5

Generalized Minkowski Spaces and Killing Symmetries

In the first Part of this book, we discussed the physical foundations of the DSR in four dimensions. This second Part will be devoted to dealing in detail with the mathematical features and properties of DSR. In this framework, the isometries of the deformed Minkowski space $\widetilde{M}$ play a basic role. The mathematical tool needed to such a study are the Killing equations, whose solution will allow us to determine both the infinitesimal and the finite structure of the deformed chronotopical groups of symmetries [41–43]. An important result we shall report at the end of this Part – due to its physical implications – is the geometrical structure of $\widetilde{M}$ as a generalized Lagrange space [12, 13, 44].

5.1 Generalized Minkowski Spaces

The structure of the deformed space–time $\widetilde{M}$ of DSR can be generalized to what we shall call *generalized Minkowski space* $\widetilde{M_N}(\{x\}_{\text{n.m.}})$. We define $\widetilde{M_N}(\{x\}_{\text{n.m.}})$ as a N-dimensional Riemann space with a global metric structure determined by the (in general nondiagonal) metric tensor $g_{\mu\nu}(\{x\}_{\text{n.m.}})$ $(\mu, \nu = 1, 2, ..., N)$, where $\{x\}_{\text{n.m.}}$ denotes a set of $N_{\text{n.m.}}$ nonmetrical coordinates (i.e., different from the N coordinates related to the dimensions of the space considered) [41]. The interval in $\widetilde{M_N}(\{x\}_{\text{n.m.}})$ therefore reads

$$\mathrm{d}s^2 = g_{\mu\nu}(\{x\}_{\text{n.m.}})\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{5.1}$$

We shall assume the signature (T, S) (T time-like dimensions and $S = N - T$ space-like dimensions). It follows that $\widetilde{M_N}(\{x\}_{\text{n.m.}})$ is *flat*, because all the components of the Riemann–Christoffel tensor vanish.

Of course, an example is just provided by the 4D deformed Minkowski space $\widetilde{M}(E)$. In the following, in order to comply with the notation adopted for generalized Minkowski spaces, we shall denote the DSR deformed space–time with $\widetilde{M}(x^5)$, where the coordinate x^5 has to be interpreted as the energy. The index 5 explicitly refers to the already mentioned fact that the deformed Minkowski space can be "naturally" embedded in a 5D (Riemannian) space (see Parts IV and V).

5.2 Maximal Killing Group of a N-Dimensional Generalized Minkowski Space

5.2.1 *Lie Groups, P.B.W. Theorem and the Transformation Representation*

Let us recall the essential content of the Poincaré-Birkhoff–Witt (P.B.W.) theorem and of the Lie theorems: Given a Lie group G_{L} of order M, it is always possible to build up an exponential representative mapping of any finite element g of G_{L}:

$$\begin{aligned}&\forall g \text{finite} \in G_{\text{L}} \\ &\quad \Rightarrow \exists \{\alpha_i\}_{i=1\ldots M} \in R^M \;\; (\{\alpha_i\} = \{\alpha_i(g)\}) : g = \exp\left(\sum_{i=1}^{M} \alpha_i T^i\right),\end{aligned} \tag{5.2}$$

where $\{T^i\}_{i=1\ldots M}$ is the generator basis of the Lie algebra of G_{L} and $\{\alpha_i = \alpha_i(g)\}_{i=1\ldots M}$ is a set of M real parameters (of course depending on $g \in G_{\text{L}}$).

Therefore, by a series development of the exponential

$$g = \exp\left(\sum_{i=1}^{M} \alpha_i T^i\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right)^k \quad \forall g \text{ finite} \in G_{\text{L}}, \tag{5.3}$$

we get, for an infinitesimal element $(g \to \delta g)$ $(\Leftrightarrow \{\alpha_i(g)\}_{i=1\ldots M} \in R^M \to \{\alpha_i(g)\}_{i=1\ldots M} \in I_0 \subset R^M\ )$:[1]

$$\delta g = 1 + \sum_{i=1}^{M} \alpha_i(g) T^i + O(\{\alpha_i^2(g)\}) \quad \forall \delta g \text{ infinitesimal} \in G_{\text{L}}. \tag{5.4}$$

[1] $I_0 \subset R^M$ is a generic neighborhood of $0 \in R^M$.

