Generalized Finite Element Method in Mixed Variational Formulation: A Study of Convergence and Solvability

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Abstract. The Generalized Finite Element Method (GFEM) is first applied to hybrid-mixed stress formulations (HMSF). Generalized shape approximation functions are generated by means of polynomials of three independent approximation fields: stresses and displacements in the domain and displacements field on the static boundary. Firstly, the enrichment can independently be conducted over each of the three approximation fields. However, solvability and convergence problems are induced mainly due to spurious modes generated when enrichment is arbitrarily applied. With the aim of efficiently exploring enrichments in HMSF, an extension of the patch-test is proposed as a necessary condition to ensure enrichment, thus preserving convergence and solvability. In the present work, the inf-sup test based on Babuška–Brezzi condition was used to demonstrate the effectiveness of the Patch-Test. In particular, the inf-sup test was applied over selectively enriched quadrilateral bilinear and triangular finite element meshes. Numerical examples confirm the Patch-Test as a necessary but not sufficient condition for convergence and solvability.

Key words: Generalized finite element method, hybrid-mixed stress formulation, Babuška–Brezzi condition, inf-sup test.

1 Introduction

Boundary value problems (BVPs) can be variationally expressed using different principles. Depending on the variables involved in the variational principle, a specific weak form results and the FEM can then be applied to generate approximated solutions.

Among the non-conventional weak forms, three variants of the Hybrid formulation can be emphasized: the Hybrid-Mixed, Hybrid and Hybrid-Trefftz forms, all of which are detailed in [6]. In the present work, the Hybrid-Mixed Stress Model Formulation (HMSF) is addressed.

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This non-conventional form is called mixed for the reason that two incompatible stress and displacements fields are approximated in the domain. Since a second displacement field is approximated independently on the static boundary (where surface forces are imposed), the formulation is also typified as a hybrid one. Finally, the model is referred to as a stress model in view of the fact that continuity is primarily imposed over the stress approximation field.

Concerning numerical tools to solve BVPs, meshless methods provide approximations which are totally or relatively independent of the finite element mesh concept. The hp-Clouds Method [5], is distinguished among the meshless methods by the enrichment alternative of a basic approximation (partition of unity) without the definition of any additional nodal points in the domain. The Generalized Finite Element Method (GFEM) [8], allows combining the enrichment scheme of the hp-cloud method and advantageous features such as the strong imposition of boundary conditions of the conventional FEM.

Pimenta et al. [9] and Góis [7] applied the nodal enrichment technique to HMSF, resulting in a new application of the GFEM. Apparently, since displacements and stresses can be independently approximated, the enrichment could also be unconditionally imposed to each of those fields. However, that is not true because not all the combinations of enrichment supply stable and convergent solutions. Actually, in spite of noticeably good convergent responses, sometimes bad combinations of enrichments furnish unsatisfactory values of strain energy or stress and displacement fields as well.

Based on the above, in the present work, two numerical tests are proposed to verify the stability of HMSF solutions provided by GFEM. The first, a kind of Patch Test is linked to solvability, while the second is used to verify whether the (inf-sup) condition is satisfied by GFEM approximation functions obtained from clouds composed by four nodes quadrilaterals.

2 Hybrid-Mixed Stress Formulation in Plane Elasticity

The basis of the Hybrid-Mixed stress model in isotropic linear elasticity employed in the present work is founded on the variational principle of Reissner–Hellinger expressed as:

$$\Pi(u, \sigma, u_\Gamma) = -\int_\Omega \frac{1}{2}\sigma^T f \sigma \, \mathrm{d}\Omega - \int_\Omega u^T (L\sigma + b) \, \mathrm{d}\Omega + \int_{\Gamma_t} u_\Gamma^T (N\sigma - t) \, \mathrm{d}\Gamma + \int_{\Gamma_u} u^T (N\sigma) \, \mathrm{d}\Gamma. \tag{1}$$

The above expression comprises stress (σ) and displacement (u) fields defined in the domain (Ω) and a displacement field (u_Γ) defined on the static part (Γ_T) of the boundary. In addition, L is the differential divergent operator; b the vector of body forces; N the matrix formed by the components of the unit vector normal to the boundary; u the vector of prescribed displacements on Γ_u and t the vector of applied superficial forces on Γ_t. The treatment will be restricted to plane elasticity. Thus, f represents the flexibility matrix for linear and isotropic elastic materials.

