures, as the next result will show. Another instance of the same phenomenon is given by the fact that

$$\mathbf{M} \equiv_{Q_0} \mathbf{M}' \Leftrightarrow \mathbf{M} \equiv \mathbf{M}'$$

(this is an easy consequence of Theorem 10), but $EL < L(\mathbf{Q}_0)$ (even on monadic structures).

The following theorem holds in general, but it is also true if only monadic structures are considered.

THEOREM 13.

(a) $L(\mathbf{most}) < L(\mathbf{more})$.

(b) $L(\mathbf{most}) \equiv L(\mathbf{more})$ on finite structures.

(c) $L(\mathbf{more}) \equiv L(\mathbf{most}, \mathbf{Q}_0)$.

Proof.

(a) Clearly $L(\mathbf{most}) \leqq L(\mathbf{more})$. That $L(\mathbf{more}) \nleqq L(\mathbf{most})$ follows from (6) and Theorem 5.

(b) This follows from the fact that, if $A \cap B$ is finite, then $\mathbf{more}_M AB \Leftrightarrow |A| > |B| \Leftrightarrow |A - B| > |B - A| \Leftrightarrow \mathbf{most}_M A \oplus BA$.

(c) We must show that $L(\mathbf{more}) \leqq L(\mathbf{most}, \mathbf{Q}_0)$. Take any M and $A, B \subseteq M$. If $A \cap B$ is finite, $\mathbf{more}_M AB$ is expressed as in (B). If $A \cap B$ is infinite, then $|A| = max(|A - B|, |A \cap B|)$ and $|B| = max)|B - A|, |A \cap B|)$. It follows that

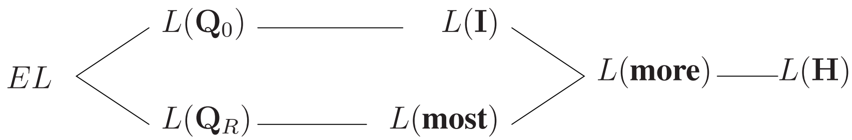
$$|A| > |B| \Leftrightarrow |A - B| > |B - A| \& |AB| > |A \cap B|,$$

and the right hand side of this is again expressible with *most* (since only disjoint sets are compared). Moreover, $\mathbf{Q}_0$ allows us to distinguish the two cases, in one sentence of $L(bf most, \mathbf{Q}_0)$.

■

This theorem tells us that the difference between $L(\mathbf{more})$ and $L(\mathbf{most})$ is *precisely* that the former, but not the latter, can distinguish between infinite and finite sets.

The results of this section allow us to extend Theorem 5 to the following picture:



Here each logic is strictly stronger than its immediate predecessor(s), and logics not on the same branch are incomparable.

REMARK 14. The only thing in the figure above not proved with the simple methods used here is the fact that $L(\mathbf{H})$ is strictly stronger than $L(\mathbf{more})$. However, if we consider the logic L^{po}, where not only the Henkin prefix but *all* partially ordered prefixes with $\forall$ and $\exists$ are allowed, then it follows from (4) that $L^{po} \nleq L(\mathbf{more})$. For,

$$|A| \text{ is even } \Leftrightarrow \exists X \subseteq A(|X| = |A - X|),$$

which can be expressed as a Σ^1_1 sentence, and is shown in [Enderton, 1970] and [Walkoe, 1970] that all such sentences are expressible in L^{po}.

Is '$|A|$ is even $|$' expressible in $L(*\mathbf{H})$? More generally, is $L(\mathbf{H})$ strictly stronger than $L(\mathbf{more})$ if we restrict attention to monadic structures/ I don't know the answer to these questions, but it may be noted that it follows from Theorem 12 and a result in [Lachlan and Krynicki, 1979] that $\equiv_{\mathbf{more}}$ and $\equiv_{\mathbf{H}}$ coincide for monadic structures.

Bibliographical remark: The theorems in this section have not, to my knowledge, appeared in the literature, although no doubt they belong to the folklore in some circles. Most of the applications to particular logics are known, but it should be noted that the methods used here are much more elementary than the ones that have been used in the literature the proof of (5) in [Barwise and Cooper, 1981] is an exception). For example, it is proved in [Hauschild, 1981] and [Weese, 1981] that $L(\mathbf{more})$ is strictly stronger than $L(\mathbf{I})$ by establishing that these logics have different properties w.r.t. the decidability of certain theories formulated in them. The same result follows from the simple observation 93); in a sense, (3) gives more, since it concerns monadic structures, whereas the theories just mentioned use non-monadic languages.

2 NATURAL LANGUAGE QUANTIFIERS

A main objective of Montague's PTQ [Montague, 1974] was to show that intensional phenomena, such as quantification into intensional contexts, could be handled rigorously with model-theoretic methods. But even if one completely disregards the intensional aspects of PTQ, its approach to quantification was novel. Although it had no category 'quantifier' or 'determiner', a general pattern is discernible from its treatment of the three quantifier expressions (*every, a*, and *the*) it in fact did account for. The basic idea is that *quantifier expressions occur as determiners in noun phrases*. By the close correspondence between syntax and semantics in Montague Grammar, this also determines the interpretation of such expressions.

In this section, I will describe this idea in somewhat more detail, and its later development in [Barwise and Cooper, 1981] and [Keenan and Stavi, 1986], within the generalised quantifier framework of Section 1.

2.1 Determiners

Suppose that the expressions of the categories *common noun* (N) and *noun phrase* (NP) have somehow been (roughly) identified.[14] Since we are disregarding intensions,the semantic types of these expressions are such that Ns are interpreted, in a model $\mathbf{M} = \langle M, \| \ \| \rangle$ with universe M and interpretation function $\| \ \|$, as *subsets* of M an NPs as *sets of subsets* of M. Here are three examples from PTQ:

$$\|\textit{every man}\| = \{X \subseteq M : \|\textit{man}\| \subseteq X\},$$
$$\|\textit{a man}| = \{X \subseteq M : \|\textit{man}\| \cap X \neq \varnothing\},$$
$$\|\textit{the man}\| = \{X \subseteq M : |\|\textit{man}\|| = 1 \& \|\textit{man}\| \subseteq X\}.$$

Many NPs, like the above ones, are naturally regarded as the result of applying a syntactic *operator* to Ns. We introduce the syntactic category *determiner* (*DET*) for this sort of operator:

(*DET*) *DETs form NPs from Ns.*

This is a rough criterion, but, in a Montagovian framework, it is enough to fix the syntax and semantics of determiners. In particular, *DET*s are interpreted as *functions* from N denotations to NP denotations. For example,

$$\|\textit{every}\|(A) = \{X \subseteq M : A \subseteq X\},$$
$$\|\textit{a}\|(A) = \{X \subseteq M : A \cap X \neq \varnothing\},$$
$$\|\textit{the}\|(A) = \{X \subseteq M : |A| = 1 \& A \subseteq X\}.$$

Another thing, of course, is to apply the criterion to identify simplex and complex English *DET*s; we will return to this in Section 2.4.

2.1.1 Three apparent problems

As noted, the basic idea of the present Montague-style treatment of quantification is this:

(*Q*) *Quantifier expressions are DETs.*

[14]We don't need to assume that proper *definitions* of these categories exist, only that there is agreement about them in a large number of cases.

This may not yet seem very exciting, but note at least that it differs, syntactically as well as semantically, from the standard predicate logic treatment of quantification. The import of (Q) will become clear as we go along. For the moment, however, let us look at a few apparent *counter-instances* to (Q) that come to mind.

I. In sentences like

(1) All cheered,

(2) Some like it hot,

(3) Few were there to meet him,

the words *all, some, few* are not applied to arguments of category N. Isn't the standard predicate logic analysis more plausible here? No, it is very natural to assume that the *DET*s have 'dummy' arguments in these sentences (what context-given interpretations); in this case (Q) still holds (cf. 2.4.5).

II. Words like *something, everything, nothing, nobody*, etc. look like quantifier expressions but are certainly not *DET*s. We have two options here. The first is to regard them as simplex *NP*s, denoting quantifiers of type $\langle 1 \rangle$ in the sense of 1.4. They would then correspond (roughly) to the standard logical $\forall$ and $\exists$. The other option, which we will take here, is to regard them as *complex: something = some(thing), nothing = no(thing)*, etc.; i.e. obtained by applying a *DET* to a (perhaps logical) N like *thing*. In this way, (Q) can be maintained.

III. In 1.4 we defined the binary quantifier **more**. The word *more*, however, is not a *DET* by our criterion; compare

(4) Some boys run,

(5) Most boys run,

(6) *More boys run.[15]

Still, *more* does occur in quantified sentences, for example,

(7) There are more girls than boys,

which in generalised quantifier notation becomes

(8) *more* $x(girl(x), boy(x))$.

[15]Even if there are contexts where (6) might be uttered, it is unreasonable to interpret *more* as an independent *DET*: the standard of comparison is missing, and has to be supplied to get at the meaning. So *more* in (6) would then stand for something like *more than 10, more ... than the number of girls*, etc. These are *DETs* by our criterion, but not the single *more*.

This is an objection to (Q) that must be taken seriously. It involves (i) finding a semantic distinction between the quantifiers **more** and, say, **most**, which explains why one but not the other is a *DET* denotation; (ii) the analysis of 'there are'-sentences; (iii) the semantics of words like *more*. These matters will be taken up in Section 2.2.

2.1.2 Determiner interpretations as generalised quantifiers

Following Montague, we interpreted *DET*s as *functions* from subsets of the universe M to sets of such subsets. From now on, however, we return to the generalised quantifier framework of Section 1, where quantifiers on M are *relations* between subsets of M. Thus, to each n-place function $\mathbf{D}$ from $(P(M))^n$ to $P(P(M))^n$ we associate the following $(n+1)$-ary quantifier on M:

$$\mathbf{Q}_M A_1 \ldots A_n B \Leftrightarrow B \in \mathbf{D}(A_1, \ldots, A_n).$$

In what follows, *DET* interpretations will be such monadic quantifiers on the universe.

The functional interpretation of *DET*s emphasises similarity of structure between syntax and semantics, which is one of the characteristics of Montague Grammar. From the present semantic perspective, however, relations turn out to be easier to work with. But keep in mind that the relational approach increases the number of arguments by one: n-place *DET*s will denote $(n+1)$-ary quantifiers (so far we have only seen 1-place *DET*s, but cf. 2.2). It should also be noted that for *some* semantic issues, the functional framework seems more natural; cf. [Keenan and Moss, 1985].

Terminological Remark: The use of words 'determiner' and 'quantifier' is rather shifting in the literature. Here, the idea is to use 'determiner' and '*DET*' *only* for syntactic objects, and 'quantifier' *only* for semantic objects. The extension of 'quantifier' was given in Section 1.4, and a criterion for *DET*-hood at the beginning of 2.1.

2.1.3 Determiners as constants

In a Montague-style model $\mathbf{M} = \langle M, \| \, \| \rangle$, *DET*s are on a par with expressions of other categories. Nothing in principle prevents, for example, that a determiner D is interpreted as **every** in one model and as **most** in another. But there is usually no point in allowing this generality. Moreover, there is a clear intuition, I think, that determiners are *constants*. We therefore lay down the following *methodological postulate*:

(*MP*) *Simplex DETs are constants: each one denotes a fixed quantifier* (modulo, of course, lexical ambiguity, vagueness, etc.; cf. 2.4).

(*MP*) allows us to dispense with the interpretation function for (simples) *DET*s and to resume the notation from 1.4, using boldface letters for quantifiers: Q denotes **Q**, *most* denotes **most**, *some* denotes **some**, etc.

What about complex *DET*s? In case such a *DET* contains a non-constant expression, there seems to be a choice. We can either persist in treating them as constants, or let their interpretation depend on the interpretation of the non-constant expressions occurring in them. To take a simple example, consider *some red*. This expression *can* be construed as giving an *NP* when applied to an N, thus *can* be classified as a *DET* by our criterion. As a constant, it would denote the quantifier defined by

$$\textbf{some red}_M AB \Leftrightarrow A \cap B \cap \{a \in M : a \text{ is (in fact) red}\} \neq \varnothing,$$

for each universe M. As an expression consisting of a constant and a non-constant symbol, i.e. of the form *some* P, it is interpreted in a model **M** as

$$\|\textit{some } P\| AB \Leftrightarrow A \cap B \cap \|P\| \neq \varnothing.$$

Given **M**, this is a quantifier on M, but the expression does not denote a fixed quantifier on each universe.

No doubt many readers will find the latter option more natural, but we need not take a stand on this methodological issue here. Our model-theoretic machinery provides adequate semantic objects for both cases, quantifiers, and quantifiers *on* universes, respectively.

Note, however, that our decision to treat simplex *DET*s as constants does not necessarily imply that they are *logical* constants. It can be argued that logicality requires a lot more; this theme will be resumed in 3.4 and 4.4 (cf. also [Westerståhl, 1985a]). For example, the quantifier **some red** defined above is not logical, the reason being that it violates the condition *ISOM* from 1.4.

In Appendix B we will indicate what happens if the postulate (*MP*) is dropped.

2.1.4 Global vs. local perspective

To study quantifiers from a *global* perspective means to concentrate on properties which are *uniform* over universes. A typical example is first-order definability: **Q** is first-order definable if there is some first-order sentence which defines it on *every* universe. Sometimes, however, it is natural to take a *local* viewpoint: fix a universe M and restrict attention to quantifiers on M. Then other definability notions become interesting as well, involving parameters from M in an essential way.

Our perspective here will be predominantly global. The main reason for this is that global definitions and results are more general: they usually have an immediate 'local version'. The converse, however, does not hold. Quantifiers from a local perspective are studied extensively in [Keenan and Stavi, 1986]. Some of their results will be reviewed in Section 4.6.

2.2 The Interpretation of Determiners

The basic quantifier postulate (Q) from 2.1.1 can be split into a syntactic and a semantic part as follows:

(Q_{syn}) *Quantifier expressions are DETs.*

(Q_{sem}) *DETs denote $(n+1)$-ary quantifiers, $n \geqq 1$.*

In contrast with standard predicate logic, there are no unary quantifiers on this approach. And although some binary *DET* denotations (e.g. Montague's **every, a, the**) are definable in standard predicate logic, others are not: we saw in 1.7 that **most** is an example. Consequently, *EL* is inadequate for formalising even the pure quantificational part of natural languages.

However, (Q) is not yet quite satisfactory. In particular, we need to account for the apparent counter-examples mentioned in 2.1.1, III. Nothing so far precludes **more** from being a *DET* denotation.

The starting-point of a systematic study of natural language quantification was the isolation, in [Barwise and Cooper, 1981], and independently in [Keenan and Stavi, 1986] (although the latter paper was published much later, they were written at about the same time), or a purely model-theoretic property characteristic of those quantifiers that are *DET* denotations. This is the property of *conservativity*, defined below (Barwise and Cooper used a different terminology, in terms of an NP denotation *living on* a given set). Actually, the property (and the term) first appeared in [Keenen, 1981], but in the two first-mentioned papers it was proposed as a significant semantic universal for determiners (although with rather different motivations; cf. below).

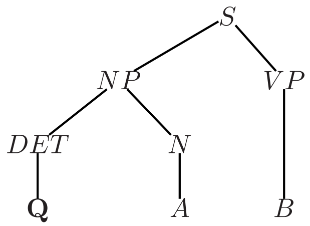
2.2.1 Conservativity

A binary quantifier **Q** is called *conservative* if the following holds:

(*CONSERV*) *for all M and all $A, B \subseteq M$, $\mathbf{Q}_M AB \Leftrightarrow \mathbf{Q}_M A\ A \cap B$.*

It is easily checked that **most** is conservative, but **more** is not. As we will see in 2.4, practically all English *DET*s denote conservative quantifiers (a few possibly doubtful cases will be noted).

CONSERV gives the *first* argument of **Q** a privileged role: only the part of B which is common to A matters for whether $\mathbf{Q}_M AB$ holds or not. This semantic difference between the arguments A and B matches the syntactic difference between the corresponding expressions:



Conservativity is a very fruitful postulate, as well be seen in Sections 3 and 4. Still, one may ask what, if any, is the idea or intuition behind it. As for Barwise and Cooper, they seem to regard it mainly as a successful empirical generalisation. Keenan and Stavi, on the other hand, give an interesting theoretical motivation: they prove that, on a given (finite) universe M, the conservative quantifiers on M are precisely those which can be generated from certain initial quantifiers by means of a few natural closure operations; an exact statement (and proof) will be given in Section 4.6. Yet another motivation, discussed in [Westerståhl, 1985a], is that *CONSERV* is related to the notion of *restricted domains of quantification*: an *NP* 'restricts' the universe to the denotation of the N; this will be formulated in Section 3.2.

CONSERV resolves the first doubt concerning (Q) expressed in 2.1.1, III. We still have to deal with 'there are'-sentences and with the semantics of *more*.

2.2.2 'there are'-sentences

Consider sentences such as

(1) There are no flowers,

(2) There are many patients waiting outside,

(3) There are some philosophers who like logic,

(4) There are a few errors in the text.

Without commitment to their syntactic form, let us write such sentences

(5) *There are* $\mathbf{Q}_M A$,

where **Q** is the quantifier denoted by the *DET* and A is the set contributed, in a model **M**, by the expression following the *DET*.[16] There are in fact two questions here. The first is to interpret quantified sentences of the form (5) in a way consonant with the basic postulate (Q). The second concerns the fact that certain *DET*s do not fit in (5): *all, most, not all*, for example. Is there a semantic explanation for this phenomenon?

[16]The 'hybrid' form (5) is used in order to avoid discussion of the syntactic structure of "there are"-sentences. This structure is quite varied, as already (1)–(4) indicate, and there may be divergent opinions about it, but it still seems that (5) is *semantically* adequate in a large number of cases.

We shall review the answers proposed by Barwise and Cooper to both of these questions, first, because they show a way to handle ‘there are’-sentences, and second, because this case can serve as a model of the kind of linguistic explanation one may expect from the present theory of quantifiers.

The first proposal is simple: interpret (5) as

(6) $\mathbf{Q}_M AM$.

This interpretation *works* in the sense that it gives (1)–(3) the right truth conditions. Moreover, one can argue that it accounts for the idea that the phrase ‘there are’ serves to ascribe *existence*, i.e. the property that everything in the universe has, to the rest of the sentence.

But why are some choices of Q apparently forbidden in (5)? First, a definition. Call a *DET strong*, if its denotation, as a binary relation, is either reflexive or irreflexive; otherwise the *DET* is *weak*. Now observe that the *DET*s that fit in (5) are weak, whereas the exceptions are strong.[17] This is still no explanation, but it is a *fact* which may point to one. The next move is theoretical: we *prove* in our theory that (6) is equivalent to

(7) $\mathbf{Q}_M AA$;

this is actually an immediate consequence of CONSERV. It follows that

If Q is strong, (5) is either trivially true or trivially false.

Thus, the connection between the strong/weak distinction and our problem has not merely been *described*; it has been *explained*, given the plausible assumption that it is in general ‘strange’ to utter trivial truths or falsities.

This simple but instructive model of explanation shows the typical interplay between linguistic facts, theoretical concepts, and results in the theory. Here the results used were quite trivial,but this may not always be the case.

Let me hasten to add that the above by no means exhausts the many interesting problems connected with ‘there are’-sentences. Moreover, Keenan and Stavi [1986] argue against the explanation in terms of the strong/weak distinction; the propose another semantic characterisation of the relevant class of determiners (a detailed discussion of these matters can be found in [Keenan, 1989]). But it is the *type* of explanation that I have tried to illustrate here.

2.2.3 $(n+1)$*-ary conservative quantifiers*

Now, what about *more*? We noted that

(8) There are more P than Q

(9) *more* $x(Px, Qx)$.

[17] Actually, *most*, as we have interpreted it, is not reflexive, since $\mathbf{most}_m AA$ is false when $A = \varnothing$. One remedy is to redefine it for this argument.

Observe further that *more P than Q* is very naturally considered as *NP*, obtained by applying the 2-place *DET more ... than* to two Ns, and typically occurring in sentences such as

(10) More men than women voted for Smith.

(10) means that the number of men who voted for Smith is greater than the number of women who voted for Smith. So *more ... than* denotes a *ternary* quantifier:

$$\textbf{more ... than}_M A_1 A_2 B \Leftrightarrow |A_1 \cap B| > |A_2 \cap B|.$$

Other examples of such ternary quantifiers are

$$\textbf{fewer ... than}_M A_1 A_2 B \leftrightarrow |A_1 \cap B| < |A_2 \cap B|,$$
$$\textbf{as many ... as}_M A_1 A_2 B \Leftrightarrow |A_1 \cap B| = |A_2 \cap B|.$$

We now see that (8) can be written as a generalisation of (5) to ternary quantifiers:

(11) *There are* $\mathbf{Q}_M A_1 A_2$.

Furthermore, (11) can be interpreted on exactly the same principle as (5), namely, as

(12) $\mathbf{Q}_m A_1 A_2 M$.

For example, if P denotes A and Q denotes B, the interpretation of (8) is

$$\textbf{more ... than}_M ABM,$$

which is equivalent to

$$|A| > |B|$$

i.e. to

$$\textbf{more}_M AB,$$

as predicted. So the previous analysis of 'there are'-sentences with binary quantifiers extends naturally to ternary (in fact, $(n + 1)$-ary) quantifiers. (The reader might wish to ponder whether the characterisation in terms of the strong/weak distinction also generalises; cf. [Keenan, 1989]).

Finally, the notion of conservativity also extends to $(n + 1)$-ary quantifiers: the set to which the VP denotation can be restricted is then the *union* of the n denotations. We get the following general version of *CONSERV* for $(n + 1)$-ary quantifiers:

(*CONSERV*) *For all M and all $A_1, \ldots, A_n, B \subseteq M$,*
$\mathbf{Q}_M A_1 \ldots A_n B \leftrightarrow \mathbf{Q}_m A_1 \ldots A_n (A_1 \cup \ldots \cup \_n) \cap B.$

It is easily verified that *more ... than, fewer ... than, as many ... as* are all conservative, in contrast with the binary operator **more**.

In conclusion, our findings about the use of *more* do not contradict the basic idea (Q), on the contrary, they support it. A final formulation of this idea goes as follows (cf. the beginning of 2.2):

(Q_{syn}) *Quantifier expressions are DETs.*

(Q_{sem}) *n-place DETs denote $(n+1)$-ary conservative quantifiers,* $n \geqq 1$.

We should perhaps note that there are other uses of *more* in determiners, for example, *more than ten* or *six or more*. These are ordinary (complex) 1-place *DET*s, and denote binary conservative quantifiers, just as (Q_{sem}) predicts (cf. also 2.4.7).

2.3 Subject-Predicate Logic

As in Montague Grammar, Barwise and Cooper use an intermediate logical language, called $L(GQ)$, into which a fragment of English is translated. $L(GQ)$ has two unusual features;

(i) Quantified sentences have $NP - VP$ form (subject-predicate form).

(ii) Quantifier symbols are not used as variable-binding operators.

The well-formed expressions in $L(GQ)$ are of two kinds: *formulas* and *set terms*. A set term is either a unary predicate symbol or an expression of the form

$$\hat{x}[\psi],$$

where x is a variable and ψ a formula; in models, set terms denote subsets of the universe. Variable-binding is done with the *abstraction operator* ˆ. Quantifier symbols are (certain) 1-place *DET*s and quantified formulas are of the form

(*) $D(\eta)(\delta)$,

where D is a *DET* and η, δ are set terms. There are the usual atomic formulas, plus formulas of the form $\eta(t)$, where η is a set term and t an individual term, and the formulas are closed under sentential connectives. *DET*s are interpreted as binary conservative quantifiers; the truth condition for (*) in a model is then obvious.

The result is that logical form in $L(GQ)$ corresponds more closely to syntactic form in the fragment than usual. (*) can be said to have $NP - vP$ form with $D(\eta)$ as the NP and δ as the VP (the formation rules actually give (*) this structure). Another pleasant feature is that some unnecessary uses of bound variables are avoided. For example,

(1) Some boys run

is translated

(1′) *some*(*boy*)(*run*)

instead of the usual

(1″) $\exists x(boy(x) \wedge run(x))$.

the example also shows that certain unnecessary sentential connectives in the standard formalisation are avoided. In more complex cases, e.g. with transitive verbs or relative clauses, $L(GQ)$ must introduce variables and connectives (though English often can avoid them): consider

(2) Most women who love Harry have a cat,

(2′) *most* $(\hat{x}[woman(x) \wedge love(x, Harry)])(\hat{x}[a(cat)(\hat{y}[have(x,y)])])$,

(2″) *most* $x(woman(x) \wedge love(x,Harry), \exists y(cat(y) \wedge have(x,y)))$.

These examples should make it plausible that there is no deep difference between $L(GQ)$ and the standard language for generalised quantifiers as in 1.4. In fact, they are even syntactically intertranslatable in a rather obvious way. Still, quantified formulas in $L(GQ)$ have subject-predicate form. It is hard to avoid the conclusion that the importance of the issue of whether subject-predicate form occurs in logic has been greatly over-estimated, from Russell and onwards.

2.4 Some Natural Language Quantifiers

A quantifier **Q** will be called a (*simple*) *natural language quantifier*, if it is denoted by some (simplex) natural language *DET*.

This notion is somewhat loose,but its serves our purposes. A more exact specification would presuppose, among other things, (i) that the class of *DET*s has been more precisely delimited; (ii) that it has been decided how to treat complex non-logical *DET*s (2.1.3); (iii) that a global or a local perspective has been chosen (2.1.4). We may think of the notion of a natural language quantifier as having various *parameters*, which can be set at different values. It turns out that, for many of the things we shall have to say about natural language quantifiers, the value of these parameters is immaterial. This is why the above ‘loose’ notion is useful. And in other cases, we will indicate how a particular observation on natural language would depend on different parameter settings.

To take a first and crude example, consider the assertion that *not all binary quantifiers are natural language quantifiers*. From a global perspective, or from a local perspective with a given infinite universe M, this is true for cardinality reasons: there are uncountably many binary quantifiers (on M), but at most countably many natural language quantifiers. But, even from a finite local perspective, the assertion is true for another reason, namely, the conservativity universal (e.g. **more** or $\mathbf{more}_M$ is not a natural language quantifier). The other parameter settings are clearly irrelevant her, so the assertion is true however the parameters are set. An example of an assertion whose truth does depend on the parameters is this: *All natural language quantifiers satisfy ISOM*. We will see in section 3.3 that this is in fact a candidate for a quantifier universal, but only under a certain delimitation of the class of *DET*s.

In the remainder of this section, I will present a list of examples of natural language quantifiers. Some of them will be used later on, but the list is also intended to give the reader a feeling for the perhaps surprising richness of the class of natural language quantifiers.

The method is simply to list the various English *DET*s, together with their semantic interpretations (when these are not obvious). The *DET*s are selected by using the criterion for *DET*-hood in Section 2.1 as liberally as possible, but with some 'common sense' (standard co-occurrence criteria for constituenthood, etc.). Thus I will be listing *possible DET*s — there may be syntactic, semantic, or methodological reasons for discarding several of them from a more definitive list. In fact, some such reasons will be discussed in what follows.

The main sources for the list that follows are [Keenan and Stavi, 1986] and [Keenan and Moss, 1985]. The reader is referred to these works for further examples, and for detailed arguments that most of the expressions listed really belong to the category *DET*.

