9

Deformed Minkowski Space as Generalized Lagrange Space

9.1 Generalized Lagrange Spaces

We want now to show that the deformed Minkowski space $\widetilde{M}$ of DSR does possess another well-defined geometrical structure, besides the deformed metrical one. Precisely, we will show (following [44]) that $\widetilde{M}$ is a *generalized Lagrange space*.

Let us give the definition of generalized Lagrange space [12], since usually one is not acquainted with it.

Consider a N-dimensional, differentiable manifold $\mathcal{M}$ and its (N-dimensional) tangent space in a point, $T\mathcal{M}_{\mathbf{x}}$ ($\mathbf{x} \in \mathcal{M}$). As is well known, the union

$$\bigcup_{\mathbf{x}\in\mathcal{M}} T\mathcal{M}_{\mathbf{x}} \equiv T\mathcal{M} \tag{9.1}$$

has a fiber bundle structure. Let us denote by $\mathbf{y}$ the generic element of $T\mathcal{M}_{\mathbf{x}}$, namely a vector tangent to $\mathcal{M}$ in $\mathbf{x}$. Then, an element $u \in T\mathcal{M}$ is a vector tangent to the manifold in some point $\mathbf{x} \in \mathcal{M}$. Local coordinates for $T\mathcal{M}$ are introduced by considering a local coordinate system $(x^1, x^2, ..., x^N)$ on $\mathcal{M}$ and the components of y in such a coordinate system $(y^1, y^2, ..., y^N)$. The $2N$ numbers $(x^1, x^2, ..., x^N, y^1, y^2, ..., y^N)$ constitute a local coordinate system on $T\mathcal{M}$. We can write synthetically $u = (\mathbf{x}, \mathbf{y})$. $T\mathcal{M}$ is a $2N$-dimensional, differentiable manifold.

Let π be the mapping (*natural projection*) $\pi : u = (\mathbf{x}, \mathbf{y}) \longrightarrow \mathbf{x}$. ($\mathbf{x} \in \mathcal{M}$, $\mathbf{y} \in T\mathcal{M}_{\mathbf{x}}$). Then, the tern $(T\mathcal{M}, \pi, \mathcal{M})$ is the *tangent bundle* to the base manifold $\mathcal{M}$. The image of the inverse mapping $\pi^{-1}(\mathbf{x})$ is of course the tangent space $T\mathcal{M}_{\mathbf{x}}$, which is called the *fiber corresponding to the point* $\mathbf{x}$ *in the fiber bundle* One considers also sometimes the manifold $\widetilde{T\mathcal{M}} = T\mathcal{M}/\{0\}$, where 0 is the zero section of the projection π. We do not dwell further on the theory of the fiber bundles, and refer the reader to the wide and excellent literature on the subject [46].

The natural basis of the tangent space $T_u(T\mathcal{M})$ at a point

$$u = (\mathbf{x}, \mathbf{y}) \in T\mathcal{M} \text{is} \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right\}, i, j = 1, 2, ..., N.$$

A local coordinate transformation in the differentiable manifold $T\mathcal{M}$ reads

$$\begin{cases} x'^i = x'^i(\mathbf{x}), \qquad \det\left(\dfrac{\partial x'^i}{\partial x^j}\right) \neq 0, \\ y'^i = \dfrac{\partial x'^i}{\partial x^j} y^j. \end{cases} \tag{9.2}$$

Here, y^i is *the Liouville vector field* on $T\mathcal{M}$, i.e., $y^i \dfrac{\partial}{\partial y^i}$.

On account of (9.2), the natural basis of $T\mathcal{M}_{\mathbf{x}}$ can be written as:

$$\begin{cases} \dfrac{\partial}{\partial x^i} = \dfrac{\partial x'^k}{\partial x^i}\dfrac{\partial}{\partial x'^k} + \dfrac{\partial y'^k}{\partial x^i}\dfrac{\partial}{\partial y'^k}, \\ \dfrac{\partial}{\partial y^j} = \dfrac{\partial y'^k}{\partial y^j}\dfrac{\partial}{\partial y'^k}. \end{cases} \tag{9.3}$$

Second (9.3) shows therefore that the vector basis $(\partial/\partial y^j)$, $j = 1, 2, ..., N$, generates a distribution $\mathcal{V}$ defined everywhere on $T\mathcal{M}$ and integrable, too (*vertical distribution on* $T\mathcal{M}$).