Concerning numerical approximations and assuming the domain (Ω) and boundary (Γ) of the solid discretized by nodal points, interpolations of nodal values can be used to approximate the three independent fields as indicated below:

$$\hat{\sigma} = S_\Omega\, s_\Omega; \quad \hat{u} = U_\Omega\, q_\Omega; \quad \hat{u}_\Gamma = U_\Gamma\, q_\Gamma. \tag{2}$$

In the preceding relations, s_Ω represents the vector of nodal stress variables while q_Ω and q_Γ are vectors of displacement nodal degrees of freedom. S_Ω, U_Ω and q_Γ are respectively the matrices collecting the approximation functions for stress and displacement fields. Taking into consideration the approximated fields and imposing the stationary condition to the functional given in Equation (1), the following system of linear equations can be generated:

$$\begin{bmatrix} F & A_\Omega & -A_\Gamma \\ A_\Omega^T & 0 & 0 \\ -A_\Gamma^T & 0 & 0 \end{bmatrix} \begin{bmatrix} s_\Omega \\ q_\Omega \\ q_\Gamma \end{bmatrix} = \begin{bmatrix} e_\Gamma \\ -Q_\Omega \\ -Q_\Gamma \end{bmatrix}. \tag{3}$$

The system matrix can be rearranged using the following sub-matrices:

$$F = \int_\Omega S_\Omega^T\, f S_\Omega\, \mathrm{d}\Omega, \tag{4a}$$

$$A_\Omega = \int_\Omega (L S_\Omega)^T U_\Omega\, \mathrm{d}\Omega, \tag{4b}$$

$$A_\Gamma = \int_{\Gamma_t} (N S_\Omega)^T U_\Gamma\, \mathrm{d}\Gamma, \tag{4c}$$

$$e_\Gamma = \int_{\Gamma_u} (N S_\Omega)^T u\, \mathrm{d}\Gamma, \tag{4d}$$

$$Q_\Omega = \int_\Omega U_\Omega^T b\, \mathrm{d}\Omega, \tag{4e}$$

$$Q_\Gamma = \int_{\Gamma_t} U_\Gamma^T (t)\, \mathrm{d}\Gamma. \tag{4f}$$

In this work body forces (b) are not considered and the vector of displacements u is prescribed as null on the boundary (Γ_u) (then $e_\Gamma = Q_\Omega = 0$).

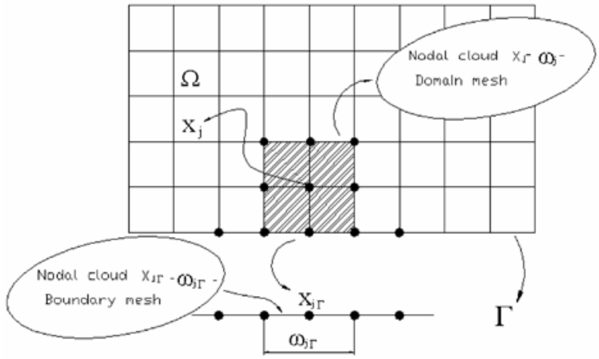

Fig. 1. Nodal clouds on domain and boundary nodes.

3 Bilinear Quadrilateral Elements with Nodal Enrichment

Consider a plain domain covered by a mesh of four nodes quadrilateral elements. The covering mesh is employed to define (ω) clouds in the domain, Figure 1. The boundary mesh is composed of linear elements connecting the nodes positioned on the boundary. Each cloud on both the domain and boundary is then formed by the elements sharing a common node. For those meshes composed of quadrangular elements, conventional bilinear Lagrangian functions are used as interpolating partition of unity for both stresses and displacements fields in the domain. Linear partitions of unit are used to interpolate displacements on boundary.

In order to provide enrichment to any approximation field attached to the nodes of the covering mesh, polynomial functions h_{kj}, $k = 1, \ldots, I(j)$ can be adopted in the domain Ω. Here $I(j)$ is the number of chosen functions at each node of index j. The following GFEM approximation family then results:

$$\Im_N^2 = \{\{S_{\Omega_j}\}_{j=1}^N \cup \{S_{\Omega_j}\, h_{kj}\}_{j=1}^N : j = 1, \ldots, N; k = 1, \ldots, I(j)\}. \qquad (5)$$

For instance the enriched stress field can be constructed as:

$$\hat{\sigma} = \sum_{j=1}^{N} S_{\Omega_j} \left\{ s_{\Omega_j} + \sum_{i=1}^{I(j)} h_{ij}\, b_{ij} \right\}, \qquad (6)$$

where N is the total number of nodes in the domain, s_{Ω_j} are the stress degrees of freedom associated with the original shape functions and b_{ij} are new nodal parameters introduced by each of the enrichment portions. With respect to the fields of displacements in domain and boundary, analogous procedure can be used to generate enriched approximations.