2.4.1 Some simplex DETs

(1) *all, every, each, some, a, no, zero, most*

(2) *both, neither*

(3) *one, two, three, ...*

(4) *many, few, several, a few*

(5) *the*

(6) *this, that, these, those*

(7) *more ... than, fewer ... than, as many ... as*

Here are some interpretations, a few of which have already been given

$$\begin{aligned}
&\mathbf{all}_M AB \Leftrightarrow \mathbf{every}_M AB \Leftrightarrow \mathbf{each}_M AB \Leftrightarrow A \subseteq B,\\
&\mathbf{some}_M AB \Leftrightarrow \mathbf{a}_M AB \Leftrightarrow A \cap B \neq \varnothing,\\
&\mathbf{no}_M AB \Leftrightarrow \mathbf{zero}_M AB \Leftrightarrow A \cap B = \varnothing,\\
&\mathbf{most}_M AB \Leftrightarrow |A \cap B| > |A - B|,\\
&\mathbf{both}_M AB \Leftrightarrow \mathbf{all}_M AB \& |A| = 2,\\
&\mathbf{neither}_M AB \Leftrightarrow \mathbf{no}_M A \& |A| = 2,\\
&\mathbf{one} = \mathbf{some},\\
&\mathbf{two}_M AB \Leftrightarrow |A \cap B| \geqq 2,\\
&\mathbf{three}_M AB \Leftrightarrow |A \cap B| \geqq 3, \ldots
\end{aligned}$$

So **n** is interpreted as **at last n** here, although it can be argued that it sometimes means **exactly n**. As for (4)–(6), cf. 2.4.2–6 below. The denotation of the 2-place *DET*s in (7) were given in 2.2.3.

2.4.2 Vague DETs

Vagueness in the sense of the occurrence of *borderline cases* (in some suitable sense) pertains to *DET*s as well as to other expressions. We do not incorporate a theory of vagueness here, but choose idealised precise versions instead.

Two examples of vague *DET*s are *several* and *a few*. Here one may, following Keenan and Stavi, stipulate that

several = three,
a few = some.

2.4.3 Context-dependent DETs

The *DET*s *many* and *few* are not only vague but also context-dependent in the sense that the 'standard of comparison' may vary with the context. For example, in

(8) Many boys in the class are right-handed,

(9) Lisa is dating many boys in the class,

some 'normal' standard for the least number considered to be many is used,but probably different standards in the two cases. Even within one sentence different standards may occur, as in the following example (due to Barbara Partee):

(10) Many boys date many girls.

Other, complex, *DET*s with a similar behaviour are, for example,

a large number of, unexpectedly few, unusually many.

Westerståhl [1985a] discusses various interpretations of *many*. Basically, there are two possible strategies. Either one excludes this type of *DET*s from extensional treatments such as the present one (this is what Keenan and Stavi do), or one tries to capture what *many* might mean in a *fixed* context (this is the approach of Barwise and Cooper). Here are some suggestions for the second strategy:

$$\mathbf{many}^1_M AB \Leftrightarrow |A \cap B| \geqq k|M| \qquad (0 < k < 1),$$
$$\mathbf{many}^2_M AB \Leftrightarrow |A \cap B| \geqq k|A| \qquad (0 < k < 1),$$
$$\mathbf{many}^3_M AB \Leftrightarrow |A \cap B| \geqq (|B|/|M|)|A|.$$

$\mathbf{many}^1$ relates the standard to the size of the universe: in a universe of 10, 5 may be many, but not in a universe of 1000. $\mathbf{many}^2$ is a *frequency* interpretation: the number of As that are B, compared to the total number of As, is at least as great as a 'normal' frequency of Bs, given by k. In both cases, k has to be supplied by the context. But in $\mathbf{many}^3$, the 'normal' frequency of Bs is just the actual frequency of Bs in the universe.

Notice that $\mathbf{many}^1$ and $\mathbf{many}^3$ make essential reference to the *universe* of the model. As we shall see, this is in contrast with most other natural language quantifiers. Also notice that $\mathbf{many}^3$ is *not conservative*. Since the conservativity universal is so central, this observation gives a (methodological) argument for discarding $\mathbf{many}^3$ as an interpretation of *many*.

As for *few*, we may simply interpret it as *not many*.

2.4.4 Ambiguous DETs

Ambiguity in the sense of a small number of clearly distinguishable meanings of a *DET* is another phenomenon than context-dependence. We have already noted that the *DET*s *one, two, three, ...* may be ambiguous with respect to the quantifiers **at least n** and **exactly n**. Another possibly ambiguous *DET* is *most*: it can be argued that aside from the interpretation we have given, *most* can also mean something like *almost all*; cf [Westerståhl, 1985a].

The fact that certain *DET*s may be ambiguous is not a problem in the present context, as long as we make sure to include each of their interpretations among the natural language quantifiers.

2.4.5 Pronominal DETs

Most 1-place *DET*s can occur without their N arguments, as was noted in 2.1.1. Such *DET*s may be called *pronominal*. The natural analysis of sentences with pronominally occurring *DET*s is that the argument (or the set it denotes) is given by the context. So

All cheered

is interpreted as

$\mathbf{all}_M X \|cheered\|$,

where the set X is provided by the context. The use of such *context sets* is studied further in [Westerståhl, 1985b].

The only non-pronominal 1-place *DET*s encountered so far are, as the reader can check,

a, every, no, the.

Moreover, *DET*s taking two or more arguments are *never* pronominal, it seems.

Note that the pronominal *all* and the non-pronominal *every* denote the same quantifier. So pronominality is not a semantic property of *DET*s in the present framework.

2.4.6 Definites

By the *simple definites* we shall understand here

(i) the definite article *the*,

(ii) the *simple possessives*, like *John's, Susan's, my, his, their*,

(iii) the demonstratives: *this, that, these, those*.

We have already given an interpretation for *the*:

$$\textbf{the}_M AB \Leftrightarrow \textbf{all}_M AB \& |A| = 1.$$

This is the *singular the*, as in

(11) The boy is running.

For a sentence like

(12) The boys are running

we must use instead

$$\textbf{the}^{\text{pl}}_M AB \Leftrightarrow \textbf{all}_M AB \& |A| > 1.$$

Thus *the* is ambiguous on this analysis. Demonstratives can be interpreted similarly; there we have singular and plural forms and thus no ambiguity. but the simple possessives exhibit the same ambiguity as *the*:

(13) John's car is clean,

(14) John's cars are clean

can be interpreted, respectively, with the quantifiers

$$\textbf{John's}_M AB \leftrightarrow \textbf{all}_M P_{\text{John}} \cap AB \& |P_{\text{John}} \cap A| = 1,$$
$$\textbf{John's}_M AB \leftrightarrow \textbf{all}_M P_{\text{John}} \cap AB \& |P_{\text{John}} \cap A| > 1,$$

where P_{John} is the st of things possessed by John; a possession relation is then supposed to be given in the model. there are also *relational* uses of possessives, where the relation is given explicitly, as in

(15) John's friends are nice.

Here it is doubtful whether *John's* applies to an N, and thus whether it is a *DET* in our sense. (In any case, the truth condition for sentences like (15) can be given by

$$\textbf{John's}^{\text{pl}}_M RB \Leftrightarrow \textbf{all}_M R_{\text{John}} B \& |R_{\text{John}}| > 1,$$

where R is a binary relation on M and $R_a = \{b \in M : Rab\}$ — here we have a generalised quantifier of type $\langle 2, 1 \rangle$.)

We see that the definites come with a *number condition*, concerning the number of elements in a certain set. It is also possible to let sentences with definites *presuppose* that the number condition is satisfied, instead of making them false when it isn't, as we did above. This could be effected by extending the model-theoretic framework to allow *partial* quantifiers **the**, $\mathbf{the}^{\mathrm{pl}}$, **John's**, $\mathbf{John's}^{\mathrm{pl}}$ would then be *undefined* when the number condition is not met. We return to this in 3.7.

2.4.7 *Complex* DET*s with definites*

There are several ways to construct complex *DET*s with definites in English, in particular with partitive constructions. I will present a rather uniform way of interpreting such *DET*s. The starting-point is the observation that one function of the simple definites is to indicate the occurrence of *context sets* (cf. 2.4.5). For simple possessives, this is usually the set of things possessed by the individual (it may also be a subset of this set). But also *the* and the demonstratives need context sets to make the interpretation come out right. For example, in (11) or (12) we are usually *not* talking about the set of all boys in the universe M (as the interpretations given in 2.4.6 would have us believe), but a context-given subset of it (in the singular case, this set has one element).

Consider sentences (with *DET*s as indicated) like

(16) *Some of the seven* men survived,

(17) *Most of John's few* books were stolen.

We interpret these on the following scheme:

(18) $(\mathbf{Q}_1 \textbf{ of Def } \mathbf{Q}_2)\, BC \Leftrightarrow \mathbf{Q}_1 X \cap BC \& \mathbf{Q}_2 X \cap BM$,

where $\mathbf{Q}_1, \mathbf{Q}_2$ are quantifiers and *Def* is a simple definite with X as associated context set (the subscript 'M' is omitted for readability). note that the second conjunct in (18) can be written, as in 2.2.2,

There are $\mathbf{Q}_2 X \cap B$,

expressing the condition that, in (17), John's books were few, and, in (16) that the set of men under consideration has (exactly?) seven elements.

Some other constructions with definites can be obtained as special cases of (18). We define

(19) $\mathbf{Def}\ \mathbf{Q}_2)BC \Leftrightarrow (\textbf{all of Def } \mathbf{Q}_2)BC$,

(20) $(\mathbf{Q}_1 \textbf{ of Def})BC \Leftrightarrow (\mathbf{Q}_1 \textbf{ of Def all})BC$
$\Leftrightarrow \mathbf{Q}_1 X \cap BC$ (by (19) with $\mathbf{Q}_2 = \textbf{all}$),

(21) $\mathbf{Def} BC \Leftrightarrow (\textbf{all of Def})BC$
$\Leftrightarrow \mathbf{all} X \cap BC$ (by (20) with $\mathbf{Q}_1 = \textbf{all}$).

(19) takes care of complex *DET*s such as

the five, these few, John's several, etc.

(20) deals with partitives such as

some of Susan's, many of these, at least five of the, etc.

And (21) returns to the simple definites: the truth conditions are essentially the same as in 2.4.6, except that context sets are mentioned.

(18)–(21) can be seen to give the right truth conditions for sentences of these forms, *except* that we have, for readability, omitted the number conditions belonging to these interpretations: in (18) and (20) a *plural condition*, i.e. that $|X \cap B| > 1$, should be added, and in (19) and (21) the cases with singular and plural conditions should be distinguished (syntactically they are distinguished by the singular or plural form of the N denoting B).

More complicated *DET*s with definites can be treated along similar lines. For example, there are *DET*s which quantify over the *possessor* a in a simple possessive

a'**s**$BC \Leftrightarrow$ **all**$P_a \cap BC$

(we continue to leave out the number condition, and assume for simplicity, in the rest of this subsection, that everything is in the plural). One example is with *DET*s like

some students', most boys', several girls', etc.,

as in

(22) Some students' books were stolen.

The interpretation of these *DET*s is given by

(23) $(\mathbf{Q}_1 A\text{'}\mathbf{s})\mathbf{C} \Leftrightarrow \mathbf{Q}_1 A\{a \in M : a\text{'}\mathbf{s}BC\}$.

Another example is with *iterated* definites. Here is one scheme, which generalises (20):

(24) $(\mathbf{Q}_1$ **of Def** $A\text{'}\mathbf{s})BC \Leftrightarrow \mathbf{Q} - 1X \cap A\{a \in M : a\text{'}\mathbf{s}BC\}$

(we could have generalised (18) similarly, but examples of this form seem rare). This covers *DET*s like

most of the students', some of these boys', three of John's cars', etc.

It could be argued that a sentence like

(25) Most of the students' books were stolen

is ambiguous; then (24) gives the sense where *most* takes *students* as argument, whereas the sense where it takes *books* as arguments is given by

(24) $(\mathbf{Q}_1 \textbf{ of DEF } A\textbf{'s})BC \Leftrightarrow \mathbf{all} X \cap A\{a \in M : (\mathbf{Q}_1 \textbf{ of } a\textbf{'s})BC\}.$

As before, if *the* is replaced by *John's* in (25), $X = P_{\text{John}}$ (or a subset of it) n (24) and (26). Also as before, we get *DET*s like

the students' those boys', Susan's cars', etc.

as a special case of (24):

(27) $(\textbf{Def } A\textbf{'s})BC \Leftrightarrow (\textbf{all of Def} A\textbf{'s})BC,$

and similarly for *DET*s like

the five students', those few boys', Susan's two cars', etc.

We have given uniform truth conditions for a number of sentences with complex *DET*s by proposing a semantics for the *DET constructions* involved there. This is one task of a theory of natural language quantification. Another is to describe and if possible explain the *restrictions* that often belong to such constructions (cf. 2.2.2).

Consider, for example, the construction in (18). One can see that only *pronominal DET*s can be in the $\mathbf{Q}_1$ position here. As for the **Def** position, the definites, and no others, will work. And there are restrictions on $\mathbf{Q}_2$ too: e.g. *most, all, every, no, some* sound strange here. This last restriction can actually be explained by combining the Barwise and Cooper explanation of the restrictions on 'there are'-sentences (2.2.2) with the *plural condition* holding for (18): the exceptions will then once more be those quantifiers making the truth condition trivial. This and other restrictions at work here are discussed further in [Westerståhl, 1985b].

There is one notable feature of the constructions with definites given here: although the analysis is compositional, it does not use the quantifiers taken to interpret the simple definites in 2.4.6. The function of simple definites was merely to provide context sets. If our analysis is viable, it opens the possibility to leave out the definites from the class of *DET*s, i.e. to treat them as not denoting quantifiers. This move has in fact been viewed desirable for independent reasons which I will not discuss here. My point is merely that such a move can be accommodated in the present quantifier framework.

Likewise, it is not strictly necessary to regard the constructions in this subsection as giving new *DET*s and thereby new natural language quantifiers. Instead, the definitions (18)–(21), (23)–25), (27) *could* be seen as uniform truth conditions for *sentences* involving (among other things) quantifiers $\mathbf{Q}_1$ and $\mathbf{Q}_2$, but not as defining new quantifiers. the class of natural language quantifiers will then become correspondingly smaller.

If, on the other hand, these constructions are regarded as quantifier definitions, it should be noted that they always yield *conservative* quantifiers, provided $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are conservative.

Clearly we have merely scratched the surface of the many problems pertaining to the analysis of definites, possessives, partitives, etc. It seems, however, that the present quantifier framework can be applied quite fruitfully to these well known linguistic questions; cf. for example [Keenan and Stavi, 1986; Partee, 1984a; Partee, 1984b; Thijsse, 1983].

2.4.8 Numerical DETs

There are many variations of the simplex numerical *DET*s *one, tow, three*, ..., e.g.

> *at least five, at most five, exactly five, five or more, between five and ten, more than five, fewer than five, infinitely many, at most finitely many, an even number of, an infinite number of, every other, every third, around ten, almost ten, nearly ten, approximately ten, ...;*

the interpretations are more or less obvious. A particular group of numerical expressions is

> *half, more than half, less than half, at least half, not more than half, two thirds, at least two thirds, ...*

These are not really *DET*s by our criterion (they don't apply to Ns), but if a phrase of the form *of Def* is appended to hem (after *half*, the *of* is optional), the resulting expressions are quite similar to those in (20): *more than half of the, two thirds of John's, not more than half of these*, ... the interpretation give in (20) fits well here,but to use it we must have suitable quantifiers $\mathbf{Q}_1$ available. Thus, it seems reasonable, even if the above expressions are not *DET*s, to include the quantifiers

$$\textbf{at least m/n}\ AB \Leftrightarrow |A \cap B| \geqq m/n|A|$$

$(n > m > 0)$ among the natural language quantifiers (Boolean combinations of these will then give the other quantifiers needed here).

2.4.9 Comparative DETs

The words *more, fewer, less*, ... can be used in *DET*s for comparison with a fixed number or proportion, as in 2.4.8. We also have the 2-place simplex *DET*s *more ... than, fewer ... than,* etc. Some complex variants of these are

> *more than twice as many ... as, less than half as many ... as, etc.*

Keenan and Stavi discuss other comparative *DET*s, e.g. those in

(28) *More male than female* students stayed home,

(29) *More* students attended *than stayed home*,

(30) *More* students attended *than teachers who stayed home*;

the respective 1-place *DET*s are italicised. That they are putative *DET*s follows by our criterion (nothing prevents a 1-place *DET* from being syntactically discontinuous!). However, it is also possible to analyse (28)–(30) with the 2-place *more ... than*: rewrite them as

(28′) *More* male students *than* female students stayed home,

(29′) there are *more* students who attended *than* students who stayed home,

(30′) there are *more* students who attended *than* teachers who stayed home.

The last two 'there are'-sentences are then treated as in 2.2.3.

These examples illustrate nicely that more than one structural analysis of an NP is often possible. Since no semantic ambiguity is involved here, one would like to make a choice. For a further illustration, consider

(31) *More* men *than* women voted for Smith,

(31′) *More* men *than women* voted for Smith.

(31) uses *more ... than*, whereas (31′) uses the 1-place *more than women*. but this latter *DET* is not conservative, as one easily sees, so we have a good reason to prefer (31). The *DET*s in (18)–(30), on the other hand, are all conservative. For example,

$$\begin{aligned}&\textbf{more than stayed home}_M AB \Leftrightarrow\\&\quad\Leftrightarrow |A \cap B| > |A \cap \|\textit{stayed home}\||\\&\quad\Leftrightarrow |A \cap (A \cap B)| > |A \cap \|\textit{stayed home}\||\\&\quad\Leftrightarrow \textbf{ more than stayed home}_M AA \cap B.\end{aligned}$$

Still, there are reasons to prefer (28′)–(30′). One is that they are simpler and more uniform. Another will be given in Section 3.3.

Keenan and Stavi also consider comparatives with definites, such as

more of John's than of Susan's, fewer of the male than of the female, etc.

These can be dealt with, if one wishes, by combining the simplex 2-place comparatives with the treatment of definites in 2.4.6 an d2.4.7; we omit details.

2.4.10 "Only"

Consider the sentence

(32) Only women voted for Smith.

If *only* is a *DET* here, its interpretation is

$$\textbf{only}_M AB \leftrightarrow B \subseteq A.$$[18]

This is *not* a conservative quantifier (indeed, $\textbf{only}_M AA \cap B$ is trivially true for all A, B). So let us look for alternatives. Now, *only* can modify many other things besides Ns, e.g. *NP*s:

(33) Only Susan voted for Smith.

An alternative analysis is then to treat *women* in (32) as a full em NP (a 'bare plural'); then *only* is not a *DET* at all.

there are also complex *DET*s with *only*. Consider the following example (essentially from Keenan and Stavi):

(34) Only liberal students voted for Smith.

This sentence is three ways ambiguous: (i) as an answer to 'Which students voted for Smith?'; (ii) as an answer to 'Which liberals voted for Smith?'; and (iii) as an answer to 'Who voted for Smith?'. Writing (34) in the form *only* ABC, we can represent its three meanings as

(i) *only* $ABC \Leftrightarrow B \cap C \subseteq A$,

(ii) *only* $ABC \Leftrightarrow A \cap C \subseteq B$,

(iii) *only* $ABC \Leftrightarrow C \subseteq A \cap B$.

There are various possibilities here. One is to treat *only* as a 2-place *DET* with three possible interpretations, as in (i)–(iii). One readily verifies that (i) and (ii), but not (iii), are conservative. Or, if one wants to analyse (34) with a 1-place *DET*, we have, in case (i),

$$\textbf{only liberal}_M AB \Leftrightarrow A \cap B \subseteq \|\textit{liberal}\|;$$

in case (ii),

$$\textbf{only \ldots students}_M AB \Leftrightarrow A \cap B \subseteq \|\textit{student}\|$$

(but *only . . . students* isn't really a *DET* since it applies to an adjective); and in case (iii) the ordinary *only*, as in (32). Again, the first two are conservative, but not the third.

Only can also combine with numerical expressions, as in

(35) Only five students voted for Smith.

[18] One may argue that (32) also says that *some* women voted for Smith. We ignore the possible existence implications of *only* here, but they could easily be added without affecting the discussion.

This time, there is no analysis with a 2-place *DET*, and there are just two possible meanings: (i) as an answer to 'How many students voted?'; and (ii) as an answer to 'How many voted?'. So, writing (35) as *only five* AB, we get

(i) *only five* $AB \Leftrightarrow$ **exactly five**$_M AB$,[19]

(ii) *only five* $AB \Leftrightarrow$ **exactly five**$_M AB \& B \subseteq A$.

In case (ii), *only five* would be a non-conservative *DET*, but it is more natural to treat *only* as an NP-modifier here. In case (i), on the other hand, *only five* works fine as a conservative *DET*. Here one would like to see a uniform treatment of *DET*s of the form

(36) *only* Q;

we have already seen that *only* 'transforms' n into *exactly* n, but when Q is a definite, things get more complicated, as the reader can check by considering the example

(36) Only John's students voted for Smith

(three possible readings). Also, one would like to explain the restrictions on Q in (36). For example, *a few, between five and ten, around ten* are fine, but not *several, all, most*.

These are just a few hints about some phrases with *only*, and nothing like a uniform semantics analysis. For further discussion, cf. Keenan and Stavi [1986], Rooth [1984; 1985].

2.4.11 Exception DETs

This term is used by Keenan and Stavi for *DET*s like

all but three, all but at most five, all but finitely many, ...

As for interpretations, we have

all but three$_M AB \leftrightarrow |A - B| = 3$,
all but at most five$_M AB \Leftrightarrow |A - B| \leqq 5$,
all but finitely many$_M AB \Leftrightarrow A - B$ is finite.

The construction *all but Q* apparently obeys certain restrictions — we will return to these in 3.4. It can create ambiguities similar to the ones discussed for *only* in 2.4.10; cf.

(38) All but five liberal students voted for Smith.

[19]There is also the idea that five is unexpectedly few here. It would be possible to add $\mathbf{few}_M AB$ as a further condition.

There are also exception *DET*s with proper names and with definites:

every but John, no but John, every but John's, all but the liberal, ...

Some of these are discontinuous

(39) *Every* student *but John* voted for Smith,

(40) *Every* car *but John's* was stolen,

(41) *Every* book *but this* (one) was returned.

If we were to treat proper names as definites in the sense of 2.4.7, i.e. as providing suitable sets (in this case: the unit set of the denoted individual), we could interpret these on the uniform scheme

(42) $\textbf{every but DEF}_M AB \Leftrightarrow |X \cap A| = 1\ \&$
$\&\ \textbf{every}_M A - XB \&\ \textbf{no}_M A \cap XB,$

where, in (39), $X = \{\text{John}\}$, and, in (40), $X = P_{\text{John}}$; note that e.g. (39) says that John is a student, that he didn't vote for Smith, but that all other students voted for Smith. Note also that (42) gives conservative quantifiers.

2.4.12 Boolean combinations

First, *negation*, as in

not every, not all, not many, note more than five, not fewer than there, not more than half (of the), ...

The semantics of negated quantifiers is obvious,

$$(\textbf{not Q})_M \Leftrightarrow \neg \mathbf{Q}_M AB,$$

but *not* cannot stand in front of all *DET*s: e.g. *not some, not most, not at most five* are not well-formed. It is not clear that there is a semantic explanation for this. An interesting question, however, is whether the class of natural language quantifiers is *closed under negation*. For example, even though *not most* is not a *DET*, we can express the intended quantifier with another *DET*:

$$\begin{aligned} \neg\, \textbf{most}_M AB &\Leftrightarrow |A \cap B| \leqq |A - B| \\ &\Leftrightarrow |A \cap B| \leqq 1/2|A| \text{ (on finite sets, of course)} \\ &\Leftrightarrow \textbf{not more than half (of the)}_M AB \end{aligned}$$

Likewise, we have ¬(**at most five**) = **more than five**. But there are other cases which seem more doubtful, for example, the exception *DET*s: what *DET* would express the negation of *all but three* or *every but John*? We return to this question in 3.4.

As for conjunction and disjunction, we have *some but not all, some but not many, most but not all, at least five and at most ten, either exactly five or more than ten, neither less than five nor more than ten, John's but not Susan's, neither John's nor Susan's, both John's and Susan's*, ...

Again the semantics is clear. It is tempting to claim that *any* two 1-place *DET*s can in principle be conjoined with *and* or *or* (another matter is that many such conjunctions and disjunction would be long and cumbersome, express trivial or otherwise 'strange' quantifiers, etc.) n-place *DET*s for $n > 1$ are discontinuous, which makes the claim less plausible in this case.[20] But the class of binary natural language quantifiers would, if the claim is correct, be closed under conjunction and disjunction.

Boolean operators can also be used to create n-place *DET*s for $n > 1$, e.g. the 2-place

every ... and, some ... or,

as in

(44) Every businessman and lawyer knows this,

(45) Some mother or father will react

Note that (43) is ambiguous. In general, there are two possible readings of sentences of the form QA *and/or* BC:

(45) $Q^1 A$ *and* $BC \leftrightarrow QA \cap BC$,
$Q^2 A$ *and* $BC \leftrightarrow QAC \,\& QBC$

(46) $Q^1 A$ *or* $BC \leftrightarrow QA \cup BC$,
$Q^2 A$ *or* $BC \leftrightarrow QAC \vee QBC$

In the one sense of (43) we have the ordinary *every* applied to the complex N *businessman and lawyer*, and in the other we have *every*^{2} applied to the two Ns *businessman* and *lawyer*. Of course, it is not absolutely necessary to use 2-place *DET*s here, since the interpretations are definable with 1-place *DET*s. For several arguments that 2-place *DET*s are in fact the natural choice, and for more examples, we refer to Keenan and Moss [1985].

We may note that

(47) *every*$^2 A$ *and* $BC \Leftrightarrow$ *every*$^1 A$ *or* BC,

(48) *some*$^1 A$ *or* $BC \Leftrightarrow$ *some*$^2 A$ *or* BC.

(47) explains why the second reading of (43) can also be expressed by

[20] We had a few examples of discontinuous 1-place *DET*s too, e.g. *every but John*, and here the claim is more dubious. But note that in all these cases, an alternative analysis was proposed, which eliminates the need for the *DET*s in question.

(49) Every businessman or lawyer knows this.

(48) explains why (44) isn't in fact ambiguous.

The same method as above can be used to create n-place *DET*s for all $n > 1$; cf.

(50) Every professor and assistant and secretary and student has a key.

This 4-place *DET* would be interpreted by a 5-ary quantifier similarly to (45) (the second reading seems to be preferred here, which again is manifested in the fact that *and* can be replaced by *or* in (50)).

3 QUANTIFIER CONSTRAINTS AND SEMANTIC UNIVERSALS

A natural way to approach the class of natural language quantifiers is to study the effect of linguistically motivated *constraints*, such as conservativity, on the class of all quantifiers. These constraints are related to *semantic universals*, i.e. general statements about semantic interpretation true for all natural languages. In this section we discuss some such constraints; a number of *possible* semantic universals will be noted along the way.

3.1 The Restriction to Monadic Quantifiers

In Section 2 we tacitly assumed that natural language quantifiers are monadic, i.e. of type $\langle 1, 1, \ldots, 1\rangle$. Is there some reason natural language should not employ non-monadic generalised quantifiers like those used in mathematical logic?