If $\mathcal{H}$ is a distribution on $T\mathcal{M}$ supplementary to $\mathcal{V}$, namely

$$T_u(T\mathcal{M}) = \mathcal{H}_u \oplus \mathcal{V}_u \quad , \quad \forall u \in T\mathcal{M}, \tag{9.4}$$

then $\mathcal{H}$ is called a *horizontal distribution*, or a *nonlinear connection* on $T\mathcal{M}$. A basis for the distributions $\mathcal{H}$ and $\mathcal{V}$ are given, respectively, by $\delta/\delta x^i$ and $\partial/\partial y^j$, where the basis in $\mathcal{H}$ explicitly reads

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - H^j_{\ i}(\mathbf{x}, \mathbf{y})\frac{\partial}{\partial y^j}. \tag{9.5}$$

Here, $H^j_i(\mathbf{x}, \mathbf{y})$ are the *coefficients* of the nonlinear connection $\mathcal{H}$. The basis

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right\} = \left\{ \delta_i, \dot{\partial}_j \right\}$$

is called the *adapted basis.*

The dual basis to the adapted basis is $\left\{\mathrm{d}x^i, \delta y^j\right\}$, with

$$\delta y^j = \mathrm{d}y^j + H^j_i(\mathbf{x},\mathbf{y})\mathrm{d}x^i. \tag{9.6}$$

A *distinguished tensor* (or *d-tensor*) *field of (r,s)-type* is a quantity whose components transform like a tensor under the first coordinate transformation (9.2) on $T\mathcal{M}$ (namely they change as tensor in $\mathcal{M}$). For instance, for a d-tensor of type (1, 2):

$$R'^i_{jk} = \frac{\partial x'^i}{\partial x^s}\frac{\partial x^r}{\partial x'^j}\frac{\partial x^p}{\partial x'^k}R^s_{rp}. \tag{9.7}$$

In particular, both $\delta/\delta x^i$ and $\partial/\partial y^j$ are d-(covariant) vectors, whereas $\mathrm{d}x^i$, δy^j are d-(contravariant) vectors.

A *generalized Lagrange space* is a pair $\mathcal{GL}^N$=($\mathcal{M}$, $g_{ij}(\mathbf{x},\mathbf{y})$), with $g_{ij}(\mathbf{x},\mathbf{y})$ being a d-tensor of type (0,2) (covariant) on the manifold $T\mathcal{M}$, which is symmetric, nondegenerate[1] and of constant signature.

A function

$$L : (\mathbf{x},\mathbf{y}) \in T\mathcal{M} \to L(\mathbf{x},\mathbf{y}) \in \mathcal{R} \tag{9.8}$$

differentiable on $\widehat{T\mathcal{M}}$ and continuous on the null section of π is named a *regular Lagrangian* if the Hessian of L with respect to the variables y^i is nonsingular. A generalized Lagrange space $\mathcal{GL}^N$=($\mathcal{M}$, $g_{ij}(\mathbf{x},\mathbf{y})$) is reducible to a *Lagrange space* $\mathcal{L}^N$ if there is a regular Lagrangian L satisfying

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} \tag{9.9}$$

on $\widehat{T\mathcal{M}}$. In order that $\mathcal{GL}^N$ is reducible to a Lagrange space, a necessary condition is the total symmetry of the d-tensor $(\partial g_{ij}/\partial y^k)$. If such a condition is satisfied, and g_{ij} are 0-homogeneous in the variables y^i, then the function $L = g_{ij}(\mathbf{x},\mathbf{y})y^i y^j$ is a solution of the system (9.9). In this case, the pair $(\mathcal{M}, L)$ is a *Finsler space*[2] defined for $\mathbf{x} \in \mathcal{M}$ and $\boldsymbol{\xi} \in T_{\mathbf{x}}\mathcal{M}$ such that $\Phi(\mathbf{x},\cdot)$ is a possibly nonsymmetric norm on $T_{\mathbf{x}}\mathcal{M}$.

Notice that every Riemann manifold ($\mathcal{M}$, $\mathbf{g}$) is also a Finsler space, the norm $\Phi(\mathbf{x},\boldsymbol{\xi})$ being the norm induced by the scalar product $\mathbf{g}(\mathbf{x})$.

A finite dimensional Banach space is another simple example of Finsler space, where $\Phi(\mathbf{x},\boldsymbol{\xi}) \equiv \|\boldsymbol{\xi}\|$. $(\mathcal{M},\Phi)$, with $\Phi^2 = L$. One says that $\mathcal{GL}^N$ is reducible to a Finsler space.

Of course, $\mathcal{GL}^N$ reduces to a pseudo-Riemannian (or Riemannian) space $(\mathcal{M}, g_{ij}(\mathbf{x}))$ if the d-tensor $g_{ij}(\mathbf{x},\mathbf{y})$ does not depend on $\mathbf{y}$. On the contrary, if $g_{ij}(\mathbf{x},\mathbf{y})$ depends only on $\mathbf{y}$ (at least in preferred charts), it is a generalized Lagrange space which is locally Minkowski.