Since any Lie group admits a representation as a group of transformations acting on a N-dimensional manifold S_N ("N-dimensional vector space of transformation representation," not to be confused with thegroup manifold V_M), given $x \in S_N$, one has, for the action of a finite and infinitesimal element of G_{L}, respectively:

$$gx = \left[\exp\left(\sum_{i=1}^{M}\alpha_i(g)T^i\right)\right]x = \left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{i=1}^{M}\alpha_i(g)T^i\right)^k\right]x = x' \in S_N; \tag{5.5}$$

$$\left.\begin{array}{c}(\delta g)\,x = \left[1+\sum_{i=1}^{M}\alpha_i(g)T^i\right]x = x + \left(\sum_{i=1}^{M}\alpha_i(g)T^i\right)x = x' \in S_N; \\ \delta g : S_N \ni x \rightarrow x' = x + \delta x_{(g)}(x) \in S_N\end{array}\right\} \Rightarrow$$
$$\Rightarrow \delta x_{(g)}(x) = \left(\sum_{i=1}^{M}\alpha_i(g)T^i\right)x. \tag{5.6}$$

5.2.2 Killing Equations in a N-Dimensional Generalized Minkowski Space

In general S_N is endowed with a metric structure we shall assume in the following to be at most Riemannian. The interval in S_N is therefore:

$$\varphi(x) \equiv \mathrm{d}s^2(x) = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{5.7}$$

with $g_{\mu\nu}(x)$ being the symmetric, rank-two metric tensor. By carrying out an infinitesimal transformation of the type

$$x^{\mu\prime} = x^\mu + \xi^\mu(x), \tag{5.8}$$

one has:

$$\begin{aligned}\delta\mathrm{d}x^\mu(x) \overset{[\delta,d]=0}{=} \mathrm{d}\delta x^\mu(x) &= \frac{\partial\xi^\mu(x)}{\partial x^\gamma}\mathrm{d}x^\gamma; \\ \delta g_{\mu\nu}(x) &= \frac{\partial g_{\mu\nu}(x)}{\partial x^\beta}\zeta^\beta(x),\end{aligned} \tag{5.9}$$

and therefore

$$\begin{aligned}\delta\varphi(x) &\equiv \delta\mathrm{d}s^2(x) = \delta(g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu) \\ &= (\delta g_{\mu\nu}(x))\,\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{\mu\nu}(x)\,(\delta\mathrm{d}x^\mu(x))\,\mathrm{d}x^\nu + g_{\mu\nu}(x)\mathrm{d}x^\mu\,(\delta\mathrm{d}x^\nu(x))\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial g_{\mu\nu}(x)}{\partial x^{\beta}} \zeta^{\beta}(x) \right) \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} + g_{\mu\nu}(x) \left(\frac{\partial \xi^{\mu}(x)}{\partial x^{\gamma}} \mathrm{d}x^{\gamma} \right) \mathrm{d}x^{\nu} \\
&\quad + g_{\mu\nu}(x) \mathrm{d}x^{\mu} \left(\frac{\partial \xi^{\nu}(x)}{\partial x^{\chi}} \mathrm{d}x^{\chi} \right) \\
&= \frac{\partial g_{\mu\nu}(x)}{\partial x^{\beta}} \zeta^{\beta}(x) \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} + g_{\nu\beta}(x) \frac{\partial \xi^{\beta}(x)}{\partial x^{\mu}} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} \\
&\quad + g_{\mu\beta}(x) \frac{\partial \xi^{\beta}(x)}{\partial x^{\nu}} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} \\
&= \left(\frac{\partial g_{\mu\nu}(x)}{\partial x^{\beta}} \zeta^{\beta}(x) + g_{\nu\beta}(x) \frac{\partial \xi^{\beta}(x)}{\partial x^{\mu}} + g_{\mu\beta}(x) \frac{\partial \xi^{\beta}(x)}{\partial x^{\nu}} \right) \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}
\end{aligned}
\tag{5.10}
$$