Towards an answer to this, recall first that generalised quantifiers are *second-order* properties or relations (cf. 1.2.1 and 1.4). Thus, *any* sentence which attributes, say, a (second-order) property to a (first-order) property can in principle be formalised as a quantified sentence. For example, consider

(1) Red is a colour.

Even in our extensional framework we *could* define a quantifier C of type $\langle 1\rangle$ by

$$C_M = \{X \subseteq M : X \text{ is the extension in } M \text{ of some colour}\}.$$

So C_M would contain the set of all blue things in M, the set of all red things in M, etc. Then (1) can be formalised as

(2) $Cx\ \mathit{red}(x)$,

which is true in a model **M** iff the set which *red* denotes in **M** is (the extension of) a colour. This quantifier is monadic, but a similar story could be told for properties of binary relations, i.e. generalised quantifiers of type $\langle 2\rangle$.

But from our perspective, (2) is clearly an *unreasonable* formalisation of (1). It is useful to understand why. Compare (2) with

(3) $\exists x\ red(x)$,

which formalises

(4) Something is red.

There is a match in *logical form* between (3) and (4),[21] which is lacking between (1) and (2). Roughly, the difference is that *some* and *colour* are of completely different syntactic categories (*some* is an operator and *colour* is a predicate). In a natural language context, such matching appears to be essential. It is now always essential in mathematical contexts; cf. the quantifier **W**, where

$WxyPxy$

expresses that

P is a wellordering.

These remarks are really just another way of putting our basic idea that, in natural language, quantifier expressions are *DET*s. So the question is this: are there *DET*s denoting non-monadic quantifiers? Put differently, are there *DET*s whose corresponding quantifier symbols bind more than one variable in the succeeding formula(s)?

The following example was suggested by Hans Kamp:

(5) Most lovers will eventually hate each other.

This sentence makes good sense,[22] and, looking closely, one sees that it does not talk about the *set* of people who love and are loved by someone, but instead about *pairs*[23] of people who love each other: most such pairs will end up as pairs whose members hate each other. In other words, (5) is *not* equivalent to

(6) Most people who love and are loved by someone will eventually hate and be hated by everyone (or someone) they love.

This follows from the observation that one person may belong to different 'loving pairs'; using this it is easy to construct models where (5) and (6) (in either version) differ in truth value.[24]

[21] The match would be even better if we had used the binary **some** instead of the usual existential quantifier.

[22] Other similar sentences are harder to make sense of, for example,

Most schoolboys tease each other.

Is this about pairs of schoolboys, or does it mean that most schoolboys tease some other schoolboy, or most other schoolboys, . . . ? The problem seems to be that *schoolboy* denotes a set but *each other* indicates a relation.

[23] I take the pairs to be ordered, but this doesn't really matter.

[24] In other cases equivalence would obtain. Consider, for example,

Most twins like each other.

Since everyone is the twin of at most one other person, there are as many individual twins as there are ordered twin pairs, and thus the same proportion of 'liking' twin pairs as that of twins who like their other twin.

In the terminology of Section 1.4 we would formalise (5) as

(7) $\textit{most}^{(2)}xy(\textit{love}^*(x,y),\ \textit{will eventually hate}^*(x,y))$,

where $R^*(x,y)$ means $R(x,y) \wedge R(y,x)$ and

$$\mathbf{most}^{(2)}_M = \{\langle R_1, R_2\rangle : R_1, R_2 \subseteq M^2 \& \\ \& |R_1 \cap R_2| > |R_1 - R_2|\},$$

a generalised quantifier of type $\langle 2, 2\rangle$.

Another suggestion to use quantification over pairs instead of individuals appears in Fenstad *et al.* [1987]. They consider sentences like

(8) Every boy who owns a dog kicks it.

There is a question as to the meaning of this, but the preferred reading appears to be that every boy who owns a dog kicks every dog he owns; in other words, using the binary *every* and *some*,

(9) *every* x(*boy*$(x)\wedge$ *some* y (*dog*(y), *owns*(x,y)), *every* y(*dog*$(y)\wedge$ *owns* (x,y) *beats*(x,y)).

The traditional problem here has been to get (9) (or something equivalent to it) from a compositional analysis of (8); note that *it* refers back to *a dog*, but does not correspond to a bound variable in (9)! Fenstad *et al.* propose a way to do this; their analysis (whose details need not concern us here) leads, essentially, to the formalisation

(10) *every*$^{(2)}xy$(*boy*$(x)\wedge$ *dog*$(y)\wedge$ *owns*(x,y), *beats*(x,y))

where *every*$^{(2)}$ denotes the type $\langle 2, 2\rangle$ generalised quantifier

(11) $\mathbf{every}^{(2)}_M = \{\langle R_1, R_2\rangle : R_1, R_2 \subseteq M^2 \& R_1 \subseteq R_2\}$.

Note that (10) and (9) are equivalent. (Note also, however that, as Johan van Benthem has pointed out, this analysis does not seem to work for all quantifiers: consider

(12) Most boys who own a dog kick it.

Here, the sentence obtained from (9) by replacing the first occurrence of *every* with *most* is *not* equivalent to the sentence obtained from (10) by replacing *every*$^{(2)}$ with *most*$^{(2)}$. Moreover, the former sentence appears to give the preferred reading.[25])

A third and final example that *could* be construed as quantification over pairs in natural language is branching quantification as discussed in Section 1.5. To take an example from Barwise [1979], consider

[25]Consider a situation with two boys, one of whom owns and kicks two dogs, the other owning, but not kicking, one dog. The formalisation with $\mathbf{most}^{(2)}$ would be true in this case, which seems counter-intuitive.

(13) Most boys in my class and most girls in your class know each other.

The preferred reading of this sentence (which has a *conjoined NP*) can be formalised as

$$(14) \qquad \begin{array}{l} \textit{most } x \textit{ boy-in-my-class}(x) \\ \qquad\qquad\qquad\qquad\qquad \textit{know}^*(x,y), \\ \textit{most } y \textit{ girl-in-your-class}(y) \end{array}$$

where this is taken to mean that there is a subset X of the boys in my class, containing most of these boys, and a subset Y of the girls in your class, containing most of those girls, such that if $a \in X$ and $b \in Y$ then a knows$^*b$ (cf. Appendix A).

(14) involves branching of the ordinary monadic **most**. But, as noted in 1.5, it is *possible* to 'simulate' branching of two (or more) quantifiers by means of one generalised quantifier. That generalised quantifier will be non-monadic — in the present case, it has type $\langle 1, 1, 2\rangle$, since it relates two sets (the set of the boys and the set of the girls) and one binary relation (know*).

What can be concluded from these examples? Two things should be noted. The first is that the *logical power of expression increases* if the constructions in the examples are included. Consider the logic $L(\mathbf{most}^{(2)})$. It is easy to see that **most** is expressible in this logic, so $L(\mathbf{most} \leqq L(\mathbf{most}^{(2)})$. But the converse does not hold; the following result was pointed out by Per Lindström:

THEOREM 15. $L(\mathbf{most}) < L(\mathbf{most}^{(2)})$ *(*even on finite models*).*

Proof. [Cf. Section 1.7] Given a natural number d, choose two finite models $\mathbf{M}= \langle M, A_0, A_1, A_2\rangle$ and $\mathbf{M}' = \langle M', A_0', A_1', A_2'\rangle$ such that the $A_i(A_i')$ are pairwise disjoint sets whose union is $M(M')$, and, if $|A_0| = k, |A_1| = m, |A_2| = n$, then $|A_0'| = k-1, |A_1'| = m, |A_2'| = n$, and

(a) $(k-1)m \leqq n < km$,

(b) $k < m < n$ and $k, m-k, n-m > 2d$.

Now, consider the sentence

$$\textit{most}^{(2)}xy((P_0x \wedge P_1y) \vee (P_2x \wedge x = y), P_0x \wedge P_1y).$$

In $\mathbf{M}$, this expresses that

$$km > n$$

(note that $P_2x \wedge x = y$ denotes $\{(a,a) \in M^2 : a \in A_2\}$, whose cardinal is n). Likewise, it expresses in $\mathbf{M}'$ that $(k-1)m > n$, so, by (a), it is true in $\mathbf{M}$ but false in $\mathbf{M}'$. On the other hand, using (b) and Theorem 10 it is easily seen that $\mathbf{M} \equiv_{d,\mathbf{most}} \mathbf{M}'$. Thus, since d was arbitrary, $L(\mathbf{most}^{(2)}) \nleqq L(\mathbf{most})$. ■

The same holds for the branching of **most**. Let $L_b(\textbf{most})$ be the logic which extends $L(\textbf{most})$ by allowing formulas of the form (14), interpreted as indicated for that example. It can be shown that '$|A|$ is even' is expressible in $L_b(\textbf{most})$. Thus, by (4) in Section 1.7, we get the

THEOREM 16. $L(\textbf{most}) < L_b(\textbf{most})$ *(even on finite models).*

The second observation to make, however, is that there are clear senses in which the non-monadic quantification considered here is *reducible* to monadic quantification. Thus, branching maybe seen as a linguistic construction on its own, making monadic quantifiers as *arguments*. And as for the first two examples, $\textbf{most}^{(2)}$ is really just the old **most** applied to the new universe M^2:

$$\textbf{most}^{(2)}_M = \textbf{most}_{M^2},$$

and similarly for $\textbf{every}^{(2)}$. Here we have *lifted* a relation on sets to a relation on binary relations. In general, any k-ary monadic quantifier **Q** can be lifted to any $n > 1$: define $\mathbf{Q}^{(n)}$, of type $\langle n, n, \ldots, n\rangle$ by letting, for all $R_1, \ldots, R_k \subseteq M^n$,

$$\langle R_1, \ldots, R_k\rangle \in \mathbf{Q}^{(n)}_M \Leftrightarrow \mathbf{Q}_{M^n} R_1, \ldots, R_k.$$

In view of the foregoing discussion we have a possible semantic universal of the form

($U1$) *Natural language quantifiers are either monadic or reducible to monadic quantifiers*,

where 'reducible' may be specified along the lines suggested above.

NB. This universal has been challenged recently, however, in [Keenan, 1987]. He considers sentences like

(15) Every boy read a different book

and shows that, although this may seem as simple *iteration* of two monadic quantifiers, the truth conditions for (15) cannot be so obtained, nor can they be obtained by branching or lifting monadic quantifiers. For further discussion of this matter, cf. also [van Benthem, 1987b]. In what follows, however, we will restrict attention to monadic quantifiers.

3.2 The Universe of Quantification

Recall the definition of conservativity for an $(n+1)$-ary quantifier **Q**:

$$\textit{CONSERV}\ \mathbf{Q}_M A_1 \ldots A_n, B \Leftrightarrow \mathbf{Q}_M A_1 \ldots A_n, (A_1 \cup \ldots \cup A_n) \cap B$$

(for all M and all $A_1, \ldots, A_n, B \subseteq M$; we will usually omit this). We have put a comma before 'B' here to indicate that '$\mathbf{Q}_M A_1, \ldots A_n$' corresponds to the NP and 'B' to the VP. *CONSERV* says that the VP denotation can be restricted to (the union of) the N denotation(s). Another way to put this is

(*) *If B and C have the same intersections with all the A_i, then* $\mathbf{Q}_M A_1 \dots A_n, B \Leftrightarrow \mathbf{Q}_M A_1 \dots A_n, C$.

It is easily checked that *CONSERV* and (*) are equivalent conditions.

It is *almost* true that *CONSERV* restricts the universe of quantification to (the union of) the first (n) argument(s); cf. the discussion in 2.2.1. But not quite: the *DET* denotation may depend essentially on the universe M. The following condition, which we formulate for arbitrary n-ary quantifiers, expresses the requirement of 'universe-independence' for quantifiers ('*EXT*' for 'extension'):

EXT *If* $A_1, \dots, A_n \subseteq M \subseteq M'$
then $\mathbf{Q}_M A_1 \dots A_n \Leftrightarrow \mathbf{Q}_{M'} A_1 \dots A_n$.

This has nothing to do with *CONSERV*; rather, it is a strengthening of the postulate, discussed in 2.1.3; that quantifier expressions are *constants*. For example, *EXT* excludes a quantifier which is $\mathbf{all}_M$ when M has fewer than 10 elements and $\mathbf{some}_M$ otherwise. But *together* with *CONSERV, EXT* gives the exact sense in which *DET*s can be said to restrict the universe of quantification:

UNIV $\mathbf{Q}_M A_1 \dots A_n, B \Leftrightarrow$
$\mathbf{Q}_{A_1 \cup \dots \cup A_n} A_1 \dots A_n, (A_1 \cup \dots \cup A_n) \cap B$.

It is an easy exercise to show

PROPOSITION 17. *UNIV is equivalent to CONSERV + EXT.*

Some further discussion of universe-restriction can be found in Westerståhl [1985a; 1983].

CONSERV and *EXT* are related to the logician's notion of *relativisation* (Sections 1.4 and 1.6). Let us first note

PROPOSITION 18. *If* $\mathbf{Q}^i$ *satisfies EXT for* $i \in I$, *then* $L(\mathbf{Q}^i)_{i \in I}$ *relativises.*

Proof. Since *EXT* implies that

$$(Q^i)^r x(Px, P_1x, \dots, P_nx) \leftrightarrow$$
$$\leftrightarrow Q^i x(Px \wedge P_1x, \dots, Px \wedge P_nx)$$

is valid. ■

If in addition *CONSERV* holds we can say more: the *binary* quantifiers satisfying *CONSERV* and *EXT* are precisely the relativized ones. Moreover, the *sentences* (in any logic) with two unary predicate symbols which satisfy *CONSERV* and *EXT* (in the obvious sense) are precisely the ones equivalent to the relativised sentences. This is the content of the next result:

THEOREM 19.

(a) A binary quantifier $\mathbf{Q}$ *satisfies CONSERV and EXT iff* $\mathbf{Q} = (\mathbf{Q}')^r$, *for some unary* $\mathbf{Q}'$.

(b) *A sentence $\phi(P_1, P_2)$ with two unary predicate symbols in a logic L satisfies CONSERV and EXT iff it is equivalent to $\psi^{(P_1)}$, for some L-sentence ψ.*

Proof. We prove (b); (a) then follows (it is also easily proved directly). Recall the basic property of relativised sentences from 1.6, in this case, with **M**$= \langle M, A, B\rangle$,

$$(REL) \quad \langle M, A, B\rangle \vDash \psi^{(P_1)} \Leftrightarrow \langle A, A \cap B\rangle \vDash \psi.$$

From this it is immediate that $\psi^{(P_1)}$ satisfies *CONSERV* and *EXT*. Conversely, if $\phi(P_1, P_2)$ satisfies *CONSERV* and *EXT*, let $\psi = \phi(x = x, P_2)$.

Then

$$\begin{aligned} \langle M, A, B\rangle \vDash \psi^{(P_1)} &\Leftrightarrow \langle A, A \cap B\rangle \vDash \psi && (REL) \\ &\Leftrightarrow \langle A, A, A \cap B\rangle \vDash \phi(P_1, P_2) && \text{(by def. of } \psi) \\ &\Leftrightarrow \langle M, A, B\rangle \vDash \phi(P_1, P_2) && \text{(by } UNIV). \end{aligned}$$

■

The interest of (b) is that it relates a semantic notion (*CONSERV* and *EXT*) to a syntactic property of sentences — a typical sort of logical result.

Notice that, for *unary* quantifiers, *CONSERV* makes no sense, and *EXT*, although it can be formulated, is *not true* for e.g. the standard universal quantifier $\forall$. This is another aspect of the advantage of binary quantifiers. Any unary quantifier can be replaced by a binary one (its relativisation) which does (at least) the same work and has the additional property of restriction the universe of quantification to the first argument. As Theorem 19 shows, this moves give us *all* the binary quantifiers with that property, in particular, it gives us all the binary natural language quantifiers (provided ($U2$) and ($U3$) below hold).

For n-ary quantifiers with $n > 1$, it is also possible to secure *CONSERV* and *EXT* by raising the number of arguments, though not quite as simply as when $n = 1$. The next proposition surveys the possibilities.

PROPOSITION 20. *Let $Q\mathbf{Q}$ be an n-ary quantifier. then*

(i) *there is an $(n{+}1)$-ary quantifier $\mathbf{Q}'$ satisfying CONSERV such that $\mathbf{Q}_M A_1 \ldots A_n \Leftrightarrow \mathbf{Q}'_M A_1 \ldots A_n, M$;*

(ii) *there is an $(n{+}2)$-ary quantifier $\mathbf{Q}''$ satisfying EXT such that $\mathbf{Q}_M A_1 \ldots A_n \Leftrightarrow \mathbf{Q}''_M A_1 \ldots A_n, M$;*

(iii) *there is an $(n + 1)$-ary quantifier $\mathbf{Q}^+$ satisfying both CONSERV and EXT such that $\mathbf{Q}_M A_1 \ldots A_n, \Leftrightarrow \mathbf{Q}^+_M A_1 \ldots A_n M, M$.*

Proof.

(i) Define $\mathbf{Q}'_M A_1 \ldots A_n, B \Leftrightarrow \mathbf{Q}_M A_1 \ldots A_n \cap B \ldots A_n \cap B$. The verification of *CONSERV* is immediate.

(ii) Let $\mathbf{Q}''_M A_1 \ldots A_n, B \Leftrightarrow \mathbf{Q}_B A_1 \cap B \ldots A_n \cap B$; again *EXT* is immediate.

(iii) Define $\mathbf{Q}''$ as in (ii), and then form $\mathbf{Q}^+$ from $\mathbf{Q}''$ as in (i); the result follows from (i) and (ii).

■

For the record, we formulate the semantic universals corresponding to *CONSERV* and *EXT*:

$(U2)$ Natural language quantifiers are conservative.

$(U3)$ Natural language quantifiers satisfy *EXT*.

We saw in 2.4 that the few apparent exceptions to $(U2)$ could be accounted for by reasonable methodological decisions (2.4.3, 2.4.9–10). As for $(U3)$, the only exceptions found were certain interpretations of context-dependent *DET*s like *many*. For example, if

$$\mathbf{Q}_M AB \Leftrightarrow |A \cap B| \geqq 1/3|M|,$$

$\mathbf{Q}$ violates *EXT*. Again, it is mainly a methodological question whether one wants to allow this kind of context-dependence or not.

3.3 Quantity

The condition *ISOM*, repeated below, was formulated for generalised quantifiers of any type $\langle k_1, \ldots, k_n \rangle$:

ISOM *If f is a bijection from M to M' then $\mathbf{Q}_M R_1 \ldots R_n \Leftrightarrow \mathbf{Q}_{M'} f[R_1] \ldots f[R_n]$.*

The idea is that $\mathbf{Q}$ does not distinguish between different elements of the universe, or even across two universes. This requirement, which is a version of what is sometimes called *topic-neutrality*, can be formulated for arbitrary syntactic categories (cf. [van Benthem, 1983b]). It is a general requirement of *logical constants*.

For monadic quantifiers, *ISOM* has a particularly conspicuous formulation. Roughly, it says that quantifiers deal only with *quantities*. The latter assertion can be made precise with the terminology from Section 1.7 as follows:

QUANT *If $\mathbf{M} = \langle M, A_0, \ldots, A_{k-1} \rangle$, $\mathbf{M}' = \langle M', A'_0, \ldots, a'_{k-1} \rangle$, and $|P_s^{\mathbf{M}}| = |P_s^{\mathbf{M}'}|$ for all $s \in 2^k$, then $\mathbf{Q}_M A_0 \ldots A_{k-1} \Leftrightarrow \mathbf{Q}_{M'} A'_0 \ldots A'_{k-1}$.*

This means that the truth value of $\mathbf{Q}_M A_0 \ldots A_{k-1}$ depends only on w^k quantities, namely, the number of elements in the partition sets.

A bijection from M to M' splits into bijections of the respective partition sets, and, conversely, bijections between these sets can be joined to one from M to M'. Thus we have that

PROPOSITION 21. *ISOM and QUANT are equivalent (for a monadic* $\mathbf{Q}$*).*

If we consider only one universe M in *ISOM* (letting $M' = M$), and thus *permutations* on M, we get a slightly weaker version, called *PERM*.[26] From a local perspective on quantifiers (2.1.4), *PERM* is the natural notion. Our global condition *EXT*, however, says that the choice of universe is unimportant. Indeed, it is straightforward to prove

PROPOSITION 22. *Under EXT, ISOM and PERM are equivalent.*

All the simplex *DET*s from 2.4.1–3 denote quantitative quantifiers. To see this, it is sufficient to check that the defining conditions can be expressed as conditions on the cardinalities of the relevant sets. For example, $\mathbf{all}_M AB \Leftrightarrow |A - B| = 0, \mathbf{some}_M AB \Leftrightarrow |A \cap B| \neq 0, \mathbf{most}_M AB \Leftrightarrow |A \cap B| > |A - B|, \mathbf{both}_M AB \Leftrightarrow |A - B| = 0 \& |A \cap B| = 2, \mathbf{many}^2_M AB \Leftrightarrow |A \cap B| \geqq k(|A \cap B| + |A - B|)$, etc.

As for complex *DET*s, there are just a few of the constructions in 2.4.6–12 which yield non-quantitative quantifiers. One example is *DET*s with fixed adjective phrases, or similar expressions, such as *more male than female, some red, only liberal*. We saw, however, that sentences with such expressions can also be interpreted using only quantitative quantifiers (2.4.9–10). Another major example are the possessives, either simple ones such as *John's*, or complex constructions with possessives. The quantifier **John's** from 2.4.6 violates ISOM since the ownership relation need not be preserved under permutations of the objects in the universe. For example, John may own two white shirts but no red tie, even though it is possible to permute the shirts and the ties, and the white things and the red things in a one-one fashion. Then

John's shirts are white

is true, but not

John's ties are red,

as *ISOM* would require.

In 2.4.7, we mentioned an alternative analysis of definites, and thus in particular of possessives. Under this analysis, one can dispense with quantifiers denoted by simple possessives, also in various complex constructions. Quantitative quantifiers would suffice, it seems, for all of these constructions (the same holds for *every but John* (2.4.11), another counter-instance to *ISOM*). It would then be possible to propose the following rather appealing universal:

($U4$) *Natural language quantifiers are quantitative.*

If one does not want to take this methodological step, on the other hand, one will settle for the more modest

($U4'$) *Simple natural language quantifiers are quantitative.*

[26] To get a 'quantity version' of *PERM*, let $M' = M$ in *QUANT*.

3.4 Logical Quantifiers, Negations and Duals

Whichever version of the last universal one prefers, the following class of quantifiers is a natural object of study:

DEFINITION 23. If n-ary quantifier $(n > 1)$ is *logical* then it satisfies *CONSERV, EXT* and *QUANT*.

The terminology is meant to suggest that these three requirements are *necessary* for logicality; further conditions will be discussed in 4.4.

For *binary* quantifiers, logicality means that the truth value of $\mathbf{A}_M AB$ depends only on the two numbers $|A - B|$ and $|A \cap B|$:

PROPOSITION 24. *A binary quantifier* $\mathbf{Q}$ *is logical iff, for all* M, M' *and all* $A, B \subseteq M$ *and* $A', B' \subseteq M', |A - B| = |A' - B'|$ *and* $|A \cap B| = |A' \cap B'|$ *implies that* $\mathbf{Q}_M AB \Leftrightarrow \mathbf{Q}_{M'} A'B'$.

Proof. If $\mathbf{Q}$ is logical and $|A - B| = |A' - B'|$ and $|A \cap B| = |A' \cap B'|$, then, by *QUANT*, $\mathbf{Q}_A AA \cap B \Leftrightarrow \mathbf{Q}_{A'} A'A' \cap B'$, and so, by *UNIV* (Proposition 17), $\mathbf{Q}_M AB \Leftrightarrow \mathbf{Q}_{M'} A'B'$. Conversely, if the right-hand side of the equivalence holds, *QUANT* is immediate. Take M and $A, B \subseteq M$ and let $M' = A' = A$ and $B' = A \cap B$. Thus, $\mathbf{Q}_M AB \Leftrightarrow \mathbf{Q}_A AA \cap B$, i.e. *UNIV* holds. ■

This means that a logical binary relation between *sets* can be replaced by a binary relation between *cardinal numbers*; we exploit this in 4.2. Proposition 24 can be generalised to n-ary logical quantifiers: *QUANT* transforms an n-ary $\mathbf{Q}$ to a relation between 2^n cardinal numbers, and *CONSERV* + *EXT* eliminate the dependence of *two* of these.

The class of logical quantifiers has some nice closure properties. It is straightforward to verify that if $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are *CONSERV and EXT* (*QUANT*), then so are $\mathbf{A}_1 \wedge \mathbf{Q}_2, \mathbf{Q}_1 \vee \mathbf{Q}_2$, and $\neg\mathbf{Q}_1$. Thus,

PROPOSITION 25. *For each* $n > 1$*, the class of* n*-ary logical quantifiers is closed under the usual Boolean operations.*

In a natural language context, there are also *inner* Boolean operations. We noted in 2.4.12 that from a binary $\mathbf{Q}$ one can construct two $(n + 1)$-ary inner conjunctions:

$$\mathbf{Q}_M^{\wedge 1} A_1 \ldots A_n, B \Leftrightarrow \mathbf{Q}_M A_1 \cap \ldots \cap A_n B,$$
$$\mathbf{Q}_M^{\wedge 2} A_1 \ldots A_n, B \Leftrightarrow \mathbf{Q}_M A_1 B \& \ldots \& \mathbf{Q}_M A_n B.$$

Inner disjunctions $\mathbf{Q}^{\vee 1}$ and $\mathbf{Q}^{\vee 2}$ are defined similarly. As for negation, we make the

DEFINITION 26. If $\mathbf{Q}$ is $(n + 1)$-ary, the *inner negation* of $\mathbf{Q}$ is the quantifier $\mathbf{Q}\neg$, defined by

$$(\mathbf{Q}\neg)_M A_1 \ldots A_n B \Leftrightarrow \mathbf{Q}_M N A_1 \ldots A_n, M - B.$$

Also, the *dual* of $\mathbf{Q}$ $\check{\mathbf{Q}}$, is the quantifier $\neg(\mathbf{Q}\neg)(= (\neg\mathbf{Q})\neg)$.

Outer and inner negation correspond to sentence negation and *VP* negation, respectively; cf.

Not many boys are lazy,
Many boys are not lazy,

with the respective truth conditions

$$(\neg\mathbf{many})_M \|boy\|\ \|lazy\|,$$
$$(\mathbf{many}\neg)_M \|boy\|\ \|lazy\|.$$

PROPOSITION 27. *The class of logical quantifiers is closed under inner conjunctions and disjunctions (both kinds), and inner negation (hence also duals).*

Proof. This is again a routine check; let us take one case and verify that $\mathbf{Q}\neg$ satisfies *EXT* if $\mathbf{Q}$ satisfies *CONSERV* and *EXT*. Suppose $A_1, \ldots, A_n, B \subseteq M \subseteq M'$. Then

$$\begin{aligned}
(\mathbf{Q}\neg)_M A_1 \ldots A_n, B &\Leftrightarrow \mathbf{Q}_M A_1 \ldots A_n, M - B \\
&\Leftrightarrow \mathbf{Q}_M A_1 \ldots A_n, (A_1 \cup \ldots \cup A_n) - B && (\textit{CONSERV}) \\
&\Leftrightarrow \mathbf{Q}_{M'} A_1 \ldots A_n, (A_1 \cup \ldots \cup A_n) - B && (\textit{EXT}) \\
&\Leftrightarrow \mathbf{Q}_{M'} A_1 \ldots A_n, M' - B && (\textit{CONSERV}) \\
&\Leftrightarrow (\mathbf{Q}\neg)_{M'} A_1 \ldots A_n, B.
\end{aligned}$$

■

It should be noted that other inner negations than *VP* negation do *not* preserve logicality. For example, if we define,for a binary $\mathbf{Q}$,

$$\mathbf{Q}^*_M AB \Leftrightarrow \mathbf{Q}_M M - AB,$$

then *CONSERV* will not be preserved.