Thus, the interpolation matrices of the HSMF stress and displacement fields in domain can be represented as:

$$S_{\Omega_e} = [\phi_1\Delta_1 \;\; \phi_2\Delta_2 \;\; \phi_3\Delta_3 \;\; \phi_4\Delta_4], \tag{7}$$

$$U_{\Omega_e} = [\phi_1\Delta_1 \;\; \phi_2\Delta_2 \;\; \phi_3\Delta_3 \;\; \phi_4\Delta_4], \tag{8}$$

where ϕ_j, $j = 1, \ldots, 4$ represents the Lagrangian bilinear functions connected to node j. The interpolation matrix for the two nodes element employed on the boundary mesh is given by:

$$U_\Pi = [\psi_1\Delta_{\Gamma_1} \;\; \psi_2\Delta_{\Gamma_2}], \tag{9}$$

where ψ_{j_Γ}, $j_\Gamma = 1, 2$ being the Lagrangian linear functions connected to node j_Γ of the element.

In Equation (7) through Equation (9), Δ_j and $\Delta_{\Gamma_{j_\Gamma}}$ are named the polynomial enrichment matrices attached respectively to node j of the domain and node j_Γ of the boundary element. Such matrices are given by:

$$\Delta_j = [I_3 \;\; h_{1j}I_3 \quad \ldots \quad h_{kj}I_3 \quad \ldots \quad h_{i(j)j}I_3] \tag{10}$$

when enrichment is over domain stress fields;

$$\Delta_j = [I_2 \;\; h_{1j}I_2 \quad \ldots \quad h_{kj}I_2 \quad \ldots \quad h_{I(j)j}I_2] \tag{11}$$

when enrichment is over domain displacements fields; and

$$\Delta_{\Gamma_{j_\Gamma}} = [I_2 \;\; h_{1j_\Gamma}I_2 \quad \ldots \quad h_{kj_\Gamma}I_2 \quad \ldots \quad h_{I(j)j_\Gamma}I_2] \tag{12}$$

when enrichment is over boundary displacements fields.

Clearly, if the functions h_{kj} and h_{kj_Γ} are null the conventional structure of the FEM is recovered. In this work the functions and are such that they are null in the enriched nodes ("bubble like functions"). The advantage of this procedure is that the original physical meaning of nodal degrees of freedom s_Ω, q_Ω and q_Γ is preserved.

The forms selected for the bubbles functions h_{kj} (later referred to as levels 2, 3 and 4 enrichment) and h_{kj_Γ} (later referred to as level 2 enrichment) are:

$$h_{kj} = (Y - Y_j)^2, (X - X_j)^2, (X - X_j)(Y - Y_j)^2, \ldots \text{ and } h_{kj_\Gamma} = (\xi - \xi_{j_\Gamma})^2, \ldots, \tag{13}$$

where X_j, Y_j are dimensionless coordinates to the finite element in the domain covering mesh and ξ_{j_Γ} is a dimensionless coordinate for the one-dimensional finite element of the boundary covering mesh.

Obviously the set of approximating functions involved in the HMSF can be limited not only to polynomials but also extended to include customized functions. On the other hand, with respect to the number of Gauss points necessary to carry out a numerical integration of the enriched matrices of each quadrilateral element, the following cases are distinguished:

- for the domain elements where the maximum degree of the h_{kj} polynomials in a direction is g_{ap}, $g_{ap} + 2$ Gauss points are necessary in each direction for integrating while;
- for the boundary elements where the maximum degree of the h_{kj_Γ} polynomials is $g_{ap\Gamma}$, the number of Gauss points necessary for integrating is $g_{ap\Gamma} + 2$;

4 On the Conditions of Convergence of HMSF with GFEM Spaces

A first step towards the study of the conditions of convergence of GFEM solutions for the case of HMSF presented here is founded in the work of Zienkiewicz et al. [11]. Initially a simple algebraic condition called Patch Test is suggested to ensure solvability (non-singularity condition) of the discrete linear system. Then a numerical study is proposed following the Babuška–Brezzi (*inf-sup*) sufficient condition to solvability and convergence.