The invariance of the infinitesimal interval under transformation (5.8) requires therefore

$$
\delta \mathrm{d}s^2(x) = 0 \Leftrightarrow \left(\frac{\partial g_{\mu\nu}(x)}{\partial x^{\beta}} \zeta^{\beta}(x) + g_{\nu\beta}(x) \frac{\partial \xi^{\beta}(x)}{\partial x^{\mu}} + g_{\mu\beta}(x) \frac{\partial \xi^{\beta}(x)}{\partial x^{\nu}} \right) = 0; \tag{5.11}
$$

$$
a_{\mu}(x) \equiv g_{\mu\eta}(x) a^{\eta}(x) \overset{g_{\mu\eta}(x) g^{\eta\chi}(x) = \delta_{\mu}^{\chi} \quad \forall x \in S_N}{\Leftrightarrow} a^{\eta}(x) = g^{\eta\mu}(x) a_{\mu}(x); \tag{5.12}
$$

$$
\begin{aligned}
\frac{\partial a^{\eta}(x)}{\partial x^{\nu}} &\equiv a^{\eta}(x)_{,\nu} \\
&= \frac{\partial \left(g^{\eta\mu}(x) a_{\mu}(x) \right)}{\partial x^{\nu}} = \frac{\partial g^{\eta\mu}(x)}{\partial x^{\nu}} a_{\mu}(x) + g^{\eta\mu}(x) \frac{\partial a_{\mu}(x)}{\partial x^{\nu}}.
\end{aligned}
\tag{5.13}
$$

Let us introduce the covariant derivative on S_N, defined by

$$
a_{\mu}(x)_{;\nu} \equiv a_{\mu}(x)_{,\nu} - \Gamma^{\lambda}_{\mu\nu}(x) a_{\lambda}(x) \tag{5.14}
$$

with $\Gamma^{\lambda}_{\mu\nu}(x)$ being the affine connection

$$
\Gamma^{\lambda}_{\mu\nu}(x) = \frac{1}{2} g^{\rho\lambda}(x) \left(\frac{\partial g_{\nu\rho}(x)}{\partial x^{\mu}} + \frac{\partial g_{\mu\rho}(x)}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}(x)}{\partial x^{\rho}} \right). \tag{5.15}
$$

Since the covariant derivative of the metric tensor vanishes ($g_{\mu\eta;\rho}(x) = 0$), it is possible to rewrite (5.11) as:

$$
\delta \mathrm{d}s^2(x) = 0 \Leftrightarrow \xi_{\mu}(x)_{;\nu} + \xi_{\nu}(x)_{;\mu} = 0, \tag{5.16}
$$

or, in compact form:

$$
\xi_{[\mu}(x)_{;\nu]} = 0, \tag{5.17}
$$

where the bracket [..] means symmetrization with respect to the enclosed indices.

As is well known, the $N(N+1)/2$ (5.17) in the N components of the covariant N-vector $\xi_\mu(x)$ are *the Killing equations* of the space S_N. As is clearly seen from their derivation, the contravariant Killing vectors correspond to directions along which the infinitesimal interval – and therefore the metric tensor – remains unchanged. Then, they determine the infinitesimal isometries of S_N. Another very useful property of Killing vectors is that they are associated to constants of motion, namely to quantities which keep their value along any geodesic. Any N-dimensional Riemannian space admits a Killing group with at most $N(N+1)/2$ parameters; in this latter case, the space is called "maximally symmetric." It can be shown that a Riemann space is maximally symmetric *iff* its scalar curvature R is constant.