The following propositions list some de Morgan-like laws for inner Boolean operations on quantifiers:

PROPOSITION 28.

(a) $(\neg\mathbf{Q})^{\wedge 1} = \neg(\mathbf{Q}^{\wedge 1}), (\neg\mathbf{Q})^{\wedge 2} = \neg(\mathbf{Q}^{\vee 2})$,

(b) $(\neg\mathbf{Q})^{\vee 1} = \neg(\mathbf{Q}^{\vee 1}), (\neg\mathbf{Q})^{\vee 2} = \neg(\mathbf{Q}^{\wedge 2})$,

(c) $(\mathbf{Q}_1 \wedge \mathbf{Q}_2)\neg = \mathbf{Q}_1\neg \wedge \mathbf{Q}_2\neg$,

(d) $(\mathbf{Q}_1 \vee \mathbf{Q}_2)\neg = \mathbf{Q}_1\neg \vee \mathbf{Q}_2\neg$,

(e) $(\mathbf{Q}^{\wedge i})\neg = (\mathbf{Q}\neg)^{\wedge i}(i = 1, 2)$,

(f) $(\mathbf{Q}^{\vee i})\neg = (\mathbf{Q}\neg)^{\vee i}(i = 1, 2)$,

In 2.4.12 we considered the suggestion that the class of binary natural language quantifiers is closed under (outer) conjunction and disjunction, i.e. that the following universal holds:

($U5$) *If* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *are binary natural language quantifiers then so are* $\mathbf{Q}_1 \wedge \mathbf{Q}_2$ *and* $\mathbf{Q}_1 \vee \mathbf{Q}_2$.

The case of negation was more doubtful. In the table opposite, some examples of *DET*s for negations and duals in English are given. '-' means that it seems hard to find a *DET*, simplex or complex, denoting the negation or dual in question. Of course these quantifiers are always expressible by some suitable paraphrase, but the question here is whether there are *determiners* denoting them.

This table suggests certain questions. When is the (inner or outer) negation of a simple quantifier again simple? Barwise and Cooper have several proposals here, e.g. that the negations of the cardinal quantifiers **at least n** and **exactly n** re never simple, and that if a language has a pair of simple duals, that pair consists of **every** and **some**; cf. also 3.6.

Here we shall look a bit closer at the '-' signs for the binary quantifiers in the table. Note that if these signs are correct, the class of binary natural language quantifiers is not closed under inner or outer negation. Discussing this question will give us an occasion to look at some typical issues, and to introduce a few useful notions. The purpose, as usual, is to illustrate problems and ideas, rather than making definite empirical claims.

Table 1.

$\mathbf{Q}$	$\neg\mathbf{Q}$	$\mathbf{Q}\neg$	$\breve{\mathbf{Q}}$
some	*no*	*not every*	*every*
every	*not every*	*no*	*some*
no	*some*	*every*	*not every*
most	*at most half*	*less than half*	*at least half*
many	*few*	-	*all but a few*
infinitely many	*at most finitely many*	-	*all but finitely many*
(at least) n	*less than n*	-	*all but less than n*
at most n	*more than n*	*all but at most n*	-
(exactly) n	*not exactly n*	*all but n*	-
more ... than	*at most as many ... as*	-	-
fewer ... than	*at least as many ... as*	-	-

Note first that part of what Table 1 claims is that certain expressions of the form *all but Q* are anomalous. Thus, while *all but five, all but at most five, all but finitely many* are fine, *all but at least five, all but not exactly five, all but (infinitely) many* are not. It might be claimed that the anomaly in the latter cases is pragmatic rather than semantic. I will not argue about this directly, but instead try to see if there are in fact significant semantic differences between the normal and the anomalous cases.

Exception *DET*s of the form *all but Q* (cf 2.4.11) are interpreted on the scheme

(1) **all but Q** $= \mathbf{Q}\neg$.

When is $\mathbf{Q}\neg$ a natural language quantifier? Before trying to give some answers to this, we need to introduce a new concept.

DEFINITION 29. A binary quantifier $\mathbf{Q}$ is *VP-positive* (*VP-negative*) if, for all M, M' and all $A, B \subseteq M, A'B' \subseteq M'$ such that $A \cap B = A' \cap B' (A - B = A' - B'), \mathbf{Q}_M AB \leftrightarrow \mathbf{Q}_{M'} A; B'$.[27]

As the terminology indicates, *VP*-positivity means that $\mathbf{Q}$ amounts solely to a condition on the *VP* denotation (intersected with the N denotation, since we assume *CONSERV*), whereas a *VP*-negative quantifier reduces to a condition on the *complement* of the *VP* denotation. For example, **some, no, many**,[28] **few, infinitely many, at least n, at most n, exactly n** are *VP*-positive, whereas **every, not every, all but n, all but at most n** are *VP*-negative. **most, at least half**, and other 'proportional' quantifiers are neither *VP*-positive nor *VP*-negative, and the same holds for the interpretations of the definites (because of the *number condition* on the N denotation; cf. 2.4.6–7).

For a *conservative* $\mathbf{Q}$, *VP*-positivity (-negativity) is related to inner and outer negation as follows:

(2) $\mathbf{Q}$ is *VP*-positive (-negative) $\Leftrightarrow \neg\mathbf{Q}$ is *VP*-positive (-negative)
$\Leftrightarrow \mathbf{Q}\neg$ is *VP*-negative (-positive).

The next result, essentially due to Barwise and Cooper shows that *VP*-positivity is in fact a simple relational property of quantifiers. A binary quantifier is *symmetric* if it is symmetric as a relation, i.e. iff for all M and all $A, B \subseteq M$,

$$\mathbf{Q}_M AB \Rightarrow \mathbf{Q}_M BA.$$

PROPOSITION 30. *If* $\mathbf{Q}$ *satisfies CONSERV and EXT the following are equivalent:*

(a) $\mathbf{Q}$ *is* VP-*positive.*

(b) $\mathbf{Q}$ *is symmetric.*

(c) $\mathbf{Q}_M AB \leftrightarrow \mathbf{Q}_M A \cap B A \cap B$ *(for all* M *and all* $A, B \subseteq M$*).*

---

[27] VP-positivity is related to the notions of *existential* and *cardinal* quantifiers in [Keenan and Stavi, 1986]. In fact, under *CONSERV*, VP-positivity is equivalent to existentiality, and cardinality is equivalent to VP-positivity $+$*QUANT*.

[28] This is for $\mathbf{many}^1$ (2.4.3); $\mathbf{many}^2$ is neither VP-positive nor VP-negative.

Proof. (a) $\Rightarrow$ (b): Suppose $\mathbf{Q}_M AB$. Let $A' = B$ and $B' = A$. Thus $A \cap B = A' \cap B'$, so, by *VP*-positivity, $\mathbf{Q}_M AB'$, i.e. $\mathbf{Q}_M BA$.

(b)$\Rightarrow$ (c): Suppose $\mathbf{Q}$ is symmetric. Then $\mathbf{Q}_M AB \Leftrightarrow \mathbf{Q}_M AA\cap B$ (*CONSERV*) $\Leftrightarrow \mathbf{Q}_M A \cap BA$ (symmetry) $\Leftrightarrow \mathbf{Q}_M A \cap BA \cap B$ (*CONSERV*).

(c) $\Rightarrow$ (a): If (c) holds and $A\cap B = A'\cap B'$, where $A, B \subseteq M$ and $A', B' \subseteq M'$, then $\mathbf{Q}_M AB \Leftrightarrow \mathbf{Q}_M A\cap BA\cap B \Leftrightarrow \mathbf{Q}_M A'\cap B'A'\cap B' \Leftrightarrow \mathbf{Q}_{M'} A'\cap B'A'\cap B$; (by *EXT*) $\Leftrightarrow \mathbf{A}_{M'} A'B'$ (by (c)). ■

The following corollary is easy using (2):

COROLLARY 31. *Under CONSERV and EXT the following are equivalent:*

(a) $\mathbf{Q}$ *is VP-negative.*

(b) $\mathbf{Q}\neg$ *is symmetric.*

(c) $\mathbf{Q}_M AB \Leftrightarrow \mathbf{Q}_M A - B\varnothing$.

From our list of English *DET*s in 2.4, it appears much easier to find *VP*-positive quantifiers than *VP*-negative ones. Moreover, it seems that for each *DET* giving a condition on the complement of the *VP* denotation, there is another *DET* giving the *same* condition on the *VP* denotation itself. For example, if the first *DET* is of the form *all but q*, the corresponding positive condition is given by Q, and if the first *DET* is *every* or *not every*, the second is *no* or *some*, respectively. This lets us propose the following universal:

($U6$) *If* $\mathbf{Q}$ *is a VP-negative natural language quantifier, then* $\mathbf{Q}\neg$ *is also a natural language quantifier.*

A related observation is that when Q denotes a *VP*-negative quantifier, the form *all but Q* is not allowed: *all but every, all but not every, all but all but five*, etc. are ruled out. The reason, one imagines, is that this would be a very cumbersome way of expressing a 'double *VP* negation', which in any case is equivalent to the more easily expressed positive condition.

($U6$) gives one (partial) answer to our question about when $\mathbf{Q}\neg$ is a natural language quantifier. But, to return to Table 1, the most interesting case concerns *VP*-positive quantifiers: all the '-' signs (for binary quantifiers) are examples of failure of $\mathbf{Q}\neg$ to be a natural language quantifier for *VP*-positive $\mathbf{Q}$. What, then, is wrong with a *DET* such as *all but at least five*?

Here is one suggestion: sentences of the form *all but Q A B* imply the *existence* of As that are B (in contrast with *all A B*). More precisely, let us say that a quantifier $\mathbf{Q}$ has *existential import*, if

(3) for sufficiently large A (and M), $\mathbf{Q}_M AB \Rightarrow \mathbf{some}_m AB$.

(3) holds for **all but five, all but at most five, all but finitely many**, etc., but fails for (**at least five**)¬, (**not exactly five**)¬, (**infinitely many**)¬, etc. E.g.

$$(\textbf{at least five})\neg_M AB \Leftrightarrow |A - B| \geqq 5,$$

so for each A with at least five elements, we have (**at least five**)$\neg_M A\varnothing$ but not **some**$_M A\varnothing$. Note that the qualification 'for sufficiently large A' is necessary: **all but at most five**$_M AB$ implies **some**$_M AB$ only when $|A| > 5$, and **all but finitely many**$_M AB$ implies **some**$_M AB$ only when A is infinite.

What condition on **Q** corresponds to the fact that **Q**¬ has existential import/ For *VP*-positive quantifiers, the answer is as follows. Call **Q** *bounded*, if

(4) there is an n such that for all M and all $A, B \subseteq M$, $\mathbf{Q}_M AB \Rightarrow |A \cap B| \leqq n$.

PROPOSITION 32. *Suppose* **Q** *is VP-positive and satisfies CONSERV and EXT. Then* **Q**¬ *has existential import iff* **Q** *is bounded.*

Proof. If **Q** is bounded by n, then $|A| > n \& \mathbf{Q}\neg_M AB \Rightarrow |A| < n \& |A - B| \leqq n \Rightarrow A \cap B \neq \varnothing$, so (3) holds for **Q**¬. On the other hand, if **Q** is not bounded, it follows from proposition 30 that there are arbitrarily large A (and M) such that $\mathbf{Q}_M AA$. But this means that $\mathbf{Q}\neg_M A\varnothing$, so (6) fails for **Q**¬. ■

From these observations it is tempting to suggest the universal: for *VP*-positive **Q**, **Q**¬ is a natural language quantifier only if **Q** is bounded. But this would be premature. The universal concerns arbitrary quantifiers **Q**¬, whereas the above discussion concerned the interpretations of *DET*s of the form *all but* Q'. In fact, there is a simple counter example to this universal: **some** is *VP*-positive, **some**¬ = **not every** is a natural language quantifier, but **some** is not bounded!

Of course we cannot require in the universal that **Q**¬ be the interpretation of a *DT all but* Q'; that would make **Q**¬ *trivially* a natural language quantifier! But all is not lost: it seems that if we require **Q** to be *non-simple*, the universal holds; possibly, the simple **some** was the *only* counter-example.

What about the converse statement, i.e. if **Q** is bounded, does it follow that **Q**¬ is a natural language quantifier? Here we can say something more definite:

PROPOSITION 33. *If* **Q** *is logical, VP-positive, and bounded, then* **Q** *is a finite disjunction of quantifies of the form* **exactly n**.

(The proof is best postponed until Section 4.2.) Thus if **Q** is as in this proposition, **Q** is clearly a natural language quantifier, and so is **Q**¬, which by Proposition 28 is a finite disjunction of quantifiers of the form **all but n**.

Some of the last observations are collected in the following tentative universal:

($U7$) *If* **Q** *is a VP-positive, non-simple, logical quantifier,then* **Q**¬ *is a natural language quantifier iff* **Q** *is bounded.*

This universal, then, would be an explanation of the empty spaces (for the binary quantifiers) in Table 1.

3.5 Non-Triviality

Call an n-ary quantifier $\mathbf{Q}$ *trivial on M*, if $\mathbf{Q}_M$ is either the empty or the universal n-ary relation on $P(M)$. Consider the condition

NONTRIV $\mathbf{Q}$ *is non-trivial on some universe.*

Quantifiers violating *NONTRIV* are not very interesting: either any sentence beginning with a *DET* denoting such a quantifier (satisfying *EXT*) is true in each model, or any such sentence is false in each model. Nevertheless, natural language permits the construction of such *DET*s, for example, *at least zero, fewer than zero, at least ten and at most nine, more than infinitely many*, as pointed out in [Keenan and Stavi, 1986]. But the following universal seems true:

($U8$) *Simple natural language quantifiers satisfy NONTRIV.*

Note that the *NONTRIV* quantifiers are *not* closed under Boolean operations: for any $\mathbf{Q}$, the quantifier $\mathbf{Q} \vee \neg\mathbf{Q}$ is trivial on every universe.

NONTRIV requires a very modest amount of 'activity' of $\mathbf{Q}$; a stronger variant is

ACT $\mathbf{Q}$ *is non-trivial on each universe.*

ACT holds for many natural language quantifiers, but there are exceptions even among the simple ones, e.g. **both, two, three, four**, ... (if M has less than 4 elements $\mathbf{four}_M AB$ is always false).

van Benthem [1984a] considers an even stronger requirement of activity, called 'variety', for binary quantifiers. Here is a generalisation to $(n+1)$-ary quantifiers:

VAR *For all M and all $A_1, \ldots, A_n \subseteq M$ such that $A_1 \cap \ldots \cap A_n \neq \varnothing$, there are $B_1, B_2 \subseteq M$ such that $\mathbf{Q}_M A_1, \ldots, A_n, B_1$ and $\neg\mathbf{Q}_M A_1, \ldots, A_n, B_2$.*

In the binary case, we could say that *VAR* transfers the requirement of activity to each non-empty first argument. For quantifiers satisfying *CONSERV* and *EXT*, this seems a reasonable strengthening of *ACT*.

Clearly,

$$VAR \Rightarrow ACT \Rightarrow NONTRIV;$$

the implications cannot be reversed: an example of a (logical) quantifiers satisfying *ACT* but not *VAR* is

$$\mathbf{Q}_M AB \Leftrightarrow |A| = 1.$$

Note that this does not seem to be a natural language quantifier. In fact, inspection of the *DET*s in 2.4 shows that the *ACT* ones — e;g; **some, no, all, not all, most, more ... than, fewer ... than, every ... and/or, some ... and/or** (both interpretations) — also satisfy *VAR*. So one may propose

($U9$) *Natural language quantifiers satisfying ACT also satisfy VAR.*

3.6 Monotonicity

The monotonicity behaviour of a quantifier **A** concerns the preservation of other truth value of $\mathbf{Q}_M A_1 \ldots, A_n$ when the arguments are decreased or increased. For simplicity, we shall only consider *binary* quantifiers here, although many of the definitions and results below can easily be extended to $(n+1)$-ary quantifiers.

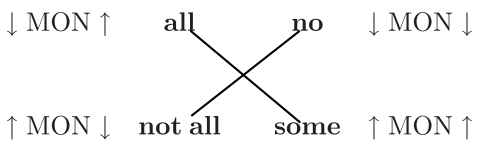
DEFINITION 34. A binary quantifier **Q** is

$$\begin{aligned}
&MON\uparrow, \text{ if } \mathbf{Q}_M AB \& B \subseteq B' \Rightarrow \mathbf{Q}_M AB',\\
&MON\downarrow, \text{ if } \mathbf{Q}_M AB \& B' \subseteq B \Rightarrow \mathbf{Q}_M AB',\\
&\uparrow MON, \text{ if } \mathbf{Q}_M AB \& A \subseteq A' \Rightarrow \mathbf{Q}_M A'B,\\
&\downarrow MON, \text{ if } \mathbf{Q}_M AB \& A' \subseteq A \Rightarrow \mathbf{Q}_M A'B.
\end{aligned}$$

Also, **Q** is *RIGHT MON* (*LEFT MON*) if it is *MON*↑ or *MON*↓ (↑*MON* or ↓*MON*), and **Q** is ↑*MON*↑ if it is both *MON*↑ and ↑*MON*; similarly for ↑*MON*↓, ↓*MON*↑, and ↓*MON*↓.

Barwise and Cooper call *RIGHT MON* monotonicity, ↑*MON persistence* and ↓*MON anti-persistence*.

Many natural language quantifiers have simple monotonicity properties. The four types of *double monotonicity* are exemplified by the square of the opposition:



Other doubly monotone quantifiers are **at least n, infinitely many**, which are ↑*MON*↑, and **at most n, at most finitely many, only liberal** (cf 2.4.10), which are ↓*MON*↓. **most** is *MON*↑ but not *LEFT MON*, as is easily seen, and the same holds for simple definites like **the** and **John's** (as defined in 2.4.6). Of the interpretations of *any* from 2.4.3, $\mathbf{many}^1$ is ↑*MON*↑, $\mathbf{many}^2$ is *MON*↑ but not *LEFT MON*, and $\mathbf{many}^3$ is neither *LEFT* nor *RIGHT MON*. Other examples of neither *LEFT* nor *RIGHT MON* quantifiers are **exactly n, all but n, between five and ten**.

The monotonicity behaviour of **Q** determines that of its negations and dual:

PROPOSITION 35.

(a) Outer negation reverses the direction of both RIGHT and LEFT MON.

(b) Inner negation reverses RIGHT MON but preserves LEFT MON.

(c) Dual-formation preserves RIGHT MON but reverses LEFT MON.

For example, from the monotonicity behaviour of one column of Table 1, we can infer that of all the other columns (for the binary quantifiers).

For doubly monotone quantifiers, we have the following pleasing result from van Benthem [1983c]. The proof is a nice demonstration of the strength and flexibility of the quantifiers constraints we are using.

THEOREM 36 (van Benthem). *Under CONSERV and VAR, the only doubly monotone quantifiers are those in the square of opposition.*

Proof. Suppose $\mathbf{Q}$ is $\downarrow$ *MON* $\downarrow$. We prove that $\mathbf{Q} = \mathbf{no}$; the theorem then follows from Proposition 35. Take a universe M and $A, B \subseteq M$. First assume that $A \cap B = \varnothing$. We claim that there is C such that $\mathbf{Q}_M AC$. This is immediate from *VAR* if $A \neq \varnothing$; otherwise, note that $\mathbf{Q}_M \varnothing\varnothing$ holds by $\downarrow$ *MON* $\downarrow$ and the fact that $\mathbf{Q}$ is non-trivial on M. By *MON* $\downarrow$ it then follows that $\mathbf{Q}_M A\varnothing$, i.e. $\mathbf{Q}_M AA \cap B$. Thus, by *CONSERV*, $\mathbf{Q}_M AB$. Conversely, suppose that $\mathbf{Q}_M AB$ holds. By $\downarrow$ *MON* $\downarrow$, $\mathbf{Q}_M A \cap BA \cap B$. But then $\mathbf{Q}_M A \cap BC$ holds for *all* $C \subseteq M$, since, for any such C, it suffices (by *CONSERV*) to show $\mathbf{Q}_M A \cap BA \cap B \cap C$, and this holds by *MON* $\downarrow$. Hence, *VAR* tells us that $A \cap B = \varnothing$, and the proof is finished. ■

For logical quantifiers, we can replace double monotonicity by *LEFT MON*:

THEOREM 37 (van Benthem). *The only logical and LEFT MON quantifiers satisfying VAR are the ones in the square of the opposition.*

A convenient method to prove this for *finite* universes (the case van Benthem considers) will be given in 4.2; actually, the result holds for all universes. Note the use of *VAR* here; without it, room is left for many other *LEFT MON* quantifiers, as is clear from the examples above.

Barwise and Cooper propose several universals involving monotonicity. One of them is the following:

($U10$) *Simple binary natural language quantifiers are either RIGHT MON or conjunctions of RIGHT MON quantifiers.*

Note that **exactly n** (which probably is simple) is the conjunction of the *RIGHT MON* **at least n** and **at most n**. This and other examples of neither *LEFT* nor *RIGHT MON* quantifiers suggest a weaker notion of monotonicity, which well be called *continuity*:

DEFINITION 38. A binary quantifier $\mathbf{Q}$ is

$$\begin{aligned}&\textit{RIGHTCONT},\ \text{if } \mathbf{Q}_M AB \& \mathbf{Q}_M AB'' \& \\ &\quad \& B \subseteq B' \subseteq B'' \Rightarrow \mathbf{Q}_M AB', \\ &\textit{LEFTCONT},\ \text{if } \mathbf{Q}_M AB \& \mathbf{Q}_M A''B \& \\ &\quad \& A \subseteq A' \subseteq A'' \Rightarrow \mathbf{Q}_M A; B.\end{aligned}$$

Let us further call a quantifier *STRONG RIGHT* (*LEFT*) *CONT* if both it and its outer negation are *RIGHT* (*LEFT*) *CONT*. We have

$$\begin{aligned}&\textit{RIGHT(LEFT)MON} \Rightarrow \\ &\quad \rightarrow \textit{STRONG RIGHT (LEFT) CONT} \Rightarrow \\ &\quad \Rightarrow \textit{RIGHT(LEFT)CONT}.\end{aligned}$$

None of the implications can be reversed: for example, **exactly n** is *RIGHT* (and *LEFT*) *CONT*, but not *STRONG RIGHT* (or *LEFT*) *CONT*.

Thijsse [1983] observes that the property of quantifiers identified in ($U10$) is in fact *RIGHT CONT*:

PROPOSITION 39. *A binary quantifiers is* RIGHT CONT *iff it is the conjunction of a* MON $\uparrow$ *and a* MON $\downarrow$ *quantifier.*

The proof is similar to the proof of Proposition 41(b) below.

Our use of the conservativity constraint on binary quantifiers gives the right and the left arguments quite different roles, so it is not surprising that right monotonicity and left monotonicity are very different properties. This is clear from Theorem 37, and will become even more apparent in Section 4.3. A further illustration of the difference is afforded by the following model-theoretic characterisation of the left monotonicity properties. Note first that any quantifier $\mathbf{Q}$ can be identified with a *class of structures*: in the binary case,

$$\mathbf{Q} = \{\langle M, A, B\rangle : \mathbf{Q}_M AB\}.$$

Call such a class *sub-closed* (*ext-closed*) if it is closed under substructures (extensions), and *inter-closed* if, whenever two structures, one a substructure of the other, are in $\mathbf{Q}$, then so is every structure 'between' these two. It is straightforward to verify that

PROPOSITION 40. *Under CONSERV and EXT, a binary quantifier is sub-closed (ext-closed, inter-closed) iff it is* $\downarrow$ MON *(*$\uparrow$ MON, LEFT CONT*).*

For *first-order definable* quantifiers, the semantic property of being subclosed has a well known syntactic counterpart, namely, definability by a *universal* sentence (cf. [Chang and Keisler, 1973, p. 128]). Thus, among first-order definable quantifiers satisfying *CONSERV* and *EXT*, the $\downarrow$ *MON* ones are precisely those definable by universal sentences. Corresponding results for $\uparrow$ *MON* and *LEFTCONT* quantifiers follow from the previous proposition and

PROPOSITION 41. *For any binary quantifier* $\mathbf{Q}$*,*

(a) $\mathbf{Q}$ *is ext-closed* $\Leftrightarrow \neg\mathbf{Q}$ *is sub-closed,*

(b) $\mathbf{Q}$ *is inter-closed* $\Leftrightarrow \mathbf{Q} = \mathbf{Q}' \wedge \mathbf{Q}''$*, for some sub-closed* $\mathbf{Q}'$ *and some ext-closed* $\mathbf{Q}''$*.*

Proof. (a) is obvious. As for (b), a conjunction of the sort indicated is clearly inter-closed. Conversely if $\mathbf{Q}$ is inter-closed, define

$$\mathbf{Q}'_M AB \Leftrightarrow \mathbf{Q}_{M'} A'B', \text{ for some extension } \langle M', A', B'\rangle \text{ of } \langle M, A, B\rangle,$$
$$\mathbf{Q}''_M AB \Leftrightarrow \mathbf{Q}_{M'} A'B', \text{ for some substructure } \langle M', A', B'\rangle \text{ of } \langle M, A, B\rangle;$$

then $\mathbf{Q}'$ and $\mathbf{Q}''$ are as desired. ■

Another syntactic characterisation of monotonicity from first-order logic is the following. Call a sentence $\phi(P)$, containing the unary P among its non-logical symbols, *upward monotone* (*in* P), if

$$\phi(P) \wedge \forall x(Px \to P'x) \to \phi(P')$$

is valid, and similarly for downward monotonicity. For example, sentences defining *LEFT* or *RIGHT MON* quantifiers will be monotone in certain predicate symbols. An occurrence of P in ϕ is said to be *positive* (*negative*), if it is within the scope of an even (odd) number of negations, when $\to$ and $\leftrightarrow$ have been eliminated. The next result is well known from first-order model theory (the proof is an application of Lyndon's interpolation theorem; cf. [Chang and Keisler, 1973, p. 90]).

PROPOSITION 42. *A first-order sentence $\phi(P)$ (which may contain other predicate symbols but no function or constant symbols) is upward (downward) monotone iff it is equivalent to a sentence where P occurs only positively (negatively).*

Monotonicity properties have been quite useful in describing and explaining linguistic phenomena; cf. [Barwise and Cooper, 1981; Keenan and Stavi, 1986], and, in connection with so-called polarity items, [Ladusaw, 1979; Zwarts, 1986]. We will have several further uses of monotonicity in Section 4. In mathematical logic, monotone quantifiers have been studied in model theory and recursion theory. The beginnings of the model theory for montone quantifiers will be given in Appendix B; further information can be found in [Barwise and Feferman, 1985]. On the more recursion-theoretic side, cf. for example, [Aczel, 1975] and [Barwise, 1978], and the references therein.