4.1 The Patch Test Applied to HMSF with GFEM

Taking into account the system of equations given by Equation (3), limited to the case without enrichment and based on the proposal of Zienkiewicz et al. [11], the following algebraic conditions are necessary for the existence of a solution:

$$s_\Omega \geq q_\Omega, \tag{14}$$

expressed in terms of degrees of freedom attached to domain patches and

$$s_\Omega \geq q_\Gamma, \tag{15}$$

expressed in terms of degrees of freedom of stress and displacements on boundary patches. Both conditions are essentially important to assure good properties to the sub-matrices, A_Ω and A_Γ composed of the system given by Equation (3).

When enrichment is considered, the patch test is extended by including the new set of degrees of freedom introduced at each node. Then, Equation (14) and Equation (15) become:

$$a_\Omega + b_{ij} \geq q_\Omega + c_{mn}, \tag{16}$$

$$s_\Omega + b_{ij} \geq q_\Gamma + d_{kl}, \tag{17}$$

where c_{mn} and d_{kl} are the additional degrees of freedom introduced by displacement enrichments in the domain and boundary respectively.

Now, analyzing the enrichment possibilities and considering the conditions expressed in formuae (16) and (17), it can be concluded initially that:

- Only stress field enrichment in the domain is unrestricted since both conditions would always be satisfied;
- The simultaneous enrichment of stress and displacement fields in the domain is also effective;
- Simultaneous enrichment of stress and displacement fields (in domain and boundary) is allowed provided enrichment applies to those nodes in the domain coincident with nodes of the enriched mesh on the boundary;

- The previous conclusion can be extended to enrichment including stress fields in the domain and displacements on the boundary;
- Enrichment restricted to displacement fields is ineffective since Equations (16) and (17) would not be satisfied.

It is important to note once more that the patch test conditions are necessary but not sufficient for solvability and consequently for the stability of the linear system.

4.2 Study of the Babuška–Brezzi (inf-sup) Condition Apllied to the HMSF with GFEM

The Babuška–Brezzi or *inf-sup* condition [1, 3] is a necessary and sufficient condition to ensure stability and convergence of any linear numerical approach supplied by the MEF. Based on Babuška [2] and Chapelle and Bathe [4], a numerical verification of this condition extended to GFEM in HMSF is presented here.

4.2.1 On the Well-Posedness of a Boundary Value Problem: The Babuška–Brezzi (inf-sup) Condition

A boundary value problem (BVP) expressed in its variational form consists in finding a solution $u_0 \in H_1$ such that:

$$B(u_0, v) = F(v) \quad \forall v \in H_2 \tag{18}$$

for every continuous linear functional F in H_2. Well-posedness of the problem is assured if the following conditions are verified: Continuity of the bilinear form $B(\ldots)$ and the Babuška–Brezzi (inf-sup) condition.

Continuity of the bilinear form

Let H_1 and H_2 be Hilbert spaces endowed with norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_{H_2}$ respectively. The bilinear form, $B(\cdot) : H_1 x H_2 \to R$ is continuous if for a given positive scalar C_B, the inequality below is valid:

$$|B(u, v)| < C_B \|u\|_{H_1} \|v\|_{H_2} \quad \forall\, u \in H_1 \quad \text{and} \quad v \in H_2. \tag{19}$$

The Babuška–Brezzi (inf-sup) condition

The class of problems governed by the addressed continuous bilinear form is well-posed if:

$$\inf_{u \neq 0 \in H_1} \sup_{v \neq 0 \in H_2} \frac{B(u, v)}{\|u\|_{H_1} \, \|v\|_{H_2}} \geq \lambda > 0, \tag{20}$$

$$\sup_{u \in H_1} B(u, v) > 0 \quad \forall\, v \neq 0 \in H_2. \tag{21}$$

In the above equations, ν represents the space of test functions. Furthermore, by definition:

$$H_1 = H_2 = \left\{ \nu | \nu \in L^2(\Omega); \frac{\partial \nu_i}{\partial x_j} \in L^2(\Omega), \right.$$
$$\left. (i, j = 1, 2, 3); \nu_{i|\Gamma_u} = 0, (i = 1, 2, 3) \right\}. \tag{22}$$

Accordingly, any element ν in H_2 must satisfy boundary conditions which are homogeneous on Γ_u. Moreover,

$$\int_\Omega (\nu)^2 \, \mathrm{d}\Omega < \infty, \tag{23}$$

$$\int_\Omega \left(\frac{\partial \nu_i}{\partial x_j} \right)^2 \mathrm{d}\Omega < \infty, \quad i, j = 1, 2, 3. \tag{24}$$

In fact, the preceding conditions are related to the existence of a solution to the problem in question. Essentially, it can be demonstrated that the conditions given by Equation (20) and Equation (21) are necessary and sufficient to ensure the existence and uniqueness of the BVP solution.

