25

Complete Solutions of Geodesic Equations for the 5D Metrics

We discussed in Chap. 24 the solutions of the geodesic equations for the four phenomenological metrics of the fundamental interactions, obtained as special cases of the classes of solutions of the vacuum Einstein equations in the Power Ansatz. However, it is easily seen that they hold only for the energy ranges where the metrics are not Minkowskian (namely below threshold for the electromagnetic and weak metrics, and above threshold for the strong and gravitational ones). Moreover, in most cases the value of the parameter r was fixed (as functions of the other coefficients q_μ, $\mu = 0, 1, 2, 3$) by the structure of the Einstein equations. We want now to give the general solutions of the geodesic equations for the four interactions, starting from the general form of the metrics (20.21)–(20.23), obtained by the 5D embedding of the 4D DSR phenomenological metrics in the DR5 framework. As already stressed, such a procedure leaves undetermined the fifth metric coefficient $f(x^5)$, and therefore yields r-parametrized metrics.

The general expression of the geodesic generating function $F_\pm(\zeta; \widetilde{\mathbf{q}}, A_2)$, which determines the geodesic motions in the Power Ansatz, is given by (24.7). On account of it, and of the exponent sets $\widetilde{\mathbf{q}}_{\text{int.}}$ (int.=e.m.,weak,strong,grav.), (20.41)–(20.43), one gets, in correspondence to the 5D metrics of the four fundamental interactions in the Power Ansatz:

$$
\begin{aligned}
&F_{\pm,\text{e.m./weak}}(\zeta; r, A_2, x^5_{0,\text{e.m./weak}}) \\
&= \zeta^{r/2} \left\{ A_2 \left(x^5_{0,\text{e.m./weak}} \right)^{-r} \mp C^2_{02} \right. \\
&\left. \pm \left[\left(x^5_{0,\text{e.m./weak}} \right) \zeta^{-1} \right]^{1/3\widehat{\Theta}_{\mathrm{L}}\left(x^5_{0,\text{e.m./weak}} - x^5 \right)} \left(C^2_{12} + C^2_{32} \right) \right\}^{-1/2} ;
\end{aligned}
\tag{25.1}
$$

$$
\begin{aligned}
&F_{\pm,\text{strong}}(\zeta; r, A_2, x^5_{0,\text{strong}}) \\
&= \zeta^{r/2} \left\{ A_2 \left(x^5_{0,\text{strong}} \right)^{-r} \pm \left(C^2_{12} + C^2_{22} \right) + \right. \\
&\left. \pm \left[\left(x^5_{0,\text{strong}} \right) \zeta^{-1} \right]^{2\widehat{\Theta}_{\mathrm{L}}\left(x^5 - x^5_{0,\text{strong}} \right)} \left(C^2_{32} - C^2_{02} \right) \right\}^{-1/2} ;
\end{aligned}
\tag{25.2}
$$

$$
\begin{aligned}
&F_{\pm,\text{grav.}}(\zeta; r, A_2, x^5_{0,\text{grav.}}) \\
&= \zeta^{r/2} \left\{ \left(x^5_{0,\text{grav.}} \right)^{-r} A_2 \pm \left(x^5_{0,\text{grav.}} \right)^{?} \zeta^{-?} \left(C^2_{12} + C^2_{22} \right) + \right. \\
&\left. \pm \left[\left(x^5_{0,\text{grav.}} \right) \zeta^{-1} \right]^{\widetilde{2\widehat{\Theta}_{\mathrm{L}}}\left(x^5 - x^5_{0,\text{grav.}} \right)} \left(C^2_{32} - C^2_{02} \right) \right\}^{-1/2} ,
\end{aligned}
\tag{25.3}
$$

$$\tag{25.4}$$

where the tilde and the question marks in $F_{\pm,\text{grav.}}(\zeta; r, A_2, x^5_{0,\text{grav.}})$ have the meaning clarified in Sect. 20.2.3.

The solutions for the geodesic equations are still given by (24.4), (24.6). Let us distinguish the two cases of Minkowskian and non-Minkowskian behavior.