In index notation, (5.6) can be written as:

$$\delta x^\mu_{(g)}(x) = \left[\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) x\right]^\mu, \mu = 1, ..., N. \tag{5.18}$$

Let us denote simply by α the parametric M-vector $\{\alpha_i\}_{i=1...M}$ of the representation (5.2) of the element $g \in G_{\mathrm{L}}$. Then, from (5.6) and (5.8) one gets

$$\delta x^\mu_{(g)}(x,\alpha) = \xi^\mu_{(g)}(x,\alpha); \tag{5.19}$$

$$\xi^\mu_{(g)}(x,\alpha) = \left[\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) x\right]^\mu, \tag{5.20}$$

namely $\delta x^\mu_{(g)}(x,\alpha)$ is the contravariant N-vector of the infinitesimal transformation associated – in the transformation representation of the Lie group G_{L} – to the infinitesimal element δg.

We can now define the mixed second-rank N-tensor $\delta\omega^\mu_\nu(g)$ of an infinitesimal transformation (associated to $\delta g \in G_{\mathrm{L}}$) as:

$$\delta x^\mu_{(g)}(x) = \left[\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) x\right]^\mu \equiv \delta\omega^\mu_\nu(g) x^\nu. \tag{5.21}$$

The number of independent components of the tensor $\delta\omega^\mu_\nu(g)$ is equal to the order M of the Lie group; in general, nothing can be said about its symmetry properties. Notice that, in the context of generalized N-dimensional Minkowski spaces, the infinitesimal mixed tensor depends in general on the set of nonmetric variables, i.e., $\delta\omega^\mu_\nu = \delta\omega^\mu_\nu(g, \{x\}_{\mathrm{n.m.}})$. From (5.18)–(5.20) it follows

$$\xi^\mu_{(g)}(x) = \delta\omega^\mu_\nu(g) x^\nu, \tag{5.22}$$

showing that $\delta\omega^\mu_\nu(g)$ is *the tensor of the "rotation" parameters* in S_N. Let us stress that (5.20)–(5.21), associating the *global* tensor $\delta\omega^\mu_\nu(g)$ ($\neq \delta\omega^\mu_\nu(g,x)$) to $\delta x^\mu_{(g)}(x)$ (in general *local*), imply a reductive assumption on the possible

Lie groups considered. Actually, as is easily seen, the introduction of $\delta\omega^{\mu}_{\nu}(g)$ (independent of x) is possible only *iff* $\delta x^{\mu}_{(g)}(x)$ is a *linear and homogeneous* function of x. Of course, this imposes severe restrictions on the possible types of Lie groups under consideration.

Indeed, let us stress that the transformation representation of the M-order Lie group G_{L} we considered above *is not a group representation of* G_{L} (in the usual meaning of the term). Indeed, although G_{L} can be interpreted as a suitable transformation group acting on S_N (with $M \neq N$ in general), such coordinate transformations *are not necessarily linear.* Otherwise speaking, G_{L} *does not admit, in general, a N-order matrix representation.* In other words, its M infinitesimal generators T^i $(i = 1, ..., M)$ *cannot in general be represented by* $N \times N$ *matrices acting on* S_N. Although (5.3) for g in terms of the generators $\{T^i\}$ can be linearized with respect to the group parameters $\{\alpha_i\}$ (by means of a "parametric MacLaurin development" in the neighborhood of the null M-vector of parameters), thus getting the infinitesimal element δg (5.4), *δgx is not necessarily linear in* x, due to the possible dependence of some of the T^i's on x.

Therefore, introducing the tensor $\delta\omega^{\mu}_{\nu}(g)$ amounts to consider only those Lie groups admitting a $N \times N$ matrix representation on S_N (corresponding to linear and homogeneous coordinate transformations).