Once the original problem is well-posed, the performance of the formulation using finite elements is dependent on the choice of n-dimensional linear subspaces $S_1^n \subset H_1$ and $S_2^n \subset H_2$ defined as:

$$s_1^n = s_2^n \tag{25}$$
$$= \left\{ \nu_n | \nu_n \in L^2(\Omega); \frac{\partial \nu_{n_i}}{\partial x_j} \in L^2(\Omega), \ (i, j = 1, 2, 3); \nu_{n_i}|_{\Gamma_u} = 0, (i = 1, 2, 3) \right\}.$$

Usually, ν_u is a polynomial function of degree n. As a rule, it is assumed that there are $\nu_k \in S^k$ such that the sequence $\nu_k (k = 1, 2, \ldots)$ converges to u_0.

The partition of unity features used in the approximation spaces, S_1^n and S_2^n, ensures continuity of the bilinear form involved in HMSF and also the verification of Equation (21). Hence, the Babuška–Brezzi (inf-sup) condition is the main focus of what follows.

Assuming that $B(u, \nu)$ satisfies Equation (19), the discrete form of Equation (21) can be written as:

$$\inf_{u \neq 0 \in s_1^n} \sup_{\nu \neq 0 \in S_2^n} \frac{B(u_n, \nu_n)}{\|u_n\|_{H_1} \|\nu_n\|_{H_2}} \geq \lambda(n) > 0. \tag{26}$$

It can be perceived that $\lambda(n)$ now depends on the dimension of the approximation spaces. However, if the problem is well-posed, convergence to a value λ_0 should be verified provided mesh refinement or enrichment of the approximation space is carried out.

Let then $u_0 \in H_1$ be the exact solution derived from the strong formulation, for instance, and $u_n \in S_1^n$ a numerical estimate. Then, since the exact solution is also a solution for the weak form, both conditions below are valid:

$$B(u_0, \nu) = F(\nu), \quad \forall \nu \in H_2, \tag{27}$$

$$B(u_n, \nu_n) = F(\nu_n), \quad \forall \nu_n \in S_2^n. \tag{28}$$

Subtracting Equation (27) from Equation (28) and using the continuity property of the bilinear form, we have:

$$\|u_0 - u_n\|_{h_1} \leq \left(1 + \frac{C_B}{\lambda(n)}\right) \inf_{\chi \in S_1^n} \|u_0 - \chi\|_{H_1}. \tag{29}$$

Hence, if the inequality Equation (26) is satisfied one can conclude that convergence and unicity of the numerical solution is also ensured. In practical terms, in a first stage, the solution exists if the Babuška–Brezzi condition gives ($\lambda(n) > 0$) for a certain approach. In a second stage, by considering successively more refined approaches, the result $\lambda(n) \to \lambda_0$ confirms convergence and unicity.

Numerical determination of $\lambda(n)$

Babuška [2] presents a mathematical development showing that the determination of $\lambda(n)$ is equivalent to finding the square root of the nonzero smallest eigenvalue of the following generalized eigenvalue problem:

$$B^T A_2^{-1} B x = \mu A_1 x. \tag{30}$$

In the above relation, B can be obtained from

$$B(u_n, \nu_n) = v^{\mathrm{T}} B u, \tag{31}$$

while v and u are vectors with components $u_n \in S_1^n$ and $\nu_n \in S_2^n$. Finally, A_1 and A_2 are symmetric and positive-definite matrices associated to the norms:

$$\|u\|_{H_1}^2 = \mathrm{u}^{\mathrm{T}} A_1 \mathrm{u};\ \|\nu\|_{H_2}^2 = \mathrm{v}^{\mathrm{T}} A_2 \mathrm{v}. \tag{32}$$

The inf-sup test applied to HMSF with GFEM

In the present work, we extend the numerical inf-sup test to a hybrid mixed formulation. An important aspect of this is the identification of all matrices involved in Equation (30).