---

[1] Namely it must be $rank\, \|g_{ij}(\mathbf{x},\mathbf{y})\| = N$.

[2] Let us recall that a Finsler space [7] is a couple ($\mathcal{M}$, Φ), where $\mathcal{M}$ is be an N-dimensional differential manifold and $\Phi : T\mathcal{M} \Rightarrow \mathcal{R}$ a function $\Phi(\mathbf{x},\boldsymbol{\xi})$

Since, in general, a generalized Lagrange space is not reducible to a Lagrange one, it cannot be studied by means of the methods of symplectic geometry, on which – as is well known – analytical mechanics is based.

A linear $\mathcal{H}-$connection on $T\mathcal{M}$ (or on $\widehat{T\mathcal{M}}$) is defined by a couple of geometrical objects $\mathcal{C}\Gamma(\mathcal{H}) = (L^i{}_{jk}, C^i{}_{jk})$ on $T\mathcal{M}$ with different transformation properties under the coordinate transformation (9.2). Precisely, $L^i{}_{jk}(\mathbf{x}, \mathbf{y})$ transform like the coefficients of a linear connection on $\mathcal{M}$, whereas $C^i_{jk}(\mathbf{x}, \mathbf{y})$ transform like a d-tensor of type (1,2). $\mathcal{C}\Gamma(\mathcal{H})$ is called *the metrical canonical* $\mathcal{H}-$*connection* of the generalized Lagrange space $\mathcal{GL}^N$.

In terms of L^i_{jk} and C^i_{jk} one can define two kinds of covariant derivatives: a *covariant horizontal (h-) derivative*, denoted by "$_{\shortmid}$," and a *covariant vertical (v-) derivative*, denoted by "$|$." For instance, for the d-tensor $g_{ij}(\mathbf{x}, \mathbf{y})$ one has

$$\begin{cases} g_{ij\shortmid k} = \dfrac{\delta g_{ij}}{\delta x^k} - g_{sj}L^s{}_{ik} - g_{is}L^s{}_{jk}; \\ g_{ij|k} = \dfrac{\partial g_{ij}}{\partial x^k} - g_{sj}C^s{}_{ik} - g_{is}C^s{}_{jk}. \end{cases} \tag{9.10}$$

The two derivatives $g_{ij\shortmid k}$ and $g_{ij|k}$ are both d-tensors of type (0,3).

The coefficients of $\mathcal{C}\Gamma(\mathcal{H})$ can be expressed in terms of the following *generalized Christoffel symbols*:

$$\begin{cases} L^i{}_{jk} = \frac{1}{2}g^{is}\left(\dfrac{\delta g_{sj}}{\delta x^k} + \dfrac{\delta g_{ks}}{\delta x^j} + \dfrac{\delta g_{jk}}{\delta x^s}\right); \\ C^i{}_{jk} = \frac{1}{2}g^{is}\left(\dfrac{\partial g_{sj}}{\partial x^k} + \dfrac{\partial g_{ks}}{\partial x^j} + \dfrac{\partial g_{jk}}{\partial x^s}\right). \end{cases} \tag{9.11}$$

Moreover, by means of the connection $\mathcal{C}\Gamma(\mathcal{H})$ it is possible to define a *d-curvature* in $T\mathcal{M}$ by means of the tensors $R_j{}^i{}_{kh}$, $S_j{}^i{}_{kh}$ and $P_j{}^i{}_{kh}$ given by

$$\begin{aligned} R_j{}^i{}_{kh} &= \frac{\delta L^i{}_{jk}}{\delta x^h} - \frac{\delta L^i{}_{jh}}{\delta x^k} + L^r{}_{jk}L^i{}_{rh} - L^r{}_{jh}L^i{}_{rk} + C^i{}_{jr}R^r{}_{kh}; \\ S_j{}^i{}_{kh} &= \frac{\partial C^i{}_{jk}}{\partial y^h} - \frac{\partial C^i{}_{jh}}{\partial y^k} + C^r{}_{jk}C^i{}_{rh} - C^r{}_{jh}C^i{}_{rk}; \\ P_j{}^i{}_{kh} &= \frac{\partial L^i{}_{jk}}{\partial y^h} - C^i{}_{j\shortmid h} + C^i{}_{jr}P^r{}_{kh}. \end{aligned} \tag{9.12}$$

Here, the d-tensor R^i_{jk} is related to the bracket of the basis $\delta/\delta x^i$:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta xj}\right] = R^s{}_{ij}\frac{\partial}{\partial y^s} \tag{9.13}$$

and is explicitly given by[3]

$$R^i{}_{jk} = \frac{\delta H^i{}_j}{\delta x^k} - \frac{\delta H^i{}_k}{\delta x^j}. \tag{9.14}$$

The tensor $P^i{}_{jk}$, together with $T^i{}_{jk}$, $S^i{}_{jk}$, defined by

$$\begin{aligned} P^i{}_{jk} &= \frac{\partial H^i{}_j}{\partial y^k} - L^i{}_{jk}; \\ T^i{}_{jk} &= L^i{}_{jk} - L^i{}_{kj}; \\ S^i{}_{jk} &= C^i{}_{jk} - C^i{}_{kj} \end{aligned} \tag{9.15}$$

are *the d-tensors of torsion of the metrical connection* $\mathcal{C}\Gamma(\mathcal{H})$.