5.2.3 Maximal Killing Group of $\widetilde{M_N}$

To our present aims, we have to impose two further conditions. First, we assume that the Lie groups under consideration, in the related transformation representation, are Killing groups of S_N (not necessarily maximal), namely (from (5.17), (5.18), and (5.20)):

$$\begin{aligned}
&\xi_{(g)\mu}(x)_{;\nu} + \xi_{(g)\nu}(x)_{;\mu} = 0 \Leftrightarrow \delta x_{(g)\mu}(x)_{;\nu} + \delta x_{(g)\nu}(x)_{;\mu} = 0\\
&\Leftrightarrow \left[\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) x\right]_{\mu;\nu} + \left[\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) x\right]_{\nu;\mu} = 0\\
&\Leftrightarrow \left(\delta\omega_{\mu\rho}(g) x^{\rho}\right)_{;\nu} + \left(\delta\omega_{\nu\rho}(g) x^{\rho}\right)_{;\mu} = 0\\
&\Leftrightarrow \left(\delta\omega_{[\mu\rho}(g) x^{\rho}\right)_{;\nu]} = 0
\end{aligned} \tag{5.23}$$

Last (5.23) can be derived from the first one on account of (5.22) and of

$$\xi_{(g)\mu}(x) = g_{\mu\nu}\xi^{\nu}_{(g)}(x) = g_{\mu\nu}\delta\omega^{\nu}_{\rho}(g) x^{\rho} = \delta\omega_{\mu\rho}(g) x^{\rho}. \tag{5.24}$$

Moreover, S_N is assumed to be endowed with a *global* metric structure, independent of $x \in S_N$ (but dependent, in general, on a set $\{x\}_{\mathrm{n.m.}}$ of nonmetric coordinates), i.e.,:

$$g_{\mu\nu}(\{x\}_{\mathrm{n.m.}}) \neq g_{\mu\nu}(x). \tag{5.25}$$

This second requirement entails that in S_N all components of the Riemann–Christoffel tensor vanish (so that the covariant derivative reduces to the ordinary derivative), and then it is a flat manifold. In other words, we are assuming that S_N is a *N-dimensional generalized Minkowski space*, as defined in Sect. 5.1. Therefore, we shall henceforth use the notation $\widetilde{M_N}$ instead of S_N.

Notice that although, in general, $\delta\omega^{\mu}_{\nu}(g,\{x\}_{\text{n.m.}})$ does depend on possible nonmetric variables, its completely covariant form *does not*, due to the dependence of $g_{\mu\nu}$ on $\{x\}_{\text{n.m.}}$:

$$\delta\omega_{\mu\rho}(g) = g_{\mu\sigma}(\{x\}_{\text{n.m.}})\delta\omega^{\sigma}_{\rho}(g,\{x\}_{\text{n.m.}}) \neq \delta\omega_{\mu\rho}(\{x\}_{\text{n.m.}}). \tag{5.26}$$

On the contrary, its completely contravariant form does depend on $\{x\}_{\text{n.m.}}$:

$$\delta\omega^{\mu\rho}(g,\{x\}_{\text{n.m.}}) = g^{\mu\sigma}(\{x\}_{\text{n.m.}})\delta\omega^{\rho}_{\sigma}(g,\{x\}_{\text{n.m.}}). \tag{5.27}$$

We can therefore state that, in a generalized Minkowski space, any form of the N-tensor $\delta\omega(g)$ is global (i.e., independent of all metric variables), but its completely covariant expression is independent of possible nonmetric variables, too. This independence is related to the fact that (as it will be seen in the following: see (5.26)) $\delta\omega_{\alpha\beta}(g)$ is nothing but the antisymmetric tensor of the space–time rotation parameters. Thus the dependence of $\delta\omega_{\alpha\beta}$ on the physical theory concerned is reducible to its very dependence on the element g of the space–time rotation group of the N-d generalized Minkowski space under consideration. That is why, in the following, parentheses will be sometime used in the covariant components of $\delta\omega$ (e.g., in the form $\delta\omega_{\alpha\beta,(\text{DSR})}(g)$ or $\delta\omega_{\text{cov.},(\text{DSR})}(g)$).