Consider then the governing equations of the HMSF derived from the stationary condition of the variational principle expressed by Equation (1):

$$\int_\Omega \delta\sigma^T \; f\sigma \; \mathrm{d}\Omega + \int_\Omega u^T (L\delta\sigma) \; \mathrm{d}\Omega - \int_{\Gamma_t} u_\Gamma^T (N\delta\sigma) \; \mathrm{d}\Gamma = \int_{\Gamma_u} u^T (N\delta\sigma) \; \mathrm{d}\Gamma, \tag{33}$$

$$\int_\Omega \delta u^T (L\sigma) \; \mathrm{d}\Omega = - \int_\Omega \delta u^T \; b \; \mathrm{d}\Omega, \tag{34}$$

$$\int_{\Gamma_t} \delta u^T (N\sigma) \; \mathrm{d}\Gamma = \int_{\Gamma_t} \delta u^t \; t \; \mathrm{d}\Gamma. \tag{35}$$

It follows that the bilinear form $B(\ldots)$ and the linear form $F(\cdot)$ can be written as:

$$\begin{aligned} B(U, V) = & \int_\Omega \delta\sigma^T \; f\sigma \; \mathrm{d}\Omega + \int_\Omega u^T (L\delta\sigma) \; \mathrm{d}\Omega - \int_{\Gamma_t} u_\Gamma^T (N\delta\sigma) \; \mathrm{d}\Gamma \\ & + \int_\Omega \delta u^T (L\sigma) \; \mathrm{d}\Omega + \int_{\Gamma_t} \delta u^T (N\sigma) \; \mathrm{d}\Gamma, \end{aligned} \tag{36}$$

$$F(V) = \int_{\Gamma_u} u^T (N\delta\sigma) \; \mathrm{d}\Gamma - \int_\Omega \delta u^T \; b \; \mathrm{d}\Omega + \int_{\Gamma_t} \delta u^T t \; \mathrm{d}\Gamma. \tag{37}$$

It can be assumed [10] that the spaces $U = (\sigma, u, u_\Gamma)$ and $V = (\delta\sigma, \delta u, \delta u_\Gamma)$ can be defined in XxY as:

$$X = \{(\sigma, u, u_\Gamma) : \sigma \in H_1; \sigma, u, u_\Gamma \in L^2(\Omega)\}, \tag{38}$$

$$Y = \{(\delta\sigma, \delta u, \delta u_\Gamma) : \delta\sigma \in H_1; \delta\sigma, \delta u, \delta u_\Gamma \in L^2(\Omega)\}. \tag{39}$$

Furthermore, the space U is endowed with the norm:

$$\begin{aligned} \|U\|_x^2 &= \|(\sigma, u, u_\Gamma)\|_x^2 \\ &= \int_\Omega \sigma^2 \; \mathrm{d}\Omega + \int_\Omega (L\sigma)^2 \; \mathrm{d}\Omega + \int_\Omega u^2 \; \mathrm{d}\Omega + \int_{\Gamma_t} u_\Gamma^2 \; \mathrm{d}\Gamma. \end{aligned} \tag{40}$$

Finally, from Equations (36) and (37), all matrices in Equation (30) can be derived.

5 Inf-sup Test: Numerical Results

Here the methodology of the inf-sup test is now applied considering the problem depicted in Figure 2. For reasons of simplification, no units for the elastic parameters and dimensions are adopted in both problems. Furthermore, the Young's modulus and Poisson's coefficient are assumed to be respectively: $E = 1000$ and $\nu = 0.3$.

The problem considers a 5 units side square plate and stressed by a uniformly distributed force of $p = 10$ units applied along its length. Essential boundary conditions are imposed over the left vertical side by constraining the plane displacement components ($u_x = 0$ and $u_y = 0$). Essential boundary conditions are also imposed over

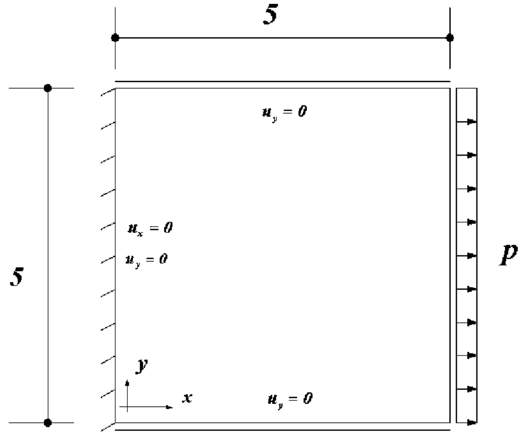

Fig. 2. Uniformly tensioned plate.

the upper and lower horizontal sides by constraining the displacements in direction $y(u_y = 0)$. The reference value of the strain energy in this problem is: 1.14.