From the curvature tensors one can get the corresponding Ricci tensors of $\mathcal{C}\Gamma(\mathcal{H})$:

$$\begin{cases} R_{ij} = R_{i\ js}^{\ s}; & S_{ij} = S_{i\ js}^{\ s}; \\ \overset{1}{P}_{ij} = P_{i\ js}^{\ s} & \overset{2}{P}_{ij} = P_{i\ sj}^{\ s}, \end{cases} \tag{9.16}$$

and the scalar curvatures

$$R = g^{ij} R_{ij}; \quad S = g^{ij} S_{ij}. \tag{9.17}$$

Finally, *the deflection d-tensors associated to the connection* $\mathcal{C}\Gamma(\mathcal{H})$ are

$$\begin{cases} D^i{}_j = y^i_{|j} = -H^i{}_j + y^s L^i{}_{sj}; \\ d^i{}_j = y^i|_j = \delta^i{}_j + y^s C^i_{sj}, \end{cases} \tag{9.18}$$

namely the h- and v-covariant derivatives of the Liouville vector fields.

In the generalized Lagrange space $\mathcal{GL}^N$ it is possible to write the Einstein equations with respect to the canonical connection $\mathcal{C}\Gamma(\mathcal{H})$ as follows:

$$\begin{cases} R_{ij} - \frac{1}{2} R g_{ij} = \kappa \overset{H}{T}_{ij}; & \overset{1}{P}_{ij} = \kappa \overset{1}{T}_{ij}; \\ S_{ij} - \frac{1}{2} S g_{ij} = \kappa \overset{V}{T}_{ij}; & \overset{2}{P}_{ij} = \kappa \overset{2}{T}_{ij}, \end{cases} \tag{9.19}$$

where κ is a constant and $\overset{H}{T}_{ij}$, $\overset{V}{T}_{ij}$, $\overset{1}{T}_{ij}$, $\overset{2}{T}_{ij}$ are the components of the energy-momentum tensor.

---

[3] R^i_{jk} plays the role of a curvature tensor of the nonlinear connection $\mathcal{H}$. The corresponding tensor of torsion is instead

$$t^i_{jk} = \frac{\partial H^i_j}{\partial y^k} - \frac{\partial H^i_k}{\partial y^j}.$$

9.2 Generalized Lagrangian Structure of $\widetilde{M}$

On the basis of the previous considerations, let us analyze the geometrical structure of the deformed Minkowski space of DSR $\widetilde{M}$, endowed with the by now familiar metric $g_{\mu\nu,\mathrm{DSR}}(E)$. As explained in Part I, E is the energy of the process measured by the detectors in Minkowskian conditions. Therefore, E is a function of the velocity components, $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, where τ is the (Minkowskian) proper time:

$$E = E\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right). \tag{9.20}$$

The derivatives $\mathrm{d}x^\mu/\mathrm{d}\tau$ define a contravariant vector tangent to M at x, namely they belong to $TM_{\mathbf{x}}$. We shall denote this vector (according to the notation of Sect. 9.1) by $\mathbf{y} = (y^\mu)$. Then, $(\mathbf{x},\mathbf{y})$ is a point of the tangent bundle to M. We can therefore consider the generalized Lagrange space $\mathcal{GL}^4 = (M, g_{\mu\nu}(\mathbf{x},\mathbf{y}))$, with

$$\begin{cases} g_{\mu\nu}(\mathbf{x},\mathbf{y}) = g_{\mu\nu,\mathrm{DSR}}(E(\mathbf{x},\mathbf{y})), \\ \qquad E(\mathbf{x},\mathbf{y}) = E(\mathbf{y}). \end{cases} \tag{9.21}$$

Then, it is possible to prove the following theorem:

The pair $\mathcal{GL}^4 = (M, g_{\mu\nu,\mathrm{DSR}}(\mathbf{x},\mathbf{y})) \equiv \widetilde{M}$ is a generalized Lagrange space which is not reducible to a Riemann space, or to a Finsler space, or to a Lagrange space.