In $\widetilde{M_N}$, the following formulae hold $\forall\delta g$ infinitesimal $\in G_{\text{L}}$:

$$\delta g : \widetilde{M_N} \ni x \to x'(\{x\}_{m.},\{x\}_{\text{n.m.}}) = x + x_{(g)}(\{x\}_{m.},\{x\}_{\text{n.m.}}) \in \widetilde{M_N}; \tag{5.28}$$

$$\begin{aligned} \delta x^{\mu}_{(g)}(x,\{x\}_{\text{n.m.}}) &= \left[\left(\sum_{i=1}^{M}\alpha_i(g)T^i(\{x\}_{\text{n.m.}})\right)x\right]^{\mu} \\ &= \delta\omega^{\mu}_{\nu}(g,\{x\}_{\text{n.m.}})x^{\nu} = \xi^{\mu}_{(g)}(x,\{x\}_{\text{n.m.}}); \end{aligned} \tag{5.29}$$

$$\xi_{(g)\mu}(x)_{,\nu} + \xi_{(g)\nu}(x)_{,\mu} = 0, \tag{5.30}$$

where "$,\mu$" denotes ordinary derivation with respect to x^{μ}. From (5.21) and (5.27) it follows also:

$$\begin{aligned} &\xi_{(g)\mu}(x)_{,\nu} + \xi_{(g)\nu}(x)_{,\mu} = 0 \\ \Leftrightarrow\ &(\delta\omega_{\mu\nu}(g)x^{\rho})_{,\nu} + (\delta\omega_{\nu\rho}(g)x^{\rho})_{,\mu} = 0. \end{aligned} \tag{5.31}$$

The last equation entails the antisymmetry of $\delta\omega_{\mu\nu}(g)$:

$$\delta\omega_{\mu\nu}(g) + \delta\omega_{\nu\mu}(g) = 0, \tag{5.32}$$

which therefore has $N(N-1)/2$ independent components (such a number, as stressed earlier, is also equal to to the order M of G_{L}, $M = N(N-1)/2$), i.e., the (rotation) transformation group related to the tensor $\delta\omega_{\mu\nu}(g)$ is a $N(N-1)/2$-parameter Killing group.

Let us observe that a N-dimensional, generalized Minkowski space, being (as noted earlier) a special case of a Riemann space with constant curvature, admits a maximal Killing group with $N(N+1)/2$ parameters. Since the (rotation) transformation group related to the tensor $\delta\omega_{\mu\nu}(g)$ is a $N(N-1)/2$-parameter Killing group, we have still to find another N-parameter Killing group of $\widetilde{M_N}$ (because $N + N(N-1)/2 = N(N+1)/2$).

This is easily done by noting that the $N(N+1)/2$ Killing equations in such a space

$$\xi_\mu(x)_{,\nu} + \xi_\nu(x)_{,\mu} = 0 \equiv \frac{\partial \xi_\mu(x)}{\partial x^\nu} + \frac{\partial \xi_\nu(x)}{\partial x^\mu} = 0 \tag{5.33}$$

are trivially satisfied by constant covariant N-vectors $\xi_\mu \neq \xi_\mu(x)$, to which there corresponds the infinitesimal transformation

$$\delta g : x^\mu \to x^{\mu\prime}(x, \{x\}_{\text{n.m.}}) = x^\mu + \delta x^\mu_{(g)}(\{x\}_{\text{n.m.}}) = x^\mu + \xi^\mu_{(g)}(\{x\}_{\text{n.m.}}) \tag{5.34}$$

with $\delta x^\mu_{(g)}(\{x\}_{\text{n.m.}})$, $\xi^\mu_{(g)}(\{x\}_{\text{n.m.}})$ constant (with respect to x^μ).