25.1 Minkowskian Behavior

This is the case of the electromagnetic and weak interactions above threshold, i.e., for $x^5 \geq x^5_{0,\text{e.m./weak}}$, and of the strong and gravitational interactions below threshold, i.e., for $x^5 \leq x^5_{0,\text{strong}}$ and $x^5 \leq x^5_{0,\text{grav.}}$ In these cases, the exponent sets of the metrics reduce to

$$\widetilde{\mathbf{q}}_{\text{e.m./weak}} \left(x^5 \geq x^5_{0,\text{e.m./weak}} \right) = (0, 0, 0, 0, r) ; \tag{25.5}$$

$$\widetilde{\mathbf{q}}_{\text{strong}} \left(0 < x^5 \leq x^5_{0,\text{strong}} \right) = (0, (0, 0), 0, r) ; \tag{25.6}$$

$$\widetilde{\mathbf{q}}_{\text{grav.}} \left(0 < x^5 \leq x^5_{0,\text{grav.}} \right) = (0, (0, 0), 0, r) . \tag{25.7}$$

These coefficient sets are identical to that of Class VIII we already discussed in Sect. 24.4. The corresponding solution for the fifth coordinate is therefore

$$x^5_{\pm,\text{int.}}(\tau; x^5_{0,\text{int.}}, r_{\text{int.}}) = \left[\pm\frac{r_{\text{int.}}+2}{2K_{1,\pm,\text{int.}}}\left(x^5_{0,\text{int.}}\right)^{r_{\text{int}}/2}(\tau + A_1)\right]^{2/(r_{\text{int}}+2)}, \tag{25.8}$$

$$K_{1,\pm,\text{int.}} = \left\{\pm\left[C^2_{12} + C^2_{22} + C^2_{32} - C^2_{02} \pm A_2\left(x^5_{0,\text{int.}}\right)^{-r_{\text{int.}}}\right]\right\}^{-1/2}, \tag{25.9}$$

whereas the space–time coordinates are given by (ESC off)

$$\begin{aligned} x^\mu_{\pm,\text{int.}}(\tau) &= C_{\mu 1} + C_{\mu 2}\int \mathrm{d}\tau \left(x^5(\tau)\right)^{-q_\mu} \\ &= C_{\mu 1} + C_{\mu 2}\left(\tau + \chi_\mu\right) = \widetilde{C}_{\mu 1} + C_{\mu 2}\tau, \\ &\mu = 0,1,2,3, \quad \widetilde{C}_{\mu 1} = C_{\mu 1} + C_{\mu 2}\chi_\mu. \end{aligned} \tag{25.10}$$

25.2 Non-Minkowskian Behavior

Let us consider separately the different cases.

25.2.1 Electromagnetic and Weak Interactions under Threshold

(a.1) $\mp C^2_{02} + A_2\left(x^5_{0,\text{e.m./weak}}\right)^{-r} = 0,\ C^2_{12} + C^2_{22} + C^2_{32} \neq 0$:

(a.1.1) $3r + 7 \neq 0$:

$$\begin{aligned} &x^5_{\pm,\text{e.m./weak.}}(\tau) \\ &= \left[\pm\frac{3r+7}{6}\sqrt{\pm\left(C^2_{12} + C^2_{22} + C^2_{32}\right)}\left(x^5_{0,\text{e.m./weak}}\right)^{(3r+1)/6}(\tau + A_1)\right]^{6/(3r+7)}; \end{aligned} \tag{25.11}$$

(a.1.2) $r = -\dfrac{7}{3}$:

$$\begin{aligned} &x^5_{\pm,\text{e.m./weak.}}(\tau) \\ &= \exp\left[\pm\sqrt{\pm\left(C^2_{12} + C^2_{22} + C^2_{32}\right)}\left(x^5_{0,\text{e.m./weak}}\right)^{(3r+1)/6}(\tau + A_1)\right]; \end{aligned} \tag{25.12}$$

(a.2) $\mp C_{02}^2 + A_2 \left(x^5_{0,\mathrm{e.m./weak}}\right)^{-r} \neq 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 = 0$:

(a.2.1) $r \neq -2$:

$$x^5_{\pm,\mathrm{e.m./weak.}}(\tau) = \left[\pm\frac{r+2}{2}\sqrt{\mp C_{02}^2 + A_2 (x^5_{0,\mathrm{e.m./weak}})^{-r}}(x^5_{0,\mathrm{e.m./weak}})^{r/2}(\tau + A_1)\right]^{2/(r+2)}; \tag{25.13}$$

(a.2.2) $r = -2$:

$$x^5_{\pm,\mathrm{e.m./weak.}}(\tau) = \exp\left[\pm\sqrt{\mp C_{02}^2 + A_2 \left(x^5_{0,\mathrm{e.m./weak}}\right)^{-r}} \left(x^5_{0,\mathrm{e.m./weak}}\right)^{r/2} (\tau + A_1)\right]; \tag{25.14}$$

(a.3) $\mp C_{02}^2 + A_2 \left(x^5_{0,\mathrm{e.m./weak}}\right)^{-r} \neq 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 \neq 0$:

(a.3.1) $3r + 7 \neq 0$:

$$\begin{aligned}\tau + A_1 \quad = \quad & \pm\frac{6\left(x^5_{0,\mathrm{e.m./weak}}\right)^{(-3r+2)/9}}{(3r+7)\sqrt{\pm\left(C_{12}^2 + C_{22}^2 + C_{32}^2\right)}} \\ & \times \left[\frac{-\left(C_{12}^2 + C_{22}^2 + C_{32}^2\right)}{C_{02}^2 \mp A_2 \left(x^5_{0,\mathrm{e.m./weak}}\right)^{-r}}\right]^{(3r+7)/6} \int \frac{\mathrm{d}t}{\sqrt{t^{\frac{2}{3r+7}} + 1}},\end{aligned} \tag{25.15}$$

where

$$t = \left[\frac{-C_{02}^2 \pm A_2 (x^5_{0,\mathrm{e.m./weak}})^{-r}}{(x^5_{0,\mathrm{e.m./weak}})^{1/3}(C_{12}^2 + C_{22}^2 + C_{32}^2)}\right]^{(3r+7)/2} (x^5_{\pm,\mathrm{e.m./weak.}}(\tau))^{(3r+7)/6}; \tag{25.16}$$

(a.3.2) $r = -\dfrac{7}{3}$:

$$x^5_{\pm,\mathrm{e.m./weak.}}(\tau) = x^5_{0,\mathrm{e.m./weak}} \left[\frac{C_{12}^2 + C_{22}^2 + C_{32}^2}{-C_{02}^2 \pm A_2 \left(x^5_{0,\mathrm{e.m./weak}}\right)^{-r}}\right]^3 \tag{25.17}$$

$$\times \sinh^{-6}\left[\pm\frac{1}{6}\sqrt{\pm\left(C_{12}^2 + C_{22}^2 + C_{32}^2\right)} \left(x^5_{0,\mathrm{e.m./weak}}\right)^{(3r+1)/6} (\tau + A_1)\right]. \tag{25.18}$$

25.2.2 Strong Interaction above Threshold

(b.1) $\pm\left(C_{12}^2+C_{22}^2\right)+A_2\left(x^5_{0,\text{strong}}\right)^{-r}=0,\ C_{02}^2-C_{32}^2\neq 0$:

(b.1.1) $r\neq -4$:

$$x^5_{\pm,\text{strong}}(\tau)=\left[\pm\frac{r+4}{2}(\tau+A_1)(x^5_{0,\text{strong}})^{(r+2)/2}\sqrt{\mp(C_{02}^2-C_{32}^2)}\right]^{2/(r+4)}; \tag{25.19}$$

(b.1.2) $r=-4$:

$$x^5_{\pm,\text{strong}}(\tau)=\exp\left[\pm(\tau+A_1)\left(x^5_{0,\text{strong}}\right)^{(r+2)/2}\sqrt{\mp\left(C_{02}^2-C_{32}^2\right)}\right]; \tag{25.20}$$