We already proved the first statement in Sect. 2.2 of Part I, on account of the dependence of the deformed metric tensor on E (and therefore on $\mathbf{y}$) only: $g_{\mu\nu,\mathrm{DSR}}(\mathbf{x},\mathbf{y}) \equiv g_{\mu\nu,\mathrm{DSR}}(\mathbf{y})$. To prove that $\mathcal{GL}^4$ is reducible neither to a Lagrange space nor to a Finsler one, it is sufficient (as stated in Sect. 9.1) to show that the d-tensor field $(\partial g_{\mu\nu,\mathrm{DSR}}/\partial y^\rho)$ is not totally symmetric, i.e., the equation

$$\frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial y^\rho} = \frac{\partial g_{\mu\rho,\mathrm{DSR}}}{\partial y^\nu} \tag{9.22}$$

does not hold. Let us assume *by absurdum* that (9.22) is satisfied. Then, for $\mu = \nu \neq \rho$, one gets

$$\frac{\partial g_{\mu\mu,\mathrm{DSR}}}{\partial y^\rho} = \frac{\partial g_{\mu\rho,\mathrm{DSR}}}{\partial y^\mu} \tag{9.23}$$

whence

$$\frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial y^\rho} = 0, \qquad \mu \neq \rho \tag{9.24}$$

(since $g_{\mathrm{DSR},\mu\nu}$ is diagonal). Equation (9.24) implies

$$\frac{\partial g_{\mu\mu,\mathrm{DSR}}}{\partial E}\frac{\partial E}{\partial y^\rho} = 0, \quad \forall \mu,\rho; \mu \neq \rho \Longrightarrow \quad \frac{\partial E}{\partial y^\rho} = 0, \tag{9.25}$$

which is impossible. This proves the theorem. Notice that such a result is strictly related to the fact that the deformed metric tensor of DSR is diagonal, and therefore it does not hold, in general, for the generalized Minkowski spaces we defined in Chapter 5.

If an external electromagnetic field $F_{\mu\nu}$ is present in the Minkowski space M, in $\widetilde{M}$ the deformed electromagnetic field is given by $\widetilde{F}^{\mu}_{\ \nu}(\mathbf{x},\mathbf{y}) = g^{\mu\rho}_{\mathrm{DSR}} F_{\rho\nu}(\mathbf{x})$ (see Sect. 3.5). Such a field is a d-tensor and is called *the electromagnetic tensor of the generalized Lagrange space.* Then, the nonlinear connection $\mathcal{H}$ is given by

$$H^{\mu}_{\ \nu} = \left\{ \begin{array}{c} \mu \\ \nu\rho \end{array} \right\} y^{\rho} - \widetilde{F}^{\mu}_{\ \nu}(\mathbf{x},\mathbf{y}), \tag{9.26}$$

where $\left\{ \begin{array}{c} \mu \\ \nu\rho \end{array} \right\}$, the Christoffel symbols of the Minkowski metric $g_{\mu\nu}$, are zero, so that

$$H^{\mu}_{\ \nu} = -\widetilde{F}^{\mu}_{\ \nu}(\mathbf{x},\mathbf{y}). \tag{9.27}$$

The adapted basis of the distribution $\mathcal{H}$ reads therefore

$$\frac{\delta}{\delta x^{\mu}} = \frac{\partial}{\partial x^{\mu}} + \widetilde{F}^{\nu}_{\ \mu}(\mathbf{x},\mathbf{y}).\frac{\partial}{\partial y^{\nu}}. \tag{9.28}$$

The local covector field of the dual basis (cf. (9.6)) is given by

$$\delta y^{\mu} = \mathrm{d}y^{\mu} - \widetilde{F}^{\mu}_{\ \nu}(\mathbf{x},\mathbf{y})\mathrm{d}x^{\nu}. \tag{9.29}$$

9.3 Canonical Metric Connection of $\widetilde{M}$

The derivation operators applied to the deformed metric tensor of the space $\mathcal{GL}^4 = \widetilde{M}$ yield

$$\frac{\delta g_{\mu\nu,\mathrm{DSR}}}{\delta x^{\rho}} = \frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial x^{\rho}} + \widetilde{F}^{\sigma}_{\ \rho}\frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial y^{\sigma}} = \widetilde{F}^{\sigma}_{\ \rho}\frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial E}\frac{\partial E}{\partial y^{\sigma}}, \tag{9.30}$$

$$\frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial y^{\sigma}} = \frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial E}\frac{\partial E}{\partial y^{\sigma}}. \tag{9.31}$$