3.7 Partial and Definite Quantifiers

In 2.4.6 we mentioned that the number conditions belonging to the definites have been taken to indicate that the corresponding quantifiers are *partial*. This is the approach of Barwise and Cooper, who furthermore identify a semantic property of partial quantifiers, called *definiteness*, characteristic of the interpretation of the definites.[29]

Consider (in this subsection) binary quantifiers which are *partial in the first argument* (i.e. for certain A, $\mathbf{Q}_M AB$ may be *undefined* for all B). For example, the partial quantifier **the** coincides with the total **the** when $|A| = 1$, but is undefined when $|A| \neq 1$.

DEFINITION 43. $\mathbf{Q}$ is *definite*, if, for all M and all $A \subseteq M$ for which $\mathbf{Q}$ is defined, there is a non-empty set B_A such that, for all $B \subseteq M$, $\mathbf{Q}_M AB \Leftrightarrow B_A \subseteq B$.

The simple definites of 2.4.6 all have this property, when treated as partial quantifiers: e.g. for **the**, $B_A = A$ (or $B_A = X \cap A$ for some context set X), and for

[29]They consider (the singular) *the, both*, and *DET*s of the form *the n*, but not possessives.

John's, $B_A = P_{\text{John}} \cap A$. That the use of partial quantifiers is necessary here follows from

PROPOSITION 44. *Under CONSERV, no definite quantifier is total.*

Proof. This follows from the fact that a definite and conservative quantifier must be undefined for $A = \varnothing$: suppose $\mathbf{Q}$ is defined for $\varnothing$ and consider $B_\varnothing$ that exists by definiteness. Since $B_\varnothing \subseteq B_\varnothing$ we have $\mathbf{Q}_M \varnothing B_\varnothing$ and thus, by *CONSERV*, $\mathbf{A}_M \varnothing\varnothing$. But then $B_\varnothing \subseteq \varnothing$, by definiteness, contradicting the stipulation that $B_\varnothing$ is non-empty. ■

In view of this proof it is natural to weaken the requirements in Definition 43 slightly. Call $\mathbf{Q}$ *universal*, if it is as in 43, *except* that $B_\varnothing$ is allowed to be empty (i.e. that B_A is required to be non-empty only when A is). All definite quantifiers are universal, but not conversely, since **all** is universal. This is indeed the prime example of a universal quantifier, as the next result shows.

THEOREM 45. *Suppose* $\mathbf{Q}$ *is logical. Then* $\mathbf{Q}$ *is universal iff* $\mathbf{Q} =$ **all** *whenever defined.*

Proof. If $\mathbf{Q}$ coincides with **all** whenever defined it is clearly universal (with $B_A = A$). Conversely, suppose $\mathbf{Q}$ is universal and defined for A. We need to show that $B_A = A$. If $A = \varnothing$ we get $B_A = \varnothing$ just as in the proof above. Suppose, then, that $A \neq \varnothing$. then $B_A \neq \varnothing$ by universality. Also, $B_A \subseteq A$; this follows from *CONSERV*, since $\mathbf{Q}_M A B_A$, whence $\mathbf{Q}_M A A \cap B_A$, and thus $B_A \subseteq A \cap B_A$ by universality. Now assume that $B_A \neq A$. Take $a \in B_A$ and $a' \in A - B_A$. Let f be a function which permutes a and a' but leaves everything else in M as it is. By *ISOM*, $\mathbf{Q}_M f[A] f[B_A]$, i.e. $\mathbf{Q}_M A (B - \{a\}) \cup \{a'\}$. Thus, by universality, $B_A \subseteq (B_A - \{a\}) \cup \{a'\}$. But this contradicts $a \in B_A$. ■

Thus the logical universal quantifiers, and in particular the definite ones, are just partial versions of **all**. This is one reason to restrict attention to total quantifiers, as we have done in preceding sections and shall continue to do in what follows. Another reason is that partial quantifiers make the model theory more cumbersome, and that many results for total quantifiers can rather easily be extended to the partial case by inserting phrases of the form 'whenever ... is defined' in suitable places.

Note finally that even if partial quantifiers are admitted in principle,, the alternative treatment of definites suggested in 2.4.7 makes it possible to propose the universal.

($U11$) *Natural language quantifiers are total,*

while still preserving the intuition that statements involving definites lack truth value when the corresponding number conditions are not met.

3.8 Finite Universes

Many *DETs* more or less presuppose that the N and *VP* denotations under consideration are finite sets. Examples are *more than half, 30 percent of, many*, but also *DETs* like *most, more ... than, fewer ... than*, where the interpretations we gave actually work for infinite sets as well. It seems that in many natural language contexts we can make the blanket assumption

FIN *Only finite universes are considered.*

For *DETs* like *infinitely many* or *all but finitely many*, on the other hand, infinite models seem to be needed. So our strategy will be to keep track of those results that need *FIN* and those that don't. Interestingly, it turns out that *FIN* is a very natural constraint for the quantifier theory in the next section, in the sense that it *simplifies results and proofs*. Most of the results have generalisations to the case when *FIN* is dropped, but the added information does not appear to be very exciting from a natural language point of view.

This should be contrasted with the situation in mathematical logic. There infinite sets are crucial, and finite models are often just a nuisance. Consider the effect *FIN* would have in classical model theory. Most standard methods of constructing models (compactness, ultraproducts, etc.) would become ineffective, and many of the usual logical questions would become pointless. For example, the four properties of logics mentioned in Section 1.6 lose their interest. This is clear for the Tarski and the Löwenheim property, and for compactness and completeness it follows from

PROPOSITION 46. *Under FIN, no logic is compact or complete.*

Proof. Under *FIN*, the set $\{\exists_{\geq n} x(x = x) : n = 1, 2, \ldots\}$ has no model, does *EL* (and hence all its extensions) fail to be compact. The statement about completeness follows from a result by Trakhtenbrot, by which the set of all finitely valid *EL*-sentences (i.e. the set of valid sentences under *FIN*) is not recursively enumerable. For any logic $L = L(\mathbf{Q}^i)_{i \in I}$, this set is the intersection of the set of finitely valid L-sentences with the (recursive) set of *EL*-sentences. It follows that the set of finitely valid L-sentences is not recursively enumerable. ∎

Some standard logical questions remain, though. For example, we may still compare the *power of expression* of various logics under *FIN*, though some of the facts may change: we showed in 1.7 that $L(\mathbf{most}) < L(\mathbf{more})$ in general, but that $L(\mathbf{most}) \equiv L(\mathbf{more})$ under *FIN*. Likewise, *definability* issues are affected by *FIN*; for example, **all but finitely many** is not first-order definable in general, although it is trivially first-order definable under *FIN*.

It should be noted, however, that the main definability results in Section 1.7 (Theorem 10 and Corollary 11) continue to hold in the presence of *FIN*.

4 THEORY OF BINARY QUANTIFIERS

Binary quantifiers are the most common ones in natural language; they are also the most manageable relations, and we restrict attention to them from now on. A similar study of $(n+1)$-ary quantifiers appears quite feasible, cf. [Keenan and Moss, 1985]. The important step is abandoning unary quantifiers: most of the results in this section have no counterpart for the unary case.

If nothing else is said, we assume in what follows that *all quantifiers involved are logical and satisfy NONTRIV*. Other constraints, such as *ACT, VAR* and *FIN*, will be stated explicitly.

As a consequence of the assumption that *EXT* holds, we can often skip reference to the universe M, and write

$$\mathbf{Q}AB$$

instead of $\mathbf{Q}_M AB$. More precisely, let $\mathbf{Q}AB$ mean that, for *some* M such that $A, B \subseteq M$, $\mathbf{Q}_M AB$. *EXT* then guarantees that this is well defined.

Most of the results in 4.1–5 below originate from [van Benthem, 1984a; van Benthem, 1983c].

4.1 Relational Behaviour

We have already encountered standard properties of binary relations, such as (ir)reflexivity (2.2.2) and symmetry (3.4), in the context of natural language quantification. A first start in quantifier theory is to exploit this perspective systematically. As we shall see, this turns out to be both rewarding in itself and useful for other purposes. Here are a few common properties of relations, and some quantifiers exemplifying them:

One project is to find informative characterisations of (logical) quantifiers having such properties. As for symmetry, two useful equivalent formulations were given in Proposition 30. To deal with the other properties, we first state a

LEMMA 47. *If* $\mathbf{Q}AB$ *holds, there exists* B' *such that* $\mathbf{Q}AB'$ *and* $\mathbf{Q}B'A$*.*

Proof. Choose B' such that $A \cap B = B' \cap A$ and $|A - B| = |B' - A|$ (this may require extending the original universe, which is permitted by *EXT*. Since $\mathbf{Q}AB$, we get $\mathbf{Q}AB'$ by *CONSERV*, and then $\mathbf{Q}B'A$ by *QUANT*. ■

Note the use of logicality here; the lemma fails if any of *CONSERV*, *EXT*, or *QUANT* are dropped. The following corollary is immediate (since we are assuming *NONTRIV*):

COROLLARY 48 (van Benthem). *There are no asymmetric quantifiers.*

A characterisation of antisymmetry is also forthcoming.

COROLLARY 49. $\mathbf{Q}$ *is antisymmetric iff* $\mathbf{Q}AB \Rightarrow A \subseteq B$*.*

Table 2.

Property	*Definition*	*Examples*
symmetry	$\mathbf{Q}AB \Rightarrow \mathbf{Q}BA$	**some, no, at least n, at most n**, **exactly n, between n and m**
antisymmetry	$\mathbf{Q}AB\&\mathbf{Q}BA \Rightarrow A = B$	**all**
asymmetry	$\mathbf{Q}AB \Rightarrow \neg\mathbf{Q}BA$	-
reflexivity	$\mathbf{Q}AA$	**all, at lest half, all but finitely many**
quasireflexivity	$\mathbf{Q}AB \twoheadrightarrow \mathbf{Q}AA$	**some, most, at least n**
weak reflexivity	$\mathbf{Q}AB \Rightarrow \mathbf{Q}BB$	**some, most, at least n**
quasiuniversality	$\mathbf{Q}AA \Rightarrow \mathbf{Q}AB$	**no, not all, all but n**
irreflexivity	$\neg\mathbf{Q}AA$	**not all, all but n**
linearity	$\mathbf{Q}AB \vee \mathbf{Q}BA \vee A = B$	**not all**
transitivity	$\mathbf{Q}AB\&\mathbf{Q}BC \Rightarrow \mathbf{Q}AC$	**all, but finitely many**
circularity	$\mathbf{Q}AB\&\mathbf{Q}BC \Rightarrow \mathbf{Q}CA$	-
euclidity	$\mathbf{Q}AB\&\mathbf{Q}AC \Rightarrow \mathbf{Q}BC$	-
antieuclidity	$\mathbf{Q}AB\&\mathbf{Q}CB \Rightarrow \mathbf{Q}AC$	?

Proof. If the condition holds, $\mathbf{Q}$ is clearly antisymmetric. Conversely, if $\mathbf{Q}$ is antisymmetric and $\mathbf{Q}AB$ holds, take B' as in the proof of Lemma 47. Thus $A = B'$ by antisymmetry, and $|A - B| = |B' - A| = 0$, i.e. $A \subseteq B$. ■

This also gives a characterisation of linearity, since $\mathbf{Q}$ is linear iff $\neg\mathbf{Q}$ is antisymmetric. As to the reflexivity properties and quasiuniversality, their main interest is in combination with other properties, as we shall see. The following consequences of Lemma 47 may nevertheless be noted:

COROLLARY 50. *Weak reflexivity implies quasireflexivity.*

This leaves only the properties in Table 2 involving *three* set variables. The '-' signs here are explained by the following results from van Benthem [1984a].

THEOREM 51 (van Benthem). *There are no Euclidean quantifiers.*

We omit the proof, but show how to obtain the following corollary with the aid of Lemma 47

COROLLARY 52 (van Benthem). *There are no circular quantifiers.*

Proof. Suppose $\mathbf{Q}$ is circular. If $\mathbf{Q}AB$, take B' as in Lemma 47. By circularity, $\mathbf{Q}AA$. Thus, $\mathbf{Q}AB \Rightarrow \mathbf{Q}AA\&\mathbf{Q}AB \Rightarrow \mathbf{Q}BA$ (again by circularity), i.e. $\mathbf{Q}$ is symmetric. But it is easy to see that a circular and symmetric quantifier is Euclidean, contradicting the theorem. ■

Actually, some of these results, e.g. Corollary 48 and Theorem 51, were first proposed as semantic universals, based on empirical evidence (Frans Zwarts). Only later was it realised that they are consequences of more fundamental properties of quantifiers. This provided a first illustration of the potential usefulness of quantifier theory for linguistic explanation.

We left a question mark for antieuclidity in Table 2. Here is an example though: $\mathbf{Q}AB \Leftrightarrow |A| = n$. The following result from [Westerståhl, 1984] explains the situation.

THEOREM 53. $\mathbf{Q}$ *is antiEuclidean iff* $\mathbf{Q}AB \Rightarrow \mathbf{Q}AC$ *(for all* A, B, C*).*

Two corollaries follow easily:

COROLLARY 54. $\mathbf{Q}$ *is antiEuclidean iff there is a class* X *of cardinal numbers such that* $\mathbf{Q}AB \Leftrightarrow |A| \in X$.

COROLLARY 55 (Zwarts). *Under* VAR *there are no antiEuclidean quantifiers.*

Thus antiEuclidean quantifiers put no condition at all on the *second* argument, i.e. the *VP* denotation. It seems safe to conclude that there are no antiEuclidean natural language quantifiers.

Finally, consider transitivity. Here are some examples of transitive quantifiers:

(a) **all, all but finitely many,**

(b) $\mathbf{all}_e AB \Leftrightarrow \varnothing \neq A \subseteq B$ (**all** with existential import; cf. 3.4)

(c) $\mathbf{all}_n AB \Leftrightarrow A \subseteq B \vee |A| < n$ ($n \geqq 1$; note that $\mathbf{all}_1 = \mathbf{all}$)

(d) any antiEuclidean quantifier (by Theorem 53)

(e) $\mathbf{Q}AB \Leftrightarrow (A \subseteq B \& |A| \geqq 5) \vee |A| = 3$.

Let us check (e): suppose $\mathbf{Q}AB$ and $\mathbf{Q}BC$. In case $|A| = 3$ we get $\mathbf{Q}AC$ automatically, so suppose $A \subseteq B \& |A| \geqq 5$. But then $|B| \neq 3$, so we must have $B \subseteq C \& |B| \geqq 5$, whence $A \subseteq C \& |A| \geqq 5$, i.e. $\mathbf{Q}AC$. ■

The following characterisation of transitivity from [Westerståhl, 1984] depends essentially on *FIN*. It shows that (e) above is in a sense the typical case. If X, Y are sets of natural numbers, let $X < Y$ mean that every number in X is smaller than every number in Y; this is taken to hold trivially if X or Y are empty.

THEOREM 56 (*FIN*). $\mathbf{Q}$ *is transitive iff there are sets* X, Y *of natural numbers such that* $X < Y$ *and* $\mathbf{Q}AB \Leftrightarrow |A| \in X \vee (A \subseteq B \& |A| \in Y)$.

The proof combines a result from [van Benthem, 1984a] with techniques that will be introduced in 4.2 below. Note that the transitive **all but finitely many** fails to satisfy the condition in the theorem, if infinite universes are allowed. The next corollary shows that *VAR* has drastic effects on transitivity.

COROLLARY 57 (*FIN*). *Under* VAR *the only transitive quantifiers are* **all** *and* $\mathbf{all}_e$.

Proof. This follows from the observation that *VAR* implies that either $X = \varnothing$ and $Y = N$, or $X = \{0\}$ and $Y = N = \{0\}$ in the theorem. ■

Having thus looked at single properties of quantifiers, we can go on to *combinations* of such properties. For example, using Theorem 51 and Proposition 30 we obtain the

COROLLARY 58. *No quantifiers are both*

(a) symmetric and transitive,

(b) symmetric and antiEuclidean,

(c) symmetric and (ir)reflexive,

(d) quasiuniversal and reflexive.

Reflexivity often has strong effects in combination with other properties. Note that, if $\mathbf{Q}$ is reflexive, $A \subseteq B \Rightarrow \mathbf{Q}AB$ (by *CONSERV*). From this and Corollary 49 we immediately get

COROLLARY 59. *The only reflexive and antisymmetric quantifier is* **all**.[30]

Furthermore, it is not hard to see that reflexivity together with the condition in Theorem 56 implies that, for some $n \geqq 1, X = \{0, \ldots, n-1\}$ and $Y = \{k : k \geqq n\}$. This gives

COROLLARY 60 (van Benthem (*FIN*)). *The only reflexive and transitive quantifies are* $\mathbf{all}_n$*, for* $n \geqq 1$.

Again, **all but finitely many** is a counterexample if *FIN* is dropped.

COROLLARY 61 (*FIN*). *Under ACT, the only reflexive and transitive quantifier is* **all**.

Proof. Suppose that $\mathbf{Q} = \mathbf{all}_n$, for some $n \geqq 2$. Let M be a universe with exactly one element. It follows that $\mathbf{Q}$ is trivial on M, contradicting *ACT*. ■

The next result connects our simple properties of relations with the monotonicity properties of Section 3.6.

THEOREM 62 (Zwarts).

(a) If $\mathbf{Q}$ *is reflexive and transitive, then* $\mathbf{Q}$ *is* $\downarrow$ MON $\uparrow$.

(b) If $\mathbf{Q}$ *is symmetric, then*

  (i) $\mathbf{Q}$ *is quasireflexive iff* $\mathbf{Q}$ *is* MON $\uparrow$*,*

  (ii) $\mathbf{Q}$ *is quasiuniversal iff* $\mathbf{Q}$ *is* MON $\downarrow$.

[30] Actually, only *CONSERV* is needed for this result [van Benthem, 1984a].

Proof. We prove (a); (b) is similar. If $\mathbf{Q}AB$ and $A' \subseteq A$, then $\mathbf{Q}A'; A$, by reflexivity and *CONSERV*, and hence $\mathbf{Q}A; B$ by transitivity. Similarly, if $\mathbf{Q}AB$ and $B \subseteq B'$, then $\mathbf{Q}BB'$ and hence $\mathbf{Q}AB'$. ■

From this and Theorem 36 we get the following variant of Corollary 61.

COROLLARY 63. *Suppose that* **Q** *satisfies* CONSERV *and* VAR *(but not necessarily* EXT *or QUANT), and is reflexive and transitive. Then* $\mathbf{Q} =$ **all**.

Proof. It suffices to note that neither Theorem 36 nor Theorem 62 uses *EXT* or *QUANT*. ■

Instead of characterising properties in terms of which quantifiers satisfy them, one may turn the question around and ask for characterisations of our most common quantifiers in terms of their properties. For the quantifier **all** and its variants, such characterisations were in fact obtained in Corollaries 57, 59–61, and 63. We end by giving a corresponding result for **some**. Let, for each cardinal κ, $\mathbf{some}_\kappa$ be the quantifier **at least** κ, i.e.

$$\mathbf{some}_\kappa AB \Leftrightarrow |A \cap B| \geqq \kappa$$

(so $\mathbf{some}_1 = \mathbf{some}$).

THEOREM 64 (van Benthem). **Q** *is symmetric and quasireflexive iff* $\mathbf{Q} = \mathbf{some}_\kappa$*, for some* $\kappa \geqq 1$.

A proof will be given in Section 4.2. The following corollary is obtained similarly to Corollary 61.

COROLLARY 65. *Under ACT, the only symmetric and quasireflexive quantifier is* **some**.

4.2 Quantifiers in the Number Tree

By Proposition 24, each binary logical quantifier **Q** can be identified with a binary relation between cardinal numbers. We use the same notation for this relation, which is thus defined by

(1) $\mathbf{Q}xy \Leftrightarrow$ for some A, B with $|A - B| = x$ and $|A \cap B| = y, \mathbf{Q}AB$.

Inversely, given any binary relation **Q** between cardinal numbers, we get the corresponding logical quantifier by

(2) $\mathbf{Q}AB \Leftrightarrow \mathbf{Q}|A - B|\,|A \cap B|$.

With (1) and (2) we can switch back and forth between a *set-theoretic* and a *number-theoretic perspective* on quantifiers. The latter perspective is the subject of the present subsection.

Here are the number-theoretic versions of a few well known quantifiers:

all $xy \Leftrightarrow x = 0$,
no $xy \Leftrightarrow y = 0$,
some$_n xy \Leftrightarrow y \geqq n$,
all$_n xy \Leftrightarrow y = 0 \vee x + y < n$,
most $xy \Leftrightarrow y < x$,
infinitely many $xy \Leftrightarrow y$ is infinite,
all but finitely many $xy \Leftrightarrow x$ is finite.

Properties of quantifiers also have their number-theoretic versions. In the case of universal properties, such as those in Table 2, there is a simple translation from the set-theoretic to the number-theoretic framework. Details can be found in [Westerståhl, 1984]; here we just consider a few examples. If two sets A, B are involved, let x correspond to $|A - B|$, y to $|A \cap B|$, and z to $|B - A|$. Then, for example,

(3) *quasireflexivity* is the property: $\mathbf{Q}xy \Rightarrow \mathbf{Q}0x + y$ (for all x, y),

(4) *symmetry* is the property: $\mathbf{Q}xy \Rightarrow \mathbf{Q}zy$ (for all x, y, z), or, equivalently, $\mathbf{Q}xy \Leftrightarrow \mathbf{Q}0y$ (for all x, y);

the last equivalence follows from Proposition 30 (it is also easy to see directly).

Sometimes proofs are simpler to carry out in the number-theoretic framework. This holds for several of the results in 4.1, in particular Theorems 53 and 56. As an illustration, we give the following

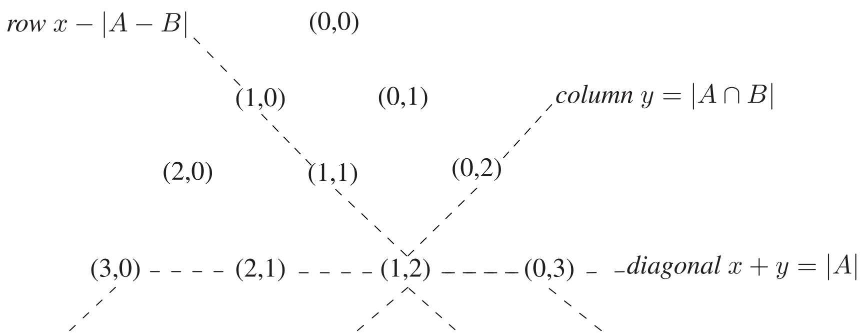
Proof.[of Theorem 64] Let κ be the least cardinal x such that $\mathbf{Q}0x, \kappa$ exists, by *NONTRIV* and (4). Also, $\kappa > 0$; otherwise, for any x, y, we get $\mathbf{Q}y0$ (from $\mathbf{Q}00$ by (4)), whence $\mathbf{Q}0y$ (by (3)), and so $\mathbf{Q}xy$ (by (4)), contradicting *NONTRIV*. We claim that $\mathbf{Q} =$ **some**$_\kappa$. Clearly, $\mathbf{Q}xy$ implies $y \geqq \kappa$, by 94). Conversely, given x, y such that $y \geqq \kappa$, take x' such that $\kappa + x' = y$. By (4) and the definition of κ, $\mathbf{Q}x'\kappa$. Thus, by (3), $\mathbf{Q}0x' + \kappa$ i.e. $\mathbf{Q}0y$ so $\mathbf{Q}xy$ by (4). ■

An operation that becomes nicely represented in the number-theoretic framework is *inner negation*, since we have

PROPOSITION 66. $(\mathbf{Q}\neg)xy \Leftrightarrow \mathbf{Q}yx$.

The number-theoretic perspective becomes particularly attractive if *FIN* is assumed. Quantifiers are then subsets of N^2. N^2 can be represented as a *number tree*, where each point (x, y) has two *immediate successors* $(x+1, y)$ and $x, y+1)$, which in turn are the *immediate predecessors* of the point $(x + 1, y + 1)$:[31]

[31] Without *FIN* one may represent logical quantifiers as subsets of $Card^2$ (*Card* = the class of cardinal numbers). This is not as easy to visualise as N^2. For example, diagonals and columns get mixed up:$(0, \aleph_0), (1, \aleph_0), \ldots$ are in the column given by $\aleph_0$, but also in the diagonal $\{(x, y) : x + y = \aleph\}$.



Quantifiers and their properties can be visualised in the number tree, and proofs can often be carried out directly in it. For an illustrative example, the reader is invited to carry out the above proof of theorem 64 in the number tree (assuming *FIN*). Note that symmetry (quasi-reflexivity) means that if a point is in **Q** then so are all the points on the column (so is the rightmost point on the diagonal) though it. Another illustration, also left to the reader, is the proof of Proposition 33 in the number tree.

When representing a quantifier **Q** in the tree it is often practical to write a '+' on the points in **Q** and a '-' on the other points. For example,

```
        +                 -                 -
      -   +             -   +             -   +
    -   -   +         -   +   -         -   -   +
  -   -   -   +     -   +   -   -     -   -   +   +
-   -   -   -   + -   +   -   -   - -   -   -   +   +
       all           exactly one          most
```

With this technique we can give our *non-triviality* conditions the following perspicuous formulations (we assume *FIN* for the rest of this subsection):

(5) *NONTRIV* $\Leftrightarrow$ there is at least one + and one -in the tree,

(6) *ACT* $\Leftrightarrow$ there is at least one + and one - *in the top triangle* (0,0),(1,0), (0,1),

(7) *VAR* $\Leftrightarrow$ there is at least one +and one - *on each diagonal* (except (0,0)).

This illustrates that *VAR* is a much stronger assumption than *ACT*, i.e. that the universal ($U9$) in 3.5 really has content.

Monotonicity properties turn out to be particularly suited to number tree representation. Beginning with the *RIGHT* monotonicity properties, we can easily verify that

(8) *MON* $\uparrow \Leftrightarrow$ each + fills the diagonal to its right with +s,

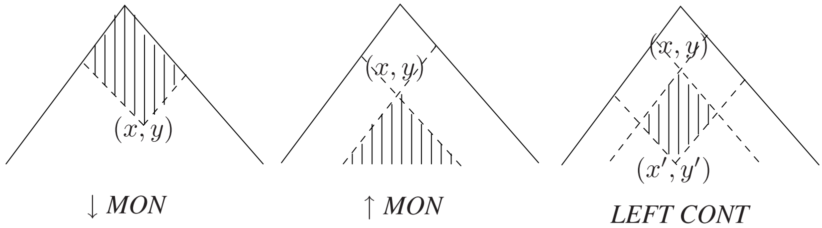
(9) *MON* $\downarrow \Leftrightarrow$ each + fills the diagonal to its left with +s,

(10) *RIGHT CONT* $\Leftrightarrow$ between two +s on a diagonal there are only +s.

Also observe that *STRONG RIGHT CONT*, i.e. *RIGHT CONT* for both **Q** and ¬**Q**, amounts to (10) together with the same condition with '+' replaced by '-'. It follows that

(11) *STRONG RIGHT CONT* $\Leftrightarrow$ on each diagonal there is at most one change of sign.