The inf-sup test was applied to the HMSF by considering sequences of regular quadrilateral element meshes indicated as (1× 1), (2× 2), (4× 4), (8×8) and (16× 16). The numbers in the preceding nomenclature represent the amount of regular divisions in the x ey directions respectively. Polynomials as indicated by Equation (13) were often adopted as enrichment functions for the stress and displacement fields in the domain and displacement field on the boundary.

For each mesh, the value of $\lambda(n)$ was computed from the square root of the smallest eigenvalue determined on solving Equation (30). The results were then plotted on a $\log(\lambda(n)) - \log(1/N)$, ($N$ is the total number of degrees of freedom) scale. The inf-sup test was considered to be satisfied if the $(\log(1/N) - \log(\lambda(n)))$ curve showed an asymptotic behavior towards positive values.

5.1 Numerical Results

The results of the inf-sup test applied to the sets of quadrilateral element meshes without enrichment are presented in Figure 3.

Even though the inf-sup condition is not satisfied, we note that the patch test is verified and there is convergence in terms of strain energy.

With the enrichment restricted to the stress field in all nodes in domain, one can conclude that the quadrilateral element satisfies the *inf-sup test*, see Figure 4. This test confirms the results predicted by the patch-test. Another conclusion derived from the control of ‘spurious modes’ is that enrichment of the stress field eliminates such modes, improving the solvability of the problem.

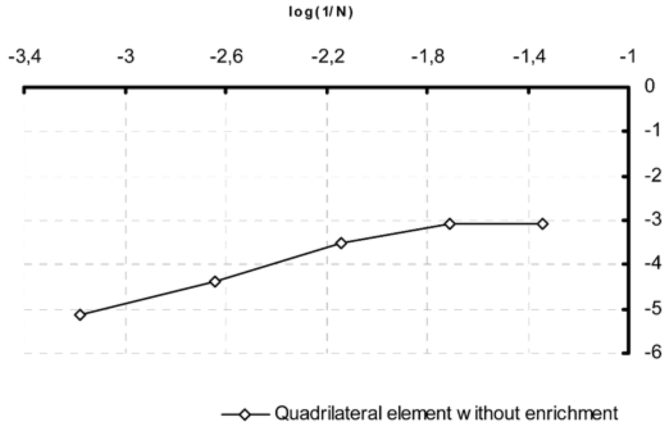

Fig. 3. Results for regular meshes without enrichment.

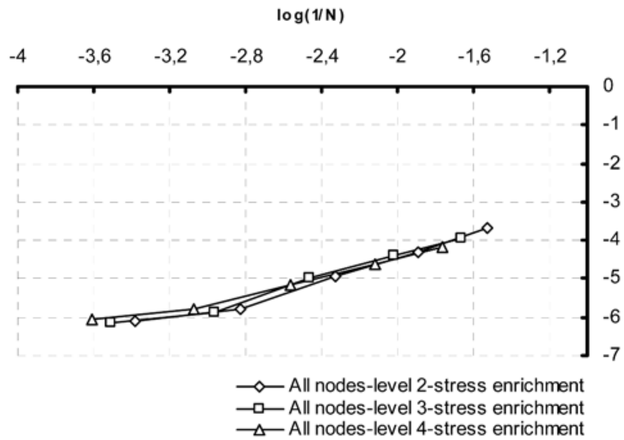

Fig. 4. Results for regular meshes with enrichment of the stress field in the domain.

Figure 5 shows results obtained by adopting selective polynomial enrichments over the stress field. It can be concluded that the number of enriched nodes, compared to the total number of nodes, affects the inf-sup test results. In fact, the responses for few enriched nodes are more comparable to the situation without enrichment.

The quadrilateral element with simultaneous enrichment of the domain fields satisfies the *inf-sup test* as illustrated in Figure 6. Spurious displacement modes are still present but not affecting solvability and convergence aspects.