Then, the coefficients of the canonical metric connection $\mathcal{C}\Gamma(\mathcal{H})$ in $\widetilde{M}$ (see (9.11)) are given by

$$\begin{cases} L^{\mu}_{\ \nu\rho} = \frac{1}{2} g^{\mu\sigma}_{\mathrm{DSR}} \frac{\partial E}{\partial y^{\alpha}} \left(\frac{\partial g_{\sigma\nu,\mathrm{DSR}}}{\partial E} \widetilde{F}^{\alpha}_{\ \rho} + \frac{\partial g_{\sigma\rho,\mathrm{DSR}}}{\partial E} \widetilde{F}^{\alpha}_{\ \nu} - \frac{\partial g_{\nu\rho,\mathrm{DSR}}}{\partial E} \widetilde{F}^{\alpha}_{\ \sigma} \right), \\ C^{\mu}_{\ \nu\rho} = \frac{1}{2} g^{\mu\sigma}_{\mathrm{DSR}} \frac{\partial E}{\partial y^{\alpha}} \left(\frac{\partial g_{\sigma\nu,\mathrm{DSR}}}{\partial E} \delta^{\alpha}_{\ \rho} + \frac{\partial g_{\sigma\rho,\mathrm{DSR}}}{\partial E} \delta^{\alpha}_{\ \nu} - \frac{\partial g_{\nu\rho,\mathrm{DSR}}}{\partial E} \delta^{\alpha}_{\ \sigma} \right). \end{cases} \tag{9.32}$$

The vanishing of the electromagnetic field tensor, $F^{\alpha}_{\ \rho} = 0$, implies $L^{\mu}_{\ \nu\rho} = 0$.

One can define the deflection tensors associated to the metric connection $\mathcal{C}\Gamma(\mathcal{H})$ as follows (cf. (9.18)):

$$\begin{aligned} D^{\mu}_{\ \nu} &= y^{\mu}_{\ \shortmid\nu} = \frac{\delta y^{\mu}}{\delta x^{\nu}} + y^{\alpha} L^{\mu}_{\ \alpha\nu} = \widetilde{F^{\mu}_{\nu}} + y^{\alpha} L^{\mu}_{\ \alpha\nu}; \\ d^{\mu}_{\ \nu} &= y^{\mu}_{\ |\nu} = \delta^{\mu}_{\ \nu} + y^{\alpha} C^{\mu}_{\ \alpha\nu}. \end{aligned} \tag{9.33}$$

The covariant components of these tensors read

$$\begin{aligned} D_{\mu\nu} &= g_{\mu\sigma,\mathrm{DSR}} D^{\sigma}_{\ \nu} = g_{\mu\sigma,\mathrm{DSR}} \left(\widetilde{F^{\sigma}_{\nu}} + y^{\alpha} L^{\sigma}_{\ \alpha\nu} \right) \\ &= F_{\mu\nu}(\mathbf{x}) + \frac{1}{2} y^{\sigma} \frac{\partial E}{\partial y^{\alpha}} \left(\frac{\partial g_{\mu\sigma,\mathrm{DSR}}}{\partial E} \widetilde{F}^{\alpha}_{\ \nu} + \frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial E} \widetilde{F}^{\alpha}_{\ \sigma} - \frac{\partial g_{\sigma\nu,\mathrm{DSR}}}{\partial E} \widetilde{F}^{\alpha}_{\ \mu} \right); \\ d_{\mu\nu} &= g_{\mu\sigma,\mathrm{DSR}} d^{\sigma}_{\ \nu} \\ &= g_{\mathrm{DSR},\mu\nu} + \frac{1}{2} y^{\sigma} \frac{\partial E}{\partial y^{\alpha}} \left(\frac{\partial g_{\mu\sigma,\mathrm{DSR}}}{\partial E} \delta^{\alpha}_{\ \nu} + \frac{\partial g_{\mu\nu,\mathrm{DSR}}}{\partial E} \delta^{\alpha}_{\ \sigma} - \frac{\partial g_{\sigma\nu,\mathrm{DSR}}}{\partial E} \delta^{\alpha}_{\ \mu} \right). \end{aligned} \tag{9.34}$$

Let us show how the formalism of the generalized Lagrange space allows one to recover some results on the phenomenological energy-dependent metrics discussed in Chap. 4.

Consider the following metric ($c = 1$):

$$\mathrm{d}s^2 = a(E)\mathrm{d}t^2 + (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{9.35}$$

where $a(E)$ is an arbitrary function of the energy and spatial isotropy ($b^2 = 1$) has been assumed. In absence of external electromagnetic field ($F_{\mu\nu} = 0$), the nonvanishing components $C^{\mu}_{\ \nu\rho}$ of the canonical metric connection $\mathcal{C}\Gamma(\mathcal{H})$ (see (9.32)) are

$$\begin{cases} C^0_{00} = \dfrac{a'}{a} y^0, \quad C^0_{01} = -\dfrac{a'}{a} y^1, \quad C^0_{02} = -\dfrac{a'}{a} y^2, \quad C^0_{03} = \dfrac{a'}{a} y^3, \\ C^1_{00} = -a' y^1, \quad C^2_{00} = -a' y^2, \quad C^0_{00} = -a' y^3, \end{cases} \tag{9.36}$$

where the prime denotes derivative with respect to E: $a' = \mathrm{d}a/\mathrm{d}E$.