(b.2) $\pm\left(C_{12}^2+C_{22}^2\right)+A_2\left(x^5_{0,\text{strong}}\right)^{-r}\neq 0,\ C_{02}^2-C_{32}^2=0$:

(b.2.1) $r\neq -2$:

$$x^5_{\pm,\text{strong}}(\tau)=\left[\pm\frac{r+2}{2}\sqrt{\pm(C_{12}^2+C_{22}^2)+A_2(x^5_{0,\text{strong}})^{-r}}(x^5_{0,\text{strong}})^{r/2}(\tau+A_1)\right]^{2/(r+2)}; \tag{25.21}$$

(b.2.2) $r=-2$:

$$x^5_{\pm,\text{strong}}(\tau)=\exp\left[\pm\sqrt{\pm\left(C_{12}^2+C_{22}^2\right)+A_2(x^5_{0,\text{strong}})^{-r}}(x^5_{0,\text{strong}})^{r/2}(\tau+A_1)\right]; \tag{25.22}$$

(b.3) $\pm\left(C_{12}^2+C_{22}^2\right)+A_2\left(x^5_{0,\text{strong}}\right)^{-r}\neq 0,\ C_{02}^2-C_{32}^2\neq 0$:

(b.3.1) $r\neq -4$:

$$\tau+A_1=\pm\frac{2\left[\mp\left(C_{02}^2-C_{32}^2\right)\right]^{(r-2)/4}}{(r+4)\left[\pm\left(C_{12}^2+C_{22}^2\right)+A_2\left(x^5_{0,\text{strong}}\right)^{-r}\right]^{(r+4)/4}}\times\int\frac{\mathrm{d}t}{\sqrt{t^{4/(r+4)}+1}}, \tag{25.23}$$

where

$$t=\left[\frac{\pm\left(C_{12}^2+C_{22}^2\right)+A_2\left(x^5_{0,\text{strong}}\right)^{-r}}{\left(x^5_{0,\text{strong}}\right)^2\left(\mp C_{02}^2\pm C_{32}^2\right)}\right]^{(r+4)/4}\left(x^5_{\pm,\text{strong}}(\tau)\right)^{(r+4)/2}; \tag{25.24}$$

(b.3.2) $r = -4$:

$$x^5_{\pm,\text{strong}}(\tau) = x^5_{0,\text{strong}} \sqrt{\frac{-C_{02}^2 + C_{32}^2}{C_{12}^2 + C_{22}^2 \pm A_2 \left(x^5_{0,\text{strong}}\right)^{-r}}} \times \sinh^{-1}\left[\pm\sqrt{\mp\left(C_{02}^2 - C_{32}^2\right)} \left(x^5_{0,\text{strong}}\right)^{(r+2)/2} (\tau + A_1)\right]. \tag{25.25}$$

25.2.3 Gravitational Interaction above Threshold

(I) ? = 0:

(c.I.1) $\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,\text{grav}}\right)^{-r} = 0,\ C_{02}^2 - C_{32}^2 \neq 0$:

(c.I.1.1) $r \neq -4$:

$$x^5_{\pm,\text{grav}}(\tau) = \left[\pm\frac{r+4}{2}(\tau + A_1)\left(x^5_{0,\text{grav}}\right)^{(r+2)/2}\sqrt{\mp\left(C_{02}^2 - C_{32}^2\right)}\right]^{2/(r+4)}; \tag{25.26}$$

(c.I.1.2) $r = -4$:

$$x^5_{\pm,\text{grav}}(\tau) = \exp\left[\pm(\tau + A_1)\left(x^5_{0,\text{grav}}\right)^{(r+2)/2}\sqrt{\mp\left(C_{02}^2 - C_{32}^2\right)}\right]; \tag{25.27}$$

(c.I.2) $\pm(C_{12}^2 + C_{22}^2) + A_2(x^5_{0,\text{grav}})^{-r} \neq 0,\ C_{02}^2 - C_{32}^2 = 0$:

(c.I.2.1) $r \neq -2$:

$$x^5_{\pm,\text{grav}}(\tau) = \left[\pm\frac{r+2}{2}\sqrt{\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,\text{grav}}\right)^{-r}}\left(x^5_{0,\text{grav}}\right)^{r/2}(\tau + A_1)\right]^{2/(r+2)}; \tag{25.28}$$

(c.I.2.2) $r = -2$:

$$x^5_{\pm,\text{grav}}(\tau) = \exp\left[\pm\sqrt{\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,\text{grav}}\right)^{-r}}\left(x^5_{0,\text{grav}}\right)^{r/2}(\tau + A_1)\right]; \tag{25.29}$$

(c.I.2) $\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,\text{grav}}\right)^{-r} \neq 0,\ C_{02}^2 - C_{32}^2 \neq 0$:

(c.I.2.1) $r \neq -4$:

$$\tau + A_1 = \pm \frac{2\left[\mp\left(C_{02}^2 - C_{32}^2\right)\right]^{(r-2)/4}}{(r+4)\left[\pm\left(C_{12}^2 + C_{22}^2\right) + A_2 (x_{0,\mathrm{grav}}^5)^{-r}\right]^{(r+4)/4}} \times \int \frac{\mathrm{d}t}{\sqrt{t^{4/(r+4)} + 1}} \tag{25.30}$$

where

$$t = \left[\frac{\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x_{0,\mathrm{grav}}^5\right)^{-r}}{\left(x_{0,\mathrm{grav}}^5\right)^2 (\mp C_{02}^2 \pm C_{32}^2)}\right]^{(r+4)/4} \left(x_{\pm,\mathrm{grav}}^5(\tau)\right)^{(r+4)/2}; \tag{25.31}$$

(c.I.2.2) $r = -4$:

$$x_{\pm,\mathrm{grav}}^5(\tau) = x_{0,\mathrm{grav}}^5 \sqrt{\frac{-C_{02}^2 + C_{32}^2}{C_{12}^2 + C_{22}^2 \pm A_2\left(x_{0,\mathrm{grav}}^5\right)^{-r}}} \times \sinh^{-1}\left[\pm\sqrt{\mp\left(C_{02}^2 - C_{32}^2\right)}\left(x_{0,\mathrm{grav}}^5\right)^{(r+2)/2}(\tau + A_1)\right]. \tag{25.32}$$

(II) $? = \tilde{2}$:

(c.II.1) $A_2\left(x_{0,\mathrm{grav}}^5\right)^{-r} = 0, \quad \mp\left(x_{0,\mathrm{grav}}^5\right)^{\tilde{2}}\left(C_{02}^2 - C_{12}^2 - C_{22}^2 - C_{32}^2\right) \neq 0$:

(c.II.1.1) $r \neq -4$:

$$x_{\pm,\mathrm{grav}}^5(\tau) = \left[\pm\frac{r+4}{2}(\tau + A_1)(x_{0,\mathrm{grav}}^5)^{(r+2)/2} \times \sqrt{\mp(x_{0,\mathrm{grav}}^5)^{\tilde{2}}(C_{02}^2 - C_{12}^2 - C_{22}^2 - C_{32}^2)}\right]^{2/(r+4)}; \tag{25.33}$$

(c.II.1.2) $r = -4$:

$$x_{\pm,\mathrm{grav}}^5(\tau) = \exp\left[\pm(\tau + A_1)\left(x_{0,\mathrm{grav}}^5\right)^{(r+2)/2}\sqrt{\mp\left(C_{02}^2 - C_{32}^2\right)}\right]; \tag{25.34}$$

(c.II.2) $A_2(x_{0,\mathrm{grav}}^5)^{-r} \neq 0, \ \mp(x_{0,\mathrm{grav}}^5)^{\tilde{2}}\left(C_{02}^2 - C_{12}^2 - C_{22}^2 - C_{32}^2\right) = 0$:

(c.II.2.1) $r \neq -2$:

$$x^5_{\pm,\mathrm{grav}}(\tau) = \left[\pm \frac{r+2}{2} \sqrt{A_2 \left(x^5_{0,\mathrm{grav}}\right)^{-r}} \left(x^5_{0,\mathrm{grav}}\right)^{r/2} (\tau + A_1)\right]^{2/(r+2)} ; \tag{25.35}$$