In Figure 7 two other possibilities of enrichment satisfying the patch test are presented. Enrichment limited to displacement field at all nodes of the domain is not recommended (patch test fails). The enrichment of the boundary displacement field

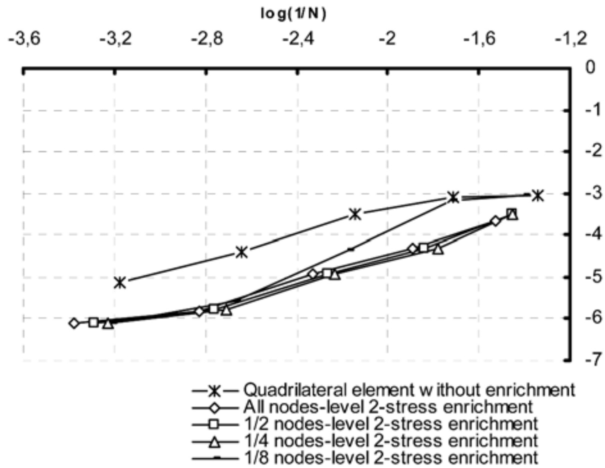

Fig. 5. Results for regular meshes with selective stress polynomial enrichment.

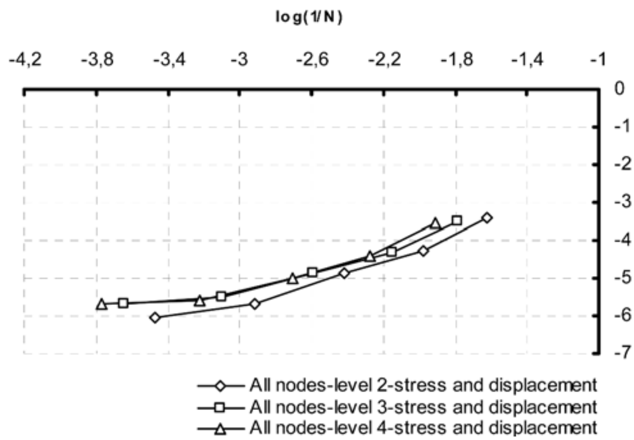

Fig. 6. Results for enrichment of the stress and displacement fields in the domain.

is effective if supplemented by enrichment of the stress fields. In such a case the patch test is verified.

Although the inf-sup test fails, the convergence strain energy is verified when the displacement field in the domain is enriched.

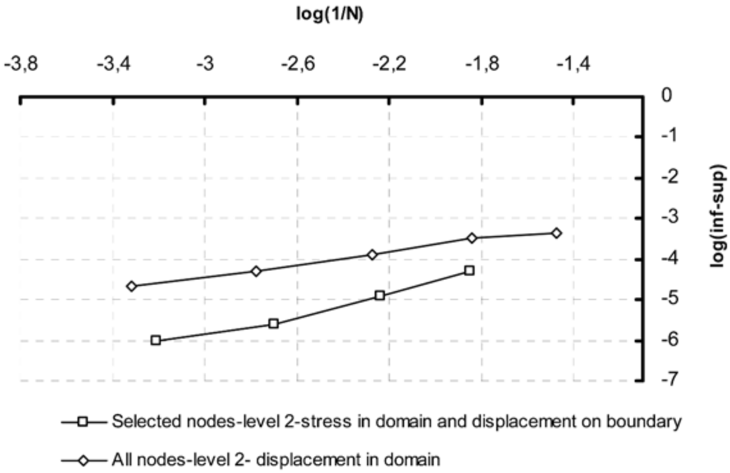

Fig. 7. Results for enrichments of displacements field in the domain or displacements on the boundary accompanied by enrichment over the stress field.

6 Conclusions

A study on the effectiveness of the Generalized Finite Element approximation spaces in the context of Hybrid Mixed Stress Formulation problems was conducted. The basic purpose was to discern appropriate enrichment combinations among multiple possibilities involving the three independent fields involved in the formulation. The issues of solvability and stability were then addressed by suggesting a patch test and by carrying out numerically the inf-sup test. Polynomial functions were selected as enrichment alternatives over regular clouds formed by quadrilateral and triangular elements.

The conclusion that the inf-sup test confirms the efficacy of the patch test as a necessary but not sufficient condition for solvability, at least in the sort of HMSF problems analyzed, is highlighted. For instance, in spite of the fact that the patch test was verified considering the basic case of partition of unity without enrichment, the inf-sup test was able to identify cinematic spurious modes and for this reason was not satisfied. Furthermore, in all situations of enrichment imposed over the basic fields wherein the patch test was not verified, the inf-sup test was not satisfied as well.

In essence, in the framework of HMSF, the enrichment over the stress field is always effective. However, enrichments over approximations for displacements in the domain and displacements on boundary always introduce spurious modes. This can be effective only if accompanied by stress field enrichment.

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