According to the formalism of generalized Lagrange spaces, we can write the Einstein equations in vacuum corresponding to the metrical connection of the deformed Minkowski space (see (9.19)). It is easy to see that the independent equations are given by

$$a' = 0; \tag{9.37}$$

$$2aa'' - (a')^2 = 0. \tag{9.38}$$

The first equation has the solution $a = \text{const.}$, namely we get the Minkowski metric. Equation (9.38) has the solution

$$a(E) = \frac{1}{4}\left(a_0 + \frac{E}{E_0}\right)^2, \tag{9.39}$$

where a_0 and E_0 are two integration constants.

This solution represents the time coefficient of an over-Minkowskian metric of the second class, (4.31), with n_0 =2. For $a_0 = 0$ it coincides with (the time coefficient of) the phenomenological metric of the strong interaction, (4.11). On the other hand, by choosing $a_0 = 1$, one gets the time coefficient of the metric for gravitational interaction, (4.18).

In other words, *considering* $\widetilde{M}$ *as a generalized Lagrange space permits to recover (at least partially) the metrics of two interactions (strong and gravitational) derived on a phenomenological basis.*

It is also worth noticing that this result shows that a space–time deformation (of over-Minkowskian type) exists even in absence of an external electromagnetic field (remember that (9.37),(9.38) have been derived by assuming $F_{\mu\nu} = 0$).

9.4 Intrinsic Physical Structure of a Deformed Minkowski Space: Gauge Fields

As we have seen, the deformed Minkowski space $\widetilde{M}$, considered as a generalized Lagrange space, is endowed with a rich geometrical structure. But the important point, to our purposes, is the presence of a physical richness, intrinsic to $\widetilde{M}$. Indeed, let us introduce the following *internal electromagnetic field tensors* on $\mathcal{GL}^4 = \widetilde{M}$, defined in terms of the deflection tensors:

$$\begin{aligned}\mathcal{F}_{\mu\nu} &\equiv \frac{1}{2}\left(D_{\mu\nu} - D_{\nu\mu}\right) \\ &= F_{\mu\nu}(\mathbf{x}) + \frac{1}{2} y^\sigma \frac{\partial E}{\partial y^\alpha}\left(\frac{\partial g_{\mu\sigma,\mathrm{DSR}}}{\partial E}\widetilde{F}^{\alpha}_{\ \nu} - \frac{\partial g_{\nu\sigma,\mathrm{DSR}}}{\partial E}\widetilde{F}^{\alpha}_{\ \mu}\right)\end{aligned} \tag{9.40}$$

(*horizontal electromagnetic internal tensor*) and

$$\begin{aligned}f_{\mu\nu} &\equiv \frac{1}{2}\left(d_{\mu\nu} - d_{\nu\mu}\right) \\ &= \frac{1}{2} y^\sigma \frac{\partial E}{\partial y^\alpha}\left(\frac{\partial g_{\mu\sigma,\mathrm{DSR}}}{\partial E}\delta^{\alpha}_{\ \nu} - \frac{\partial g_{\nu\sigma,\mathrm{DSR}}}{\partial E}\delta^{\alpha}_{\ \mu}\right)\end{aligned} \tag{9.41}$$

(*vertical electromagnetic internal tensor*).

The internal electromagnetic h- and v-fields $\mathcal{F}_{\mu\nu}$ and $f_{\mu\nu}$ satisfy the following *generalized Maxwell equations*

$$\begin{aligned} 2\left(\mathcal{F}_{\mu\nu\shortmid\rho} + \mathcal{F}_{\nu\rho\shortmid\mu} + \mathcal{F}_{\rho\mu\shortmid\nu}\right) &= y^{\alpha}\left(R^{\beta}_{\ \mu\nu} C_{\beta\alpha\rho} + R^{\beta}_{\ \nu\rho} C_{\beta\alpha\mu} + R^{\beta}_{\ \rho\mu} C_{\beta\alpha\nu}\right), \\ R^{\beta}_{\ \mu\nu} &= g^{\beta\sigma}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}}; \end{aligned} \tag{9.42}$$

$$\mathcal{F}_{\mu\nu|\rho} + \mathcal{F}_{\nu\rho|\mu} + \mathcal{F}_{\rho\mu|\nu} = f_{\mu\nu\shortmid\rho} + f_{\nu\rho\shortmid\mu} + f_{\rho\mu\shortmid\nu}; \tag{9.43}$$