The *LEFT* monotonicity properties can be illustrated as follows:

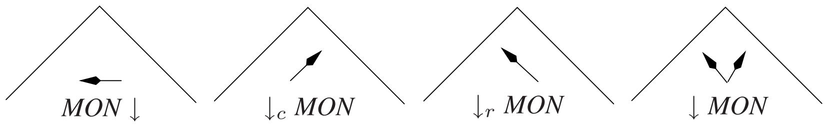


I.e. if (x, y) (and (x', y')) is in **Q** then so are all the points in the shaded area.

Working in the number tree, we can introduce several variants of the above monotonicity properties. Define $\uparrow_c$ $MON, \downarrow_c$ $MON, LEFT_c CONT$, and *STRONG LEFT*$_c$ *CONT* by replacing in (8)–(11), respectively 'diagonal' with 'column', and do the same for $\uparrow_r$ $MON, \downarrow_r$ $MON, LEFT_r CONT$, and *STRONG LEFT*$_r$ *CONT*, replacing 'diagonal' with 'row'. The terminology is motivated by the fact that

(12) $\uparrow_c$ *MON* $\Leftrightarrow$ $(\mathbf{Q}AB \& A' \subseteq A \& A \cap B = A' \cap B \Rightarrow \mathbf{Q}A'B)$,

and similarly for the other properties; in other words, they are as the previous *LEFT* properties, only we keep $A \cap B$ fixed in the 'c' case, and $A - B$ fixed in the 'r' case. To make the intuitive picture clear, her is yet another way to illustrate the downward monotone properties we have so far encountered:



In the tree it is easy to check whether particular quantifiers have such properties. For example, it is clear from the above illustrations that **most** is *MON* ↑ and $\downarrow_c$ *MON*, but not $\downarrow_r$ *MON*. It is also clear that

(13) ↑ *MON* $\Leftrightarrow \uparrow_c$ *MON*& $\uparrow_r$ *MON*,

(14) ↓ *MON* $\Leftrightarrow \downarrow_c$ *MON*& $\downarrow_r$ *MON*.

The corresponding statement for *CONT* fails, however (as can also be seen from the tree).

An interesting application of 'tree techniques' is given in [van Benthem, 1983c] to an idea in [Barwise and Cooper, 1981] concerning how hard it is (psychologically) to 'process' (verify or falsify) quantified statements. Barwise and Cooper speculated that quantifiers with monotonicity properties were easier to process and would therefore be preferred in natural language. Now, verifying a sentence of the form *all* AB in a universe with n elements takes n observations, and falsifying it takes at least 1 observation. If we look at *most* AB instead (and suppose that n is even for simplicity), the least possible number of observations it takes to verify it is $n/2 + 1$, and the corresponding number for falsification is $n/2$. In both cases the sum is $n + 1$. This holds for many basic quantifiers, but not all: e.g. *exactly one* AB requires n observations for verification and 2 for falsification.

van Benthem defines, with reference to the number tree, **Q** to be of *minimal count complexity* if, on each universe with n elements (this corresponds to the finite top triangle of the tree with the diagonal $x + y = n$ as base), there is a minimal confirmation pair (x_1, y_1) and a minimal refutation pair $(x_2, y_2)(x_i + y_i \leqq n)$ such that every pair (x, y) on the diagonal $x + y = n$ is determined by them:

$$x \geqq x_1 \& y \geqq y_1 \Rightarrow \mathbf{Q}xy,$$
$$x \geqq x_2 \& y \geqq y_2 \Rightarrow \neg\mathbf{Q}xy$$

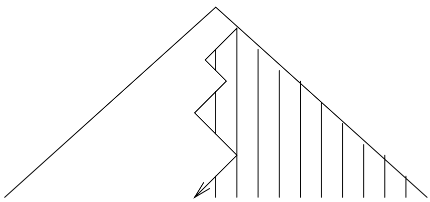
One can verify that $x_1 + y_1 + x_2 + y_2 = n + 1$, and thus that **all** and **most** are of minimal count complexity, but not **exactly one**.

Now consider the very strong continuity property:

$$SUPER\ CONT =_{\text{df}} \begin{array}{l} STRONG\ RIGHT\ CONT\& \\ \&STRONG\ LEFT_c CONT\& \\ \&STRONG\ LEFT_r CONT. \end{array}$$

In other words, *SUPER CONT* means that there are no changes of sign in any of the three main directions in the number tree. It can be seen that the *SUPER CONT* quantifiers are precisely those determined by a *branch* in the tree (which can start anywhere on the edges; not necessarily at the top) with the property that, going downward, it always contains one of the immediate successors of each point on it:

The connection with count complexity is now the following:



THEOREM 67 (van Benthem). *(FIN) Under ACT,* **Q** *is of minimal count complexity iff it is SUPER CONT.*

The proof of this consists simply in showing that the two combinatorial descriptions give the same tree pattern.

From the above description of *SUPER CONT* one also obtains the following results:

PROPOSITION 68. *SUPER CONT* $\Rightarrow$ *RIGHT MON.*[32]

PROPOSITION 69. *There are uncountably many SUPER CONT logical quantifiers (even under FIN).*

What is the relation between *SUPER CONT* and *LEFT CONT*? Using the tree it is easy to see that neither property implies the other. In the next subsection we shall find, moreover, that there are only countably many *LEFT CONT* logical quantifiers, under *FIN*.

4.3 First-order Definability and Monotonicity

We shall prove a theorem characterising the first-order definable quantifiers in terms of monotonicity, under *FIN*. The most general form of the result has nothing directly to do with logicality, so we begin by assuming that $\mathbf{Q}$ is an arbitrary k-ary quantifier $(K \geqq 1)$. We noted in 3.6 that $\mathbf{Q}$ can be identified with the class of structures $\langle M, A_0, \ldots, A_{k-1}\rangle$ such that $\mathbf{Q}_M A_0, \ldots, A_{k-1}$, and we defined the properties of being sub-closed, ext-closed, and inter-closed for classes of structures.

The key to the result is the following lemma from [van Benthem, 1984a]:

LEMMA 70 (van Benthem). *(FIN) Suppose* $\mathbf{K}$ *is a class of (finite) structures which is definable in EL by a set of monadic universal sentences. Then* $\mathbf{K}$ *is definable already by one such sentence.*

THEOREM 71. *(FIN)* $\mathbf{Q}$ *is first-order definable iff there are interclosed quantifiers* $\mathbf{Q}_1, \ldots, \mathbf{Q}_m$ *satisfying ISOM such that* $\mathbf{Q} = \mathbf{Q}_1 \vee \ldots \vee \mathbf{Q}_m$.

Proof. Suppose first $\mathbf{Q}$ is a disjunction of this kind. By Proposition 41, each $\mathbf{Q}_i$ can be written $\neg\mathbf{Q}'_i \wedge \mathbf{Q}''_i$, where $\mathbf{Q}'_i$ and $\mathbf{Q}''_i$ are sub-closed. Moreover, it easily follows from the proof of that proposition that both $\mathbf{Q}'_i$ and $\mathbf{Q}''_i$ satisfy *ISOM* if $\mathbf{Q}_i$ does. Thus it will suffice to show that every sub-closed quantifier satisfying *ISOM* is first-order definable. Assume, then, that $\mathbf{Q}$ has these properties. Under *FIN*, *any* class of structures closed under isomorphism is definable by a *set* of *EL*-sentences, by a standard argument: a finite structure can be completely described (up to isomorphism) by *one EL*-sentence, and the relevant set consists of all negated descriptions of models *not* in the class. If the class is in addition sub-closed, a variant of this argument shows that the sentences can be taken universal (one takes the negations of the existentially quantified diagrams of structures not in the class).[33] Since in our case the class is also monadic, $\mathbf{Q}$ is first-order definable by Lemma 70.

[32]This does not need *FIN*.

[33]This observation is also from [van Benthem, 1984a]. For the notion of a diagram, cf. [**?**, p. 68].

Now suppose $\mathbf{Q}$ is definable by an *EL*-sentence $\psi = \psi(P_0, \ldots, P_{k-1})$. By Corollary 11 (with $L = EL$), there is a natural number n such that $\mathbf{Q}$ is closed under the relation $\approx_n$ (cf. Section 1.7). Consider sentences expressing conditions

$$|P_s^{\mathbf{M}}| = i,$$

for some $i < n$, *or*

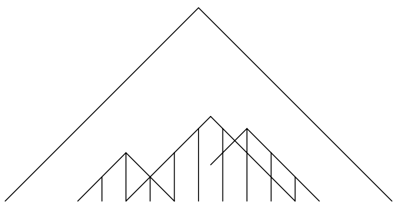
$$|P_s^{\mathbf{M}}| \geqq n.$$

It follows that any conjunction of such sentences where s runs through all the functions from k to 2, is a complete description of a model $\langle M, A_0, \ldots, A_{k-1}\rangle$, as far as $\mathbf{Q}$ is concerned. There are finitely many such descriptions, and ψ must be equivalent to the disjunction of all complete descriptions of structures in $\mathbf{Q}$. Moreover, each disjunct defines a quantifier, which, by the form of the definition, is easily seen to be inter-closed. Since any *EL*-definable quantifier satisfies *ISOM*, the theorem is proved. ■

Returning now to the case of binary logical quantifiers, we get from the theorem and Proposition 40 that

COROLLARY 72. *(FIN) If* $\mathbf{Q}$ *is binary and logical, then* $\mathbf{Q}$ *is first-order definable iff* $\mathbf{Q}$ *is a finite disjunction of LEFT CONT (binary and logical) quantifiers.*

There is a simpler direct proof of the corollary. This is because we can work in the number tree. In one direction, it suffices to show that $\uparrow$ *MON* quantifiers are first-order definable. If $\mathbf{Q}$ is $\uparrow$ *MON*, each point in $\mathbf{Q}$ generates an infinite downward triangle. From a given triangle within $\mathbf{Q}$, only finitely many steps can be taken towards the edges of the tree. It follows that $\mathbf{Q}$ is a finite union of such triangles,



and therefore clearly first-order definable. The proof in the converse direction, using Corollary 11, also becomes simpler in the number tree.

Corollary 72 shows, once more, that the *LEFT* monotonicity properties are much stronger than the *RIGHT* ones, due to the special role *CONSERV* gives to the left argument of a quantifier. In particular, there are only denumerably many *LEFT CONT* logical quantifiers (under *FIN*); this should be contrasted with Proposition 69.

Note that *FIN* is essential here. For example, **at most finitely many** is $\downarrow$ *MON* but not definable by any first-order sentence (or set of such sentences).[34]

Definability results such as these have not only logical interest: they also tell us something about the extent to which a certain logic — first order logic in this case — is adequate for natural language semantics. Of course, we knew already that first-order logic is not adequate, e.g. by the non-definability of **most**, but Corollary 72 places such isolated facts in a wider perspective.

The results here concern definability in the set-theoretic framework for quantifiers. What about *number-theoretic definability* (for logical quantifiers, under *FIN*)? Here we should consider formulas $\phi(x, y)$ in some suitable arithmetical language, containing at least the individual constant 0 and the unary successor function symbol S (and hence the *numerals* $\mathbf{0} = 0, \mathbf{1} = S0, \mathbf{2} = SS0$, etc.). Then ϕ *defines* $\mathbf{Q}$ iff, for all m, n,

$$\mathbf{Q}mn \Leftrightarrow \langle N, 0, S, \ldots \rangle \models \phi(\mathbf{m}, \mathbf{n}).$$

Examples of definable quantifiers, some of which in languages with the relation $<$ or the operation $+$, were given at the beginning of Section 4.2. Now which arithmetical definability notion corresponds to first-order definability in the set-theoretic sense? Notice first that even the simple formula

$$x = y$$

defines a non-first-order definable quantifiers, namely, **exactly half**. However, let the *pure number formulas* be those formulas in the language $\{0, S\}$ obtained from atomic formulas of the form

$$x = \mathbf{n}$$

by closing under Boolean connectives. Clearly every pure number formula with variables among x, y defines a first-order definable quantifier. But also conversely, for it can be seen by inspecting more closely the proofs of Theorem 71 and Corollary 72 that every first-order definable quantifier is in fact a Boolean combination of quantifiers of the form **at most n** and **all but at most n**, and the former, for example, is defined by the pure number formula

$$y = \mathbf{0} \vee \ldots \vee y = \mathbf{n}.$$

Thus, we have the

COROLLARY 73. *(FIN)* $\mathbf{Q}$ *is first-order definable iff* $\mathbf{Q}$ *is arithmetically defined by some pure number formula.*

This of course raises new definability questions. Which quantifies are defined by arbitrary formulas in $\{0, S\}$? Which are defined by formulas in $\{0, S, +\}$? It

[34] Michał Krynicki has observed (private communication) that, without *FIN*, *LEFTCONT* quantifiers are definable in logic with the cardinality quantifiers $\mathbf{Q}_\alpha$ (Section 1.3).

can be seen that **most** belongs to the second, but not the first, class. These questions are studied in connection with *computational complexity* in van Benthem [1985; 1987a]. He shows, among other things, that the second class of quantifiers mentioned above consists precisely of those computable by a push-down automaton (under *FIN*). He also characterises the first-order definable quantifiers computationally, namely, as those computable by a certain type of finite-state machine. This illustrates another aspect of the interest of definability questions: *classification* of quantifiers w.r.t. various notions of complexity. For the relevant definitions, and for several other interesting results along the same lines, we must refer to the two papers by van Benthem mentioned above.

4.4 Logical Constants

Clearly not all the $2^{\aleph_0}$ logical quantifiers deserve the title *logical constant*. We have already presented conditions that severely restrict the range of quantifiers. For example, *LEFT MON* plus *VAR* leaves only the quantifiers in the square of opposition (3.6). But there is no immediate reason why these two constraints should apply to logical constants. In this subsection, we look at some conditions which can be taken to have an independent connection with logical constanthood.

One idea seems natural enough, namely, that quantifiers that are logical constants should be *simple* natural language quantifiers (Section 2.4). Thus, the semantic universals holding for simple quantifiers apply to them. It follows that they should be logical (i.e. obey *CONSERV, EXT*, and *QUANT*) and satisfy *NONTRIV* and *RIGHT CONST* (by (U10) and Proposition 39).

As for constraints specifically related to logical constanthood, we will concentrate on one rather strong property often claimed to be characteristic of logical constants, namely, that *they do not distinguish cardinal numbers*. The idea is that such distinctions belong to mathematics, not logic. We will consider two rather different ways of making this idea precise.

FIN is used in what follows, so that we can argue in the number tree. It is possible, however, to generalise the results (with suitable changes) to infinite universes.

The first version of the above idea goes back to Mostowski [1957], although he only applied it to the infinite cardinalities. Given $\mathbf{Q}AB$, the relevant cardinality here is that of the universe, or in our case, by *CONSERV* and *EXT*, that of A. We must of course separate 0 from the other cardinalities, since distinguishing nonzero numbers from 0 is precisely what basic quantifiers such as **some** and **all** do. With these observations, we can transplant Mostowski's idea to the finite case as follows:

DEFINITION 74. Suppose $m, n > 0$. $\mathbf{Q}$ does not distinguish m and n if

(a) $\mathbf{Q}m0 \Leftrightarrow \mathbf{Q}n0$,

(b) $\mathbf{Q}0m \Leftrightarrow \mathbf{Q}0n$,

(c) if $x_1 + y_1 = m$ and $x_2 + y_2 = n$, where $x_i, y_i > 0$, then $\mathbf{Q}x_1y_1 \Leftrightarrow \mathbf{Q}x_2y_2$.

For example, **at least k** does not distinguish any $m, n < k$, but distinguishes all m, n for which at least one is $\geqq k$.

Note that no restriction at all is put on the point $(0, 0)$. To avoid trivial complications n the next result, we shall restrict attention to the number tree *minus* $(0, 0)$ (we write '-0' to indicate this). Also, we replace, in this subsection, *NONTRIV* by the slightly stronger condition that in the tree *minus*$(0, 0)$, there is at least one $+$ and one $-$.

It is not surprising that the present logicality constraint has rather drastic effects on the range of quantifiers:

THEOREM 75 (*FIN*, -0). *Suppose that* **Q** *does not distinguish any pair of non-zero natural numbers and satisfies* RIGHT CONT. *Then* **Q** *is one of the quantifiers* **some, no, all, not all***, and* **some but not all**.

Proof. There are four possible patterns for the top triangle minus $(0, 0)$.
Case 1: $++$. By the cardinality property and *RIGHT CONT*, this puts a $+$ everywhere, contradicting (our present version of) *NONTRIV*. *Case 2:* $+-$. Then the left edge will have only $+$, and the right edge only $-$. For the remaining interior triangle, there are two possibilities, giving either **no** or **not all**. *Case 3:* $-+$. This is symmetric to Case 2 an gives **some** and **all**. *Case 4:* $--$. Besides the trivial case with only $-$, there is the case with $+$ in the interior triangle and $-$ on both edges, i.e. **some but not all**. ■

We may note that *VAR*, or *STRONG RIGHT CONT*, will exclude **some but not all**, but it is not clear that we have to assume any of these. (On the other hand, it could be argued that one interpretation of the *DET some*, especially when focused or stressed, *is* **some but not all**.)

The second version of the requirement that logical constants do not distinguish cardinal numbers is from [van Benthem, 1984a]. Here the idea is that no point in the tree is special: you always *proceed downward in the same way*. Proceeding one step downward can be regarded as a thought experiment, whereby one, given A and B, adds one element to $A - B$ or $A \cup B$. The condition is then that the outcome is *uniform* in the tree, i.e. that it does not depend on the number of elements in these sets (the point $(0, 0)$ need not be excluded here, although it can be):

UNIF *The sign of any point in the tree determines the sign of its two immediate successors.*

THEOREM 76 (van Benthem (*FIN*)). *The UNIF and* RIGHT CONT *quantifiers are precisely* **some, no, all, not all** *and the quantifiers* $|A|$ **is even** *and* $|A|$ **is odd**.

Proof. Again a simple tree argument suffices. There are eight top triangles to consider; let us look at two. First consider $\begin{smallmatrix} & - & \\ - & & + \end{smallmatrix}$. Here the $-$ successors are determined, but for the right $+$ successor there is a choice, and we get two patterns

```
      -                  -
    -   +              -   +
  -   +   -          -   +   +
-   +   -   +      -   +   +   +
```

The first of these is excluded by *RIGHT CONT*, and the second is **some**.

Now consider the top triangle $\overset{+}{-\;-}$. Here the $-$successors are either both $-$ or both $+$. In the first case we get only $-$ in the rest of the tree, contradicting *NONTRIV*. In the second case we get $|A|$ **is even**. the other cases are similar. ■

The last two quantifiers in the theorem are not natural language quantifiers and should be excluded somehow. The following slight strengthening of *NONTRIV* would suffice:

*NONTRIV** *On some diagonal in the tree, there is at least one* $+$ *and at least one* $-$.

In fact, it seems that we may safely replace *NONTRIV* by *NONTRIV** in the universal ($U8$) in 3.5.

It is interesting that these two quite different implementations of the idea that logical constants are insensitive to changes in cardinal number give so similar results. There are of course other ideas than cardinal insensitivity on which one can base constraints for logical constanthood. Further ideas and results in this direction can be found in van Benthem [1984a; 1983c]. For example, he shows that by slightly weakening *UNIF* one can obtain, in addition to the quantifiers in the square of opposition, **most, not most, least** (i.e. **least** $AB \Leftrightarrow |A \cap B| < |A - B|$), **not least**, and no others, as logical constants. The number tree is an excellent testing ground for experiments in this area.

4.5 Inference Patterns

The universal properties of quantifiers we have considered can be seen as *inference schemes for quantified sentences*:

$$\frac{QAB}{QBA} \qquad \frac{\begin{array}{c}QAB\\QBC\end{array}}{QAC} \qquad \frac{\begin{array}{c}QAB\\QAC\end{array}}{QBC} \qquad \text{etc.}$$

(symmetry) (transitivity) (euclidity)

There are also schemes with fixed quantifiers, such as

$$\frac{\begin{array}{l}QAB\\ \textit{all } BC\end{array}}{QAC}$$

(*MON*↑).

In 4.1 we answered some questions of the type: which quantifiers satisfy inference scheme S? This is familiar from Aristotle's study of syllogisms, cf. Section 1.1. Aristotle aimed at systematic survey, and he answered the question for *all schemes of a certain form*.

EXAMPLE. Consider schemes with 2 premisses, 1 conclusion (all of the form QXY with distinct X, Y), at most 3 variables, and 1 quantifier symbol. There are 6 possibilities for each formula in a scheme, and hence, up to notional variants (permutations of the variables), $6^3/3! = 36$ possible schemes. Identifying schemes that differ only by the order of the premisses, and deleting the trivially valid schemes whose conclusion is among the premisses, 15 schemas will remain. Then, it can be shown, using Lemma 47 and Theorems 51–53, that for *logical* quantifiers these reduce to *symmetry, transitivity, anti-euclidity*, and the following property which we may call *weak symmetry*:

$$\begin{array}{c} QAB \\ QBC \\ \hline QBA \end{array}$$

(ignoring unsatisfiable schemes, such as euclidity). Weak symmetry is strictly weaker than symmetry; a number-theoretic characterisation of it can be found in [Westerståhl, 1984].

Thus, there are no other schemes than these (of the present form), and the results of 4.1 (e.g. Corollary 54 and Theorem 56) give us a pretty good idea of which quantifiers satisfy them.

EXAMPLE. *Aristotelian syllogisms*. The schemes are as in the first example, except that there are 3 quantifier symbols, and that one variable (the 'middle term') is required to occur in both premisses but not in the conclusion. Aristotle solved this problem in the special case that quantifiers are taken among **some, all, no, not all**. In the general case of logical quantifiers the solution is of course much more complicated.

The last example indicates that systematic survey of all possible cases is not necessarily an interesting task. In this subsection we shall consider a more specific problem: given the well known inference schemes for basic quantifiers such as **some** and **all**, are these quantifiers *determined* by the schemes, or are the schemes, as it were inadvertently, satisfied by other quantifiers as well?

The logical interest of such questions should be clear. They concern the extent to which the syntactic behaviour of logical constants determine their semantic behaviour. Negative results will tell us that inference rules of a certain type underdetermine semantic interpretation — a familiar situation in logic. Positive results, on the other hand, can be viewed as a kind of *completeness* or *characterisation* theorems.[35]

[35] One analogy is with the usual completeness theorems in logic, relating provability to truth in models. Or, one may think of the extent to which axiomatic characterisations of a relation (say) determine

These questions are also related to deeper issues in the philosophy of language, namely, whether the 'concrete manifestations' of linguistic expressions determine their meaning; cf. post-Wittgensteinian discussions of meaning and use, or Quine's idea on the indeterminacy of translation, or the debate on whether the meaning of logical constants are given by their introduction rules, and more generally on the relation between meaning and proofs (in the context of classical vs. intuitionistic logic; cf. [Prawitz, 1971; Prawitz, 1977; Dummett, 1975]).

Clearly, inference patterns concern the 'concrete' side of language, whereas model theory deals with abstract entities. It would seem that results which relate these two perspectives may be of interest regardless of one's position on the deeper philosophical issues.

A first observation is that the content of our question depends crucially on which kind of inference scheme one allows, i.e. on the choice of *inferential language*. We will look at two such languages here, with quite different properties. But then point is illustrated even more clearly by the following

EXAMPLE. Let the inferential language be predicate logic with the (binary) quantifiers **some** and **all** (this is not essential; we could use $\forall$ and $\exists$ instead). The standard rules for **some**, but with an arbitrary quantifier symbol Q in place of *some*, can be formulated as follows;

(1) $$\frac{\phi(t)\psi(t)}{Qx(\phi(x),\psi(x))} \quad \frac{\phi(x)\wedge\psi(x)\rightarrow\theta}{Qx(\phi(x),\psi(x))\rightarrow\theta} \quad (x \text{ not free in } \theta).$$

$\mathbf{Q}$ *satisfies* a rule of this type if, for each model $\boldsymbol{M}$ and each sequence $\bar{a}$ of individuals from M, if the premisses are true in $(\boldsymbol{M},\bar{a})$ (with Q interpreted as $\mathbf{Q}_M$), then so is the conclusion. But then it is practically *trivial* that

(2) $\mathbf{Q}$ satisfies the rules (1) iff $\mathbf{Q}$ = **some**.

For suppose $\mathbf{Q}$ satisfies (1). Take any M. We must show that $\mathbf{Q}_M AB \Leftrightarrow A\cap B \neq \varnothing$. If $a \in A\cap B$ then P_1x and P_2x are true in $\langle M,A,B,a\rangle$, and hence, by the first rule, so is $Qx(P_1x,P_2x)$, i.e. $\mathbf{Q}_M AB$ holds. If, on the other hand, $A\cap B=\varnothing$, let θ be a logically false sentence and b any element of M. Then $P_1x\wedge P_2x\rightarrow\theta$ is true in $\langle M,A,B,b\rangle$, and thus also $Qx(_1x,P_2x)\rightarrow\theta$, by the second rule. So $Qx(P_1x,_2x)$ is false in the model, i.e. $\mathbf{Q}_M AB$ does not hold. (Similar remarks apply to **all**.)

Why does the inferential language of this example trivialise the question of whether the rules characterise the quantifiers? One suggestion might be that rules like (1) are *circular* (in some sense to be specific) as explanations of meaning. In any case, we shall now define two other inferential languages, IL_{syll} and IL_{boole}, for which the problem has non-trivial solutions. These languages have no individual

an intended interpretation (e.g. questions of categoricity). Since the relations in the present case are basic logical constants, a third analogy suggests itself: characterisations of *EL*, such as Lindström's theorem (Section 1.6).

variables, only *set* variables. Most of the inference schemes we have seen so far can be expressed in them. The idea to pose the present charaterisation problem for quantifiers was introduced in [van Benthem, 1984a] and the results on IL_{syll} below are from [van Benthem, 1983c].

DEFINITION of IL_{syll}.

(a) *Syntax: Elementary schemes* in IL_{syll} are of the form QAB or $\neg Q'AB$, where $A, B, \ldots$ are the set variables and $Q, Q', \ldots$ quantifier symbols. A *scheme* in IL_{syll} is either an elementary scheme or has the form

(a) $\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \theta_1 \vee \ldots \vee \theta_k$,

where ϕ_i and θ_j are elementary schemes.

(c) *Semantics*: Suppose ψ is a scheme in IL_{syll} with quantifier symbols among $Q^1, \ldots, Q^m$, and with p set variables. For any quantifiers $\mathbf{Q}^1, \ldots, \mathbf{Q}^m$, a $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$-*model* (for ψ)is a model $\mathbf{M} = \langle M, A_0, \ldots, A_{p-1}\rangle$, where Q^i is interpreted as $\mathbf{Q}^i_M$. We say that

$(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$ *satisfies* the scheme ψ,

if ψ is true (in the obvious sense) in all $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$-models. Similarly, $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$ satisfies a *set* Ψ of IL_{syll}-schemes if it satisfies each element of Ψ. Finally, the *syllogistic theory* of $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$ is

$$Th_{\text{syll}}(\mathbf{Q}^1, \ldots, \mathbf{Q}^m) = \{\psi : (\mathbf{Q}^1, \ldots, \mathbf{Q}^m) \text{ satisfies } \psi\}.$$

This definition just gives more formal versions of notions we have been using all along. For example, all the properties in Table 2 (4.1), *except* antisymmetry and linearity, can be expressed in IL_{syll} (these two would be expressible if we had allowed quantifier *constants* above). That a quantifier $\mathbf{Q}$ satisfies a scheme just means that the scheme expresses a valid inference rule for $\mathbf{Q}$. For example, $\mathbf{Q}$satisfies

$$QAB \rightarrow QBA$$

just in case $\mathbf{Q}$ is symmetric. Note that more than one quantifier symbol may occur in a scheme. For instance, the scheme

$$Q^1AB \wedge Q^2CA \rightarrow Q^1CB$$

is satisfied by the pair (**no**, **all**) (this is the syllogistic inference ‘Celarent’; cf. Section 1.1).