In conclusion, a N-d generalized Minkowski space $\widetilde{M_N}(\{x\}_{\text{n.m.}})$ admits a maximal Killing group which is the (semidirect) product of the Lie group of the N-dimensional space–time rotations (or N-d generalized, homogeneous Lorentz group $\mathrm{SO}(T,S)^{N(N-1)/2}_{\text{GEN.}}$) with $N(N-1)/2$ parameters, and of the Lie group of the N-dimensional space–time translations $\mathrm{Tr.}(T,S)^N_{\text{GEN.}}$ with N parameters:

$$P(T,S)^{N(N+1)/2}_{\text{GEN.}} = \mathrm{SO}(T,S)^{N(N-1)/2}_{\text{GEN.}} \otimes_s \mathrm{Tr.}(T,S)^N_{\text{GEN.}} \tag{5.35}$$

The semidirect nature of the group product is due to the fact that, as it shall be explicitly derived (in the case $N = 4, T = 1, S = 3$ of DSR, without loss of generality) in Chap. 7, in general we have that

$$\begin{aligned} &\exists \text{ at least } 1 \ \ (\mu,\nu,\rho) \in \{1,2,...,N\}^3 : \\ &: [I^{\mu\nu}_{\text{GEN.}}(\{x\}_{\text{n.m.}}), \Upsilon^\rho_{\text{GEN.}}(\{x\}_{\text{n.m.}})] \neq 0, \ \forall \{x\}_{\text{n.m.}} , \end{aligned} \tag{5.36}$$

where $I^{\mu\nu}_{\text{GEN.}}(\{x\}_{\text{n.m.}})$ and $\Upsilon^\rho_{\text{GEN.}}(\{x\}_{\text{n.m.}})$ are the infinitesimal generators of $\mathrm{SO}(T,S)^{N(N-1)/2}_{\text{GEN.}}$ and $\mathrm{Tr.}(T,S)^N_{\text{GEN.}}$, respectively. We will refer to $P(S,T)^{N(N+1)/2}_{\text{GEN.}}$ as the *generalized* (or *inhomogeneous Lorentz*) *group* .

5.2.4 Solution of Killing Equations in a 4D Generalized Minkowski Space

We want now to solve the Killing equations in a 4D generalized Minkowski space $\widetilde{M}(\{x\}_{\text{n.m.}})$ $(S \leq 4, T = 4-S)$. A covariant Killing four-vector $\xi_\mu(x)$

must satisfy (5.17), which explicitly amounts to the system:

$$\begin{cases} (\mathrm{I}) & \xi_0(x)_{,0} = 0; \\ (\mathrm{II}) & \xi_0(x)_{,1} + \xi_1(x)_{,0} = 0; \\ (\mathrm{III}) & \xi_0(x)_{,2} + \xi_2(x)_{,0} = 0; \\ (\mathrm{IV}) & \xi_0(x)_{,3} + \xi_3(x)_{,0} = 0; \\ (\mathrm{V}) & \xi_1(x)_{,1} = 0; \\ (\mathrm{VI}) & \xi_1(x)_{,2} + \xi_2(x)_{,1} = 0; \\ (\mathrm{VII}) & \xi_1(x)_{,3} + \xi_3(x)_{,1} = 0; \\ (\mathrm{VIII}) & \xi_2(x)_{,2} = 0; \\ (\mathrm{IX}) & \xi_2(x)_{,3} + \xi_3(x)_{,2} = 0; \\ (\mathrm{X}) & \xi_3(x)_{,3} = 0. \end{cases} \tag{5.37}$$