$$f_{\mu\nu|\rho} + f_{\nu\rho|\mu} + f_{\rho\mu|\nu} = 0. \tag{9.44}$$

Let us stress explicitly the different nature of the two internal electromagnetic fields. In fact, the horizontal field $\mathcal{F}_{\mu\nu}$ is strictly related to the presence of the external electromagnetic field $F_{\mu\nu}$, and vanishes if $F_{\mu\nu} = 0$. On the contrary, *the vertical field $f_{\mu\nu}$ has a geometrical origin, and depends only on the deformed metric tensor $g_{\mu\nu,\mathrm{DSR}}(E(\mathbf{y}))$ of $\mathcal{GL}^4 = \widetilde{M}$ and on $E(\mathbf{y})$. Therefore, it is present also in space–time regions where no external electromagnetic field occurs.* As we shall see in Part III, this fact has deep physical implications.

A few remarks are in order. First, the main results obtained for the (abelian) electromagnetic field can be probably generalized (with suitable changes) to non-abelian gauge fields. Second, the presence of the internal electromagnetic h- and v-fields $\mathcal{F}_{\mu\nu}$ and $f_{\mu\nu}$, intrinsic to the geometrical structure of $\widetilde{M}$ as a generalized Lagrange space, is the cornerstone to build up a *dynamics (of merely geometrical origin) internal to the deformed Minkowski space.*

The important point worth emphasizing is that *such an intrinsic dynamics springs from gauge fields.* Indeed, the two internal fields $\mathcal{F}_{\mu\nu}$ and $f_{\mu\nu}$ (in particular the latter one) do satisfy equations of the gauge type (cf. (9.42)–(9.44)). Then, we can conclude that *the (energy-dependent) deformation of the metric of $\widetilde{M}$, which induces its geometrical structure as generalized Lagrange space, leads in turn to the appearance of (internal) gauge fields.*

Such a fundamental result can be schematized as follows:

$$\widetilde{M} = (M, g_{\mu\nu,\mathrm{DSR}}(E)) \Longrightarrow \mathcal{GL}^4 = (M, g_{\mu\nu}(\mathbf{x},\mathbf{y})) \Longrightarrow \left(\widetilde{M}, \mathcal{F}_{\mu\nu}, f_{\mu\nu}\right) \tag{9.45}$$

(with self-explanatory meaning of the notation).

We want also to stress explicitly that this result follows by the fact that, in deforming the metric of the space–time, *we assumed the energy as the physical (nonmetric) observable on which letting the metric coefficients depend* (see Chap. 2). This is crucial in stating the generalized Lagrangian structure of $\widetilde{M}$, as shown in Sect. 9.2.

As is well known, successfully embodying gauge fields in a space–time structure is one of the basic goals of the research in theoretical physics starting from the beginning of the twentieth century. The almost unique tool to achieve such objective is increasing the number of space–time dimensions. In such a kind of theories (whose prototype is the celebrated Kaluza–Klein formalism), one preserves the usual (special-relativistic or general-relativistic) structure of 4D space–time, and gets rid of the nonobservable extra dimensions by compactifying them (for example to circles). Then the motions of the extra metric components over the standard Minkowski space satisfy identical equations to gauge fields. The gauge invariance of these fields is simply a consequence of the Lorentz invariance in the enlarged space. In this framework, gauge fields are *external* to the space–time, because they are *added* to it by the hypothesis of extra dimensions.

In the case of the DSR theory, gauge fields arise from the very geometrical, basic structure of $\widetilde{M}$, namely they are a consequence of the metric deformation. The arising gauge fields are *intrinsic and internal to the deformed space–time, and do not need to be added from the outside.* As a matter of fact, *DSR is the first theory based on a 4D space–time able to embody gauge fields in a natural way.*

Such a conventional, intrinsic gauge structure is related to a *given* deformed Minkowski space $\overline{\widetilde{M}}$, in which the deformed metric is fixed:

$$\overline{\widetilde{M}} = (M, \bar{g}_{\mu\nu,\mathrm{DSR}}(E))\,. \tag{9.46}$$

On the contrary, with varying g_{DSR}, we have another gauge-like structure – as already stressed in Sect. 4.4 – namely what we called a metric gauge. In the latter case, the gauge freedom amounts to choosing the metric according to the interaction considered.

The circumstance that the deformed Minkowski space $\widetilde{M}$ is endowed with the geometry of a generalized Lagrange space testifies the richness of nontrivial mathematical properties present in the seemingly so simple structure of the deformation of the Minkowski metric. This will be further supported in Part IV, where we shall show that $\widetilde{M}$ can be naturally embedded in a 5D Riemannian space.