DEFINITION of IL_{boole}.

(a) *Syntax*: As for IL_{syll}, except that elementary schemes now have the form QXY or $\neg Q'XY$, where X, Y are (Boolean) combinations of set variables with the symbols $\cap, \cup$, and $-$.

(b) *Semantics*: As before, where the Boolean symbols have their usual meaning.

Examples of schemes in IL_{boole} but not in IL_{syll} are

$$\begin{array}{l} QAB \rightarrow AA\ A \cap B, \\ QAA\cap \quad B \rightarrow QAB, \\ QAB \rightarrow QA \cap B\ A \cap B, \\ \qquad\qquad QA \cap B\ A \cap B \rightarrow QAB; \end{array}$$

the first two together express *CONSERV*, and the other two are (together) equivalent to symmetry.

There is one last

DEFINITION 77. Let Ψ be a set of schemes in IL_{syll} (or IL_{boole}), in the quantifier symbols $Q^1, \ldots, Q^m$. Let $\mathbf{Q}^1, \ldots, \mathbf{Q}^m$ be quantifiers. We say that

Ψ *determines* $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$,

if (a) $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$ satisfies Ψ, and (b) no other sequence of m quantifiers satisfies Ψ. Also, $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$ is *determined in* IL_{syll} (IL_{boole}), if some set of schemes in IL_{syll} (IL_{boole}) determines $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$.

Note that if $(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)$ is determined in IL_{syll} (IL_{boole}),, it is determined by the set $Th_{\text{syll}}(\mathbf{Q}^1, \ldots, \mathbf{Q}^m)(Th_{\text{boole}}(\mathbf{Q}^1, \ldots, \mathbf{Q}^m))$.

As an example, consider the set consisting of two IL_{syll}-schemes expressing *symmetry* and *quasireflexivity*. **some** satisfies this set, but, by Theorem 64, the set does *not* determine **some**. The obvious question is then whether some larger set determines **some** i.e. whether **some** is determined in IL_{syll}. A negative answer follows from the next theorem.

We assume *FIN* from now on (but see the comments at the end). The quantifiers $\mathbf{some}_n$ and $\mathbf{all}_n$ were defined in Section 4.1.

THEOREM 78 (van Benthem). Th_{syll} *(***some***,* **all***) is satisfied precisely by the pairs* $(\mathbf{some}_n, \mathbf{all}_n)$*, for* $n \geqq 1$.

Thus not even (**some**, **all**) is determined in IL_{syll}. That **some** (or **all**) is not determined follows immediately, since $Th_{\text{syll}}(\mathbf{some}) \subseteq Th_{\text{syll}}(\mathbf{some}, \mathbf{all})$.

This theorem is an immediate consequence of the next two theorems, which give additional information about the pair (**some**, **all**).

THEOREM 79 (van Benthem). $Th_{\text{syll}}(\mathbf{some}, \mathbf{all}) = Th_{\text{syll}}(\mathbf{some}_n, \mathbf{all}_n)$ *for* $n \geqq 1$.

For the next result, let Φ consist of the IL_{syll}-schemes saying that Q^1 is symmetric and quasireflexive and that Q^2 is reflexive and transitive, plus the following schemes:

(4) $Q^1AB \wedge Q^2AC \rightarrow Q^1BC$,

(5) $\neg A^1AA \rightarrow Q^2AB$.

THEOREM 80 (van Benthem). *If* $(\mathbf{Q}^1, \mathbf{Q}^2)$ *satisfies* Φ*, then, for some* $n \geqq 1, \mathbf{Q}^1 =$ **some**$_n$ *and* $\mathbf{Q}^2 =$ **all**$_n$.

The proof uses Theorem 64 and Corollary 60, which tells us that $\mathbf{Q}^1 =$ **some**$_m$ and $\mathbf{Q}^2 =$ **all**$_k$, for some m, k. It can then be seen that (4) implies that $k \leqq m$, and (5) that $m \leqq k$.

As to the proof of Theorem 79, we shall indicate the basic technique that is used. The first step is reformulation. Note that the negation of a scheme of the form (3) is equivalent to

$$\phi_1 \wedge \ldots \wedge \phi_n \wedge \neg\theta_1 \wedge \ldots \wedge \neg\theta_k,$$

i.e. that *negated schemes* are (equivalent to) conjunctions of elementary schemes. Since

$$\psi \in Th_{\text{syll}}(\mathbf{Q}^1, \ldots, \mathbf{Q}^m) \Leftrightarrow \neg\psi \text{ has no } (\mathbf{Q}^1, \ldots, \mathbf{Q}^m) - \text{model},$$

we are done if any (**some**, **all**)-model for a negated scheme can be transformed into a (**some**$_n$, **all**$_n$)-model for the scheme and *vice versa*.

Now let $\boldsymbol{M} = \langle M, A_0, \ldots, A_{p-1}\rangle$ be a (**some**, **all**)-model for $\neg\psi$. Each conjunct in $\neg\psi$ expresses either that a set of the form $A_i \cap A_j$ or $A_i - A_j$ is empty, or that it is non-empty. Each $A_i \cap A_j$ or $A_i - A_j$ can be written uniformly as a union of partition sets of the form $P_s^{\mathbf{M}}$ (cf. Section 1.7). The two types of condition expressed are thus

(a) $x = x_1 + x_2 + \ldots > 0$,

(b) $x = x_1 + x_2 + \ldots = 0$,

where x is the cardinal of $A_i \cap A_j$ (or A_A_j) and the x_k are the cardinals of the relevant partition sets. Now add $n - 1$ new elements to each *non-empty* partition set. This gives a model $\boldsymbol{M}^+\langle M^+, A_0^+, \ldots, A_{p-1}^+\rangle$, where the conditions (a) and (b) are transformed into

(a$^+$) $x^+ = X_1^+, +x_2^+ + \ldots \geqq n$,

(b$^+$) $x^+ = x_1^+ + X_2^+ + \ldots = 0$.

But then it is easy to check that $\mathbf{M}^+$ is a (**some**$_n$, **all**$_n$)-model of $\neg\psi$.

Note that this method does not work if we start with a (**some**$_n$, **all**$_n$)-model and want to get a (**some**$_{n+1}$, **all**$_{n+1}$)-model, say. For example, with $n = 3$, we may have

$$x = x_1 + x_2 < 3$$

with $x_1 = x_2 = 1$; then adding 1 gives

$$x^+ = x_1^+ + x_2^+ \geqq 4,$$

which means that the schemes of the form $\neg Q^1 A_i A_j$ will not be preserved.

Nevertheless, by an ingenious elaboration of this technique, van Benthem shows that a ($\mathbf{some}_{n+1}, \mathbf{all}_{n+1}$)-model can in fact always be obtained, and, combining this with yet another construction, he also shows how to obtain a (**some**,**all**)-model from a ($\mathbf{some}_n, \mathbf{all}_n$)-model.

In view of these negative results about IL_{syll}, it is natural to ask if there is a stronger inferential language where the basic logical constants are determined. Indeed, IL_{boole} is such a language. First observe that in IL_{boole} it is sufficient to look at *one* of the quantifiers **some** and **all**. This follows from the next, easily verified, proposition.

PROPOSITION 81.

(a) $\mathbf{Q}$ *is determined in* IL_{syll} *iff* $\neg\mathbf{Q}$ *is determined in* IL_{syll}.

(b) $\mathbf{Q}$ *is determined in* IL_{boole} *iff* $\mathbf{Q}\neg$ *is determined in* IL_{boole} *iff* $(\mathbf{Q}, \check{\mathbf{Q}})$ *is determined in* IL_{boole}.

We therefore concentrate on **some**. Let Φ_0 consist of schemes saying that Q is symmetric and quasireflexive, plus the following IL_{boole}-scheme:

(vi) $\neg QAA \wedge \neg QBB \rightarrow \neg QA \cup B\ A \cup B$

THEOREM 82. Φ_0 *determines* **some**.

Proof. Clearly **some** satisfies these schemes. Now suppose $\mathbf{Q}$ is any (logical) quantifier satisfying Φ_0. As before, the first two schemes imply that $\mathbf{Q} = \mathbf{some}_n$ for some $n \geqq 1$. Since $\mathbf{Q}$ satisfies (6), we also have

$$|A| < n \& |B| < n \Rightarrow |A \cup B| < n$$

(for all *sets* A, B). But this means that $n = 1$. ■

Now let us look at the other $\mathbf{some}_n$ in KL_{boole}. From the last result, Th_{boole} $(\mathbf{some}) \neq Th_{\text{boole}}(\mathbf{some}_n)$ when $n > 1$. The proof technique for IL_{syll} works for IL_{boole} as well — indeed, it works better since conditions on (the cardinal number of) *any* Boolean combinations of $A_0, \ldots, A_{p-1}$ can be expressed there. We thus get a $\mathbf{some}_n$-model from a **some**-model as before. In fact, even from a $\mathbf{some}_2$-model we get a $\mathbf{some}_n$-model with this method: adding $n-2$ to each non-empty partition set transforms

(a) $x = x_1 + x_2 + \ldots \geqq 2$,

(b) $x = x_1 + x_2 + \ldots < 2$

into

(a)$^+$ $x^+ = x_1^+ + x_2^+ + \ldots \geqq n$,

(b)$^+$ $x^+ = x_1^+ + x_2^+ + \ldots < n$,

since at most one x_i in (b) is non-zero. This gives us

THEOREM 83. $Th_{\text{boole}}(\textbf{some}_n) \subseteq Th_{\text{boole}}(\textbf{some}_2) \subseteq Th_{\text{boole}}(\textbf{some})$, *for* $n > 2$.

No such method works if we start with a $\textbf{some}_m$-model with $m > 2$, however. This was pointed out by Per Lindström: in fact, we have the

THEOREM 84.

(a) $Th_{\text{boole}}(\textbf{some}_{n+1}) \not\subseteq Th_{\text{boole}}(\textbf{some}_n)$, *for* $n \geqq 2$.

(b) On the other hand, if $m \geqq n^2$ *then* $th_{\text{boole}}(\textbf{some}_m) \subseteq Th_{\text{boole}}(\textbf{some}_n)$.

Proof.

(a) the case $n = 3$ will give the general idea. Let $\neg\psi$ be a negated scheme in IL_{boole} expressing the conditions

$$\begin{array}{llll} x_1 + x_2 + x_3 \geqq k, & x_1 + x + 4 < k, & x_2 + x_4 < k, & x_3 + x_4 < k, \\ x_4 + x_5 + x_6 \geqq k, & x_1 + x_5 < k, & x_2 + x_5 < k, & x_3 + x_5 < k, \\ & x_1 + x_6 < k, & x_2 + x_6 < k, & x_3 + x_6 < k, \end{array} \tag{7}$$

when **Q** is interpreted as $\textbf{some}_k$ (6 partition sets are needed, so a negated scheme with 3 set variables suffices). First note that for $k = 3$, (7) is satisfied when all the x_i are 1. Thus $\neg\psi$ has a $\textbf{some}_3$-model. But (7) cannot be true when $k = 4$. For, the first two conditions would give an $x_i (1 \leqq i \leqq 3)$ and an $x_j (4 \leqq j \leqq 6)$ which both are $\geqq 2$, and this contradicts one of the remaining conditions. So $\neg\psi$ has no $\textbf{some}_4$-model.

(b) Suppose $m \geqq n^2$, and take k such that $(k-1)n \leqq m < kn$. It follows that $n \leqq k$, and hence that $k(n-1) \leqq (k-1)n < m$. Now, given conditions

(a) $x = x_1 + x_2 + \ldots \geqq n$,

(b) $x = x_1 + x_2 + \ldots \leqq n - 1$,

multiply all the x_i by k. Then, $x^+ \geqq m$ in (a)$^+$ and $x^+ < m$ in (B)$^+$; this gives the desired $\textbf{some}_m$-model.

■

As to the converse inclusions, we have the

THEOREM 85. $Th_{\text{boole}}(\textbf{some}_n) \not\subseteq Th_{\text{boole}}(\textbf{some}_m)$, *for* $1 \leqq n < m$.

Proof. Generalising (6), we can write a scheme in ψ in IL_{boole} with $n+1$ set variables which expresses

$$\bigwedge |A_{i_1} \cup \cup \ldots A_{i_n}| < K \Rightarrow |A_1 \cup \ldots \cup A_{n+1}| < k$$

(here the conjunction is taken over all subsets of $\{1, \ldots, n+1\}$ with exactly n elements), when **Q** is interpreted as **some**$_k$. Then **some**$_n$ satisfies ψ. For otherwise, there are sets $A_1, \ldots, A_{n+1}$ such that $|A_1 \cup \ldots \cup A_{n+1}| \geqq n$ and $|A_{i_1} \cup \ldots \cup A_{i_n}| < n$ for $1 \leqq i_1, \ldots i_n \leqq n+1$. it follows that, for all i,

$$A_i \not\subseteq \bigcup_{j \neq i} A_j.$$

So in every A_i there is an element not in the other A_j. But this means that $|A_1 \cup \ldots \cup A_n| \geqq n$, a contradiction.

Now let $m < n$. Choose pairwise disjoint $A_1, \ldots, A_{n+1}$ such that $|A_1| = m - n$ and $|A_i| = 1$ for $1 < i \leqq n+1$. Then, if $1 \leqq i_1, \ldots, i_n \leqq n+1$, the cardinal of $A_{i_1} \cup \ldots \cup A_{i_n}$ is either n or $m-1$, i.e. in both cases $< m$, whereas $|A_1 \cup \ldots \cup A_{n+1}| = m$. So **some**$_m$ does not satisfy ψ. ■

Summarising, we find once more that **some** behaves in a significantly different way than **some**$_n$ for $n > 1$ (and similarly for **all**):

COROLLARY 86. *Of the quantifiers* **some**$_n$*, only* **some** *is determined in* IL_{boole}.

Proof. some is determined, by Theorem 82. Further, if Ψ determines **some**$_n$, then, by Theorem 83,

$$\Psi \subseteq Th_{\mathrm{boole}}(\textbf{some}_n) \subseteq Th_{\mathrm{boole}}(\textbf{some}).$$

Thus **some** satisfies Ψ, and it follows that $n = 1$. ■

As for the quantifiers satisfying $Th_{\mathrm{boole}}(\textbf{some}_n)$, it follows from our results here that they are all of the form **some**$_k$ with $k \leqq n$, that **some**, **some**$_2$, and **some**$_n$ are always among them, but that **some**$_{n-1}$ never is if $n > 3$.

The results in this subsection depend on *FIN*. For IL_{boole}, the proof technique works without *FIN*, but the facts are different. More precisely, with the previous methods one easily proves

THEOREM 87. *For each infinite cardinal* $\kappa, Th_{\mathrm{boole}}(\textbf{some}) = Th_{\mathrm{boole}}$ *(* **infinitely many***)* = $Th_{\mathrm{boole}}(\textbf{some}_\kappa)$.

Thus, as one would expect, **some** is not determined in IL_{boole} without *FIN*.

4.6 Local Perspective

Let M be a fixed finite universe, with n elements. We can then study local quantifiers *on* M,with much the same aim as before: of all these quantifiers, which ones are 'realised' in natural language?

Most of our global constraints have local versions. *CONSERV* is the same as before (with M fixed), and so are the monotonicity properties of 3.6 and the relational properties of 4.1. *ISOM* reduces to the local *PERM* (3.3). But one constraint which lacks a local version is *EXT*. As a consequence, results not depending on *EXT* have more or less immediate local versions, but when *EXT* is used, such versions may be harder to get. For example, Theorem 36 on double monotonicity holds locally as well, whereas Corollary 48 on the non-existence of asymmetric quantifiers, which uses *EXT*, fails: $\mathbf{Q}_M AB \Leftrightarrow A = M \& B = \varnothing$ is an asymmetric quantifier on M, satisfying *CONSERV* and *PERM*. Suitably modified versions of Corollary 48 and similar results do exist, however, cf. [Westerståhl, 1983].

One advantage of a local and finite perspective is that the effects of constraints such as *CONSERV* and *PERM* can be assessed in a rather perspicuous way, namely, by the *number* of quantifiers they allow. here are some examples for binary quantifiers on M:

Table 3.

number of quantifiers on M under	no constraints	*CONSERV*	*CONSERV* & *VP*-positivity	*CONSERV* & *MON*↑
no constraints	$2w^{4^n}$	2^{3^n}	2^{2^n}	?
when $n = 2$	65536	512	16	108
PERM	$2^{\binom{n+3}{3}}$	$2^{\binom{n+2}{2}}$	$2^{\binom{n+1}{1}}$	$(n+2)!$
when $n = 2$	1024	64	8	24

There is a simple uniform calculation for the first three entries in both rows of this table (these and other calculations have appeared in [Higginbotham and May, 1981; Keenan and Stavi, 1986; Keenan and Moss, 1985; van Benthem, 1984a; Thijsse, 1983]). Consider a pair (A, B), with $A, B \subseteq M$, as a function f from M to $\{0,1\}^2 : f(x) = (1,1)$ if $x \in A \cap B$, $f(x) = (0,1)$ if $x \in B - A$, etc. There are 4^n such functions and hence 2^{4^n} quantifiers on M. *CONSERV* means that $B - A$

can be assumed to be empty, removing the value $(0, 1)$, and reducing the number of functions to 3^n. By Proposition 30, *CONSERV* + *VP*-positivity means that only the pairs (A, A) need be considered, reducing the number of functions to 2^n.

Under *PERM*, $\mathbf{Q}_M$ is a relation between 4 numbers whose sum is n. To choose such numbers is essentially to put n indistinguishable objects in 4 (distinguished) boxes; there are $\binom{n+3}{3}$ ways to do this, by standard combinatorics. As before, addition of *CONSERV* or *CONSERV* + *VP*-positivity reduces the number of boxes to 3 and 2, respectively.

PERM and *CONSERV* are defined for k-ary quantifiers on $M (k \geqq 2)$, and the above calculations extend straightforwardly to this case: just replace '4' by '2^k' (= the number of partition sets induced by $(A_0, \ldots, A_{k-1})$), '3' by '$2^k - 1$', and '2' by '$2^k - 2$' (in the exponent) in the first two columns of Table 3.

The value $(n + 1)!$ for *Perm* + *CONSERV* + *MON*↑ can be obtained by looking in the *number tree for* M, i.e. the number tree restricted to pairs (x, y) such that $x + y \leqq n$. But the corresponding value without *PERM* is unknown:[36] [Thijsse, 1983] shows that a calculation of this appears to require an explicit calculation of the number of *anti-chains* in $P(M)$; the latter is an unsolved mathematical problem. Thijsse's paper contains several further counting results for quantifiers under various constraints (e.g. the number 108 for the case $|M| = 2$), and so does the paper by Keenan and Moss.

It is rather amazing at first sight that there are 65536 possible quantifiers on a universe with only two elements. The strength of the conservativity universal appears clearly from Table 3, which indicates that counting quantifiers is not just pleasant combinatorics — see the papers by Keenan and Stavi and Keenan and Moss for linguistic applications of such counting results.

Another distinguishing feature of the local perspective on quantifiers is that new *definability* issues arise here. Suppose certain *DET* denotations are *given* in M, and likewise denotations of other expressions: proper names, common nouns, transitive and intransitive verbs, etc. (we may think of a *model* ***M*** being given, not just a universe). Suppose further that we have identified certain constructions in natural language which can be interpreted as operations producing new quantifiers

[36]Editors' note. The problem indicated is known as Dedekind's problem: give a nice formula (closed-form expression) for the number of anti-chains in $P(M)$ (or, equivalently, the number of monotone Boolean functions of n variables). As far as I know, the problem is still unsolved. These so-called Dedekind numbers form sequence A000372 in the On-line Encyclopaedia of Integer Sequences, `http://www.research.att.com/~njas/sequences/`: 2, 3, 6, 20, 168, 7581, 7828354, 2414682040998, 56130437228687557907788

The problem also pops up in areas such as tiling and graph colouring. Upper and lower bounds are known (and important for computational purposes), as well as its asymptotic behaviour. The number is well defined and it is rather easy to write a program that calculates the numbers — given sufficient resources. Before 1990 I checked the number for $n = 7$ on a simple PC (one of the values reported in the literature, viz. 2414682040998, turned out to be correct), shortly before 2000 the value for n=8 was calculated. (vide link). FYI: the listed number 108 arises as the product of powers of Dedekind numbers: $2^1 3^2 6^1$, where the exponents are binomial coefficients.

The Editors are grateful to E. Thijsse for this information.

from given denotations. We can then ask which quantifiers can be *generated* from the given denotations by means of these operations. Such generated quantifiers are 'realised' in a definite sense; in fact, if the operations and the starting-point were chosen wisely, one may expect each generated quantifier to be *denoted* by some complex *DET* expression (relative to ***M***).

This approach is pursued in [Keenan and Stavi, 1986]. We will present one of their main results, which shows that *conservativity* is a crucial invariant here. Let $\mathit{CONSERV}_M$ be the class of binary quantifiers on M. Also if K is any class of binary quantifiers on M, let $B(K)$ be the smallest class containing K which is *closed* under conjunction, disjunction, and inner and outer negation. Finally, for each $a \in M$, define the quantifiers $\mathbf{S}_a$ on M by

$$\mathbf{S}_a AB \Leftrightarrow a \in A \cap B.$$

Keenan and Stavi argue that each $\mathbf{S}_a$ can be taken as a basic, initially given quantifier. For, if b is an individual in M who *owns a* and nothing else, i.e. if $P_b = \{a\}$ (cf. Section 2.4.6), then

$$\begin{aligned}\textbf{b's one or more}_M AB &\Leftrightarrow P_b \cap A \subseteq B \& |P_b \cap A| \geqq 1 \\ &\Leftrightarrow \mathbf{S}_a AB.\end{aligned}$$

Note that the $\mathbf{S}_a$ are conservative (but *PERM* fails), and that, to regard them as *given*, we also need each element of M to be given (by proper names or other means), and enough ownership relations to guarantee that for each a in M there is a b in M such that $P_b = \{a\}$. these are not implausible assumptions, and the Boolean operations are natural enough.[37]

THEOREM 88 (Keenan and Stavi). *Suppose* $K \subseteq \mathrm{CONSERV}_M$ *and that* $\mathbf{S}_a \in K$ *for* $a \in M$. *Then* $B(K) = \mathrm{CONSERV}_M$.

Proof. We know from 3.4 that Boolean operations preserve conservativity, so $B(K) \subseteq \mathit{CONSERV}_M$. Now let $\mathbf{Q}$ be any element of $\mathit{CONSERV}_M$. We then have

$$\begin{aligned}\mathbf{Q}AB &\Leftrightarrow \mathbf{Q}A\, A \cap B \\ &\Leftrightarrow \exists X \exists y \subseteq X(\mathbf{Q}XY \wedge X = A \wedge Y = A \cap B) \\ &\Leftrightarrow \bigvee_{\substack{X \subseteq Y \subseteq M \\ \& \mathbf{Q}XY}} (X = A \wedge Y = A \cap B).\end{aligned}$$

Note that the last disjunction is finite. It only remains to show that each disjunct can be generated from the $\mathbf{S}_a$ by Boolean operations. We claim that each disjunct is equivalent to the conjunction of

[37] Cf. [Keenan and Stavi, 1986] for the plausibility of the assumptions. Unlike Keenan and Stavi, I have included inner negation in the closure operations, but this can be avoided at the cost of adding a variant of $\mathbf{S}_a$ (namely, **b's zero or more**, when $P_b = \{a\}$) to the initial quantifiers. In 3.4 I expressed some doubts as to the closure of natural language quantifiers under inner or outer negation. These doubts do not affect Theorem 88, however, for, in the proof, we only apply inner and outer negation to the quantifiers $\mathbf{S}_a$, and, as Keenan and Stavi show, $\neg\mathbf{S}_a$ and $\mathbf{S}_a\neg$ are expressible with familiar *DET*s.

(1) $\bigwedge_{a \in Y} \mathbf{S}_a AB$,

(2) $\bigwedge_{a \in X-y} (\mathbf{S}_a \neg) AB$,

(3) $\bigwedge_{a \in M-X} (\neg \mathbf{S}_a AB = \check{\mathbf{S}}_a AB)$.

For, (1) expresses that $Y \subseteq A \cap B$, (2) that $X - Y \subseteq A - B$, and (3) that $A \cap B \subseteq X$ and $A - B \subseteq X$, and it is easily verified that the conjunction of these expresses that $X = A \wedge Y = A \cap B$. ■

By this theorem, *precisely* the conservative quantifiers on M are generated from certain basic ones by Boolean operations. This lends new significance to the conservativity universal ($U2$). By ($U2$) and the theorem, precisely these quantifiers on M are 'realised', in the sense of being denoted by *DET*s (relative to a model; cf. also note 38).

Note that the complex *DET* expression resulting from the proof of the theorem depends crucially on M. That is, conservative quantifiers, such as **most**, will get *different* 'definitions' on different universes, and there is in general no way of giving a global definition working for all universes. Keenan and Stavi prove a theorem (the 'Ineffability Theorem') to the effect that no fixed *DET* expression, containing symbols for simplex *DET*s, K-place predicates, adjectives, NPs and prepositions, can be made to denote, by varying the interpretation of these symbols, an arbitrary conservative quantifier on an arbitrary universe. The reason is that the number of possible denotations of such expressions grows slower with $|M| = n$ than 2^{3^n}.[38]

5 PROBLEMS AND DIRECTIONS

A basic theme of this paper has been to point to natural language as a source for logical investigation. This theme is by no means limited to quantifiers. Thus, one main direction for further study is *extension to other categories*. Some of the constraints we have studied can be transferred to other categories, and new constraints emerge. A typical trans-categorical constraint is *ISOM*, which has significant effects in most categories. For instance [Westerståhl, 1985a] shows that, for *relations between individuals, ISOM* leaves essentially just Boolean combinations with the *identity relation*, and [van Benthem, 1983b] proves that, for arbitrary *operations on subsets* of the universe, *ISOM* leaves precisely the operations whose values are Boolean combinations of the arguments. For further results in this area, and for a broad assessment of the present approach to logical semantics, the reader is referred to [van Benthem, 1986],which also lists several topics for further research, both in the quantifier area and beyond, complementing the brief suggestions given below.

[38]This makes heavy use of the universal ($U4'$) that simplex *DET*s denote *PERM* quantifiers: $2^{(n+1)(n+2)/2}$ grows slower than w^{3^n}. Without (Ur'), a simplex *DET* symbol could denote any conservative quantifier on any M.