From equations (5.37) (I ,V,VII, and X) one gets:

$$\begin{cases} \xi_0 = \xi_0(x^1, x^2, x^3); \\ \xi_1 = \xi_1(x^0, x^2, x^3); \\ \xi_2 = \xi_2(x^0, x^1, x^3); \\ \xi_3 = \xi_3(x^0, x^1, x^2). \end{cases} \tag{5.38}$$

Solving system (5.37) is cumbersome but straightforward [41]. The final result is:

$$\begin{cases} \xi_0(x) = -\zeta^1 x^1 - \zeta^2 x^2 - \zeta^3 x^3 + T^0; \\ \xi_1(x) = \zeta^1 x^0 + \theta^2 x^3 - \theta^3 x^2 - T^1; \\ \xi_2(x) = \zeta^2 x^0 - \theta^1 x^3 + \theta^3 x^1 - T^2; \\ \xi_3(x) = \zeta^3 x^0 + \theta^1 x^2 - \theta^2 x^1 - T^3, \end{cases} \tag{5.39}$$

where ζ^i, θ^i $(i = 1, 2, 3)$ and T^μ $(\mu = 0, 1, 2, 3)$ are real coefficients.

We can draw the following conclusions:

1. In spite of the fact that no assumption was made on the functional form of the Killing vector, we got a dependence at most linear (inhomogeneous) on the metric coordinates for all components of $\xi_\mu(x)$. Therefore, in order to determine the (maximal) Killing group of a generalized Minkowski space,[2] one can, without loss of generality, consider only groups whose transformation representation is implemented by transformations at most linear in the coordinates.

2. In general, $\xi_\mu \neq \xi_\mu(\{x\}_{\text{n.m.}})$, i.e., the covariant Killing vector does not depend on possible nonmetric variables.[3] On the contrary, the *contravariant* Killing four-vector *does indeed*, due to the dependence of the fully contravariant metric tensor on $\{x\}_{\text{n.m.}}$:

$$\xi^\mu(x, \{x\}_{\text{n.m.}}) = g^{\mu\nu}(\{x\}_{\text{n.m.}})\xi_\nu(x). \tag{5.40}$$

   Such a result is consistent with the fact that $\delta\omega_{\mu\nu}(g)$, unlike $\delta\omega^\mu_\nu(g, \{x\}_{\text{n.m.}})$, is independent of $\{x\}_{\text{n.m.}}$ (cf.(5.28),(5.29)).

3. Solution (5.39) *does not depend on the metric tensor.* This entails that all 4D generalized Minkowski spaces admit the same *covariant* Killing four-vector. *It corresponds to the covariant four-vector of infinitesimal transformation of the space–time rototranslational group of* $\widetilde{M}(\{x\}_{\text{n.m.}})$. Therefore, assuming the signature $(+,-,-,-)$ (i.e., $S = 3$, $T = 1$), in a basis of "length-dimensional" coordinates, we can state that:

   (a) $\boldsymbol{\zeta} = (\zeta^1, \zeta^2, \zeta^3)$ is the three-vector of the dimensionless parameters ("rapidity") of a generalized 3D boost

   (b) $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$ is the three-vector of the dimensionless parameters (angles) of a generalized 3D rotation

   (c) $T_\mu = (T^0, -T^1, -T^2, -T^3)$ is the covariant four-vector of the ("length-dimensional") parameters of a generalized 4D translation

[2] In fact, although we discussed explicitly the 4D case, the extension to the generic N-d case is straightforward.

[3] Indeed

$$\begin{aligned}\xi_{\mu(g)}(x) &= g_{\mu\nu,DSR4}(\{x\}_{\text{n.m.}})\xi^\nu_{(g)}(x, \{x\}_{\text{n.m.}}) \\ &= g_{\mu\nu,\text{DSR4}}(\{x\}_{\text{n.m.}})\delta\omega^\nu_\rho(g, \{x\}_{\text{n.m.}})x^\rho = \delta\omega_{\mu\rho}(g)x^\rho.\end{aligned}$$