(c.II.2.2) $r = -2$:

$$x^5_{\pm,\mathrm{grav}}(\tau) = \exp\left[\pm \sqrt{A_2 \left(x^5_{0,\mathrm{grav}}\right)^{-r}} \left(x^5_{0,\mathrm{grav}}\right)^{r/2} (\tau + A_1)\right] ; \tag{25.36}$$

(c.II.3) $A_2 \left(x^5_{0,\mathrm{grav}}\right)^{-r} \neq 0$, $\mp \left(x^5_{0,\mathrm{grav}}\right)^{\tilde{2}} \left(C^2_{02} - C^2_{12} - C^2_{22} - C^2_{32}\right) \neq 0$:

(c.II.3.1) $r \neq -4$:

$$\tau + A_1 = \pm \left(x^5_{0,\mathrm{grav}}\right)^{-r/2} \frac{2}{(r+4)\sqrt{\mp \left(x^5_{0,\mathrm{grav}}\right)^{\tilde{2}} \left(C^2_{02} - C^2_{12} - C^2_{22} - C^2_{32}\right)}}$$
$$\times \left[\frac{\mp \left(x^5_{0,\mathrm{grav}}\right)^{\tilde{2}} \left(C^2_{02} - C^2_{12} - C^2_{22} - C^2_{32}\right)}{A_2 \left(x^5_{0,\mathrm{grav}}\right)^{-r}}\right]^{(r+4)/4} \int \frac{\mathrm{d}t}{\sqrt{t^{4/(r+4)} + 1}}, \tag{25.37}$$

where

$$t = \left[\frac{A_2 \left(x^5_{0,\mathrm{grav}}\right)^{-r}}{\mp \left(x^5_{0,\mathrm{grav}}\right)^{\tilde{2}} \left(C^2_{02} - C^2_{12} - C^2_{22} - C^2_{32}\right)}\right]^{(r+4)/4} \left(x^5_{\pm,\mathrm{grav}}(\tau)\right)^{(r+4)/2} ; \tag{25.38}$$

(c.II.3.2) $r = -4$:

$$x^5_{\pm,\mathrm{grav}}(\tau) = \sqrt{\frac{\mp \left(x^5_{0,\mathrm{grav}}\right)^{\tilde{2}} \left(C^2_{02} - C^2_{12} - C^2_{22} - C^2_{32}\right)}{A_2 \left(x^5_{0,\mathrm{grav}}\right)^{-r}}}$$
$$\times \sinh^{-1}\left[\pm \sqrt{\mp \left(x^5_{0,\mathrm{grav}}\right)^{\tilde{2}} \left(C^2_{02} - C^2_{12} - C^2_{22} - C^2_{32}\right)} \left(x^5_{0,\mathrm{grav}}\right)^{r/2} (\tau + A_1)\right]. \tag{25.39}$$

Let us stress that, in both special cases ? $= 0, \tilde{2}$, the treatment is quite analogous to those of the strong interaction above energy threshold.

25.3 Slicing and Dynamics

It must be by now clear, from the examples discussed in Sect. 25.2, that the explicit form of the geodesics in $\Re_5$ (namely, its dynamics) strictly depends on the sets of metric exponents $\widetilde{\mathbf{q}}_{\text{int.}}$ ((20.24)–(20.26)), which determine $x^A(\tau)$ through the knowledge of the generating function $F_\pm(\zeta; \widetilde{\mathbf{q}}, A_2)$.