Within the area of quantifiers there is, to begin with, the whole field of the *syntax* of various constructions with *DET*s, and of how to treat them semantically. We have mentioned (Section 2) constructions with *only*, the treatment of definites, of partitives, and of 'there are'-sentences, to take just a few examples. The papers [Keenan and Stavi, 1986; Keenan and Moss, 1985] provide ample evidence that these linguistic questions may be fruitfully pursued from the present model-theoretic perspective.

Another linguistic concern is the search for *universals*. As we have seen, universals can be used as basic theoretical postulates, or they can appear as empirical generalisations, sometimes amenable to explanation by means of other principles. the list of universals in Section 3 was not meant to be complete, and some of the formulations were quite tentative. Further proposals can be found in the papers by Barwise and Cooper and by Keenan and Stavi.

The use of semantic theory to explain linguistic facts, such as the privileged status of certain constants, the restrictions on various syntactic constructions, or the discrepancies between possible and actual interpretations of expressions of a certain category, can most likely be carried a lot further. Recall, for example, the discussion after Table 1 in 3.4. Other similar questions are easily found. Why are there so few simple *VP*-negative quantifiers? Why so few simple *MON* $\downarrow$ ones? Why isn't **not every** a *simple* natural language quantifier (like the other quantifiers in the square of opposition)? Such questions may warrant psychological considerations, but van Benthem's analysis of the 'count complexity' in 4.2 shows that simple model theory may be useful even in this context.

In connection with the last remark, it should be mentioned that van Benthem [1985; 1987a] carries the study of *computational complexity* in semantics much further. He shows (cf. the end of Section 4.3) that the well known complexity hierarchies of automata theory are eminently suitable for classification of quantifiers. Moreover, these investigations carry the promise of a new field of *computational semantics*, which, in addition to questions of logical and mathematical interest, has applications to *language learning* and to mental *processing* of natural language.

On the *logical* side of quantifier theory, many further questions suggest themselves. One natural direction is *generalisation* by weakening the assumptions. For example:

(a) *Drop EXT*. This allows for 'universe-dependent' quantifiers, such as some of the interpretations of *many* in 2.4.3. Some hints on how this admission affects the theory can be found in [Westerståhl, 1983].

(b) *Drop QUANT*. If possessives are allowed, this is a natural move. One can then replace *QUANT(ISOM)* by postulates of *quality*, requiring closure under 'structure-preserving' bijections. Other new constraints can also be formulated for this case, which is studied in [van Benthem, 1983b].

(c) Allow *ternary* quantifiers, or arbitrary n-ary ones ($n \geqq 2$). We did this in Section 3 for the basic concepts, but the corresponding generalisation of the theory in Section 4 is by no means straight-forward; cf. [Keenan and Moss, 1985].

Dropping *CONSERV*, on the other hand, does not seem fruitful (except for purely logical issues such as definability; cf. Section 4.3). (a)–(c) are not (only) generalisations for their own sake, but linguistically motivated. The next generalisation is more mathematical:

(d) *DROP FIN*. Many of the results using *FIN* can in fact be generalised, as we have noted from time to time. Two apparent exceptions were the results on transitivity, Theorem 56 and Corollary 60 (without *VAR*; cf. Corollary 63). Are there generalisations of these to infinite universes? But perhaps these generalisations lead in the wrong direction. It could be that *FIN*, or some similar constraint, is an essential characteristic of natural language quantification (cf Section 3.8). In any case, the assessment of some minimal model-theoretic means for handling 'natural language infinity' appears to be an interesting task. Some results in this direction can be found in [van Deemter, 1985].

But, even without generalising, the type of logical study conducted in Section 4 can be pursued further. The properties in 4.1 were chosen in a rather conventional way; there may be more interesting *properties of relations* to study. *Definability* questions need not be confined to first-order definability — as we saw in 4.3, *arithmetical definability* is a natural concept in the realm of (logical) quantifiers.

A particularly interesting aspect of definability concerns the *expressive power of natural language*. Various global notions of definability may be used here, e.g. definability *from* given quantifiers. There is also the local definability question mentioned in 4.6: of the possible denotations of expressions of a certain category, which ones are 'generated' in a given model? The conservativity theorem of Keenan and Stavi gives one answer, for *DET* denotations. Perhaps *NP* denotations are even more interesting; this aspect of expressive power is studied in [Keenan and Moss, 1985], where several results on which *NP* denotations are obtainable from quantifiers with certain properties (conservative, logical, *VP*-positive, etc.) are proved.

The study of *inferential languages* from Section 4.5 gives rise to a number of logical questions. This appears to be a recent field, though related to well-known questions on the correlation between a proof-theoretic and a model theoretic perspective on logic.[39] Note that the results of 4.5 depend crucially on our use of *binary* quantifiers instead of unary ones. As for particular questions, one would like to know which quantifiers are determined in these languages. Are *any* (non-trivial) quantifiers determined in IL_{syll}? Are any quantifiers *besides* those in the square of

[39]Zucker [1978] adopts a point of view similar to the present one. There seems to be a connection between his notion of a quantifier being *implicitly definable* and our notion of it being *determined*, even though the settings are different.

opposition determined in IL_{boole}? One can also pose 'finiteness' (compactness) questions, e.g. if **Q** is determined by Ψ, is **Q** by necessity determined by a finite subset of Ψ? This may of course be a trivial question, depending on the answer to the first two. Another compactness question is: if every finite subset of Ψ is satisfied by some quantifier (or sequence of quantifiers), must Ψ itself be satisfiable? Actually, this question can be seen to have a negative answer for IL_{boole}, but the case of IL_{syll} seems open. Other inferential languages could also be considered. In general, one would like to have a better understanding of what is required of a good inferential language. An obvious extension of IL_{syll} and IL_{boole}, however, is to add **some** and **all** as constants. This allows, e.g. monotonicity properties to be expressed in IL_{syll}, and the logical questions are reopened.

In this connection we should also mention an application of the present theory outside the domain of quantifiers: [van Benthem, 1984b] analyses *conditional* sentences *If X then Y* as relations between *sets* $\|X\|$ and $\|Y\|$ (of possible worlds, situations, etc.), i.e. as binary quantifiers, an obtains several interesting results for the logic of conditionals.

Finally, all of the logical questions mentioned so far presuppose the classical model-theoretic framework we have used in this paper. If one wants to treat such linguistically interesting phenomena as *plurals, collective quantification* (as opposed to the *distributive* quantification we have studied; cf. sentences such as *five boys lifted the piano*), or *mass terms* (with new determiners such as *much* or *a little*), this framework has to be extended. From a natural language point of view, such extension seems imperative. For some steps taken in these directions, cf. e.g. [van Benthem, 1983b; Hoeksema, 1983; Link, 1987; Lønning, 1987a; Lønning, 1987b]. An even more radical change would be the switch from the traditional 'static' model theory to a *dynamic* view on interpretation, e.g. along the lines suggested in [Kamp, 1981] or [Barwise and Perry, 1983]. It would be pleasant if the insights gained from the present quantifier perspective were preserved in such a transition. But, however that may be, standard model-theoretic semantics has already, I think, proved unexpectedly useful for a rich theory of quantifiers, and this theory is in turn a fair illustration of the possibilities of a logical study which starts not from mathematics but from natural language.

APPENDIX

A BRANCHING QUANTIFIERS AND NATURAL LANGUAGE

This appendix presents a brief summary of the main issues related to occurrence of branching quantification (Section 1.5) in natural language. A more detailed presentation is given in [Barwise, 1979].

Let us say, somewhat loosely, that a sentence exhibits *proper branching* if its formalisation requires a partially ordered quantifier prefix which is not equivalent to a linear one. There has been some debate over the following question:

(I) *Does proper branching occur in natural languages*?

The debate started with the claim in [Hintikka, 1973] that proper branching occurs in English. Here is the most well known of his examples:

(1) Some relative of each villager and some relative of each townsman hate each other.

The idea is that (1) should be analysed with the Henkin prefix. Arguing that the branching reading of (1) is preferred over linear versions requires a detailed and quite complicated analysis of what we actually mean when using such a sentence, and not all linguists agreed with Hintikka. In [Barwise, 1979], where the main arguments are summarised, it is argued that the most natural logical form of (1) does involve a branching reading, but one which is equivalent to a linear one, so that this branching is not proper. But the answer to (I) does not necessarily depend on sentences like (1). Barwise, who was sympathetic to Hintikka's general claim argued that with other quantifiers that $\forall$ and $\exists$ one can find clearer examples of proper branching. One of his examples was

(2) Most boys in your class and quite a few girls in my class have all dated each other.

It seems that (2) does *not* mean the same as

(3) Most boys in your class have dated quite a few girls in my class

or

(4) Quite a few girls in my class have dated most boys in your class.

The preferred reading of (2) is *stronger* than both of these: it says that there is a set X containing most boys in your class and a set Y containing quite a few girls in my class, such that *any* boy in X and *any* girl in Y have dated each other. Note that X and Y are *independent* of each other. This is a branching reading, which is (provably) not equivalent to any linear sentence in L(**most, quite a few**). We could formalise (2) as

(5)
$$\begin{array}{ll} \textit{most } x \ \textit{boy}(x) & \\ & > \textit{have dated each other}(x, y). \\ \textit{quite a few } y \ \textit{girl}(y) & \end{array}$$

Barwise pointed out that the above truth definition for such sentences gives the desired reading when, as in the present case, both quantifiers are *MON* $\uparrow$, and gave a similar (but different) truth condition for the case when both are *MON* $\downarrow$. He also noted that sentences of this form with one *MON* $\uparrow$ and one *MON* $\downarrow$ quantifier are anomalous.

(6) Few of the boys in my class and most girls in your class have dated each other.

Even though it seems perfectly grammatical, (6) makes no sense, and this may be explained by means of the monotonicity behaviour of the quantifiers involved. Further discussion of the circumstances under which it makes sense to branch two quantifiers can be found in [Westerståhl, 1987].

For another example, van Benthem has noted that we can have proper branching with certain first-order definable quantifiers that are *not* monotone. Consider

(7) Exactly one boy in your class and exactly one girl in my class have dated each other.

The meaning of (7) is clear and unambiguous, and it is easily seen that (7) is not equivalent to any of its 'linear versions' (or to their conjunction). (Note that we are talking about prefixes with *exactly one* here; it is in this sense the branching is proper, even though (7) is clearly equivalent to a (linear) first-order sentence.)

In conclusion, it seems that there are good arguments for an affirmative answer to (I). Then, one may ask:

(II) *What are the consequences for the 'logic of natural language' of the occurrence of proper branching*?

One of the aims of Hintikka's original paper was to use the occurrence of proper branching to give lower bounds of the complexity of this logic. From 1.5 and 1.6 it should be clear that logic with the Henkin quantifier has many affinities with *second-order logic*. In fact, it can be shown that the set of valid sentences with the Henkin quantifiers, or with arbitrary partially ordered prefixes $\forall$ and $\exists$, is recursion-theoretically just as complex as the set of valid second-order sentences, and this is an extremely complicated set. It is tempting to conclude that natural language is at least as complicated. This last inference, however, is not unproblematic. The result about second-order logic depends crucially on the fact that second-order variables vary over *all* subsets (relations) of the universe. In a natural language context, on the other hand, it may be reasonable to *restrict* the range of these variables, and thus to alter the strength of the resulting logic. More on these issues can be found in the chapter by van Benthem and Doets in this Handbook. Some other types of consequences of the occurrence of proper branching are discussed in [Barwise, 1979].

In addition to the principled questions (I) and (II), there is also the more pragmatical:

(III) *Should branching quantification be used more extensively in the analysis of logical and linguistic form*?

Both Hintikka and Barwise suggest that in many cases a branching reading may be preferable regardless of whether the branching is proper or not: the actual *order* between two (or more) quantifier expressions in a sentence sometimes seems irrelevant, syntactically *and* semantically, and a logical form where these expressions are unordered is then natural. Certain syntactic constructions appear to trigger such branching readings, in particular, conjoined noun phrases with a reciprocal object (*each other*). An even more extensive use of branching is proposed in [van Benthem, 1983a]: he suggests using branching instead of 'substitution' to explain certain well-known scope ambiguities with $\forall$ and $\exists$; cf. also [van Eijck, 1982]. There seem to be a lot of interesting possibilities in this field.

B LOGIC WITH FREE QUANTIFIER VARIABLES

Quantifier symbols have been *constants* in this paper (cf. Section 2.1.3). What happens if they are treated as free variables instead, or, more precisely, as symbols whose interpretation varies with models? From a logical perspective at least, this is a natural question. Some answers are reviewed in this appendix.

To fix ideas, consider a language L_Q, of standard first-order logic with one binary quantifier symbol Q added (for simplicity; we could have added several monadic quantifier symbols, and a fixed (countable) vocabulary of other non-logical symbols. L_Q is a language for logics like $L(\mathbf{most})$, except that this time Q does not denote a fixed quantifier. Instead, a *model* is now a pair $(\boldsymbol{M}, \mathbf{q})$, where $\boldsymbol{M}$ is as before and $\mathbf{q}$ is a binary quantifier on M. Such models are often called *weak models* (since nothing in particular is required of $\mathbf{q}$). *Truth* (satisfaction) in $(\boldsymbol{M}, \mathbf{q})$ is defined in the obvious way, with Q interpreted as $\mathbf{q}$. A *valid* sentence is thus true regardless of the interpretation of Q (and other non-logical symbols). Here is a trivial example:

$$Qx(x \neq x, \psi) \to (\exists x\phi \vee Qx(\phi, \psi))$$

(where ϕ, ψ only have x free). Are there non-trivially valid sentences in L_Q? This is answered below.

B.1 The Weak Logic

Add to a standard axiomatisation of first order logic the axioms

(1) $Qx(\phi(x), \psi(x)) \leftrightarrow Qy(\phi(y), \psi(y)$
(y free for x in $\phi(x), \psi(x)$)

(2) $\forall x(\phi_1 \leftrightarrow \phi_2) \to (Qx(\phi_1, \psi) \to Qx(\phi_2, \psi))$

(3) $\forall x(\phi_1 \leftrightarrow \phi_2) \to (Qx(\Psi, \phi_1) \to Qx(\psi, \phi_2))$

(the last two are extensionality axioms for Q).Call this the *weak logic*. Provability (from assumptions) is defined as usual, the deduction theorem holds, and the axiomatisation is obviously sound. The following completeness theorem goes back to [Keisler, 1970]:

THEOREM 89. *If Σ is a consistent set of sentences in the weak logic, then Σ has a weak model.*

Proof.[Outline] A slight extension of the usual Henkin-style proof suffices. Extend Σ to Σ' by witnessing existentially quantified sentences and then to a maximally consistent Γ. Let M consist of the usual equivalence classes $[c]$ of new individual constants, and interpret relation and constant symbols as usual. For each $\psi(x)$ with at most x free, let $\psi(x)^\Gamma = \{[c] \in M : \Gamma \vdash \psi(c)\}$. Then define **q** as follows:

$$\mathbf{q}AB \Leftrightarrow \text{ there are } \phi, \psi \text{ such that } \phi^\Gamma =, \psi^\Gamma = B, \text{ and } \Gamma \vdash Qx(\phi, \psi).$$

One then shows that, for all sentences θ,

$$(\boldsymbol{M}, \mathbf{q}) \models \theta \Leftrightarrow \Gamma \vdash \theta$$

by a straight-forward inductive argument, using (1)–(3) and properties of Γ when θ is of the form $Qx(\phi, \psi)$. ■

COROLLARY 90. *The weak logic is complete, compact, and satisfies the downward Löwenheim–Skolem theorem.*

B.2 Axiomatisable Properties of Quantifiers

By the last results, if *all* weak models are allowed, no 'unexpected' new valid sentences appear. However, it may be natural to *restrict* the interpretation of Q to, say, *conservative* quantifiers, or *transitive and reflexive* ones, or *MON* $\uparrow$ ones. Such properties are *second-order*, and hence in general not directly expressible in L_Q. Nevertheless, in many cases the resulting logic is still axiomatisable, by adding the obvious axioms to the weak logic.

Let P be a property of **q** expressible by a universal second-order sentence

(4) $\forall X_1, \ldots, \forall X_n \Psi((X_1, \ldots, X_n),$

where the X_i are unary set variables and Ψ is in L_Q (with the X_i acting as predicate symbols). Let the *corresponding set of L_Q-sentences*, Σ_P, consist of the universal closures of all formulas obtained by replacing all occurrences of $X_1, , \ldots, X_n$ in Ψ by L_W-formulas $\phi_1, \ldots, \phi_n$. For example, $\Sigma_{CONSERV}$ and $\Sigma_{MON\uparrow}$ consist, respectively, or universal closures of formulas of the form

$$Qx(\phi, \psi) \leftrightarrow Qx(\phi, \phi \wedge \psi),$$
$$Qx(\phi, \psi) \wedge \forall x(\psi \rightarrow \theta) \rightarrow Qx(\phi, \theta),$$

Let $\mathbf{K}_P$ be the class of models $(\mathbf{M}, \mathbf{q})$ such that **q** satisfies P. Clearly,

(5) $(\mathbf{M}, \mathbf{q}) \in \mathbf{K}_P \Rightarrow (\mathbf{M}, \mathbf{q}) \models \Sigma_P,$

but the converse fails in general. To $\mathbf{K}_P$ corresponds a logic, which we write $L(\mathbf{K}_P)$, where truth and validity is as for the weak logic, except that models are restricted to $\mathbf{K}_P$. then is $L(\mathbf{K}_P)$ axiomatised by Σ_P? A sufficient condition is given below.

A subset A of M is called $(\mathbf{M}, \mathbf{q})$-*definable*, if, for some L_Q-formula ψ and some finite sequence $\bar{b}$ of elements of $M, a \in A \Leftrightarrow (\mathbf{M}, \mathbf{q}) \models \psi[a, \bar{b}]$. Consider the following property of P:

(*) *If* $(\mathbf{M}, \mathbf{q}) \models \Sigma_P$ *then there is a* $\mathbf{q}'$ *satisfying* P *which agrees with* $\mathbf{q}$ *on the* $(\mathbf{M}, \mathbf{q})$-*definable sets.*

we need one more definition: $(\mathbf{M}', \mathbf{q}')$ is an *elementary extension* of $(\mathbf{M}, \mathbf{q})$, in symbols, $(\mathbf{M}, \mathbf{q}) < (\mathbf{M}', \mathbf{q}')$, if $\mathbf{M}'$ is an extension of $\mathbf{M}$ and, for all L_Q formulas ψ and all finite sequences $\bar{b}$ of elements of M, $(\mathbf{M}, \mathbf{q}) \models \psi[\bar{b}] \Leftrightarrow (\mathbf{M}', \mathbf{q}') \models \psi[\bar{b}]$. Now a straightforward induction proves the

LEMMA 91. *If* $\mathbf{q}$ *and* $\mathbf{q}'$ *agree on the* $(\mathbf{M}, \mathbf{q})$-*definable sets, then* $(\mathbf{M}, \mathbf{q}) < (\mathbf{M}, \mathbf{q}')$.

From this Lemma and Theorem 89 we immediately obtain the

THEOREM 92. *If (*) holds for* P *then each set of* L_Q-*sentences consistent with* Σ_P *in the weak logic has a model in* $\mathbf{K}_P$. *Hence,* $L(\mathbf{K}_P)$ *is complete, compact, and satisfies the Löwenheim–Skolem theorem.*

Instances of this result appear, for example, in [Keisler, 1970; Broesterhuizen, 1975; Sgro, 1977; Makowski and Tulipani, 1977; Barwise, 1978]. To see its utility we consider some examples.

EXAMPLE. Given $(\mathbf{M}, \mathbf{q})$, let M^d be the set of $(\mathbf{M}, \mathbf{q})$-definable subsets of M, and let $\mathbf{Q}^d - \mathbf{Q} \cap (M^d)^2$. If $(\mathbf{M}, \mathbf{q}) \models \Sigma_P$ then, since P is universal, $\mathbf{q}^d$ satisfies P *on* M^d. In some cases, $\mathbf{Q}^d$ actually satisfies P on the whole of $P(M)$, i.e. (*) holds with $\mathbf{Q}' = \mathbf{q}^d$. This is true for all the properties of quantifiers in Table 2 (Section 4.1), *except* reflexivity, quasiuniversality and linearity, as is easily checked. So, for example, the logic $L(\mathbf{K}_P)$, where P is the property of being a *strict partial order* (irreflexive and transitive), is axiomatisable.

EXAMPLE. $P =$ *strict linear order*. If $(\mathbf{M}, \mathbf{q}) \models \Sigma_P$, let $\mathbf{Q}^*$ be any strict linear order on $P(M) = M^d$, and let $\mathbf{q}' = \mathbf{q}^d + \mathbf{q}^*$ (order type addition). Then $\mathbf{Q}'$ is a strict linear order coinciding with $\mathbf{q}$ on M^d, so $L(\mathbf{K}_P)$ is axiomatisable. As similar construction can be used to show that each of the three properties left over in the preceding example is axiomatisable.

EXAMPLE. $P = MON\uparrow$. If $(\mathbf{M}, \mathbf{q}) \models \Sigma_{MON\uparrow}$, define $\mathbf{q}'$ by: $\mathbf{q}'AB \Leftrightarrow$ for some $C \in M^d, C \subseteq B$ and $\mathbf{q}AC$. Since $\mathbf{q}$ is $MON\uparrow$, $\mathbf{q}'$ agrees with $\mathbf{q}$ on M^d. Also, $\mathbf{q}'$ is $MON\uparrow$ (on all subsets of M). Other monotonicity (or continuity) properties can be treated similarly.

EXAMPLE. $P =$ *CONSERV*. If $(\mathbf{M}, \mathbf{q}) \models \Sigma_{CONSERV}$, let $\mathbf{q}'AB \Leftrightarrow \mathbf{q}A\ A \cap B$. Again, the verification that (*) holds is immediate.

EXAMPLE. In the following mathematical example, $\mathbf{q}$ is *unary*, and satisfies P iff $\mathbf{q}^- = P(M) = \mathbf{q}$ is a *proper, non-principal ideal* in $P(M)$, i.e. iff for all $A, B \subseteq M$, (i) $A, B \in \mathbf{q}^- \Rightarrow A \cup B \subseteq \mathbf{q}^-$; (ii) $A \in \mathbf{q}^- \& B \subseteq A \Rightarrow B \in \mathbf{q}^-$; (iii) $M \notin \mathbf{q}^-$; (iv) $\{a\} \in \mathbf{q}^-$ for all $a \in M$. In $L(\mathbf{K}_P), Qx\psi$ can be read 'for many x in the (infinite) universe, ψ'. Now suppose $(\mathbf{M}, \mathbf{q}) \models \Sigma_P$. Then $\mathbf{q}^{d-} = M^d - \mathbf{q}^d$ is a proper, non-principal ideal in M^d. Also, $\mathbf{q}^{d-}$ generates a proper, non-principal ideal $\mathbf{q}'^-$ in $P(M)$: let $A \in \mathbf{q}'^- \Leftrightarrow A \subseteq B_1 \cup \ldots \cup B_n$, for some $B_1, \ldots, B_n \in \mathbf{q}^{d-}$. Then (*) holds for $\mathbf{q}' = P(M) - \mathbf{q}'^-$, so $L(\mathbf{K}_P)$ is axiomatisable. $L(\mathbf{K}_P)$ is studied in [Bruce, 1978], mainly as a mains for obtaining results about the logic $L(\mathbf{Q}_1)$ where $\mathbf{Q}_1$ is the quantifier 'for uncountably many'.

Note that even though axiomatisability comes rather easily in these examples, other properties, such as interpolation, unions of chains, etc. may be much harder an require new methods (cf. [Bruce, 1978]).

C A NON-AXIOMATISABLE PROPERTY

In view of the above examples, one may ask if the property of *quantity* is also axiomatisable. After all, *PERM* is a universal second-order property (with a binary relation variable in addition to the unary set variables), and a corresponding Σ_{PERM} can be found much as before. However, $L(\mathbf{K}_{PERM})$ is a rather strong logic, and *not* axiomatisable. The reason is, roughly, that it can express that two sets have different cardinalities. For example, if $(\mathbf{M}, \mathbf{q}) \in \mathbf{K}_{PERM}$, and $\mathbf{q}MA$ is *not* equivalent to $\mathbf{q}MB$, it follows that either $|A| \neq |B|$ or $|M - A \neq |M - B|$. This is used in the following result, which is due to [Yasuhura, 1969].

THEOREM 93. *Then natural number ordering, $\langle N, < \rangle$, is characterisable in $L(\mathbf{K}_{PERM})$ in the sense that there is an L_Q-sentence θ such that $\langle M, R \rangle$ is isomorphic to $\langle N, < \rangle$ iff, for some $\mathbf{q}$ satisfying PERM, $(\langle M, R \rangle, \mathbf{q}) \models \theta$.*

Proof. Let θ be the conjunction of a sentence saying that $<$ is a linear ordering with immediate successors and a first but not last element, and the sentence

$$\forall x \forall y (\text{`}y \text{ is the successor of } x\text{'} \rightarrow$$
$$\neg(Qz(z = z, z < x) \leftrightarrow Qz(z = z, z < y))).$$

If $(\langle M, R \rangle, \mathbf{q}) \models \theta$, where $\mathbf{q}$ satisfies *PERM*, it is easy to see that for each $a \in M, |M_a| < |M_{a+1}|$ (where M_a is the set of predecessors to a), and thus that $\langle M, R \rangle$ is isomorphic to $\langle N, < \rangle$. Conversely, if the quantifier $\mathbf{q}$ on N is defined by $\mathbf{q}AB \Leftrightarrow A = N \& |B|$ is even, then *PERM* holds and $(\langle N, < \rangle, \mathbf{q}) \models \theta$. ■

As in Section 1.6, we obtain the

COROLLARY 94. *$L(\mathbf{K}_{PERM})$ is neither complete nor compact.*

Väänänen [1979] extends these results to show that, in terms of *implicit definability* (definability with extra non-logical symbols), $L(\mathbf{K}_{PERM})$ is equivalent to the logic $L(\mathbf{I})$ (cf. 1.6), and that its set of valid sentences is very complicated: it is neither $\Pi_1'^1$ nor $\Sigma_1'^1$ in the analytical hierarchy.

The above theorem and corollary extend, with the same proof, to the logic $L(\mathbf{K}_{PERM+CONSERV})$. They also extend to *logical* quantifiers. To see this, note that in this appendix we have used *local* quantifiers in our models, for which *ISOM* or *EXT* have no immediate meaning. An alternative procedure would be to consider models of the form $(\mathbf{M}, \mathbf{Q})$, where $\mathbf{Q}$ is a global quantifier, and interpret Q as $\mathbf{Q}_M$ on such a model. It is then easy to check that, for each model $(\mathbf{M}, \mathbf{q})$ in $\mathbf{K}_{PERM+CONSERV}$, there is a *logical* quantifier $\mathbf{Q}$ such that $\mathbf{Q}_M = \mathbf{q}$. From this it follows that a sentence is valid in $(L(\mathbf{Q}_{PERM+CONSERV})$ iff it is valid when Q varies over arbitrary logical quantifiers.

Let us remark, finally, that the results of this appendix depend on the fact that the usual universal and existential quantifier constants occur in L_Q. Anapolitanos and Väänänen [1981] show that, if we drop these, and also drop identity, then $L(\mathbf{K}_{PERM})$ becomes axiomatisable; actually it becomes *decidable*.

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