But – as is easily seen from their expressions – the exponent sets $\widetilde{\mathbf{q}}_{\text{int.}}$ are discontinuous at the threshold energy $x^5_{0,\text{int.}}$:

$$\lim_{x^5 \longrightarrow x^{5+}_{0,\text{int.}}} \widetilde{\mathbf{q}}_{\text{int.}}(x^5) \neq \lim_{x^5 \longrightarrow x^{5-}_{0,\text{int.}}} \widetilde{\mathbf{q}}_{\text{int.}}(x^5), \tag{25.40}$$

namely, for a given interaction, different sets are obtained in the two different energy ranges (below and above threshold). This entails among the others that, as done in Sect. 20.2, it is necessary to use the right and left specifications of the Heaviside function in order to write $\widetilde{\mathbf{q}}_{\text{int.}}$ in the compact form (20.41)–(20.43), valid on the whole energy range. In turn, such a discontinuity in $\widetilde{\mathbf{q}}_{\text{int.}}$ at $x^5_{0,\text{int.}}$ causes an analogous behavior in the geodesic motions. In fact, let $x^5_{\text{int.},<}(\tau)$, $x^5_{\text{int.},>}(\tau)$ denote the solutions of the geodesic equation (24.3) for the fifth coordinate under and above threshold, respectively. Then, it is possible to impose e.g., $x^5_{\text{int.},<}(\overline{\tau}) = x^5_{0,\text{int.}}$ and find the corresponding value $\overline{\tau} \in R$. However, if such a value is replaced in the geodesic solution corresponding to the other energy range, one finds in general $x^5_{\text{int.},>}(\overline{\tau}) = \overline{x^5_{\text{int.}}} \neq x^5_{0,\text{int.}}$.

The situation is exactly analogous to that we encountered in the case of the Killing symmetries (Sect. 22.4). The nontrivial "bifurcation of dynamics" in the two energy ranges is clearly related to the nature change (from parameter to coordinate) the variable x^5 undergoes in the passage DSR→DR5. Therefore, dynamic structures present in an energy range in which the space–time sector is standard Minkowskian – or at least its metric coefficients are constant – may no longer occur when (in a different energy range) the space–time of $\Re_5$ becomes Minkowskian deformed, and vice versa.

Such a change of role of energy in the geometrical embedding of $\widetilde{M}$ in $\Re_5$ implies also, in full analogy with the case of the Killing isometries, that the dynamics in a given 4D space $\widetilde{M}\left(x^5 = \overline{x^5}\right)$ is *different* from the dynamics obtained for the slice of $\Re_5$ at constant energy $x^5 = \overline{x^5}$ with space–time sector coinciding with $\widetilde{M}\left(x^5 = \overline{x^5}\right)$. Symbolically one has:

$$\text{Dynamics in } \Re_5|_{\mathrm{d}x^5=0 \Leftrightarrow x^5=\overline{x^5}} \neq \text{Dynamics in } \widetilde{M}\left(x^5 = \overline{x^5}\right). \tag{25.41}$$

In fact the change of role of x^5 causes the destruction of the nonhomogeneous linearity in τ of the geodesic motions in DSR, which is no longer recovered in the inverse process of slicing of $\Re_5$ at $\mathrm{d}x^5 = 0$. This is again at variance with the metric level, where (see (19.10)) the constant-energy sections of $\Re_5$ at $x^5 = \overline{x^5}$ are endowed with the same metric structure of $\widetilde{M}\left(\overline{x^5}\right)$. Again, as in the case of the Killing symmetries, it is possible to understand this point by remembering that one is considering sections of a genuine Riemannian space, which therefore do keep memory of the fifth coordinate.

An explicit example of the key dynamic role played by the fifth coordinate in the embedding process is provided by the results of Sect. 24.4 for the geodesics relevant to class VIII of solutions of the 5D Einstein's equations. As already noted, the 5D metric (24.80) corresponding to the exponent set $\widetilde{\mathbf{q}}_{\mathrm{VIII}} = (0, 0, 0, 0, r)$ has a standard Minkowski structure for its space–time sector. In spite of this, the embedding of such a Minkowski space in $\Re_5$ (i.e., the presence and the form of the fifth metric coefficient) makes the dynamic behavior genuinely nontrivial, because to the standard geodesic motion of M it is added the further condition that the geodesics must correspond to a minimal value of the time–energy uncertainty.

We can therefore conclude that not only Killing isometries, but dynamics, too, depends on the geometrical framework. This further supports the deep physical (not only mathematical) significance of the geometrical embedding of $\widetilde{M}$ in $\Re_5$.